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Printed: August 24, 2015

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Kuta Software
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1.1 Solving Multi-Step Equations: Combine Like Terms

Guidance

Consider this simple problem.

Suppose you bought 3 yellow peaches, 5 white peaches and 2 red apples at a fruit stand.

You could also say that you bought 8 peaches and 2 apples because:

$3 \ peaches + 5 \ peaches + 2 \ apples = 8 \ peaches + 2 \ apples.$

However, you couldn't say that you bought 10 peaches. When you determine how many peaches you bought, you can add 3 yellow peaches to 5 white peaches, but you cannot add the 2 red apples to that total. That is because apples and peaches are *different* kinds of fruit.

When you add or subtract the terms in an expression, you can only combine *like terms* .

Consider this expression:

$3p + 5p + 2a$

This expression represents the problem above. The variable p stands for peaches. The variable a stands for apples.

Just as you can combine the yellow peaches with the white peaches because they are both peaches, you can combine $3p$ and $5p$ because they are like terms. Each of those terms includes the same variable, p. However, you could not combine $5p$ with $2a$, because they are not like terms. Each of those terms has a different variable.

Like terms **are terms that contain the same variable, and these terms can be combined.**

This shows how you could simplify the expression above by combining like terms:

$3p + 5p + 2a = 8p + 2a.$

Let's take a look at how we can apply what we know about combining like terms to solving algebraic equations.

Solve for r: $5r - r - 9 = 15.$

First, combine the like terms—$5r$ and r on the left side of the equation. It may help to remember that $r = 1r$.

$$\begin{aligned}5r - r - 9 &= 15\\(5r - 1r) - 9 &= 15\\4r - 9 &= 15\end{aligned}$$

Notice that 9 cannot be combined with $4r$ because they are *not* like terms.

Now that we have combined like terms, we can solve the equation as we would solve any two-step equation.

Our next step is to isolate the term with the variable, $4r$, on one side of the equation. Since 9 is *subtracted* from $4r$, we should add 9 to both sides of the equation to isolate that term.

$$\begin{aligned}4r - 9 &= 15\\4r - 9 + 9 &= 15 + 9\\4r + (-9 + 9) &= 24\\4r + 0 &= 24\\4r &= 24\end{aligned}$$

Since $4r$ means $4 \times r$, we should divide each side of the equation by 4 to get the r by itself on one side of the equation.

$$\begin{aligned}4r &= 24\\\frac{4r}{4} &= \frac{24}{4}\\1r &= 6\\r &= 6\end{aligned}$$

The value of r is 6.

Solve for n: $6n + 3 + 8n + 2 = 33$.

First, combine the like terms on the left side of the equation. The terms $6n$ and $8n$ are like terms since each has the same variable, n. The numbers 3 and 2 are also like terms, so they can be combined as well.

Use the ***commutative property of addition*** to help you reorder the terms being added. This property states that terms can be added in any order. Then use the ***associative property of addition*** to group the terms so like terms are being added. The associative property of addition states that the grouping of terms being added does not matter.

$$\begin{aligned}6n + 3 + 8n + 2 &= 33\\6n + (3 + 8n) + 2 &= 33\\6n + (8n + 3) + 2 &= 33\\(6n + 8n) + (3 + 2) &= 33\end{aligned}$$

Now, that the like terms are grouped together with parentheses, combine them.

$$\begin{aligned}(6n + 8n) + (3 + 2) &= 33\\14n + 5 &= 33\end{aligned}$$

Now, we can solve as we would solve any two-step equation.

The next step is to isolate the term with the variable, $14n$, on one side of the equation. Since 5 is *added* to $14n$, we should subtract 5 from both sides of the equation to do this.

$$\begin{aligned} 14n+5 &= 33 \\ 14n+5-5 &= 33-5 \\ 14n+0 &= 28 \\ 14n &= 28 \end{aligned}$$

Since $14n$ means $14 \times n$, we should divide each side of the equation by 14 to get the n by itself on one side of the equation.

$$\begin{aligned} 14 &= 28 \\ \frac{14n}{14} &= \frac{28}{14} \\ 1n &= 2 \\ n &= 2 \end{aligned}$$

The value of n is 2.

Example A

$3p+5p+2=18$

Solution: $p=2$

Example B

$13x+6x+14a-9a$

Solution: $19x+5a$

Example C

$3x+5x+9x-7=44$

Solution: $x=3$

Practice

Directions: Practice combining like terms as you simplify each expression.

1. $8x+3x+2$
2. $5y-3y+8$
3. $6x+9x+x-4$
4. $9x+4x-8+2x$
5. $2y-10y+16$

6. $3x+4x+5-6+2x$

Directions: Combine like terms and solve each equation.

7. $8x+3x+2=24$
8. $5y+2y+6=48$
9. $4x-6x+3=13$
10. $7y-10y+6=9$
11. $5x+8x+4=30$
12. $9a+3a-4=44$
13. $7a+4a+6=83$
14. $12x-14x+3=19$
15. $10y-16y+5=35$

1.2 Solving Multi-Step Equations: Distributive Property

Guidance

Solving Multi-Step Equations by Using the Distributive Property

When faced with an equation such as $2(5x+9)=78$, the first step is to remove the parentheses. There are two options to remove the parentheses. You can apply the Distributive Property or you can apply the Multiplication Property of Equality. This Concept will show you how to use the Distributive Property to solve multi-step equations.

Example A

Solve for $x : 2(5x+9)=78$.

Solution: Apply the Distributive Property: $10x+18=78$.

Apply the Addition Property of Equality: $10x+18-18=78-18$.

Simplify: $10x=60$.

Apply the Multiplication Property of Equality: $10x \div 10 = 60 \div 10$.

The solution is $x=6$.

Check: Does $10(6)+18=78$? Yes, so the answer is correct.

Example B

Solve for n when $2(n+9)=6n$.

Solution:

$$\begin{aligned}
&2(n+9)=5n\\
&2\cdot n+2\cdot 9=5n\\
&2n+18=5n\\
&-2n+2n+18=-2n+5n\\
&18=3n\\
&\frac{1}{3}\cdot 18=\frac{1}{3}\cdot 3n\\
&6=n
\end{aligned}$$

Checking the answer:

$$\begin{aligned}
&2(6+9)=5(6)\\
&2(15)=30\\
&30=30
\end{aligned}$$

Example C

Solve for d when $3(d+15)-18d=0$.

Solution:

$$\begin{aligned}
&3(d+15)-18d=0\\
&3\cdot d+3\cdot 15-18d=0\\
&3d+45-18d=0\\
&-15d+45=0\\
&-15d+45-45=0-45\\
&-15d=-45\\
&-\frac{1}{15}\cdot -15d=-\frac{1}{15}\cdot -45\\
&d=3
\end{aligned}$$

Checking the answer:

$$\begin{aligned}
&3(3+15)-18(3)=0\\
&3(3+15)-18(3)=0\\
&3(18)-18(3)=0\\
&54-54=0\\
&0=0
\end{aligned}$$

Guidance

Solve for x when $3(2x+5)+2x=7$.

Solution:

Step 1: Apply the Distributive Property.

$$\begin{aligned}
&3(2x+5)+2x=7\\
&3\cdot 2x+3\cdot 5+2x=7\\
&6x+15+2x=7
\end{aligned}$$

Step 2: Combine like terms.

$$\begin{aligned}
&6x+15+2x=7\\
&8x+15=7
\end{aligned}$$

Step 3: Isolate the variable and its coefficient by using the Addition Property.

$$8x+15=7$$
$$8x+15-15=7-15$$
$$8x=-8$$

Step 4: Isolate the variable by applying the Multiplication Property.

$$8x=-8$$
$$\frac{1}{8}\cdot 8x=-8\cdot\frac{1}{8}$$
$$\frac{1}{8}\cdot 8x=-8\cdot\frac{1}{8}$$
$$x=-1$$

Step 5: Check your answer.

Substitute $x=-1$ into $3(2x+5)=7$.

$3(2(-1)x+5)+2(-1)=3(-2+5)-2=3(3)-2=9-2=7.$

Therefore, $x=-1$.

Practice

1. $3(x-1)-2(x+3)=0$
2. $7(w+20)-w=5$
3. $9(x-2)=3x+3$
4. $2\left(5a-\frac{1}{3}\right)=\frac{2}{7}$
5. $\frac{2}{9}\left(i+\frac{2}{3}\right)=\frac{2}{5}$
6. $4\left(v+\frac{1}{4}\right)=\frac{35}{2}$
7. $22=2(p+2)$
8. $-(m+4)=-5$
9. $48=4(n+4)$
10. $\frac{6}{5}\left(v-\frac{3}{5}\right)=\frac{6}{25}$
11. $-10(b-3)=-100$
12. $6v+6(4v+1)=-6$
13. $-46=-4(3s+4)-6$
14. $8(1+7m)+6=14$
15. $0=-7(6+3k)$
16. $35=-7(2-x)$
17. $-3(3a+1)-7a=-35$
18. $-2\left(n+\frac{7}{3}\right)=-\frac{14}{3}$
19. $-\frac{59}{60}=\frac{1}{6}\left(-\frac{4}{3}r-5\right)$
20. $\frac{4y+3}{7}=9$
21. $(c+3)-2c-(1-3c)=2$
22. $5m-3[7-(1-2m)]=0$

1.3 Solving Multi-Step Equations: Involving Rational Numbers

Guidance

Rational numbers include integers, fractions, and terminating decimals. Some equations may require you to work with a combination of these kinds of numbers. If you know how to solve an equation, you can apply the same rules when you work with rational numbers.

Take a look at this dilemma.

Solve for b: $-6\left(1-\frac{b}{12}\right)=\frac{2}{3}$

This problem involves two different kinds of rational numbers: integers (-6 and 1) and fractions $\left(\frac{b}{12}\right.$ and $\left.\frac{2}{3}\right)$. You will need to know how to compute with fractions as well as how to compute with integers in order to solve this.

Apply the distributive property to the left side of the equation. Multiply each of the two numbers inside the parentheses by -6 and then subtract those products.

$$\begin{aligned}
-6\left(1-\frac{b}{12}\right) &= \frac{2}{3}\\
(-6\times 1)-\left(-6\times\frac{b}{12}\right) &= \frac{2}{3}\\
-6-\left(\frac{-6}{1}\times\frac{1}{12}b\right) &= \frac{2}{3}\\
-6-\left(\frac{-6}{12}b\right) &= \frac{2}{3}\\
-6-\left(-\frac{6}{12}b\right) &= \frac{2}{3}\\
-6+\left(\frac{6}{12}b\right) &= \frac{2}{3}
\end{aligned}$$

You may recognize immediately that the variable term, $\frac{6}{12}b$, could be simplified as $\frac{1}{2}b$ or $\frac{b}{2}$. You can wait until the problem is finished before simplifying, but if you recognize this fact, it makes sense to simplify that now. It will only make the computation easier, so simplify the variable term as $\frac{1}{2}b$.

$$\begin{aligned}
-6+\frac{6}{12}b &= \frac{2}{3}\\
-6+\frac{1}{2}b &= \frac{2}{3}
\end{aligned}$$

Now, we can solve as we would solve any two-step equation. To get $\frac{1}{2}b$ by itself on one side of the equation, we can subtract -6 from both sides.

$$
\begin{aligned}
-6+\frac{1}{2}b &= \frac{2}{3}\\
-6-(-6)+\frac{1}{2}b &= \frac{2}{3}-(-6)\\
-6+6+\frac{1}{2}b &= \frac{2}{3}+6\\
0+\frac{1}{2}b &= 6\frac{2}{3}\\
\frac{1}{2}b &= 6\frac{2}{3}
\end{aligned}
$$

To get b by itself, you will need to divide each side of the equation by $\frac{1}{2}$. Remember, that is the same as multiplying each side by $\frac{2}{1}$. Also, keep in mind that you will need to rewrite the mixed number $6\frac{2}{3}$ as an improper fraction $\left(\frac{20}{3}\right)$ before multiplying by $\frac{2}{1}$.

$$
\begin{aligned}
\frac{1}{2}b &= 6\frac{2}{3}\\
\frac{1}{2}b\times\frac{2}{1} &= 6\frac{2}{3}\times\frac{2}{1}\\
\frac{\cancel{1}}{\cancel{2}}b\times\frac{\cancel{2}}{\cancel{1}} &= \frac{20}{3}\times\frac{2}{1}\\
1b &= \frac{40}{3}\\
b &= 13\frac{1}{3}
\end{aligned}
$$

The value of b is $13\frac{1}{3}$.

Now, let's solve an algebraic equation that includes both decimals and fractions.

Solve for k: $0.4k+0.2k+\frac{3}{10}=\frac{9}{10}$.

First, add the like terms $0.4k$ and $0.2k$ on the left side of the equation.

$$
\begin{aligned}
0.4k+0.2k+\frac{3}{10} &= \frac{9}{10}\\
0.6k+\frac{3}{10} &= \frac{9}{10}
\end{aligned}
$$

The next step is to isolate the term with the variable, $0.6k$, on one side of the equation. We can do this by subtracting $\frac{3}{10}$ from both sides of the equation.

$$
\begin{aligned}
0.6k+\frac{3}{10} &= \frac{9}{10}\\
0.6k+\frac{3}{10}-\frac{3}{10} &= \frac{9}{10}-\frac{3}{10}\\
0.6k+0 &= \frac{6}{10}\\
0.6k &= \frac{6}{10}
\end{aligned}
$$

Since $0.6k$ **means** $0.6 \times k$**, we should divide each side of the equation by 0.6 to get the** k **by itself on one side of the equation.** This will involve dividing a fraction, $\frac{6}{10}$, by a decimal, 0.6. To do this, you will need to convert both numbers to the same form. One way to do this would be to convert the fraction $\frac{6}{10}$ to a decimal. $\frac{6}{10}$ is read as "six tenths," so the decimal form of $\frac{6}{10}$ is 0.6.

$$\begin{aligned}
0.6k &= \frac{6}{10}\\
0.6k &= 0.6\\
\frac{0.6k}{0.6} &= \frac{0.6}{0.6}\\
1k &= 1\\
k &= 1
\end{aligned}$$

The value of k **is 1.**

Example A

$8.7n - 3.2n + 4.5 = 37.5$

Solution: $n = 6$

Example B

$\frac{x}{.9} = -72$

Solution: $x = -64.8$

Example C

$17x - 22.3x + 4 = -33.1$

Solution: $x = 7$

Now let's go back to the dilemma at the beginning of the Concept.

First, write an equation to show what you know and what you don't know.

Saturday = 90 minutes

Monday $= t -$ **missing time**

Tuesday $= \frac{1}{2}t -$ **half the time of Monday**

Total time = 3 hours

$90 + t + \frac{1}{2}t = 3\ hours$

First, convert hours to minutes.

$90 + t + \frac{1}{2}t = 180\ minutes$

Now we can solve the equation.

Monday's time = 60 minutes

Tuesday's time = 30 minutes

Guided Practice

Here is one for you to try on your own.

For a long-distance call, Guillermo's phone company charges \$0.10 for the first minute and \$0.05 for each minute after that. Guillermo was charged \$1.00 for a long distance call he made last Friday.

a. Write an algebraic equation that could be used to represent *m*, the length in minutes of Guillermo's \$1.00 long-distance call.

b. Determine how many minutes his \$1.00 long-distance call lasted.

Solution

Consider part *a* first.

You know that the phone company charges \$0.10 for the first minute and \$0.05 for each minute after that. How could you represent that? If the company charged \$0.05 for each minute the call lasted, you could represent that as $0.05 \times m$. However, the company charges \$0.10 for the first minute and \$0.05 for each minute *after* that first minute.

So, a 1-minute call will cost: $\$0.10 + (\$0.05 \times 0) = \$0.10 + \$0.00 = \$0.10$.

A 2-minute call will cost: $\$0.10 + (\$0.05 \times 1) = \$0.10 + \$0.05 = \$0.15$.

A 3-minute call will cost: $\$0.10 + (\$0.05 \times 2) = \$0.10 + \$0.10 = \$0.20$.

Notice that the number you multiply by \$0.05 is always 1 less than the length of the call, in minutes. If *m* represents the length of a call in minutes, then this could be represented as: $\$0.10 + \$0.05 \times (m-1)$.

Write an equation that could be used to represent the cost of Guillermo's \$1.00 call.

$$\begin{array}{ccccc} (\text{cost of first minute}) + & (\text{cost of each minute after first minute}) & = & (\text{total cost}) \\ \downarrow \quad\quad \downarrow & \downarrow & \downarrow & \downarrow \\ 0.10 \quad\quad + & 0.05(m-1) & = & 1.00 \end{array}$$

So, the equation $0.10 + 0.05(m-1) = 1.00$ represents the number of minutes that Guillermo's \$1.00 phone call lasted.

Next, consider part *b* .

To find the length of the \$1.00 call in minutes, solve the equation for *m*. First, apply the distributive property to the right side of the equation.

$$\begin{aligned} 0.10 + 0.05(m-1) &= 1.00 \\ 0.10 + (0.05 \times m) - (0.05 \times 1) &= 1.00 \\ 0.10 + 0.05m - 0.05 &= 1.00 \end{aligned}$$

Use the commutative property to rearrange the terms being added so it is easier to see how to combine the like terms. Then combine the like terms.

$$\begin{aligned} (0.10 + 0.05m) - 0.05 &= 1.00 \\ (0.05m + 0.10) - 0.05 &= 1.00 \\ 0.05m + (0.10 - 0.05) &= 1.00 \\ 0.05m + 0.05 &= 1.00 \end{aligned}$$

Now, solve as you would solve any two-step equation. First, subtract 0.05 from both sides of the equation.

$$\begin{aligned}
0.05m + 0.05 &= 1.00\\
0.05m + 0.05 - 0.05 &= 1.00 - 0.05\\
0.05m + 0 &= 0.95\\
0.05m &= 0.95
\end{aligned}$$

Next, divide both sides of the equation by 0.05.

$$\begin{aligned}
0.05m &= 0.95\\
\frac{0.05m}{0.05} &= \frac{0.95}{0.05}\\
1m &= 19\\
m &= 19
\end{aligned}$$

The value of m is 19, so the $1.00 call lasted for 19 minutes.

Practice

Directions: Solve each equation to find the value of the variable.

1. $7n - 3.2n + 6.5 = 17.9$
2. $0.2(3 + p) = -5.6$
3. $s + \frac{3}{5} + \frac{1}{5} = 1\frac{2}{5}$
4. $j + \frac{5}{7} - \frac{1}{7} = 9\frac{4}{7}$
5. $\frac{3}{4}\left(g - \frac{1}{2}\right) = \frac{1}{8}$
6. $-2\left(1 - \frac{a}{4}\right) = \frac{1}{8}$
7. $0.09y - 0.08y = .005$
8. $.28x + 4x = -8.56$
9. $\frac{1}{3}y + \frac{1}{3}y = 8$
10. $\frac{1}{4}x + \frac{1}{3} = \frac{2}{3}$
11. $\frac{1}{2}x = 18$
12. $.9x = 54$
13. $.6x + 1 = 19$
14. $\frac{1}{4}x + 2 = 19$
15. $9.05x = 27.15$

1.4 Solving Equations with Variables on Both Sides: Solving an Equation

Guidance

Do you remember how to solve a basic equation?

Consider the problem, $12+t=30$.

The strategy for solving this equation is to use inverse operations to isolate the variable, t, on one side of the equation. Since 12 is added to t, you would subtract 12 from both sides of the equation to get t by itself.

$$\begin{aligned} 12+t &= 30 \\ 12-12+t &= 30-12 \\ 0+t &= 18 \\ t &= 18 \end{aligned}$$

What if you needed to solve an equation like this?

$12+t=30+3t$

How do we solve an equation with variables on both sides of the equation?

To solve an equation that has the same variable on both sides of it, you will use the same basic strategy you already know. You will use inverse operations to isolate the variables on one side of the equation. You will do this by using inverse operations to get all the terms that include variables on one side of the equation and using inverse operations to get all the numerical terms on the other side. Once you do this, you will be able to solve for the variable.

Think about it logically and it makes perfect sense. You get the variables together on one side of the equation, and then you get the numbers together on the other side of the equation. Once you have done this, you can combine like terms and solve for the value of the variable.

Solve for t: $12+t=30+3t$.

The variable, t, is on both sides of the equation. We can treat terms with variables the same as we treat numbers. That is, we can use inverse operations to get all of the terms with the variable, t, on one side of the equation. So, just as we could subtract 12 from both sides of the equation to get all of the numerical terms on the right side of the equation, we could subtract t from both sides of the equation to get all of the terms with variables on the *right* side of the equation.

Alternatively, we could subtract $3t$ from both sides of the equation to get all of the terms with variables on the *left* side of the equation. It does not matter which of these steps we take. Either will result in the correct answer. However, since it is easier to subtract $3t-t$ than it is to subtract $t-3t$, let's subtract t from both sides of the equation. Remember, $t=1t$.

$$\begin{aligned} 12+t &= 30+3t \\ 12+t-t &= 30+3t-t \\ 12+0 &= 30+2t \\ 12 &= 30+2t \end{aligned}$$

Now, the only variable is on the right side of the equation. So, let's get all the numerical terms on the left side of the equation. Since 30 is added to $2t$, we can get $2t$ by itself on the right side of the equation by subtracting 30 from both sides of the equation. Remember, subtracting 30 from 12 is the same as adding -30 to 12.

$$\begin{aligned}12 &= 30+2t\\ 12-30 &= 30-30+2t\\ 12+(-30) &= 0+2t\\ -18 &= 2t\end{aligned}$$

Now, we can use inverse operations to get the t by itself on one side of the equation. Let's divide both sides by 2 to do that. Doing so involves dividing a negative integer, -18, by a positive integer, 2.

$$\begin{aligned}-18 &= 2t\\ \frac{-18}{2} &= \frac{2t}{2}\\ -9 &= 1t\\ -9 &= t\end{aligned}$$

The value of t is -9.

Sometimes, an equation will have a set of parentheses and variables on both sides of the equation. The ***distributive property*** is very helpful in solving these equations.

Solve for a: $4a+16 = 13a-(2a+3a)$

Our first step should be to simplify the expression on the right side of the equation. According the order of operations, we should combine the like terms inside the parentheses first. Then we can simplify the rest of that expression, like this:

$$\begin{aligned}4a+16 &= 13a-(2a+3a)\\ 4a+16 &= 13a-5a\\ 4a+16 &= 8a\end{aligned}$$

Now, we notice that the variable, a, is on both sides of the equation. We can use inverse operations to get all of the terms with the variable, a, on one side of the equation. Since there is a number on the left side of the equation and there is no number on the right side of the equation, it is easier to try to get all of the variable terms on the right side of the equation. We can get all of the variable terms on the right side of the equation by subtracting $4a$ from both sides.

$$\begin{aligned}4a+16 &= 8a\\ 4a-4a+16 &= 8a-4a\\ 0+16 &= 4a\\ 16 &= 4a\end{aligned}$$

Now, the only term with a variable, $4a$, is on the right side of the equation. The only numerical term, 16, is on the left side of the equation. To solve for a, we can divide both sides of the equation by 4.

$$\begin{aligned} 16 &= 4a \\ \frac{16}{4} &= \frac{4a}{4} \\ 4 &= 1a \\ 4 &= a \end{aligned}$$

The value of a is 4.

Example A

$6x+3=9x+6$

Solution: $x=-1$

Example B

$4x+x+2=10x-13$

Solution: $x=3$

Example C

$8y+2y=20y+10$

Solution: $y=-1$

Guided Practice

Here is one for you to try on your own.

$6x+1=8x+3$

Let's break down working on this problem. First, we need to move the terms with variables to the same side of the equation. Let's move the $6x$. We can do this by using an inverse operation. We subtract $6x$ from both sides of the equation.

$$\begin{aligned} 6x+6x+1 &= 8x-6x+3 \\ 1 &= 2x+3 \end{aligned}$$

Here we performed the inverse operation and then simplified the equation. Now we can solve this just as we would any other two step equation. Take a look and be sure to watch out if you end up working with negative numbers. Don't mix up the signs!

$$\begin{aligned} 1 &= 2x+3 \\ 1-3 &= 2x+3-3 \\ -2 &= 2x \\ -1 &= x \end{aligned}$$

The value of x is -1.

Practice

Directions: Solve each equation with variables on both sides.

1. $6x = 2x + 16$
2. $5y = 3y + 12$
3. $4y = y - 18$
4. $8x = 10x + 20$
5. $7x = 4x + 24$
6. $9y = 2y - 21$
7. $-6x + 22 = 5x$
8. $15y = 9y + 36$
9. $14x = 10x - 40$
10. $19y = 4y - 30$
11. $18x = 2x - 32$
12. $4x + 1 = 2x + 5$
13. $6x + 4 = 4x + 10$
14. $8x + 3 = 5x + 9$
15. $10y - 4 = 6y - 12$
16. $8x - 5 = 10x - 13$
17. $12y - 8 = 14y + 14$
18. $18x - 5 = 20x + 19$
19. $-20y + 8 = -8y - 4$

Directions: Solve each equation with variables on both sides, by simplifying each equation first by using the distributive property.

20. $2(x + 3) = 8x$
21. $3(x + 5) = -2x$
22. $9y = 4(y - 5)$

1.5 Solving Equations with Variables on Both Sides: Identities and Equations with No Solution

Guidance

An equation has no solution if no value of the variable makes the equation true. The equation $2x = 2x + 1$ has no solution. An equation that is true for every value of the variable is an identity. The equation $2x = 2x$ is an identity.

Identities and Equations with No Solutions

a. Solve $10 - 8a = 2(5 - 4a)$.

b. Solve $6m - 5 = 7m + 7 - m$.

Practice

1. 6x - 2 = x + 13 2. 5y - 3 = 2y + 12 3. 4k - 3 = 3k + 4 4. 5m + 3 = 3m + 9 5. 8 - x = 2x - 1 6. 2n - 5 = 8n + 7 7. 3a + 4 = a + 18 18. 6b + 14 = -7 - b 19. 5a - 14 = -5 + 8a 10. 3 + 4x = 3x + 6 11. 30 - 7z = 10z - 4 12. 8x - 3 = 7x + 2 13. -36 + 2w = -8w + w 14. 4p - 10 = p + 3p - 2p

1.6 Literal Equations: Rewriting Literal Equations and Formulas

Guidance

A literal equation is any equation that involves more than one variable or letter.

Examples of Literal Equations:

$A = bh$ (the formula for finding the area of a rectangle)

$E = mc^2$(Einstein's Theory of Relativity)

When given an literal equation, you will often be asked to solve the equation for a given variable. The goal is to isolate the given variable. The process is the same process used to solve linear equations; the only difference is there are more variables.

Solve $A = bh$ for b

Since h is multiplied times b, you must divide both sides by hi in order to isolate b

$$A = bh$$

$$\frac{A}{h} = \frac{b\cancel{h}}{\cancel{h}}$$

$$\frac{A}{h} = b$$

FIGURE 1.1

Solve $P = 2l + 2w$ for w

First, you subtract 2l from both sides, then divide both sides by 2 to isolate w.

Solve $Q = \frac{(c+d)}{2}$ for d

Since (c + d) is divided by 2, you must first multiply both sides of the equation by 2. Then you have to subtract c from both sides in order to isolate d.

Practice

1. V = Bh *for h*

$$P = 2l + 2w$$

$$\begin{array}{rcl} P & = & 2l + 2w \\ -2l & & -2l \\ \hline P - 2l = & & 2w \end{array}$$

$$\frac{P-2l}{2} = \frac{\cancel{2}w}{\cancel{2}}$$

$$\frac{P-2l}{2} = w$$

FIGURE 1.2

2. *P - RB for B*
3. *C = 2pr for r*
4. *V = LWH for H*
5. *I = Prt for r*
6. *S = 2prh for r*
7. *y = mx + b for x*
8. *A = P (rt -1) for t*
9. $F = \frac{9}{5}C - 32$ *for C*
10. $C = \frac{5}{9}(F - 32)$ *for F*

11. A rectangular solid has a base with length 6 cm and width 4 cm. If the volume of the solid is 72 cubic centimeters (cm^2), find the height of the solid. Use formula V = Bh

12. A principal of $2,000 was invested in a savings account for 4 years. If the interest earned for that period was $480, what was the interest rate? Use formula: I = Prt

13. If the perimeter of a rectangle is 60 ft and its length is 18 ft, find its width. Use formula P = 2l + 2w

14. A cylinder has a radius of 4 inches (in). If the volume is 144 cubic inches (in^3) what is the height of the cylinder? Use formula V = Bh

$$Q = \frac{(c+d)}{2}$$

$$2 \cdot Q = \frac{(c+d)}{\cancel{2}} \cdot \cancel{2}$$

$$2Q = c + d$$

$$\begin{array}{rcrcl} 2Q & & = & & c+d \\ & -c & & -c & \\ \hline 2Q & -c & = & & d \end{array}$$

$$2Q - c = d$$

FIGURE 1.3

1.7 Solving Proportions: Solving A Proportion Using the Multiplication Property

Guidance

Remember ratios? Think back to what you have already learned about ratios.

A *ratio* represents a comparison between two quantities. We can write ratios in fraction form, using a colon or using the word "to".

We also learned that ***equivalent ratios*** are two ratios that are equal. The numbers in the ratios may not be the same, but the comparison of quantities is the same.

Equivalent ratios are directly related to proportions.

What is a proportion?

A *proportion* states that two ratios are equivalent. Here is an example of a proportion.

$\frac{1}{2} = \frac{2}{4}$

This proportion shows that the ratios $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent. In other words, a proportion is made up of two equivalent ratios.

In the situation above, we knew all of the parts of the two ratios that made up the proportion. Sometimes, we will know three of the numbers, but not four of them. When this happens, we have to use a variable and solve for the missing number.

Look at this proportion.

$\frac{1}{2} = \frac{n}{12}$

Notice that the first term of the second ratio—its numerator—is a variable. Suppose we wanted to find the value of this variable. We could do that by using ***proportional reasoning.***

Proportional reasoning is when we figure out a missing value in a proportion by thinking about the relationship between the numbers in the two ratios.

Use proportional reasoning to solve for $n: \frac{1}{2} = \frac{n}{12}$.

To figure this out, we need to figure out a relationship between either numerators or denominators. The proportion does not show the relationship between the first terms in the ratios—the numerators of the fractions. However, we can determine the relationship between the second terms in the ratios—the denominators of the fractions.

We can ask ourselves: "what number, when multiplied by 2, results in 12?"

Since $2 \times 6 = 12$, we can multiply both the numerator and the denominator of $\frac{1}{2}$ by 6 to find the value of n.

$\frac{1}{2} = \frac{1 \times 6}{2 \times 6} = \frac{6}{12} = \frac{n}{12}$

This shows that when the second term (the denominator) of the ratio is 12, the first term (the numerator) is 6.

The value of n is 6.

Mental math is very helpful when looking at proportional reasoning. When you can figure out the relationship between numbers, then you can solve for the missing value of the variable.

Use proportional reasoning to solve for $x: \frac{15}{35} = \frac{x}{7}$.

Which relationship can we use to figure out the variable? This proportion does not show the relationship between the first terms in the ratios—the numerators of the fractions. **We need to find the relationship between the second terms in the ratios—the denominators of the fractions.**

We can ask ourselves, "what number can we divide 35 to get 7?"

Since $35 \div 5 = 7$, we can divide both the numerator and the denominator of $\frac{15}{35}$ by 5 to find the value of x.

$\frac{15}{35} = \frac{15 \div 5}{35 \div 5} = \frac{3}{7} = \frac{x}{7}$

This shows that when the second term (the denominator) of the ratio is 7, the first term (the numerator) is 3.

The value of x is 3.

Use proportional reasoning to find the value of each unknown variable.

Example A

$\frac{2}{3} = \frac{x}{6}$

Solution: $x = 4$

Example B

$\frac{12}{24} = \frac{24}{z}$

Solution: $z = 48$

Example C

$\frac{y}{5} = \frac{14}{20}$

Solution: 3.5

Guided Practice

Here is one for you to try on your own.

Use equal ratios to solve for z : $\frac{z}{9} = \frac{32}{36}$.

Answer

The problem does not show the relationship between the first terms in the ratios—the numerators of the fractions.

We need to find the relationship between the second terms in the ratios—the denominators of the fractions.

We can ask ourselves "what number, when multiplied by 9, results in 36?"

$$9 \times 4 = 36$$
$$\frac{z}{9} = \frac{z \times 4}{9 \times 4} = \frac{32}{36}.$$

From this, we can see that $z \times 4 = 32$.

We must ask ourselves, "what number, when multiplied by 4, results in 32?"

$8 \times 4 = 32$ **, so** $z = 8$.

This is our answer.

Practice

Directions: Look at each pair of ratios. Tell whether or not these ratios form a proportion.

1. $\frac{1}{2}$ *and* $\frac{4}{8}$
2. $\frac{3}{7}$ *and* $\frac{6}{14}$
3. $\frac{5}{2}$ *and* $\frac{10}{6}$
4. $\frac{3}{1}$ *and* $\frac{9}{3}$
5. $\frac{2}{9}$ *and* $\frac{1.5}{4.5}$
6. $\frac{4}{9}$ *and* $\frac{8}{10}$
7. $\frac{1}{4}$ *and* $\frac{5}{20}$
8. $\frac{3}{4}$ *and* $\frac{9}{10}$

Directions: Use proportional reasoning to find the value of the variable in each proportion.

9. $\frac{1}{4} = \frac{a}{20}$
10. $\frac{15}{30} = \frac{x}{2}$
11. $\frac{2}{9} = \frac{n}{63}$
12. $\frac{z}{7} = \frac{12}{21}$
13. $\frac{3}{5} = \frac{t}{60}$
14. $\frac{k}{72} = \frac{5}{12}$
15. $\frac{x}{32} = \frac{4}{8}$

1.8 Solving Proportions: Solving a Proportion Using the Cross Product Property

Guidance

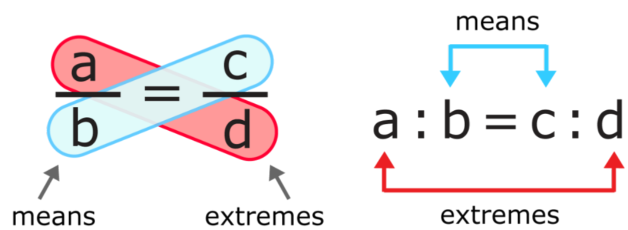
Previously we learned that a ***proportion*** **states that two ratios are equivalent.** Here are two proportions.

$\frac{a}{b} = \frac{c}{d}$ or $a : b = c : d$

In a proportion, the ***means*** are the two terms that are closest together when the proportion is written with colons. So, in $a : b = c : d$, the means are b and c.

The ***extremes*** are the terms in the proportion that are furthest apart when the proportion is written with colons. So, in $a : b = c : d$, the extremes are a and d.

The diagram below shows how to identify the means and the extremes in a proportion.



In the last lesson, you learned how to solve proportions by using proportional reasoning. We can also solve a proportion for a variable in another way. This is where the cross products property of proportions comes in.

What is the Cross Products Property of Proportions?

The *Cross Products Property of Proportions* states that the product of the means is equal to the product of the extremes. You can find these cross products by cross multiplying, as shown below.

$$\frac{a}{b} = \frac{c}{d}$$
$$b \cdot c = a \cdot d$$

$\frac{a}{4} = \frac{6}{8}$

To solve this, we can multiply the means and the extremes.

$$a \cdot 8 = 4 \cdot 6$$
$$8a = 24$$

Next, we solve the equation for the missing variable. To do this, we use the inverse operation. Multiplication is in the problem, so we use division to solve it. We divide both sides by 8.

$$\frac{8a}{8} = \frac{24}{8}$$
$$a = 3$$

Our answer is 3.

Solve for each variable in the numerator by using cross products.

Example A

$\frac{x}{5} = \frac{6}{10}$

Solution:$x = 3$

Example B

$\frac{a}{9} = \frac{15}{27}$

Solution:$a = 5$

Example C

$\frac{b}{4} = \frac{12}{16}$

Solution:$b = 3$

Guided Practice

Here is one for you to try on your own.

The ratio of boys to girls in the school chorus is 4 to 5. There are a total of 20 boys in the chorus. How many total students are in the chorus?

Answer

The ratio given, 4 to 5, compares boys to girls. However, the question asks for the total number of students in the chorus.

One way to set up a proportion for this problem would be to write two equivalent ratios, each comparing boys to total students.

The ratio of boys to girls is 4 to 5. We can use this ratio to find the ratio of boys to total students.

$\frac{boys}{total} = \frac{boys}{boys+girls} = \frac{4}{4+5} = \frac{4}{9}$

You know that there are 20 boys in the chorus. The total number of students is unknown, so represent that as x.

$\frac{boys}{total} = \frac{20}{x}$

Get those ratios equal to form a proportion. Then cross multiply to solve for x.

$$\begin{aligned}
\frac{4}{9} &= \frac{20}{x} \\
9 \cdot 20 &= 4 \cdot x \\
180 &= 4x \\
\frac{180}{4} &= \frac{4x}{4} \\
45 &= x
\end{aligned}$$

So, there are a total of 45 students in the school chorus.

Practice

Directions: Use cross products to find the value of the variable in each proportion.

1. $\frac{6}{10} = \frac{x}{5}$
2. $\frac{2}{3} = \frac{x}{9}$
3. $\frac{4}{9} = \frac{a}{45}$
4. $\frac{7}{8} = \frac{a}{4}$
5. $\frac{b}{8} = \frac{5}{16}$
6. $\frac{6}{3} = \frac{x}{9}$
7. $\frac{4}{x} = \frac{8}{10}$
8. $\frac{1.5}{y} = \frac{3}{9}$
9. $\frac{4}{11} = \frac{c}{33}$
10. $\frac{2}{6} = \frac{5}{y}$
11. $\frac{2}{10} = \frac{5}{x}$
12. $\frac{4}{12} = \frac{6}{n}$
13. $\frac{5}{r} = \frac{70}{126}$
14. $\frac{4}{14} = \frac{14}{k}$
15. $\frac{8}{w} = \frac{6}{3}$
16. $\frac{2}{5} = \frac{17}{a}$

1.9 Solving Proportions: Solving a Multi-Step Proportion

Guidance

A proportion is an equation where one ratio is equal to another. In a proportion, the cross products are equal. Therefore, to solve an proportional equation, you cross multiply.

The Property of Cross Products of a Proportion stated the following:

If $\frac{a}{b} = \frac{c}{d}$, then ad = bc

Example: $\frac{2}{3} = \frac{8}{12}$, then 2 * 12 = 3 * 8

Solve: $\frac{x+3}{4} = \frac{x+4}{6}$

The cross products are: 6 (x + 3) = 4 (x + 4)

Simplify (using the Distributive property): 6x + 18 = 4x + 16

Subtract 4x from each side: 6x - 4x + 18 = 4x - 4x + 16

Simplify: 2x + 18 = 16

Subtract 18 from each side: 2x + 18 - 18 = 16 - 18

Simplify: 2x = -2

Divide by 2: $\frac{2x}{2} = \frac{-2}{2}$

Simplify: x = -1

Practice

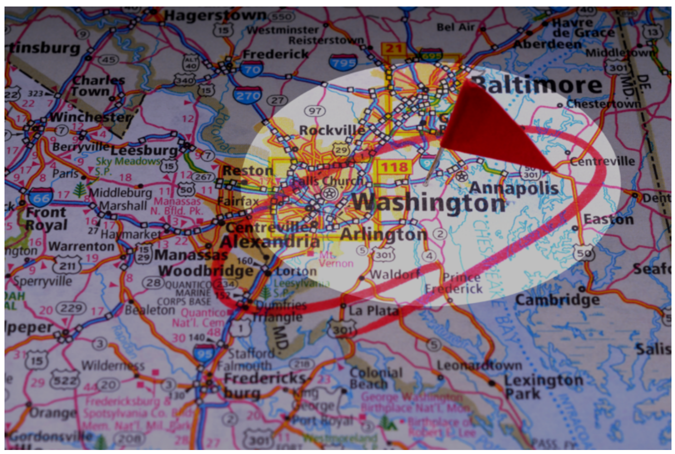
1. $\frac{2}{3} = \frac{6}{x+4}$
2. $\frac{7}{x+2} = \frac{7}{3}$
3. $\frac{10}{12} = \frac{20-x}{18}$
4. $\frac{x+3}{5} = \frac{6}{45}$
5. $\frac{x+5}{x-3} = \frac{14}{10}$
6. $\frac{x+9}{5} = \frac{x-10}{11}$
7. $\frac{4x+5}{5} = \frac{2x+7}{7}$
8. $\frac{7x-1}{6} = 5$
9. $\frac{5x-3}{4} = \frac{5x+3}{6}$
10. $\frac{2w+1}{3w-2} = \frac{5}{2}$

1.10 Solving Proportions: Using a Proportion to Solve a Problem

Guidance

As you already know, some types of problems can be solved by writing and solving a proportion. However, how you choose to solve the proportion may vary. Sometimes, it may be easier to use proportional reasoning. Other times, cross multiplying may be easier.

Let's look at two different situations where you can use a proportion.



On a map, Sonia measured the straight-line distance between Baltimore, Maryland and Washington D.C. to be 2 centimeters. The scale on the map shows that 1 centimeter = 28 kilometers. What is the actual straight-line distance between Baltimore and Washington D.C.?

This problem involves a map, which is a type of scale drawing. It makes sense to use proportions to solve it. The unit scale, 1 centimeter = 28 kilometers, can be represented as a ratio. We can also write a ratio that compares the scale distance, 2 centimeters, to the unknown actual distance, d.

$\frac{centimeters}{kilometers} = \frac{1}{28}$ $\qquad$ $\frac{centimeters}{kilometers} = \frac{2}{d}$

These are equivalent ratios, so we can use them to write a proportion.

$\frac{1}{28} = \frac{2}{d}$

Consider which strategy to use. Should we solve for d by using proportional reasoning? Or should we cross multiply?

Either strategy will work, but look at the terms in the numerators. The relationship between those two terms is easy to see—we can multiply 1 by 2 to get 2. So, the computation will probably be simpler if we use proportional reasoning and multiply both terms of the first ratio by 2.

$\frac{1}{28} = \frac{1 \times 2}{28 \times 2} = \frac{2}{56} = \frac{2}{d}$

From the work above, we can see that when the first term is 2, the second term is 56. So, $d = 56$.

The actual distance between Baltimore and Washington D.C. is 56 kilometers.

Sometimes, it is easier to use the cross product property of proportions than to use proportional reasoning. This is especially true when the relationship between a pair of terms in a proportion is not immediately obvious.

A baker uses 22 cups of flour to make 4 loaves of bread. How many cups of flour will he need to use to make 31 loaves of bread?

We can write a proportion to help us solve this problem. The first ratio can use the fact that it takes 21 cups of flour to make 4 loaves of bread. The second ratio can compare the unknown number of cups of flour needed, c, to the 31 loaves of bread the baker wants to make.

$\frac{cups}{loaves} = \frac{22}{4}$ $\qquad$ $\frac{cups}{loaves} = \frac{c}{31}$

These are equivalent ratios, so we can use them to write a proportion.

$\frac{22}{4} = \frac{c}{31}$

Consider which strategy to use. Should we solve for c by using proportional reasoning? Or should we cross multiply?

The relationship between the terms in the denominators, 4 and 31, is not immediately obvious because 31 is not a multiple of 4. So, cross multiplying is probably easier.

$$\begin{aligned}\frac{22}{4} &= \frac{c}{31}\\ 4 \cdot c &= 22 \cdot 31\\ 4c &= 682\\ \frac{4c}{4} &= \frac{682}{4}\\ c &= 170.5\end{aligned}$$

The baker will need 170.5, or $170\frac{1}{2}$, cups of flour to bake 31 loaves of bread.

Now it is time for you to try a few on your own. Use the information from the baker above.

Example A

How many cups for 6 loaves?

Solution: 33 cups

Example B

How many loaves for 55 cups?

Solution: 10 loaves

Example C

If the baker made half as many loaves to start with, how many cups would he have needed?

Solution: 11 cups

Here is the original problem again. It is time to check your answers.

After a whole year of reading, the students in Mrs. Henderson's class are ready to tally up the number of books that were read. The students read a total of 544 books. There were many different types of books that were read. A few of the students took everyone's lists and organized them into categories. Then they tallied the number of books in each category and composed a list.

History 12 books

Adventure 250 books

Romance 100 books

Mystery 120 books

Nature/Science 62 books

Then the students began comparing the number of books in each category. They compared the three largest categories with each other. They compared romance to adventure, mystery to romance, and mystery to adventure.

Given what you have learned about ratios and proportions answer each of the following questions.

What is the ratio of romance books to adventure books? Write your answer in simplest form too.

$\frac{100}{250} = \frac{2}{5}$

What is the ratio of mystery to romance books? Write your answer in simplest form.

$\frac{120}{100} = \frac{6}{5}$

What is the ratio of mystery to adventure? Write your answer in simplest form.

$\frac{120}{250} = \frac{12}{25}$

Do any of these ratios form a proportion? Why or why not?

None of these ratios form a proportion because none of them are equal.

Take a few minutes to go over your answers with a friend.

Guided Practice

Here is one for you to try on your own.

Use a proportion to solve the following problem.

If a person can run 3 miles in 18 minutes, how long will it take the same person to run 21 miles if it is at the same rate?

Answer

In this problem, we are comparing miles and time. That is our ratio. Let's set it up.

$\frac{miles}{time} = \frac{miles}{time}$

Next we fill in the given information.

$\frac{3}{18} = \frac{21}{x}$

Now we cross multiply and solve.

$3x = 378$

$x = 126$

The person would run 21 miles in 126 minutes, a little over two hours.

Practice

Directions: Use what you have learned to solve each problem. Consider more than one strategy for solving each problem. Then choose the strategy you think will work best and use it to solve the problem.

1. A jar contains only pennies and nickels. The ratio of pennies to nickels in the jar is 2 to 7. If there are 14 nickels in the jar, how many pennies are in the jar?

2. Anya charges $40 for 5 hours of babysitting. Lionel charges $14 for 2 hours of babysitting. Which babysitter charges the cheapest rate?

3. On a map, Derek measured the straight-line distance between Toronto, Canada and Niagara Falls, New York to be 2 inches. The scale on the map shows that $\frac{1}{2}\ inch = 11\ miles$. What is the actual straight-line distance between Toronto and Niagara Falls?

4. A desk is 120 centimeters long. What is the length of the desk in meters? Use this unit conversion: 1 meter = 100 centimeters.

5. On a field trip, the ratio of teachers to students is 1 : 25. If there are 5 teachers on the field trip, how many students are on the trip?

6. Kara bought 5 pounds of Brand X roast beef for $43. Cameron bought 3 pounds of Brand Y roast beef for $27. Which brand of roast beef is the better buy?

7. If two inches on a map are equal to three miles, how many miles are represented by four inches?

8. If eight inches on a map are equal to ten miles, how many miles are 16 inches equal to?

9. Casey drew a design for bedroom. On the picture, she used one inch to represent five feet. If her bedroom wall is ten feet long, how many inches will Casey draw on her diagram to represent this measurement?

10. If two inches are equal to twelve feet, how many inches would be equal to 36 feet?

11. If four inches are equal to sixteen feet, how many feet are two inches equal to?

12. The carpenter chose a scale of 6" for every twelve feet. Given this measurement, how many feet would be represented by 3"?

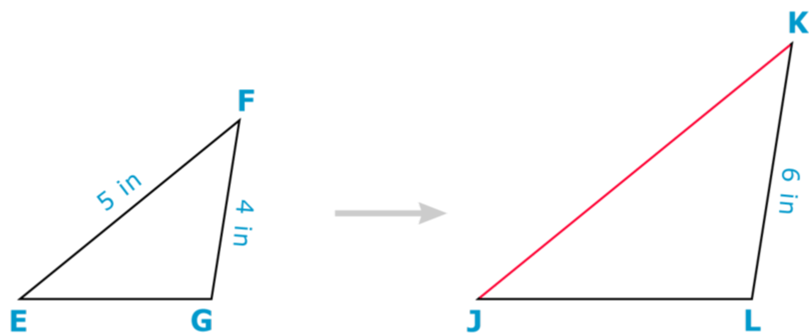
13. If 9 inches are equal to 27 feet, how many feet are equal to three inches?

14. If four inches are equal to 8 feet, how many feet are equal to two inches?

15. If six inches are equal to ten feet, how many inches are five feet equal to?

1.11 Proportions and Similar Figures: Finding the Length of a Side

Have you ever been stuck figuring out a math problem?

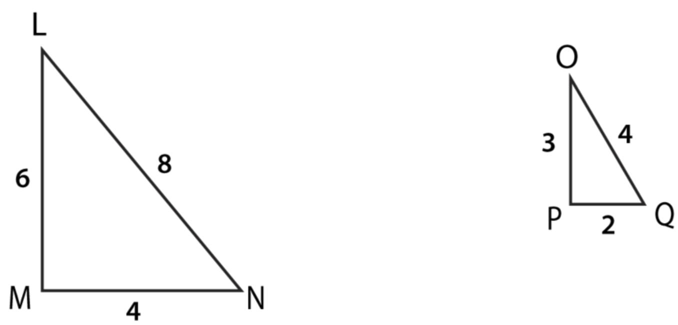


Jessica saw this problem on her homework assignment. She knows that she will need to use proportions in some way to figure out the length of the missing side, she just isn't sure how to do it.

This Concept will teach you how to tackle problems like this one.

Guidance

We can write ***ratios*** to compare the lengths of sides.



First, identify the corresponding sides of these two similar triangles.

$$\frac{LM}{OP} = \frac{LN}{OQ} = \frac{MN}{PQ}$$

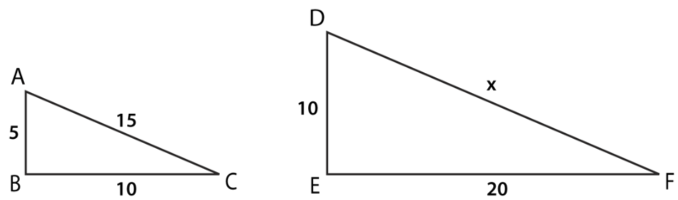
Now we have been given side lengths for each pair of corresponding sides. These have been written in a proportion or a set of three equal ratios. Remember that there is a relationship between the corresponding sides because they are parts of similar triangles. The side lengths of the similar triangles form a ***proportion***.

Let's substitute the given measurements into our formula.

$$\frac{6}{3} = \frac{8}{4} = \frac{4}{2}$$

There is a pattern with the ratios of corresponding sides. You can see that the measurement of the each side of the first triangle divided by two is the measure of the corresponding side of the second triangle.

We can use patterns like this to problem solve the length of missing sides of similar triangles.



Here we have two similar triangles. One is larger than the other, but they are similar. They have the same shape but a different size. Therefore, the corresponding sides are similar.

If you look at the side lengths, you should see that there is one variable. That is the missing side length. We can figure out the missing side length by using proportions. We know that the corresponding side lengths form a proportion. Let's write ratios that form a proportion and find the pattern to figure out the length of the missing side.

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$
$$\frac{5}{10} = \frac{15}{x} = \frac{10}{20}$$

Looking at this you can see the pattern. The side lengths of the second triangle are double the length of the corresponding side of the first triangle.

Using this pattern, you can see that the length of DF in the second triangle will be twice the length of AC. The length of AC is 15.

$\mathbf{15 \times 2 = 30}$

The length of DF is 30.

Practice solving these proportions.

Example A

$\frac{6}{12} = \frac{x}{24} = \frac{3}{6}$

Solution: $x = 12$

Example B

$\frac{12}{x} = \frac{16}{4} = \frac{20}{5}$

Solution: $x = 3$

Example C

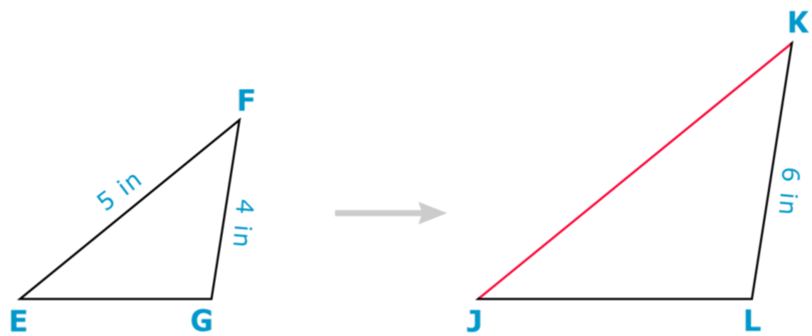
$\frac{8}{2} = \frac{16}{4} = \frac{x}{1}$

Solution: $x = 4$

Here is the original problem once again.

Jessica saw this problem on her homework assignment. She knows that she will need to use proportions in some way to figure out the length of the missing side, she just isn't sure how to do it.

Do you know?



Now the first thing that we can do is to set up a proportion to solve for the missing side. Remember that a proportion is two equal ratios. We can set up and compare the corresponding sides.

Here is our proportion.

$\frac{KJ}{5} = \frac{6}{4}$

Our proportion is written so that the corresponding sides form the two ratios of the proportion. We can say that KJ is our unknown in this proportion.

Do you remember how to solve proportions?

We can see a clear relationship between five and four, so we need to use cross products.

$$\begin{aligned} KJ \times 4 &= 4KJ \\ 5 \times 6 &= 30 \\ 4KJ &= 30 \end{aligned}$$

Now we can solve the equation for KJ by dividing both sides of the equation by 4.

$$\begin{aligned} 30 \div 4 &= 7.5 \\ KJ &= 7.5 \end{aligned}$$

The side length of KJ is 7.5.

Guided Practice

Here is one for you to try on your own.

$\frac{8}{10} = \frac{4}{5} = \frac{2}{x}$

Answer

To look at the relationships between each side length, we can begin by looking for a pattern of equal ratios.

The first ratio was divided in half to equal the second ratio.

The numerator of the second ratio was divided in half to equal the numerator of the third ratio.

The denominator of the third ratio is unknown.

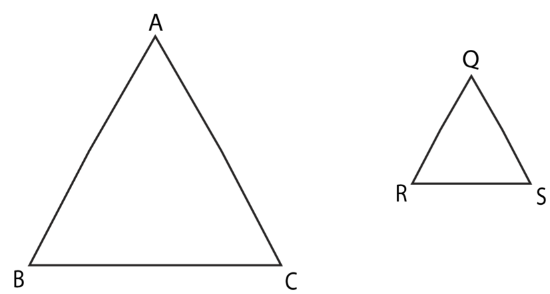
We can divide the denominator of the second ratio in half to equal the missing denominator.

$5 \div 2 = 2.5$

$x = 2.5$

Practice

Directions: Use the figures to answer the following questions.



1. Are these two triangles similar or congruent?
2. How do you know?
3. Which side is congruent to AB?
4. Which side is congruent to AC?
5. Which side is congruent to RS?
6. Look at the following proportion and solve for missing side length x.

$$\frac{7}{3.5} = \frac{x}{3.5} = \frac{6}{y}$$
$$x = \_\_\_\_\_$$

7. What is the side length for y?
8. How did you figure these out?

Directions: Figure out the missing value in each pair of ratios.

9. $\frac{6}{12} = \frac{x}{24}$
10. $\frac{8}{12} = \frac{x}{3}$
11. $\frac{9}{10} = \frac{18}{y}$
12. $\frac{4}{5} = \frac{x}{2.5}$
13. $\frac{16}{20} = \frac{4}{y}$
14. $\frac{19}{21} = \frac{x}{42}$
15. $\frac{9}{54} = \frac{6}{y}$

Prop

1.12 Proportions and Similar Figures: Applying Similarity and Scale Models

Guidance

Two figures that have the same shape are said to be **similar**. When **two figures** are **similar**, the ratios of the lengths of their corresponding sides are equal. To determine if the triangles below are **similar**, compare their corresponding sides.

Similar figures have the same shape, but might not be the same size. When two **shapes** are **similar**, their corresponding sides are **proportional** and their corresponding angles are congruent.

The figures in each pair are similar. Let's identify the corresponding sides and angles.

Example A:

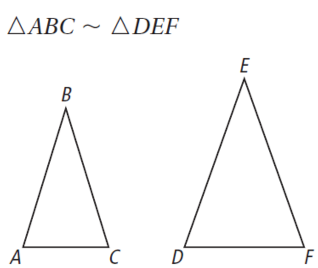


FIGURE 1.4

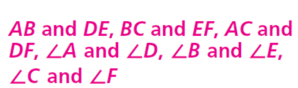


FIGURE 1.5

Example B:

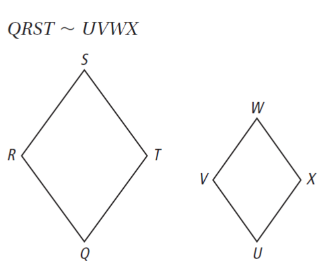
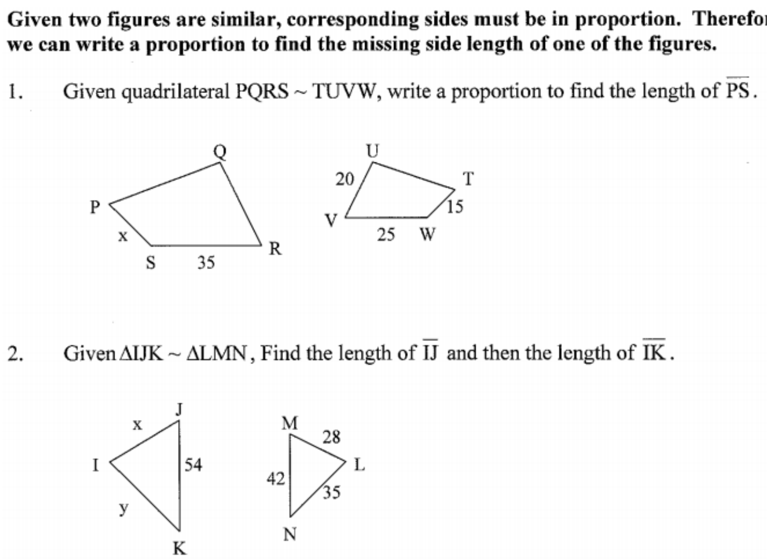


FIGURE 1.6

RS and *VW*, *ST* and *WX*, *TQ* and *XU*, *QR* and *UV*, ∠*R* and ∠*V*, ∠*S* and ∠*W*, ∠*T* and ∠*X*, ∠*Q* and ∠*U*

FIGURE 1.7

Guided Practice


Given two figures are similar, corresponding sides must be in proportion. Therefor we can write a proportion to find the missing side length of one of the figures.

1. Given quadrilateral PQRS ~ TUVW, write a proportion to find the length of $\overline{PS}$.

2. Given $\Delta IJK \sim \Delta LMN$, Find the length of $\overline{IJ}$ and then the length of $\overline{IK}$.

FIGURE 1.8

Practice

1. A flagpole casts a shadow 102 feet long. A 6-ft man casts a shadow 17 ft long. How tall is the flagpole?

2. A building casts a shadow of 26-ft. A boy standing nearby casts a 12-ft shadow. The boy's height is 4.5 ft. How tall is the building?

3. The San Antonio Wax Museum has a was sculpture of a historical village. The scale is 1.5:8. If the height of the hut in the sculpture is 5 feet, how tall was the original hut to the nearest whole foot?

4. On a map, the length of the Mississippi River is 4.75 inches. The actual length of the river is 247 miles. What is the scale of the map?

5. John is constructing a model bridge out of sticks. The actual bridge is 1320 ft long. John wants the scale of his bridge to be 1:400. How long should the model be?

6. A pizza show sells small 6 inch pizzas and medium 12 inch pizzas. Should the medium pizza cost as much as the small pizza because they are twice the size? Explain?

7. Martin has a scale model of a car. The scale is 1 inch:32 inches. If the model is 6.75 inches long, how long is the actual car?

8. Veronica is using similar triangles to find the height of a tree nearby her house. A stick that is 5 feet tall casts a shadow that is 4 feet long. A tree casts a shadow that is 22 feet long. How tall is the tree?

9. If a 36 inch yardstick casts a 21 foot shadow, how tall is a building whose shadow is 168 feet? (Draw a picture with 2 similar polygons)

10. Johnny wants to enlarge a triangle with 3, 6, and 6 inches. If the shortest side of the new triangle is 13 inches, how long with the other two sides be?

ro

1.13 Solving Multi-Step Inequalities: Adding and Subtracting Using One Step

Guidance

Inequalities Using Addition or Subtraction

To solve inequalities, you need some properties.

Addition Property of Inequality: For all real numbers a, b, and c:

If $x < a$, *then* $x + b < a + b$.

If $x < a$, *then* $x - c < a - c$.

The two properties above are also true for $\leq$ or $\geq$.

Because subtraction can also be thought of as **"add the opposite,"** these properties also work for subtraction situations.

Just like one-step equations, the goal is to **isolate the variable,** meaning to get the variable alone on one side of the inequality symbol. To do this, you will cancel the operations using inverses.

Example A

Solve for $x: \ x - 3 < 10$.

Solution: To isolate the variable x, you must cancel "subtract 3" using its inverse operation, addition.

$$x - 3 + 3 < 10 + 3$$
$$x < 13$$

Now, check your answer. Choose a number less than 13 and substitute it into your original inequality. If you choose 0, and substitute it you get:

$$0 - 3 < 10 = -3 < 10$$

What happens at 13? What happens with numbers greater than 13?

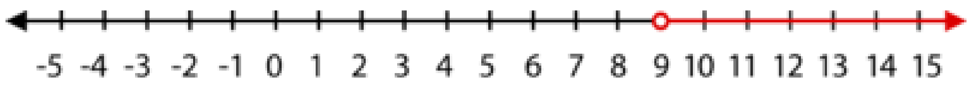
Example B

Solve for $x: \ x + 4 > 13$.

Solution:

	$x + 4 > 13$
Subtract 4 from both sides of the inequality.	$x + 4 - 4 > 13 - 4$
Simplify.	$x > 9$

The solution is shown below in a graph:

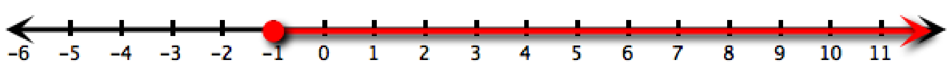


Example C

Solve for x: $x+\frac{2}{3} \geq -\frac{1}{3}$.

Solution:

$$\begin{aligned} & x+\frac{2}{3} \geq -\frac{1}{3} \\ \text{Subtract } \frac{2}{3} \text{ from both sides of the inequality.} \qquad & x+\frac{2}{3}-\frac{2}{3} \geq -\frac{1}{3}-\frac{2}{3} \\ \text{Simplify.} \qquad & x \geq -1 \end{aligned}$$



Guided Practice

Solve for y: $5.6 > y-3.4$.

Solution:

$$\begin{aligned} & 5.6 > y-3.3 \\ \text{Add 3.3 to both sides of the inequality.} \qquad & 5.6+3.3 > y-3.3+3.3 \\ \text{Simplify.} \qquad & 8.9 > y \end{aligned}$$

Practice

Solve each inequality and graph the solution on a number line.

1. $x-1 > -10$
2. $x-1 \leq -5$
3. $-20+a \geq 14$
4. $x+2 < 7$
5. $x+8 \leq -7$
6. $5+t \geq \frac{3}{4}$
7. $x-5 < 35$
8. $15+g \geq -60$
9. $x-2 \leq 1$
10. $x-8 > -20$

11. $11+q>13$
12. $x+65<100$
13. $x-32\leq 0$
14. $x+68\geq 75$
15. $16+y\leq 0$

1.14 Solving Multi-Step Inequalities: Multiplication and Division Using One Step

Guidance

Equations are mathematical sentences in which the two sides have the same "weight." By adding, subtracting, multiplying, or dividing the same value to both sides of the equation, the balance stays in check. However, inequalities begin off-balance. When you perform inverse operations, the inequality will remain off-balance. This is true with inequalities involving both multiplication and division.

Before we can begin to solve inequalities involving multiplication or division, you need to know two properties: the Multiplication Property of Inequality and the Division Property of Inequality.

Multiplication Property of Inequality: For all real positive numbers a, b, and c:

If $x < a$, *then* $x(c) < a(c)$.

If $x > a$, *then* $x(c) > a(c)$.

Division Property of Inequality: For all real positive numbers a, b, and c:

If $x < a$, *then* $x \div (c) < a \div (c)$.

If $x > a$, *then* $x \div (c) > a \div (c)$.

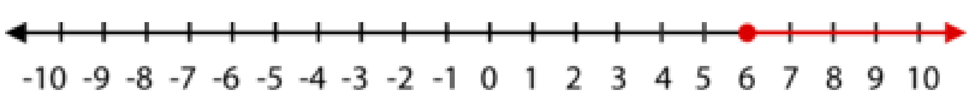
Example A

Consider the inequality $2x \geq 12$. To find the solutions to this inequality, we must isolate the variable x by using the inverse operation of "multiply by 2," which is dividing by 2.

$$2x \geq 12$$
$$\frac{2x}{2} \geq \frac{12}{2}$$
$$x \geq 6$$

This solution can be expressed in four ways. One way is already written: $x \geq 6$. Below are the three remaining ways to express this solution:

- $\{x|x \geq 6\}$
- $[6, \infty)$
- Using a number line:



Example B

Solve for $y : \frac{y}{5} \leq 3$. *Express the solution using all four methods.*

Solution: The inequality above is read, "y divided by 5 is less than or equal to 3." To isolate the variable y, you must cancel division using its inverse operation, multiplication.

$$\frac{y}{5} \cdot \frac{5}{1} \leq 3 \cdot \frac{5}{1}$$
$$y \leq 15$$

One method of writing the solution is $y \leq 15$.

The other three are:

- $(-\infty, 15]$
- $\{y|y \leq 15\}$
- 

Multiplying and Dividing an Inequality by a Negative Number

Notice that the two properties in this Concept focused on c being only positive. This is because those particular properties of multiplication and division do not apply when the number being multiplied (or divided) is negative.

Think of it this way. When you multiply a value by –1, the number you get is the negative of the original.

$$6(-1) = -6$$

Multiplying each side of a sentence by –1 results in the opposite of both values.

$$5x(-1) = 4(-1)$$
$$-5x = -4$$

When multiplying by a negative, you are doing the "opposite" of everything in the sentence, including the verb.

$$x > 4$$
$$x(-1) > 4(-1)$$
$$-x < -4$$

Example C

Solve for $r : -3r < 9$.

Solution: To isolate the variable r, we must cancel "multiply by –3" using its inverse operation, dividing by –3.

$$\frac{-3r}{-3} < \frac{9}{-3}$$

Since you are dividing by –3, everything becomes opposite, including the inequality sign.

$$r > -3$$

Example D

Solve for $p : 12p < -30$.

Solution: To isolate the variable p, we must cancel "multiply by 12" using its inverse operation, dividing by 12.

$$\frac{12p}{12} < \frac{-30}{12}$$

Because 12 is **not** negative, you do **not** switch the inequality sign.

$$p < \frac{-5}{2}$$

.

In set notation, the solution would be: $\left(-\infty, \frac{-5}{2}\right)$.

Practice

1. In which cases do you change the inequality sign?

2. $3x \leq 6$
3. $\frac{x}{5} > -\frac{3}{10}$
4. $-10x > 250$
5. $\frac{x}{-7} \geq -5$
6. $9x > -\frac{3}{4}$
7. $\frac{x}{-15} \leq 5$
8. $620x > 2400$
9. $\frac{x}{20} \geq -\frac{7}{40}$
10. $-0.5x \leq 7.5$
11. $75x \geq 125$
12. $\frac{x}{-3} > -\frac{10}{9}$
13. $\frac{k}{-14} \leq 1$
14. $\frac{x}{-15} < 8$
15. $\frac{x}{2} > 40$
16. $\frac{x}{-3} \leq -12$
17. $\frac{x}{25} < \frac{3}{2}$
18. $\frac{x}{-7} \geq 9$
19. $4x < 24$
20. $238 < 14d$
21. $-19m \leq -285$
22. $-9x \geq -\frac{3}{5}$

1.15 Solving Multi-Step Inequalities: More than One Step, Distributive Property, Variables on Both Sides

Here you will solve more complicated inequalities involving variables on both sides and the Distributive Property.

Guidance

Like multi-step equations, multi-step inequalities can involve having variables on both sides, the Distributive Property, and combining like terms. Again, the only difference when solving inequalities is the sign must be flipped when multiplying or dividing by a negative number.

Example A

Is $x = -3$ a solution to $2(3x-5) \leq x+10$?

Solution: Plug in -3 for x and see if the inequality is true.

$$\begin{aligned}
2(3(-3)-5) &\leq (-3)+10 \\
2(-9-5) &\leq 7 \\
2 \cdot -14 &\leq 7 \\
-28 &\leq 7
\end{aligned}$$

This is a true inequality statement. -3 is a solution.

Example B

Solve and graph the inequality from Example A.

Solution: First, distribute the 2 on the left side of the inequality.

$$\begin{aligned}
2(3x-5) &\leq x+10 \\
6x-10 &\leq x+10
\end{aligned}$$

Now, subtract the x on the right side to move it to the left side of the inequality. You can also add the 10's together and solve.

$$\begin{array}{c}
6x - \cancel{10} \geq \cancel{x} + 10 \\
\underline{-x + \cancel{10} \quad -\cancel{x} + 10} \\
\dfrac{\cancel{5}x}{\cancel{5}} \leq \dfrac{20}{5} \\
x \leq 4
\end{array}$$

Test a solution, $x = 0 : 2(3(0)-5) \leq 0+10 \checkmark$

$-10 \leq 10$

The graph looks like:

Example C

Solve $8x-5-4x \geq 37-2x$

Solution: First, combine like terms on the left side. Then, solve for x.

$$\begin{aligned} 8x-5-4x &\geq 37-2x \\ 4x-\cancel{5} &\geq 37-\cancel{2x} \\ \underline{+2x+\cancel{5}} & \underline{\ +5+\cancel{2x}} \\ \frac{\cancel{6}x}{\cancel{6}} &\geq \frac{42}{6} \\ x &\geq 7 \end{aligned}$$

Test a solution, $x=10: 8(10)-5-4(10) \geq 37-2(10)$

$80-5-40 \geq 37-20$

$35 \geq 17\checkmark$

Recall that the conversion formula for Celsius to Fahrenheit is $C=\frac{5}{9}(F-32)$. The temperature can be equal to or greater than $-35^\circ C$.

$$\begin{aligned} \frac{5}{9}(F-32) &\geq -35 \\ F-32 &\geq -35 \cdot \frac{9}{5} \\ F-32 &\geq -63 \\ F &\geq -31 \end{aligned}$$

So, the temperature can be equal to or higher than $-31^\circ F$.

Guided Practice

1. Is $x=12$ a solution to $-3(x-10)+18 \geq x-25$?

Solve and graph the following inequalities.

2. $-(x+16)+3x>8$

3. $24-9x<6x-21$

Answers

1. Plug in 12 for x and simplify.

$$\begin{aligned}-3(12-10)+18 &\geq 12-35\\ -3\cdot 2+18 &\geq -13\\ -6+18 &\geq -13\end{aligned}$$

This is true because $12 \geq -13$, so 12 is a solution.

2. Distribute the negative sign on the left side and combine like terms.

$$\begin{aligned}-(x+16)+3x &> 8\\ -x-16+3x &> 8\\ 2x-\cancel{16} &> 8\\ +\cancel{16} &+16\\ \hline \frac{\cancel{2}x}{\cancel{2}} &> \frac{24}{2}\\ x &> 12\end{aligned}$$

Test a solution, $x = 15$:

$$\begin{aligned}-(15+16)+3(15) &> 8\\ -31+45 &> 8\\ 14 &> 8\end{aligned}$$

3. First, add $9x$ to both sides and add 21 to both sides.

$$\begin{aligned}24-\cancel{9x} &< 6x-21\\ +\cancel{9x} &\quad +9x\\ \hline 24 &< 15x-\cancel{21}\\ +21 &\qquad +\cancel{21}\\ \hline \frac{45}{15} &< \frac{\cancel{15}x}{\cancel{15}}\\ 3 &< x\end{aligned}$$

Test a solution, $x = 10$:

$$\begin{aligned}24-9(10) &< 6(10)-21\\ 24-90 &< 60-21\\ -66 &< 39\end{aligned}$$

Practice

Determine if the following numbers are solutions to $-7(2x-5)+12 > -4x-13$.

1. $x = 4$
2. $x = 10$
3. $x = 6$

Solve and graph the following inequalities.

4. $2(x-5) \geq 16$
5. $-4(3x+7) < 20$
6. $15x-23 > 6x-17$
7. $5x+16+2x \leq -19$
8. $4(2x-1) \geq 3(2x+1)$
9. $11x-17-2x \leq -(x-23)$

Solve the following inequalities.

10. $5-5x > 4(3-x)$
11. $-(x-1)+10 < -3(x-3)$
12. Solve $5x+4 \leq -2(x+3)$ by adding the $2x$ term on the right to the left-hand side.
13. Solve $5x+4 \leq -2(x+3)$ by *subtracting* the $5x$ term on the left to the right-hand side.
14. Compare your answers from 12 and 13. What do you notice?
15. **Challenge** Solve $3x-7 > 3(x+3)$. What happens?

1.16 Compound Inequalities: Writing and Solving

Here you'll learn how to separate compound inequalities with *"and"* or *"or"* and solve them separately. You'll then learn how to combine your answers into a single solution and graph the solution set.

Guidance

When we solve compound inequalities, we separate the inequalities and solve each of them separately. Then, we combine the solutions at the end.

- **Compound inequalities** combine two or more inequalities with **"and"** or **"or."**
- **"And"** combinations mean that only solutions for *both* inequalities will be solutions to the compound inequality.
- **"Or"** combinations mean solutions to *either* inequality will also be solutions to the compound inequality.

Example A

Solve the following compound inequalities and graph the solution set.

a) $-2 < 4x - 5 \leq 11$

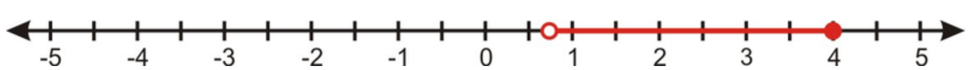
b) $3x - 5 < x + 9 \leq 5x + 13$

Solution

a) First we re-write the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$$\begin{array}{ccc} -2 < 4x - 5 & & 4x - 5 \leq 11 \\ 3 < 4x & \textit{and} & 4x \leq 16 \\ \frac{3}{4} < x & & x \leq 4 \end{array}$$

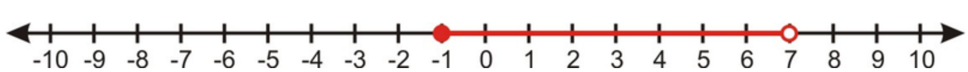
Answer: $\frac{3}{4} < x$ and $x \leq 4$. This can be written as $\frac{3}{4} < x \leq 4$.



b) Re-write the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$$\begin{array}{ccc} 3x - 5 < x + 9 & & x + 9 \leq 5x + 13 \\ 2x < 14 & \textit{and} & -4 \leq 4x \\ x < 7 & & -1 \leq x \end{array}$$

Answer: $x < 7$ and $x \geq -1$. This can be written as: $-1 \leq x < 7$.



Example B

Solve the following compound inequalities and graph the solution set.

a) $9-2x \leq 3$ or $3x+10 \leq 6-x$

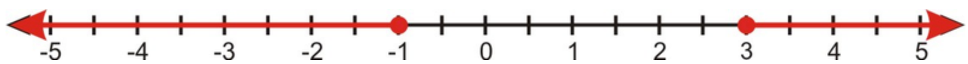
b) $\frac{x-2}{6} \leq 2x-4$ or $\frac{x-2}{6} > x+5$

Solution

a) Solve each inequality separately:

$$\begin{array}{rcr} 9-2x \leq 3 & & 3x+10 \leq 6-x \\ -2x \leq -6 & or & 4x \leq -4 \\ x \geq 3 & & x \leq -1 \end{array}$$

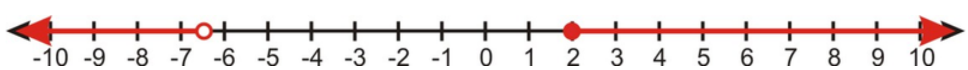
Answer: $x \geq 3$ or $x \leq -1$



b) Solve each inequality separately:

$$\begin{array}{rcr} \frac{x-2}{6} \leq 2x-4 & & \frac{x-2}{6} > x+5 \\ x-2 \leq 6(2x-4) & & x-2 > 6(x+5) \\ x-2 \leq 12x-24 & or & x-2 > 6x+30 \\ 22 \leq 11x & & -32 > 5x \\ 2 \leq x & & -6.4 > x \end{array}$$

Answer: $x \geq 2$ or $x < -6.4$



One thing you may notice in the video for this Concept is that in the second problem, the two solutions joined with "or" overlap, and so the solution ends up being the set of all real numbers, or $(-\infty, \infty)$. This happens sometimes with compound inequalities that involve "or"; for example, if the solution to an inequality ended up being "$x < 5$ or $x > 1$," the solution set would be all real numbers. This makes sense if you think about it: all real numbers are either a) less than 5, or b) greater than or equal to 5, and the ones that are greater than or equal to 5 are also greater than 1—so all real numbers are either a) less than 5 or b) greater than 1.

Compound inequalities with "and," meanwhile, can turn out to have *no* solutions. For example, the inequality "$x < 3$ and $x > 4$" has no solutions: no number is both greater than 4 and less than 3. If we write it as $4 < x < 3$ it's even more obvious that it has no solutions; $4 < x < 3$ implies that $4 < 3$, which is false.

Solve Real-World Problems Using Compound Inequalities

Many application problems require the use of compound inequalities to find the solution.

Example C

The speed of a golf ball in the air is given by the formula $v = -32t + 80$. *When is the ball traveling between 20 ft/sec and 30 ft/sec?*

Solution

First we set up the inequality $20 \leq v \leq 30$, and then replace v with the formula $v = -32t + 80$ to get $20 \leq -32t + 80 \leq 30$.

Then we separate the compound inequality and solve each separate inequality:

$$\begin{array}{rcr} 20 \leq -32t + 80 & & -32t + 80 \leq 30 \\ 32t \leq 60 & \text{and} & 50 \leq 32t \\ t \leq 1.875 & & 1.56 \leq t \end{array}$$

Answer: $1.56 \leq t \leq 1.875$

To check the answer, we plug in the minimum and maximum values of t into the formula for the speed.

For $t = 1.56,\ v = -32t + 80 = -32(1.56) + 80 = 30\ ft/sec$

For $t = 1.875,\ v = -32t + 80 = -32(1.875) + 80 = 20\ ft/sec$

So the speed is between 20 and 30 ft/sec. **The answer checks out.**

Guided Practice

William's pick-up truck gets between 18 to 22 miles per gallon of gasoline. His gas tank can hold 15 gallons of gasoline. If he drives at an average speed of 40 miles per hour, how much driving time does he get on a full tank of gas?

Solution

Let $t =$ driving time. We can use dimensional analysis to get from time per tank to miles per gallon:

$$\frac{t\ hours}{1\ tank} \times \frac{1\ tank}{15\ gallons} \times \frac{40\ miles}{1\ hour} \times \frac{40t}{15}\frac{miles}{gallon}$$

Since the truck gets between 18 and 22 miles/gallon, we set up the compound inequality $18 \leq \frac{40t}{15} \leq 22$. Then we separate the compound inequality and solve each inequality separately:

$$\begin{array}{rcr} 18 \leq \dfrac{40t}{15} & & \dfrac{40t}{15} \leq 22 \\ 270 \leq 40t & \text{and} & 40t \leq 330 \\ 6.75 \leq t & & t \leq 8.25 \end{array}$$

Answer: $6.75 \leq t \leq 8.25$.

Andrew can drive between 6.75 and 8.25 hours on a full tank of gas.

If we plug in $t = 6.75$ we get $\frac{40t}{15} = \frac{40(6.75)}{15} = 18\ miles\ per\ gallon.$

If we plug in $t = 8.25$ we get $\frac{40t}{15} = \frac{40(8.25)}{15} = 22\ miles\ per\ gallon.$

The answer checks out.

Practice

Solve the following compound inequalities and graph the solution on a number line.

1. $-5 \leq x - 4 \leq 13$
2. $1 \leq 3x + 5 \leq 4$
3. $-12 \leq 2 - 5x \leq 7$
4. $\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$
5. $-2 \leq \frac{2x-1}{3} < -1$
6. $4x - 1 \geq 7$ or $\frac{9x}{2} < 3$
7. $3 - x < -4$ or $3 - x > 10$
8. $\frac{2x+3}{4} < 2$ or $-\frac{x}{5} + 3 < \frac{2}{5}$
9. $2x - 7 \leq -3$ or $2x - 3 > 11$
10. $4x + 3 < 9$ or $-5x + 4 \leq -12$
11. How would you express the answer to problem 1 as a set?
12. How would you express the answer to problem 1 as an interval?
13. How would you express the answer to problem 6 as a set?
14. Could you express the answer to problem 6 as a single interval? Why or why not?
    a. How would you express the first part of the solution in interval form?
    b. How would you express the second part of the solution in interval form?
15. Express the answers to problems 2 through 5 in interval notation.
16. Solve the inequality "$x \geq -3$ or $x < 1$" and express the answer in interval notation.
17. How many solutions does the inequality "$x \geq 2$ and $x \leq 2$" have?
18. To get a grade of B in her Algebra class, Stacey must have an average grade greater than or equal to 80 and less than 90. She received the grades of 92, 78, 85 on her first three tests.
    a. Between which scores must her grade on the final test fall if she is to receive a grade of B for the class? (Assume all four tests are weighted the same.)
    b. What range of scores on the final test would give her an overall grade of C, if a C grade requires an average score greater than or equal to 70 and less than 80?
    c. If an A grade requires a score of at least 90, and the maximum score on a single test is 100, is it possible for her to get an A in this class? (Hint: look again at your answer to part a.)

Vocabulary:

Solving Multi-Step: Equivalent, Isolate, Like Terms, Rational Numbers

Solving Equations with Variables on Both Sides: Distributive Property, Inverse Operations, Identity

Literal Equations: Variable, Formula, Literal Equation

Solving Proportions: Cross Products, Proportion, Rate, Ratio

Proportions and Similar Figures: Scale, Scale Drawing, Scale Model, Similar Figures

Solving Multi-Step Inequalities: Equivalent Inequalities, Inequality, Solution of an Inequality

Compound Inequalities: Evaluate, Simplify, Compound Inequality

CHAPTER 2 Introduction to Functions

Chapter Outline

Using Graphs to Relate Two Quantities: A.12a

Patterns and Linear Functions: A.2c

Patterns and Non-Linear Functions: A.12c

Graphing a Function Rule: A.3c

Writing a Function Rule: A.2c

Formalizing Relations and Functions: A.12a

Using Function Notation: A.2a, A.12b

2.1 Using Graphs to Relate Two Quantities: Analyzing, Matching, and Sketching

Guidance

An important life skill is to be able to a read graph. When looking at a graph, you should check the title, the labels on the axes, and the general shape of the graph.

You will learn how to interpret characteristics of graphs to describe real-world situations. You will analyze graphs by reading the title.

The axis titles will tell you what variables are related. The graph itself represents the relationship as the variables change.

Example A

What information can you determine from the graph?

Linda's Trip

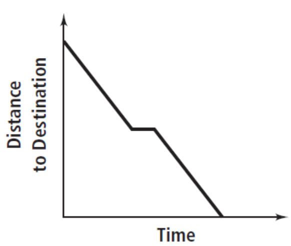


FIGURE 2.1

• The title tells you that the graph describes Linda's trip.
• The axes tell you that the graph relates the variable of time the variable of distance to the destination.
• In general, the more time that has elapsed, the closer Linda gets to her destination. In the middle of the trip, the distance does not change, showing she stops for a while.

Example B

Explain the variables for each graph. Describe how the variables are related at various points on the graph.

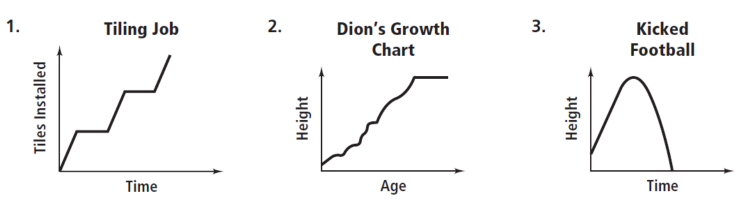


FIGURE 2.2

Example C

A graph can also show the relationship from a table. Which graph below represents the information in the table shown?

CDs Purchased	Total Cost
1	$15
2	$30
3	$45
4	$60
5	$75

+1 +1 +1 +1 / +15 +15 +15 +15

FIGURE 2.3

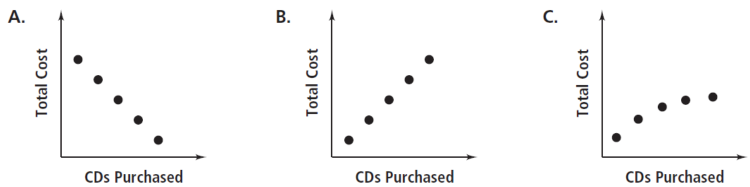


FIGURE 2.4

Note: that for each additional CD purchased, the total cost increases by $15. the points on the graph should be in a straight line that goes up from left to right. The graph that shows this trend is Graph B.

Practice:

1. Match each graph with its related table. Explain your answers.

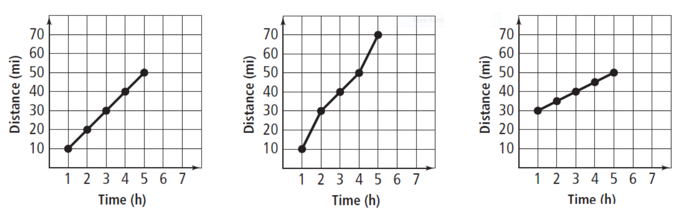


FIGURE 2.5

Time (h)	Distance (mi)
1	10
2	20
3	30
4	40
5	50

Time (h)	Distance (mi)
1	30
2	35
3	40
4	45
5	50

Time (h)	Distance (mi)
1	10
2	30
3	40
4	50
5	70

FIGURE 2.6

2. Skiing Sketch a graph of each situation. Are the graphs the same? Explain.

a. your speed as you travel from the bottom of a ski slope to the top
b. your speed as you travel from the top of a ski slope to the bottom

What is likely to be true about your speed as you go from the bottom of the ski slope to the top?

What is likely to be true about your speed as you go from the top of the ski slope to the bottom?

3. What are the variables in each graph? Describe how the variables are related at various points on the graph.

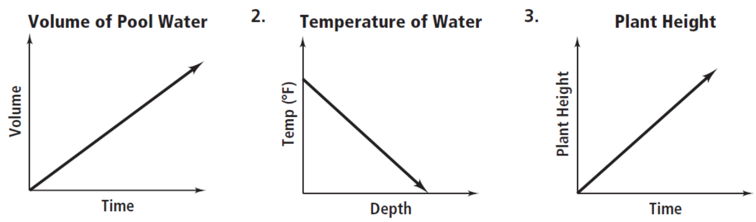


FIGURE 2.7

Sketch the following and label accordingly:

4. You buy two shirts and the third one is free.

5. You warm up for gym class, play basketball, and then cool down.

6. The temperature warms up during the day and then decreases at night.

7. During a trip, your speed increases during the first hour and decreases over the next 2 hours.

8. The average temperature steadily decreases over the course of the football season.

9. The average score of the class increased throughout the semester until it decreased slightly on the last test.

10. Error Analysis. DVDs cost \$19.99 each for the first 2 purchased. After that, they cost \$5.99 each. Describe and correct the error in sketching a graph to represent the relationship between the total cost and the number of DVDs purchased.

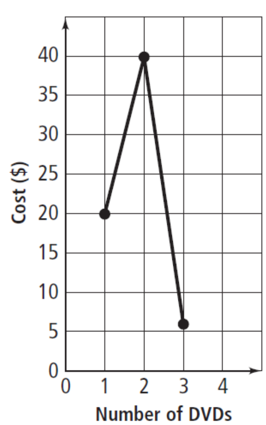


FIGURE 2.8

11. During a trip, your speed increases during the first hour and decreases over the next 2 hours.

12.

13.

The graph shows your distance from the practice field as you go home after practice. You received a ride from a friend back to his house where you ate supper. You then walked home from there. Which point represents a time when you are walking home?

A. A **B.** B **C.** C **D.** D

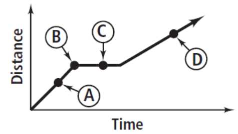


FIGURE 2.9

Which table is related to the graph at the right?

Time (h)	Temp. (°F)
1	68
2	73
3	78
4	85

H.

Time (h)	Temp. (°F)
68	1
73	2
78	3
85	4

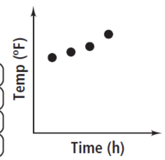


FIGURE 2.10

Temp. (°F)	Time (h)
1	85
2	78
3	73
4	68

I.

Temp. (°F)	Time (h)
85	1
78	2
73	3
68	4

14.

How are the variables related on the graph?

A. as speed decreases, height stay constant
B. as speed decreases, height increases
C. as speed increases, height decreases
D. as speed increases, height increases

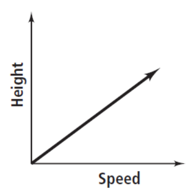


FIGURE 2.11

Using

2.2 Patterns and Linear Functions: Representing Geometric Relationships and Linear Functions

Guidance

A **function** is a relation in which, for each distinct value of the first component of the ordered pairs, there is exactly one value of the second component.

A relation is a set of ordered pairs.

A **relationship** can be represented in a table, as ordered pairs, in a graph, in words, or in a equation.

Consider the relationship between the number of squares in the pattern and the perimeter of the figure. How can this be represented in relationship as a table, as ordered pairs, in a graph, in words, and in an equation. See the following examples of each.

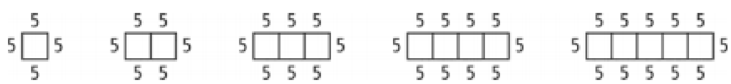


FIGURE 2.12

Table:

Number of squares	1	2	3	4	5
Perimeter	20	30	40	50	60

FIGURE 2.13

Ordered pairs:

(1, 20), (2, 30), (3, 40), (4, 50), (5, 60)

FIGURE 2.14

Graph:

Words: The pattern shows the perimeter is the number of squares times 10 plus 10.

Equation: $y = 10x + 10$

Vocabulary Support

An **independent variable** is a variable whose value determines the value of another variable, called the **dependent variable**.

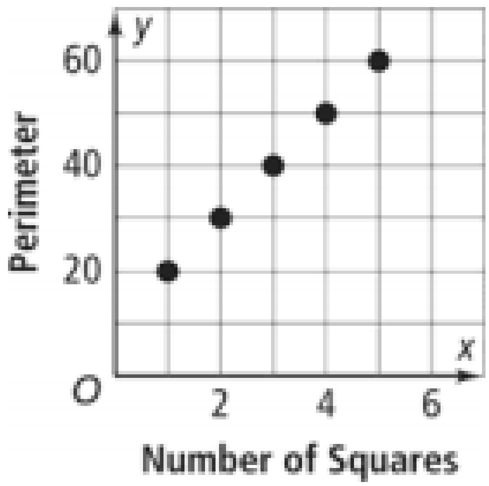


FIGURE 2.15

The independent variable can also be called the **input** and the dependent variable the **output**.

See diagram below.

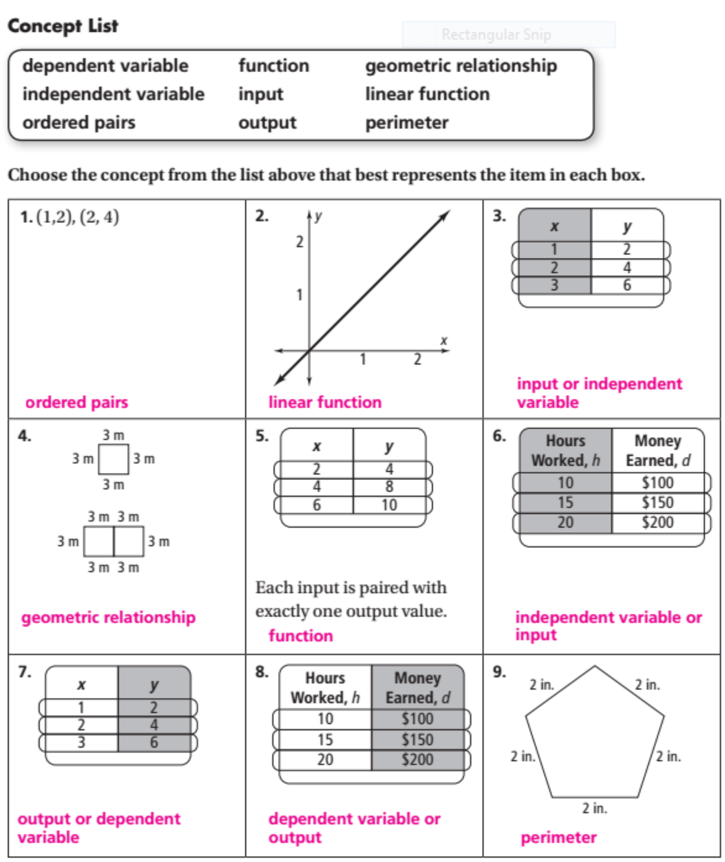


FIGURE 2.16

Practice

1.

For the diagram below, find the relationship between the number of shapes and the perimeter of the figure they form. Represent this relationship using a table, words, an equation, and a graph.

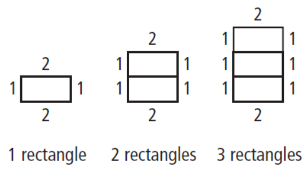


FIGURE 2.17

2.

For each table, determine whether the relationship is a function. Then represent the relationship using words, an equation, and a graph.

x	y
0	2
1	3
2	4
3	5

FIGURE 2.18

3.

For each table, determine whether the relationship is a function. Then represent the relationship using words, an equation, and a graph.

x	y
0	5
1	10
2	15
3	20

FIGURE 2.19

4.

5.

For each table, identify the dependent and independent variables. Then describe the relationship using words, an equation, and a graph.

x	*y*
0	–2
1	–1
2	0
3	1

FIGURE 2.20

For each table, identify the dependent and independent variables. Then describe the relationship using words, an equation, and a graph.

n	*m*
0	1
1	–2
2	–5
3	–8

FIGURE 2.21

6.

Reasoning Graph the set of ordered pairs (0, 6), (1, 4), (2, 2), (3, 0). Determine whether the relationship is a linear function. Explain how you know.

FIGURE 2.22

7.

Which equation represents the relationship shown in the table at the right?

A. $y = -x - 3$
B. $y = x - 3$
C. $y = 2x - 3$
D. $y = -2x + 3$

x	y
0	–3
1	–1
2	1
3	3

FIGURE 2.23

8.

In a relationship between variables, what is the variable called that changes in response to another variable?

F. function **H.** independent variable

G. input function **I.** dependent variable

FIGURE 2.24

9.

A lawn care company charges a $10 trip fee plus $0.15 per square foot of x square feet of lawn for fertilization. Which equation represents the relationship?

A. $x = 0.10y + 15$ **B.** $y = 0.15x + 10$ **C.** $y = 10x + 0.15$ **D.** $x = 10y + 0.15$

FIGURE 2.25

10.

Which equation represents the relationship shown in the graph?

F. $y = -2x$ **H.** $y = -\frac{1}{2}x$

G. $y = 2x$ **I.** $y = \frac{1}{2}x$

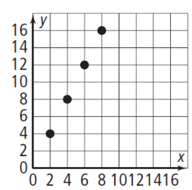
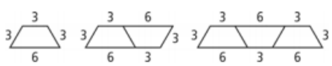


FIGURE 2.26

11.

Pa

Consider each pattern.

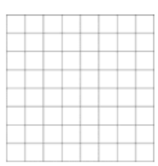


a. Make a table to show the relationship between the number of trapezoids and the perimeter.

Number of Trapezoids			
Perimeter			

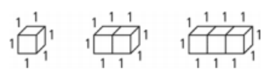
b. Write the ordered pairs for the relationship.

c. Make a graph for the relationship.



d. Use words to describe the relationship.

e. Write an equation for the relationship.

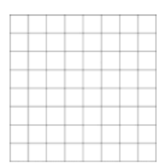


a. Make a table to show the relationship between the number of cubes and the surface area.

Number of Cubes			
Surface Area			

b. Write the ordered pairs for the relationship.

c. Make a graph for the relationship.



d. Use words to describe the relationship.

e. Write an equation for the relationship.

FIGURE 2.27

2.3 Patterns and Non-Linear Functions: Classification of Linear and Non-Linear

Guidance

In a function, each input (independent variable) is paired exactly with one output (dependent variable).

A linear function is a function whose graph is a line or part of a line. The graph of a linear function represents constant rate of change.

A non-linear function is a function whose graph is not a line or part of a line. The graph of a non-linear function can be a curve. In addition, the points do not lie on a line.

Let's compare linear and non-linear functions:

Example: Linear Functions

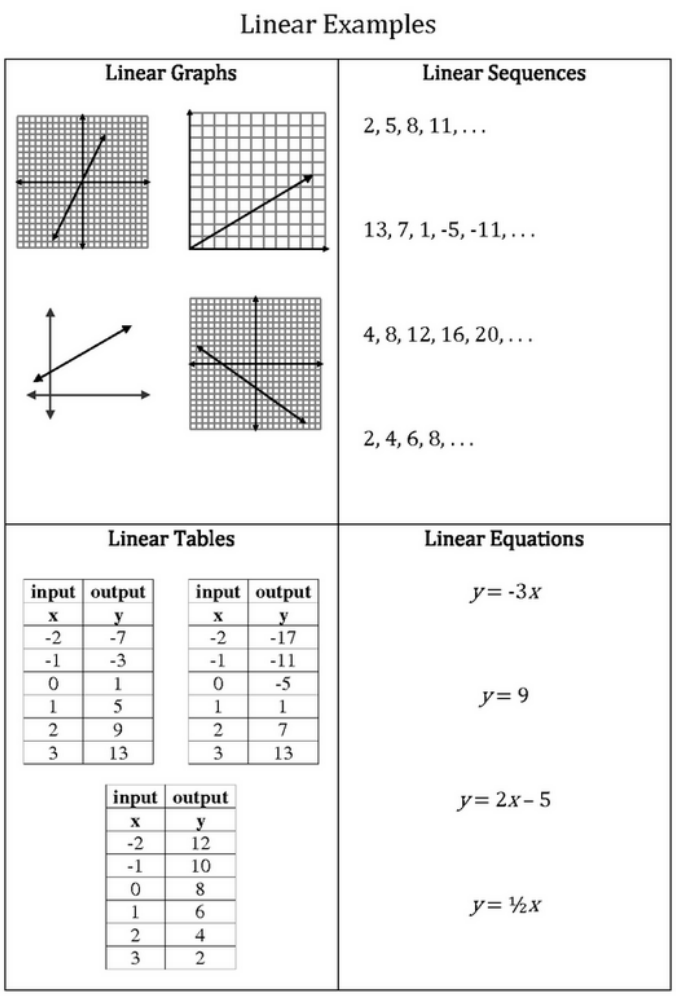
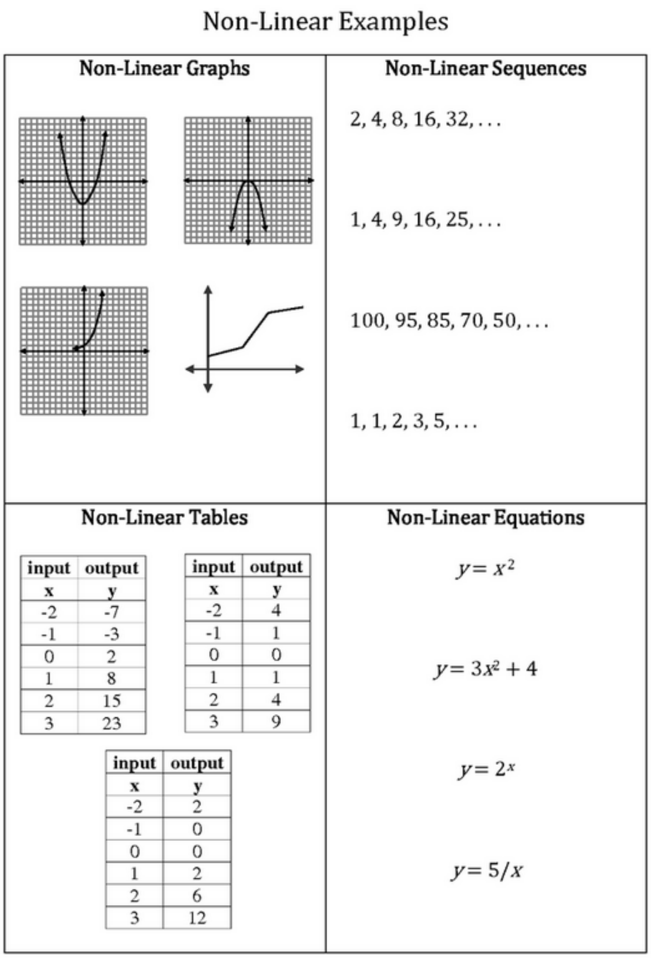


FIGURE 2.28

Example: Non-Linear Functions



Non-Linear Examples

Non-Linear Graphs

Non-Linear Sequences

2, 4, 8, 16, 32, . . .

1, 4, 9, 16, 25, . . .

100, 95, 85, 70, 50, . . .

1, 1, 2, 3, 5, . . .

Non-Linear Tables

input x	output y
-2	-7
-1	-3
0	2
1	8
2	15
3	23

input x	output y
-2	4
-1	1
0	0
1	1
2	4
3	9

input x	output y
-2	2
-1	0
0	0
1	2
2	6
3	12

Non-Linear Equations

$y = x^2$

$y = 3x^2 + 4$

$y = 2^x$

$y = 5/x$

FIGURE 2.29

Practice

1. Using graph paper, graph the function shown by each table. Tell whether the function is a linear or non-linear.
2. Using graph paper, graph the function shown by each table. Tell whether the function is a linear or non-linear.
3. A worker's wages W, in dollars, is a function of the number h of hours worked. Using graph paper, graph the function shown by the table below. Tell whether the function is linear or non-linear.
4. How can you determine if a function is linear or non-linear from the graph of the function?
5. Explain if the set of ordered pairs represents a function. Then indicate if it is a linear or non-linear function.
   (0,2), (1,1), (2,0), (3,-1), (4,-2)
6. Explain if the set of ordered pairs represents a function. Then indicate if it is a linear or non-linear function.
   (0,-1), (1,0), (2,7), (3,26), (4,63)
7. Explain if the set of ordered pairs represents a function. Then indicate if it is a linear or non-linear function.
   (0,1), (1,5), (2,9), (3,13), (4,17)
8. Draw 3 non-linear examples on a graph paper.
9. Which set of ordered pairs represents a non-linear function?

x	y
0	−1
1	0
2	3
3	8

FIGURE 2.30

x	y
0	−4
1	2
2	8
3	14

FIGURE 2.31

Hours, *h*	2	4	6	8	10
Wages ($), *W*	20	40	60	80	100

FIGURE 2.32

a. (0,0), (1,1), (2,2), (3,3), and (4,4)
b. (0,0), (1, -1), (2, -2), and (4, -4)
c. (0,-1), (1,0), (2,1), (3,2), and (4,3)
d. (0,0), (1,1), (2,8), (3,27), and (4,64)

10.

Circle each graph of a *nonlinear* function.

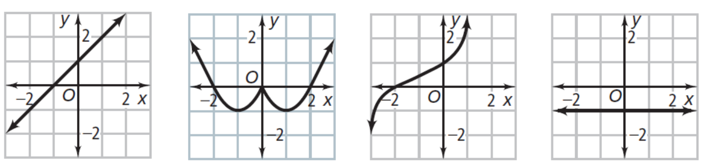


FIGURE 2.33

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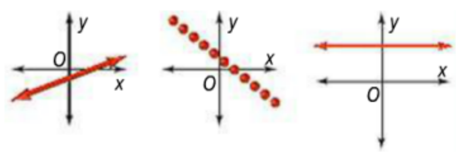
2.4 Patterns and Non-Linear Functions: Representing Patterns and Non-Linear Patterns

Guidance

There are several ways to represent patterns of linear and non-linear functions, such as table or graph.

Linear Function

** a function whose graph is a non-vertical line or part of a non-vertical line



Nonlinear Function

** a function whose graph is NOT a line or part of a line

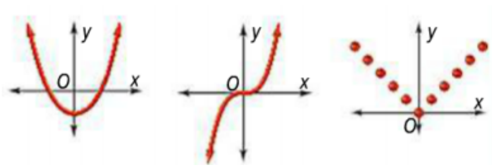


FIGURE 2.34

Guided Practice

1) The area A, in square inches, of a pizza is a function of its radius r, in inches. The Cost C, in dollars, of the sauce for a pizza is a function of the weight w, in ounces, of sauce used. Graph these functions shown by the tables below. Is each function linear or nonlinear? How do you know?

Pizza Area

Radius (in.), r	Area (in.^{2}), A
2	12.57
4	50.27
6	113.10
8	201.06
10	314.16

Sauce Cost

Weight (oz), w	Cost, C
2	$.80
4	$1.60
6	$2.40
8	$3.20
10	$4.00

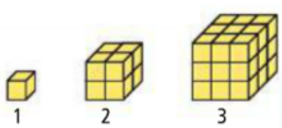
FIGURE 2.35

2) The table below shows the fraction A of the original area of a piece of paper that remains after the paper has been cut in half *n* times. Graph the function represented by the table. Is the function linear or nonlinear? Will the area A ever reach zero? Explain.

Cutting Paper				
Number of Cuts, *n*	1	2	3	4
Fraction of Original Area Remaining, *A*	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

FIGURE 2.36

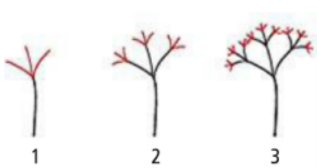
3) The table shows the total number of blocks in each figure below as a function of the number of blocks on one edge. What is the pattern you can use to complete the table? Represent the relationship using an equation and a graph.



Number of Blocks on Edge, *x*	Total Number of Blocks, *y*	Ordered Pair (*x*, *y*)
1	1	(1, 1)
2	8	(2, 8)
3	27	(3, 27)
4	■	■
5	■	■

FIGURE 2.37

4) The table shows the number of new branches in each figure of the pattern below. What is the pattern you can use to complete the table? Represent the relationship using and equation and a graph.



Number of Figure, *x*	1	2	3	4	5
Number of New Branches, *y*	3	9	27	■	■

FIGURE 2.38

Practice

Please see assigned worksheet

2.5 Graphing a Function Rule

Guidance

Some graphs are connected lines or curves called continuous graphs. Some graphs are only distinct points. They are called discrete graphs.

Example discrete graphs:

The graph on theme park attendance is an example of a discrete graph. It consists of distinct points because each year is distinct and people are counted in whole numbers only. The values between whole numbers are not included, since they have no meaning for the situation.

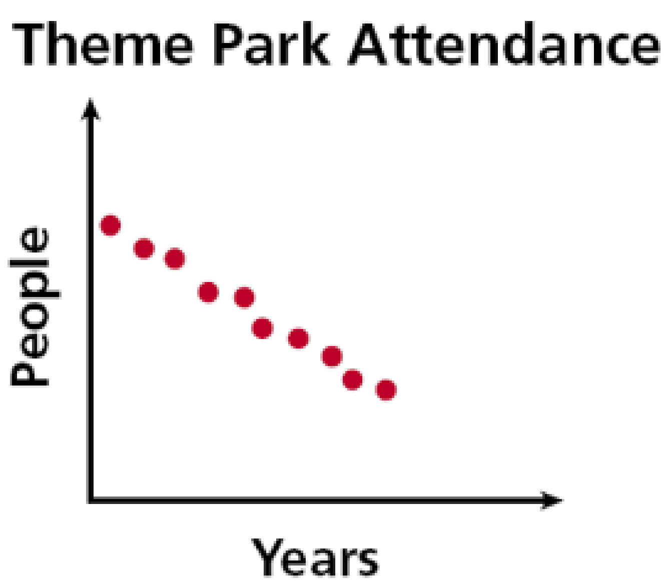


FIGURE 2.39

Example continuous graphs:

The graph of a car approaching a traffic light is an example of a continuous graph. It consists of continuous line and all the points on the line, because any point on the line has meaning. Procedures to Graph a Function Rule:

Practice

For Questions 1 to 6: Make a table of values for each function. Graph each function rule. 1. y = -x + 3 2. y = .25x 3. y = 5x - 2 4. y = 2 - x 5. y = .50x 6. y = 3x + 1 For Questions 7 to: Graph each function rule. Tell whether the graph is continuous or discrete. 7. The cost d, in dollars, for a parking pass depends on the number of whole weeks w you purchase. This situation is represented by the function rule $d = 25w$. 8. The price p, in dollars, for apples

FIGURE 2.40

Procedure: Graphing Functions	
Step 1	Use the function to generate ordered pairs by choosing several values for *x*.
Step 2	Plot enough points to see a pattern for the graph.
Step 3	Connect the points with a line or smooth curve.

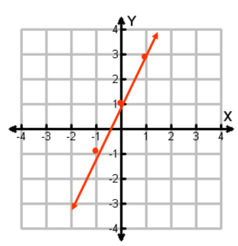
FIGURE 2.41

Example: Graphing Linear Functions

Graph $y = 2x + 1$.

Step 1 Choose **three values of *x*** and generate ordered pairs.

x	$y = 2x + 1$	(x, y)
1	$y = 2(1) + 1 = 3$	$(1, 3)$
0	$y = 2(0) + 1 = 1$	$(0, 1)$
−1	$y = 2(-1) + 1 = -1$	$(-1, -1)$



Step 2 Plot the points and connect them with a straight line.

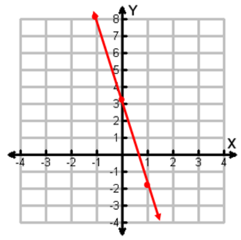
FIGURE 2.42

Example: Graphing Linear Functions

Graph $15x + 3y = 9$.

Step 1 Choose **three values of x** and generate ordered pairs

x	$y = -5x + 3$	(x, y)
1	$y = -5(1) + 3 = -2$	$(1, -2)$
0	$y = -5(0) + 3 = 3$	$(0, 3)$
−1	$y = -5(-1) + 3 = 8$	$(-1, 8)$



Step 2 Plot the points and connect them with a straight line.

FIGURE 2.43

Example: Graphing Nonlinear Functions

Graph the function $g(x) = |x| + 2$.

Step 1 Choose several values of x and generate ordered pairs.

x	$g(x) = \|x\| + 2$	$(x, g(x))$
−2	$g(x) = \|-2\| + 2 = 4$	$(-2, 4)$
−1	$g(x) = \|-1\| + 2 = 3$	$(-1, 3)$
0	$g(x) = \|0\| + 2 = 2$	$(0, 2)$
1	$g(x) = \|1\| + 2 = 3$	$(1, 3)$
2	$g(x) = \|2\| + 2 = 4$	$(2, 4)$
3	$g(x) = \|3\| + 2 = 5$	$(3, 5)$

FIGURE 2.44

depends on the weight p, in pounds, of the apples. This situation is represented by the function rule $p = 1.99w$. 9. The cost c, in dollars, for a health club membership depends on the number m of whole months you join. This situation is represented by the function rule $c = 49 + 20m$. 10. The cost c, in collars, for bananas depends on the weight w, in pounds, of the bananas. This situation is represented by the function rule $c = 0.5w$.

Step 2 Plot enough points to see a pattern.

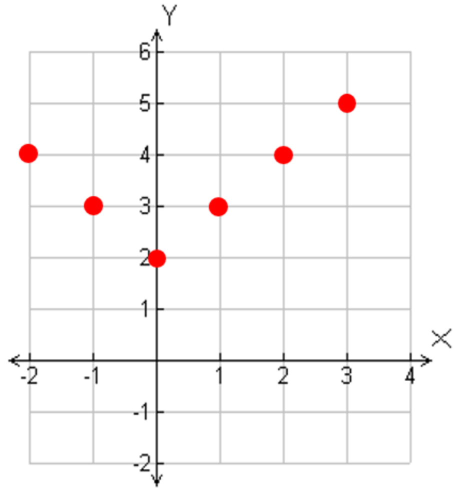


FIGURE 2.45

Step 3 The ordered pairs appear to form a v-shape. Draw lines through all the points to show all the ordered pairs that satisfy the function. Draw arrowheads on the “ends” of the “v”.

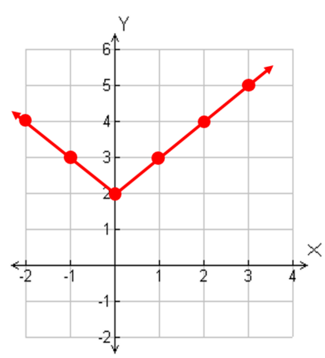


FIGURE 2.46

2.6 Writing a Function Rule

Guidance

When writing function rules for verbal descriptions, you should look for key words.

TABLE 2.1:

Words that Suggest Addition	Words that Suggest Subtraction	Words that Suggest Multiplication	Words that Suggest Division
plus	minus	times	divided by
sum	difference	product	quotient
more than	less than	of	rate
increased by	decreased by	each	rate
total	fewer than	factors	half
in all	subtracted by	twice	a third of

Example:

Twice a number n increased by 4 equals m. What is a function rule that represents the sentence?

Twice a number n —>$2n$

increased by 4 —>+4

equals —>=

m —>m

The function rule is $2n + 4 = m$.

Example:

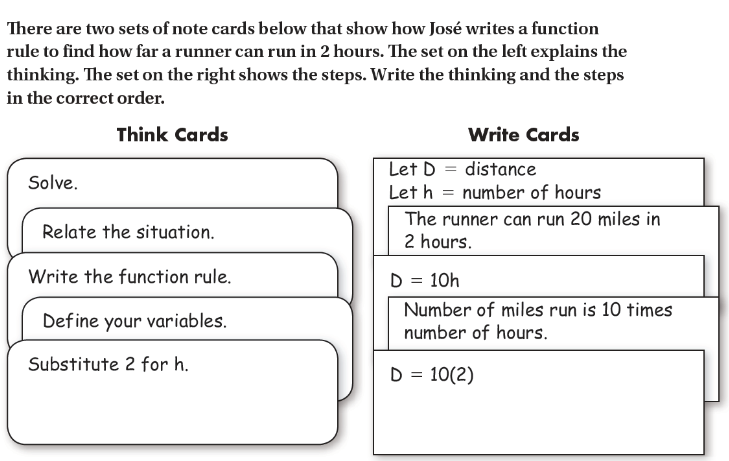


FIGURE 2.47

Think	Write
First, relate the situation.	Step 1 Number of miles run is 10 times number of hours
Second, define your variables.	Step 2 Let D = distance Let h = number of hours
Next, write the function rule.	Step 3 $D = 10h$
Then, substitute 2 for h.	Step 4 $D = 10(2)$
Finally, solve.	Step 5 The runner can run 20 miles in 2 hours.

FIGURE 2.48

Practice

1. 7 more than the quotient of a number n and 4 is 9.

2. P is 9 more than half of q.

3. 5 less than one fourth of x is y.

4. 1.5 more than the quotient a and 4 is b.

5. 8 more than 5 times a number is -27.

6. A taxicab charges \$4.25 for the first mile and \$1.50 for each additional mile. Write a rule for describing the total rate r as a function of the total miles m. What is the taxi rate for 12 miles?

7. A hot dog d costs \$1 more than one half the cost of a hamburger h.

8. The price p of an ice cream is \$3.95 plus \$0.85 for each topping t on the ice cream.

9. A plumber's fees f are \$75 for a house call and \$60 per hour h for each hour worked.

10. A babysitter's earnings e are a function of the number of hours n worked at a rate of \$7.25 an hour.

WriWr

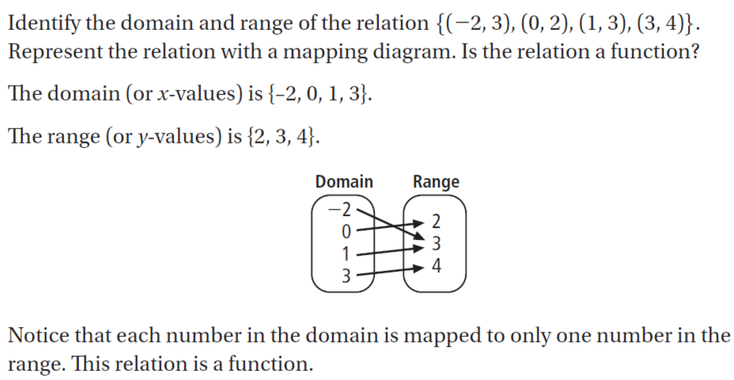
2.7 Formalizing Relations and Functions

Guidance

When a relation is represented as a set of ordered pairs, the domain of the relation is the set of x-values. The range is the set of y-values.

A relation where each value in the domain is paired with just one value in the range is called a function.

Example:


Identify the domain and range of the relation $\{(-2, 3), (0, 2), (1, 3), (3, 4)\}$. Represent the relation with a mapping diagram. Is the relation a function?

The domain (or x-values) is {-2, 0, 1, 3}.

The range (or y-values) is {2, 3, 4}.

Notice that each number in the domain is mapped to only one number in the range. This relation is a function.

FIGURE 2.49

Example:

When identifying the domain and range of each relation, use a mapping diagram to determine whether the relation is a function.

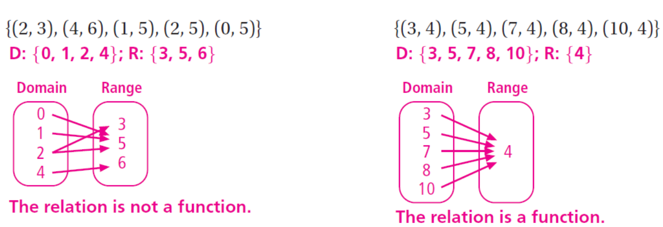


FIGURE 2.50

You can determine whether or not a relation is a function by looking at the graph of the relation. If a vertical line is drawn anywhere on the graph and passes through two points of the relation, the relation is not a function. This called the vertical line test.

Example:

Example:

A vertical line test was conducted on the following graphs.

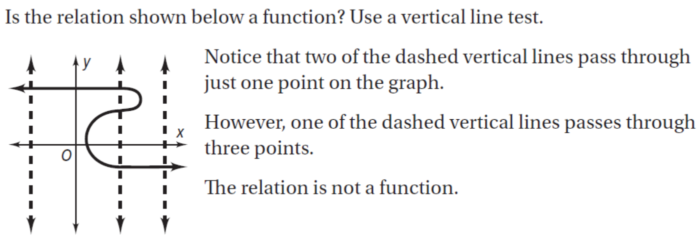


FIGURE 2.51

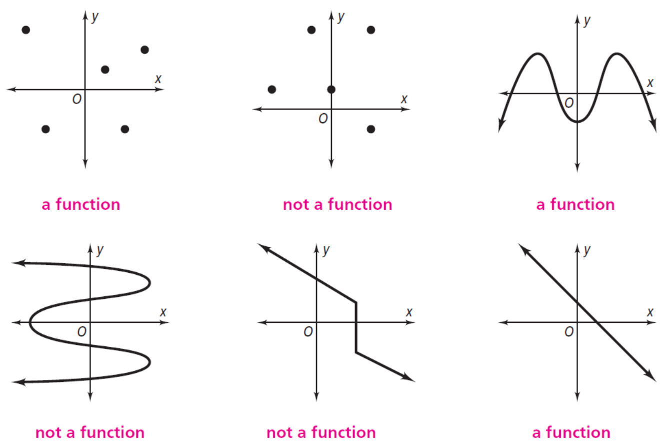


FIGURE 2.52

Guided Practice

Determine if each relation is a function:

1) $(-1,4),(0,3),(1,5),(1,7),(2,15)$
2) $y = x$
3) $(2,0),(4,-1),(2.1,4),(1,4),(4,-1)$
4) $y = 4x$
5) $x = |y|$

Answers

1) There are two different 'outputs' or y-values for the 'input' or x-value of 1. Because we cannot know whether 1 should go with 5 or 7 at any given time, this relation is **not** a function.

2) Since $y = x$, any time a number is chosen to represent x, that, and only that, number becomes y. From this it is apparent that each input has one and only one output: This relation **is** a function.

3) Don't be fooled! This is a function, there is only one unique output for each input. The fact that both x values 2.1 and 1 are associated with y value 4 does not mean that 2.1 and 1 don't have a specific associated value. Also, not matter how close two x's (2 and 2.1, for instance) may be, if they are not *exactly* the same, they don't affect the definition of a function.

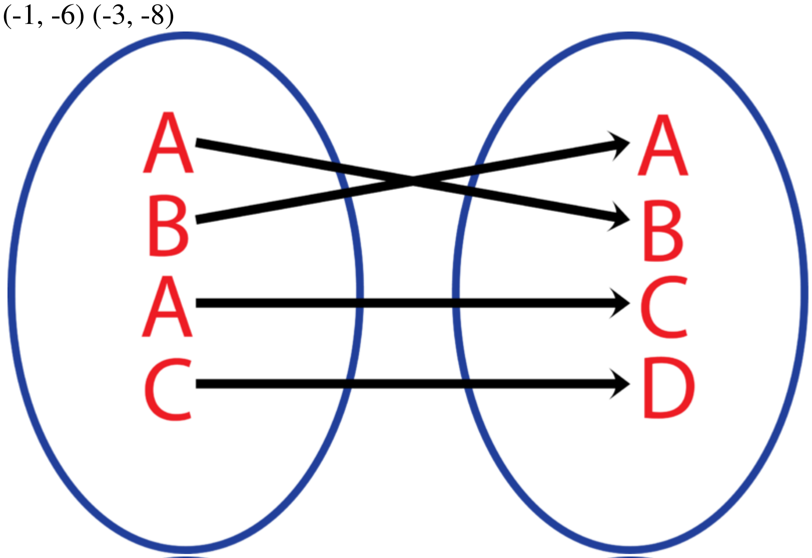
4) This is a function, very similar to #2. Any value chosen for x has one and only one associated value for y (4 times as big).

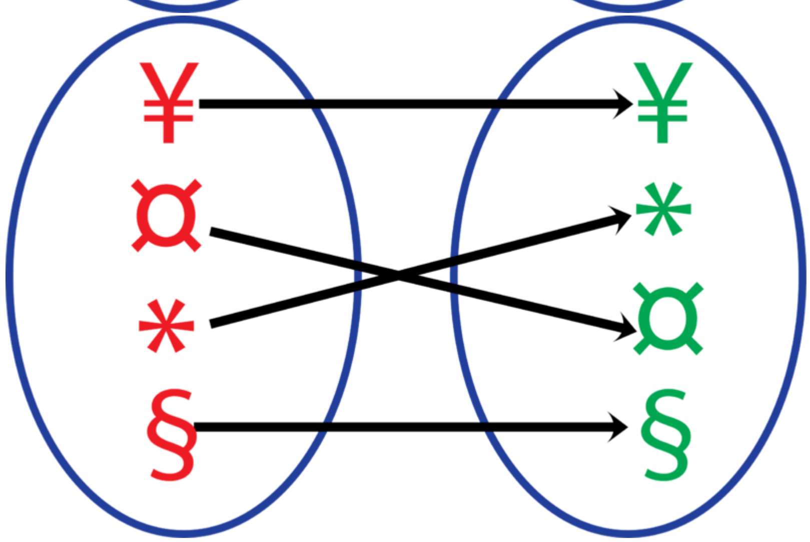
5) This is *not* a function. This graph looks like a "<", with the point on the origin. Any value chosen for x will have 2 associated y values. For instance: 4 = |-4| *and* 4 = |4|.

Practice

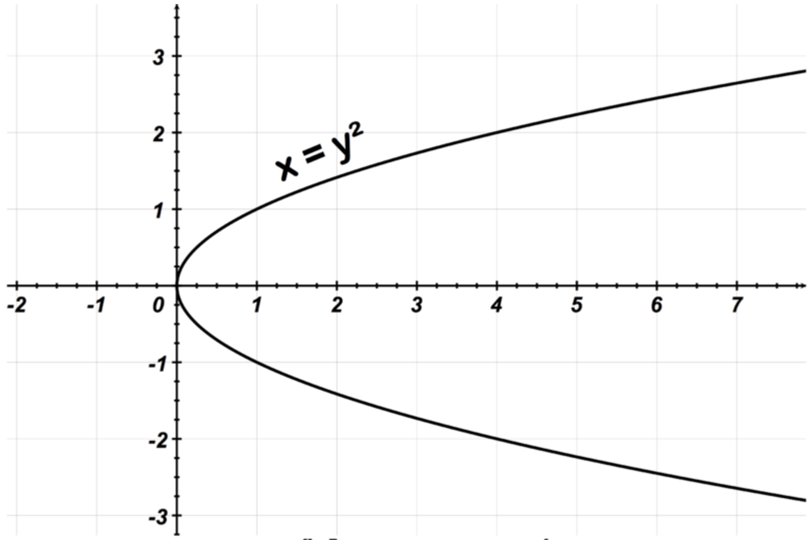
1. What is the definition of a function?
2. Can a function definition be written in the form $x = 3y$ instead of $y = 3x$?
3. Is it mandatory for a function to have both an input and an output?
4. Can a statement be a function if there is only one input and output?
5. Give an example of a relation that is not a function, and explain why it is not a function.

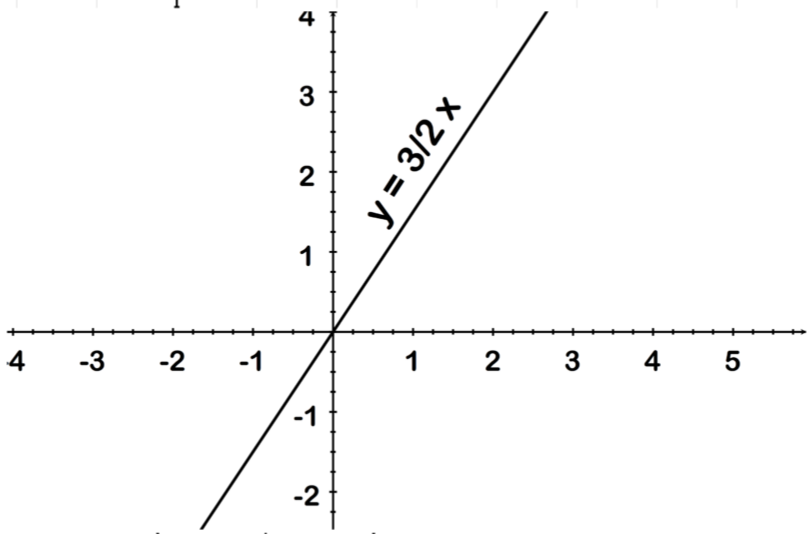
For Questions 6 - 14, identify each relation as either a function, or not a function:

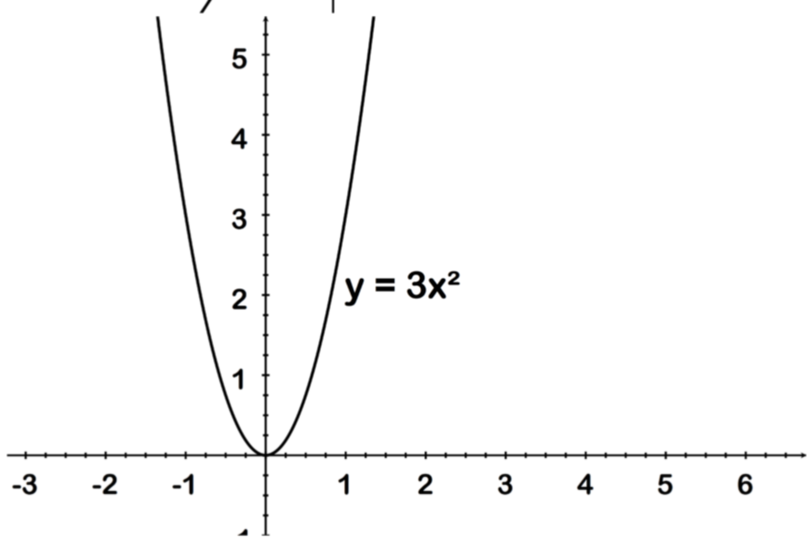
6. (2, 4) (4, 6) (6, 8) (3, 4) (5, 7) (8, 2)
7. (-1, 6) (0, 4) (-4, 0) (-1, -6) (-3, -8)
8. 

9. 

10. (Jim, Kitty) (Joe, Betty) (Brian, Alice) (Jesus, Anissa) (Ken, Kelli)
11. (Jim, Alice) (Joe, Alice) (Brian, Betty) (Jim, Kitty) (Ken, Anissa)

12. 

13. 

14. 

15. At a Prom dance, each boy pins a corsage on his date. Is this an example of a function?
16. Later, at the same dance, Cory shows up with two dates, does this change the answer?

2.8 Using Function Notation

Guidance

Instead of purchasing a one-day ticket to the theme park, Joseph decided to pay by ride. Each ride costs $2.00. To describe the amount of money Joseph will spend, several mathematical concepts can be used.

First, an expression can be written to describe the relationship between the cost per ride and the number of rides, r. An equation can also be written, if the total amount he wants to spend is known. An inequality can be used if Joseph wanted to spend less than a certain amount.

Example A

Using Joseph's situation, write the following:

a. An expression representing his total amount spent

b. An equation representing his total amount spent

c. An equation that shows Joseph wants to spend exactly $22.00 on rides

d. An inequality that describes the fact that Joseph will not spend more than $26.00 on rides

Solution: The variable in this situation is the number of rides Joseph will pay for. Call this r.

a. $2(r)$

b. $2(r) = m$

c. $2(r) = 22$

d. $2(r) \leq 26$

In addition to an expression, equation, or inequality, Joseph's situation can be expressed in the form of a function or a table.

Definition: A **function** is a relationship between two variables such that the input value has ONLY one output value.

Writing Equations as Functions

A function is a set of ordered pairs in which the first coordinate, usually x, matches with exactly one second coordinate, y. Equations that follow this definition can be written in function notation. The y coordinate represents the **dependent variable**, meaning the values of this variable depend upon what is substituted for the other variable.

Consider Joseph's equation $m = 2r$. Using function notation, the value of the equation (the money spent, represented by m) is replaced with $f(r)$. f represents the function name and (r) represents the variable. In this case the parentheses do not mean multiplication; they separate the function name from the **independent variable.**

$$\overset{\displaystyle input}{\overset{\downarrow}{\underbrace{f(x)}_{\substack{function\\ box}}}} = y \leftarrow output$$

Example B

Rewrite the following equations in function notation.

a. $y = 7x - 3$

b. $d = 65t$

c. $F = 1.8C + 32$

Solution:

a. According to the definition of a function, $y = f(x)$, so $f(x) = 7x - 3$.

b. This time the dependent variable is d. Function notation replaces the dependent variable, so $d = f(t) = 65t$.

c. $F = f(C) = 1.8C + 32$

Why Use Function Notation?

Why is it necessary to use function notation? The necessity stems from using multiple equations. Function notation allows one to easily decipher between the equations. Suppose Joseph, Lacy, Kevin, and Alfred all went to the theme park together and chose to pay \$2.00 for each ride. Each person would have the same equation $m = 2r$. Without asking each friend, we could not tell which equation belonged to whom. By substituting function notation for the dependent variable, it is easy to tell which function belongs to whom. By using function notation, it will be much easier to graph multiple lines.

Example C

Write functions to represent the total each friend spent at the park.

Solution:

$J(r) = 2r$ represents Joseph's total,

$L(r) = 2r$ represents Lacy's total,

$K(r) = 2r$ represents Kevin's total, and

$A(r) = 2r$ represents Alfred's total.

Guided Practice

Students are selling t-shirts to raise money for a school trip. The cost of printing the shirts is expressed as $100+7x$ and the revenue, has the expression $15x$, where x is the number of shirts.

a. Write two functions, one for the cost and one for revenue.

b. Express that the cost must be less than or equal to $800.

c. Express that the revenue must be equal to $1500.

d. How many shirts must the students sell in order to make $1500?

Solution:

a. The cost function we will write as $C(x) = 100+7x$ and the revenue function we will write as $R(x) = 15x$.

b. Since $C(x)$ represents the costs, we substitute in $800 for $C(x)$ and replace the equation with the appropriate inequality symbol

$$100+7x \leq 800$$

This reads that $100+7x$ is less than or equal to $800, so we have written the inequality correctly.

c. We substitute in $1500 for $R(x)$, getting

$1500 = 15x$.

d. We want to find the value of x that will make this equation true. It looks like 100 is the answer. Checking this (see below) it is clear that 100 does satisfy the equation. The students must sell 100 shirts in order to have a revenue of $1500.

$1500 = 15(100)$

$1500 = 1500$

Practice

1. Rewrite using function notation: $y = \frac{5}{6}x - 2$.
2. Rewrite using function notation: $m = n^2 + 2n - 3$.
3. What is one benefit of using function notation?
4. Write a function that expresses the money earned after working some number of hours for $10 an hour.
5. Write a function that represents the number of cuts you need to cut a ribbon in x number of pieces.
6. Jackie and Mayra each will collect a $2 pledge for every basket they make during a game. Write two functions, one for each girl, expressing how much money she will collect.

Vocabulary:

Using Graphs to Relate Two Quantities: Quantity, x-axis, y-axis

Patterns and Linear Functions: Dependent Variable, Function, Independent Variable, Input, Linear, Function, Output

Patterns and Non-Linear Functions: Function, Linear Function, Nonlinear Function

Graphing a Function Rule: Dependent Variable, Independent Variable, Continuous Graph, Discrete Graph

Writing a Function Rule: Input, Nonlinear Function, Output

Formalizing Relations and Functions: Domain, Range, Relation, Vertical Line Test

Using Function Notation: Domain, Range, Function

CHAPTER 3 Linear Equations and Functions

Chapter Outline

Rate of Change and Slope: A.2g, A.3a, A.3b

Direct Variation: A.2c, A.2d, A.3c, A.12e

Slope-Intercept Form: A.2b, A.2c, A.3a, A.3b, A.3c

Point-Slope Form: A.2b, A.2c, A.3a, A.3b, A.3c, A.12e

Standard Form: A.2a, A.2b, A.2c, A.2g, A.3a, A.3c

Parallel and Perpendicular Lines: A.2b, A.2c, A.2e, A.2f, A.3a

Transformations of Linear Functions: A.3c, A.3e, A.12b

Scatter Plots and Trend Lines: A.2b, A.2c, A.3a, A.3b, A.4a, A.4b, A.4c

3.1 Rate of Change and Slope

Guidance

Definition of Slope

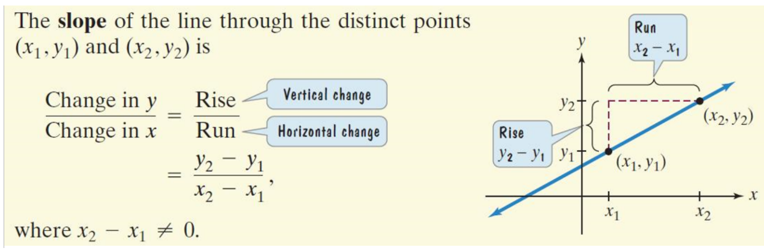


FIGURE 3.1

Finding Slope Using Points

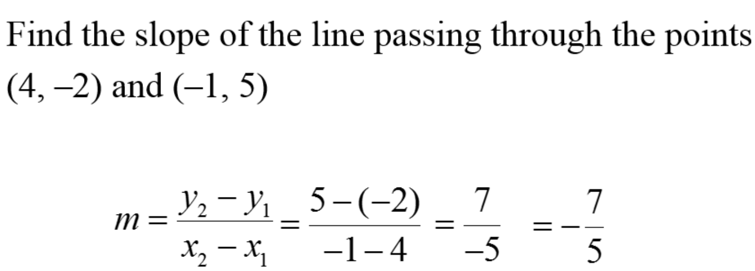
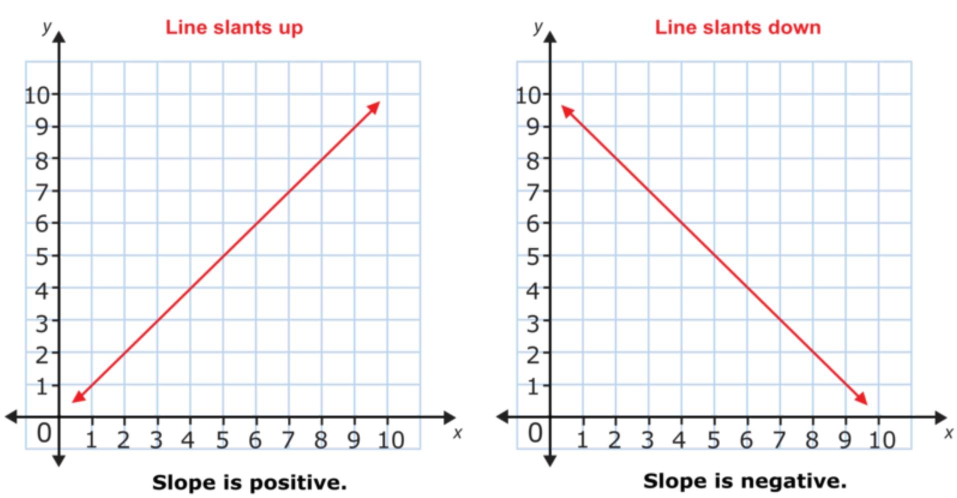


FIGURE 3.2

Knowing some basic information about the slope of a line can tell you about its slant **.**

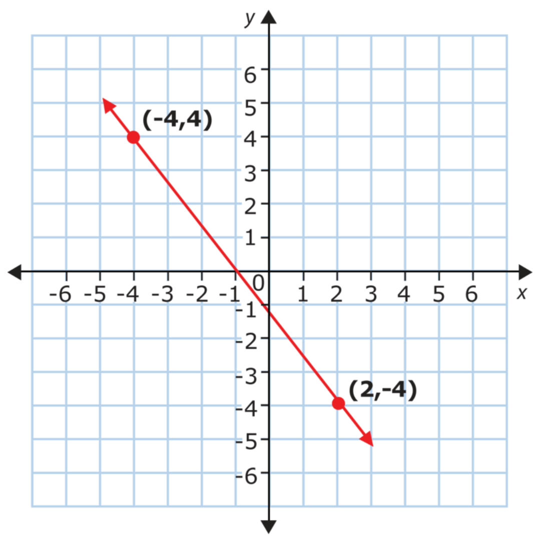


- A line that slants *up* from left to right has a ***positive slope.***
- A line that slants *down* from left to right has a ***negative slope.***

Determine if the slope of each line shown below is positive or negative.

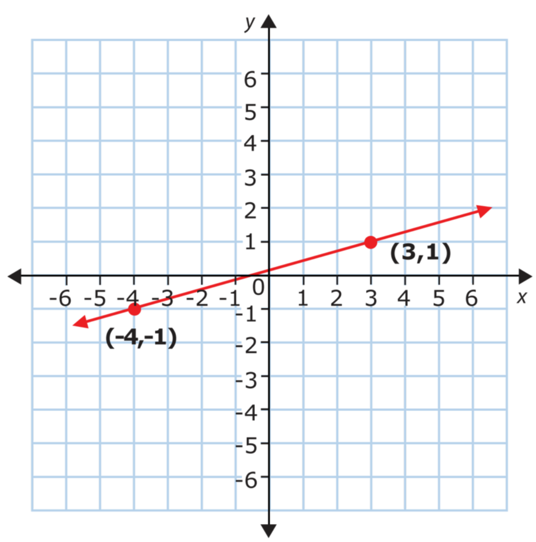
a.

Line through (-4,4) and (2,-4)



b.

Line through (-4,-1) and (3,1)



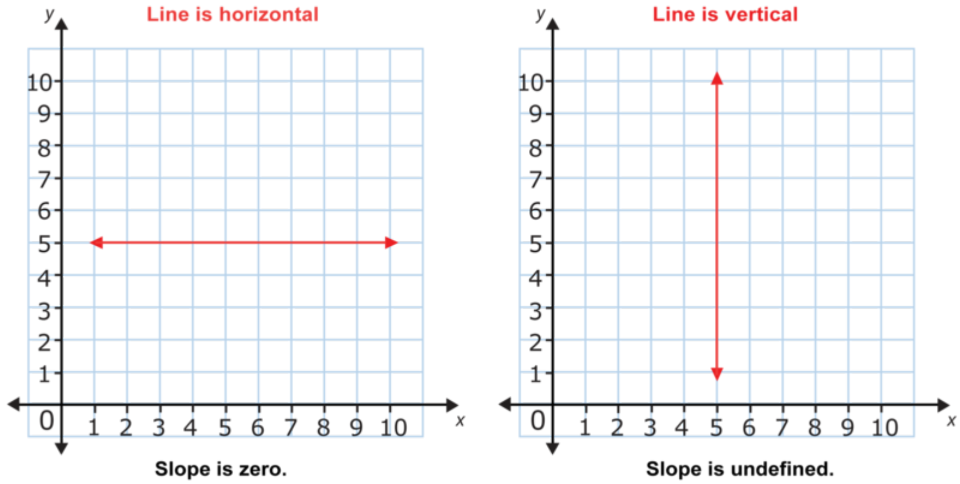
Consider the line in *a*.

The line slants *down* from left to right, so its slope is negative.

Consider the line in *b*.

The line slants *up* from left to right, so its slope is positive.

You should also know about the slopes of horizontal and vertical lines.



- A horizontal line has a run, but does not have a rise.

slope $= \frac{rise}{run} = \frac{0}{n} = 0$.

So, the slope of a horizontal line is zero.

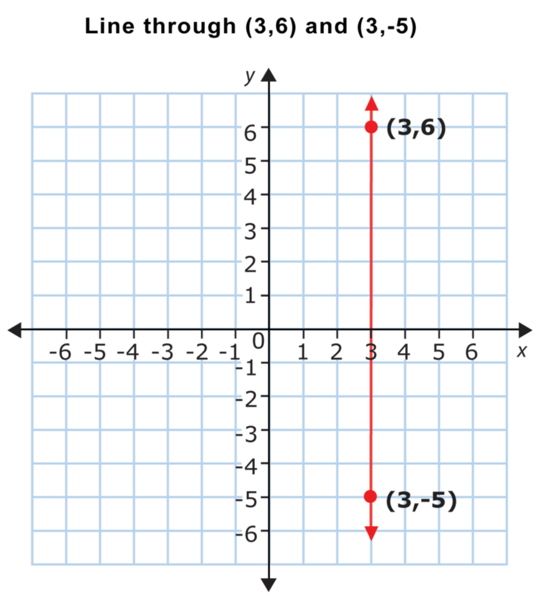
- A vertical line has a rise, but does not have a run.

slope $= \frac{rise}{run} = \frac{n}{0} = undefined$.

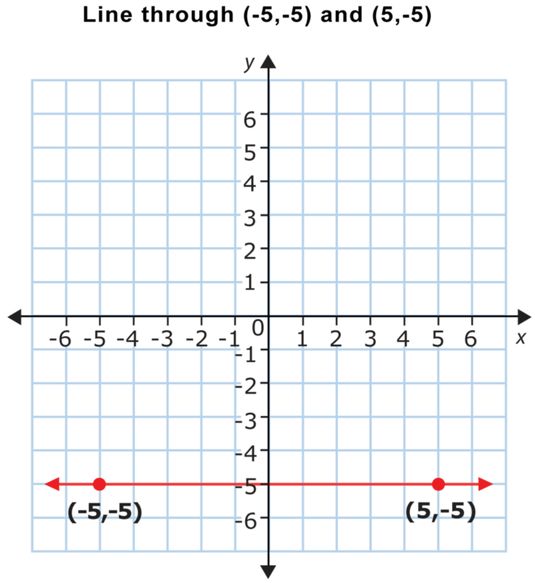
Any fraction with a zero in the denominator is undefined. So, the slope of a vertical line is undefined.

Identify the slope of each line shown below.

a.



b.



Consider the line in *a*.

The line is vertical, so its slope is undefined.

Consider the line in *b*.

The line is horizontal, so its slope is zero.

Now it is your turn to think about slope. Answer the following questions.

Example A

What is the slope of a horizontal line?

Solution: 0

Example B

What is the slope of a vertical line?

Solution: Undefined

Example C

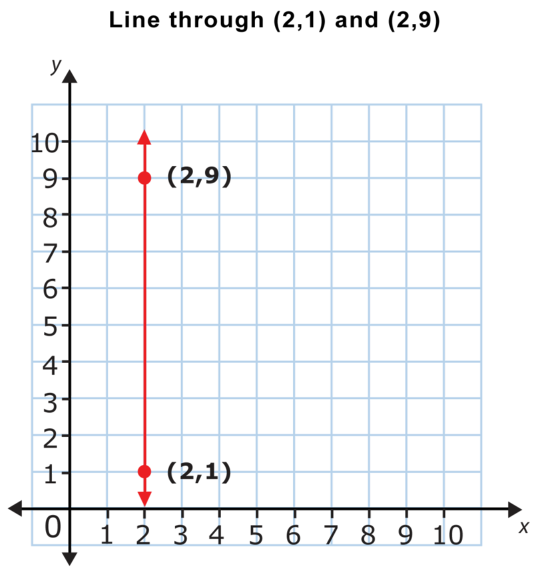
What is the slope of a line that goes up from left to right?

Solution: Positive

Guided Practice

Here is one for you to try on your own.

Is this slope positive, negative or undefined?



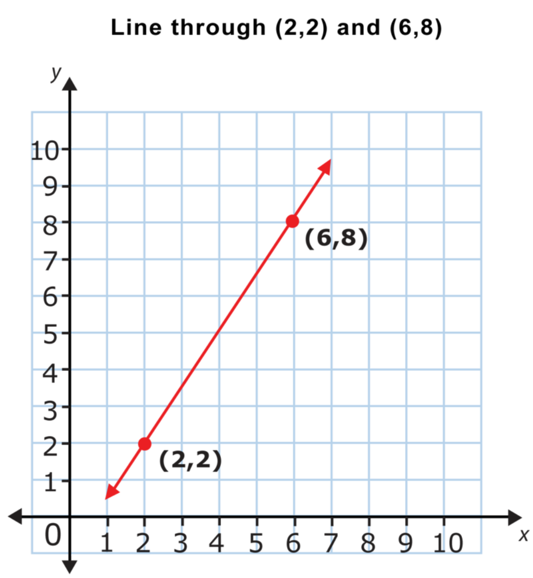
Answer

This line is vertical, therefore the slope of the line is undefined.

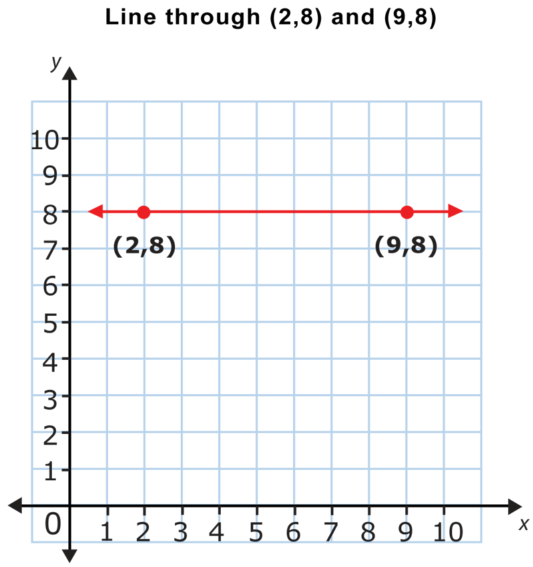
Practice

Directions: For each graph, find the slope using $\frac{rise}{run}$ and then tell if the slope of the line shown is positive, negative, zero, or undefined.

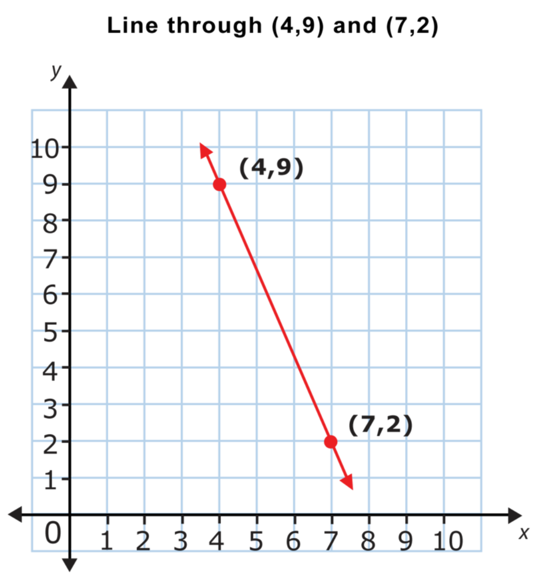
1.



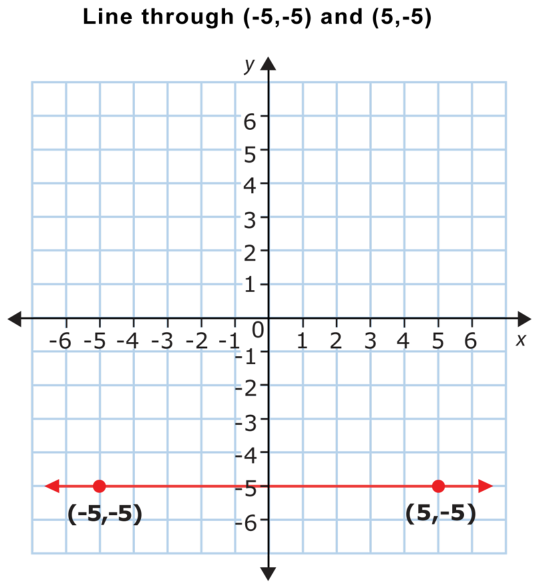
2.


Line through (2,8) and (9,8)
y
x
0
1 2 3 4 5 6 7 8 9 10
1 2 3 4 5 6 7 8 9 10
(2,8)
(9,8)

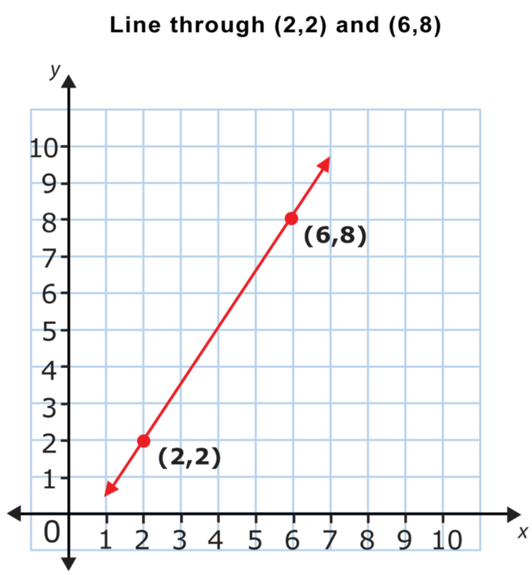
3.


Line through (4,9) and (7,2)
y
x
0
1 2 3 4 5 6 7 8 9 10
1 2 3 4 5 6 7 8 9 10
(4,9)
(7,2)

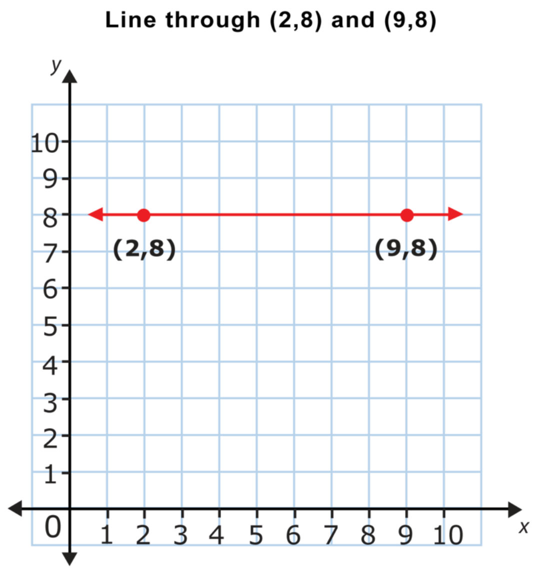
4.



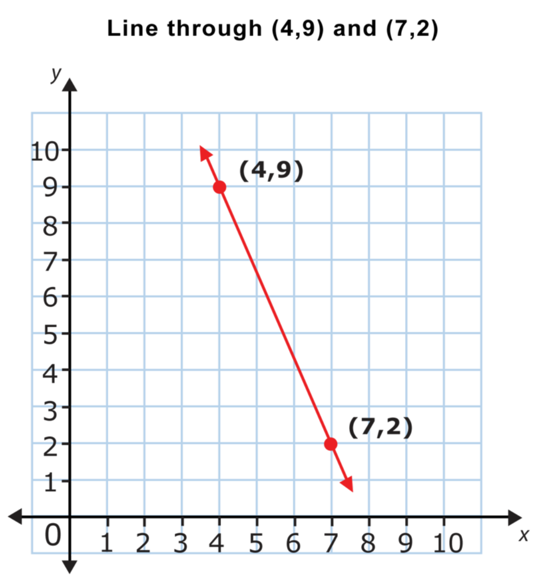
5.



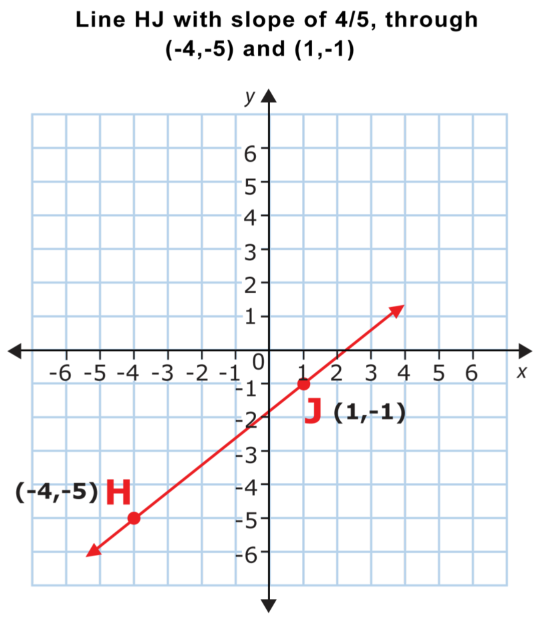
6.


Line through (2,8) and (9,8)
y
x
0
1
2
3
4
5
6
7
8
9
10
(2,8)
(9,8)

7.

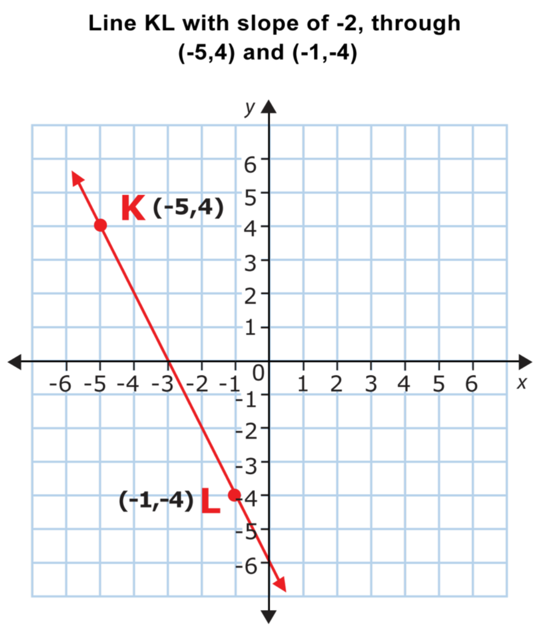

Line through (4,9) and (7,2)
y
x
0
1
2
3
4
5
6
7
8
9
10
(4,9)
(7,2)

8.



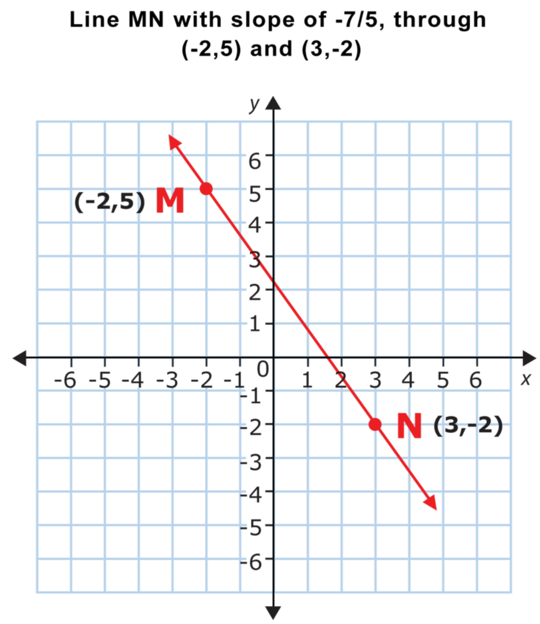
Line HJ with slope of 4/5, through
(-4,-5) and (1,-1)
y
x
J (1,-1)
(-4,-5) H

9.



Line KL with slope of -2, through
(-5,4) and (-1,-4)
y
x
K (-5,4)
(-1,-4) L

10.



Directions: Answer each question.

11. Does a positive slope have to contain positive numbers?
12. True or false. A horizontal line is undefined.
13. True or false. A negative slope goes down from right to left.
14. True or false. A vertical line has an undefined slope.
15. True or false. You can figure out any slope as long as the line has some slant to it.

3.2 Direct Variation

Guidance

At the local farmer's market, you saw someone purchase 5 pounds of strawberries and pay \$12.50. You want to buy strawberries too, but you want only 2 pounds. How much would you expect to pay?

This situation is an example of a **direct variation**. You would expect that the strawberries are priced on a "per pound" basis, and that if you buy two-fifths of the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries, or \$5.00. Similarly, if you bought 10 pounds of strawberries (twice the amount), you would pay \$25.00 (twice \$12.50), and if you did not buy any strawberries you would pay nothing.

Direct variation can be expressed as the equation $y = (k)x$, where k is called the constant of proportionality.

Direct variation occurs when:

- The fraction $\frac{rise}{run}$ or $\frac{change\ in\ y}{change\ in\ x}$ is always the same, and
- The ordered pair (0, 0) is a solution to the equation.

Example A

If y varies directly with x according to the relationship $y = k \cdot x$, and $y = 7.5$ when $x = 2.5$, determine the constant of proportionality, k.

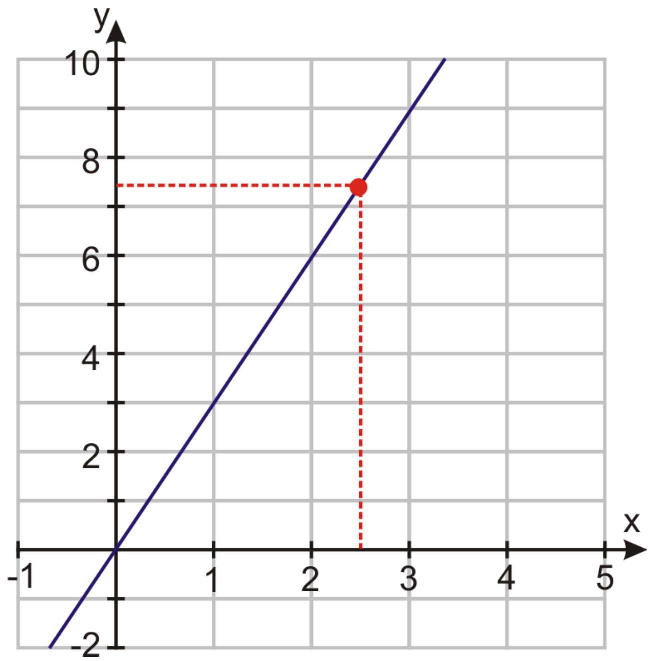
Solution: We can solve for the constant of proportionality using substitution.

Substitute $x = 2.5$ and $y = 7.5$ into the equation $y = k \cdot x$.

$$7.5 = k(2.5) \qquad \text{Divide both sides by 2.5.}$$
$$\frac{7.5}{2.5} = k = 3$$

The constant of variation (or the constant of proportionality) is 3.

You can use this information to graph this direct variation situation. Remember that all direct variation situations cross the origin. You can plot the ordered pair (0, 0) and use the constant of variation as your slope.



Example B

Explain why each of the following equations are not examples of direct variation.

$$y = \frac{2}{x}$$
$$y = 5x - 1$$
$$2x + y = 6$$

Solution: In equation 1, the variable is in the denominator of the fraction, violating the definition.

In equation 2, there is a $y-$intercept of –1, violating the definition.

In equation 3, there is also a $y-$intercept, violating the definition.

Translating a Sentence into a Direct Variation Equation

Direct variation equations use the same phrase to give the reader a clue. The phrase is either "directly proportional" or "varies directly."

Example C

The area of a square varies directly as the square of its side.

Solution: The first variable you encounter is "area." Think of this as your y. The phrase "varies directly" means $= (k)\times$. The second variable is "square of its side." Call this letter s.

Now translate into an equation: $y = (k) \times s^2$.

You've written your first direct variation equation.

Guided Practice

The distance you travel is directly proportional to the time you have been traveling. Write this situation as a direct variation equation.

Solution:

The first variable is *distance*; call it d. The second variable is the *time you have been traveling*; call it t. Apply the direct variation definition:

$$d = (k) \times t$$

Practice

1. Describe direct variation.
2. What is the formula for direct variation? What does k represent?

Translate the following direct variation situations into equations. Choose appropriate letters to represent the varying quantities.

3. The amount of money you earn is directly proportional to the number of hours you work.
4. The weight of an object on the Moon varies directly with its weight on Earth.
5. The volume of a gas is directly proportional to its temperature in Kelvin.
6. The number of people served varies directly with the amount of ground meat used to make burgers.
7. The amount of a purchase varies directly with the number of pounds of peaches.

Explain why each equation is not an example of direct variation.

8. $\frac{4}{x} = y$
9. $y = 9$
10. $x = -3.5$
11. $y = \frac{1}{8}x + 7$
12. $4x + 3y = 1$

Graph the following direct variation equations.

13. $y = \frac{4}{3}x$
14. $y = -\frac{2}{3}x$
15. $y = -\frac{1}{6}x$
16. $y = 1.75x$
17. Is $y = 6x - 2$ an example of direct variation? Explain your answer.

Mixed Review

18. Graph $3x + 4y = 48$ using its intercepts.
19. Graph $y = \frac{2}{3}x - 4$.
20. Solve for $u : 4(u + 3) = 3(3u - 7)$.
21. Are these lines parallel? $y = \frac{1}{2}x - 7$ and $2y = x + 2$
22. In which quadrant is (–99, 100)?
23. Find the slope between (2, 0) and (3, 7).
24. Evaluate if $a = -3$ *and* $b = 4$: $\frac{1+4b}{2a-5b}$.

3.3 Slope-Intercept Form

Guidance

We have seen linear equations in function form, have created tables of values and graphs to represent them, looked at their x- and y-intercepts, and studied their slopes. One of the most useful forms of a linear equation is the ***slope-intercept form*** which we will be using with standard form in this Concept.

Remember ***standard form***?

The standard form of an equation is when the equation is written in $Ax+By=C$ form.

This form of the equation allows us to find many possible solutions. In essence, we could substitute any number of values for x and y and create the value for C. When an equation is written in standard form, it is challenging for us to determine the slope and the y – intercept.

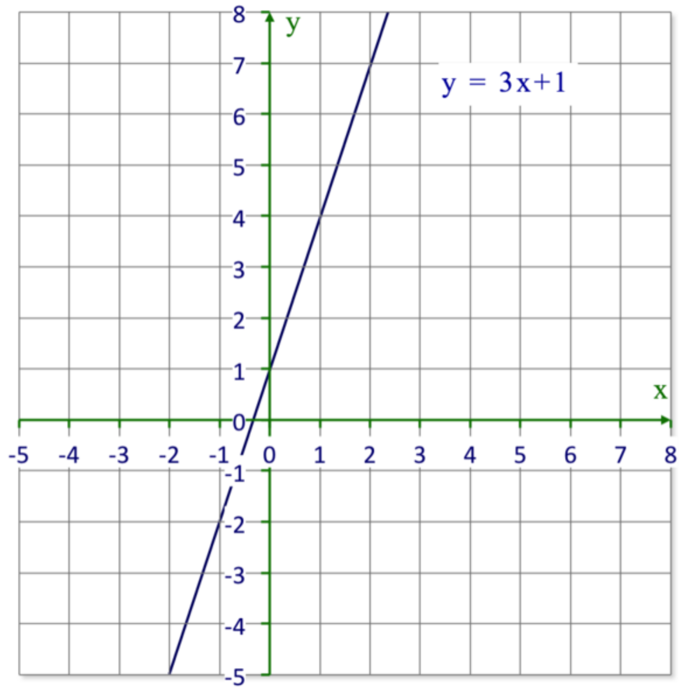
Think back, remember that the *slope* is the steepness of the line and the y – *intercept* is the point where the line crosses the y – axis.

We can write an equation in a different form than in standard form. This is when $y =$ an equation. We call this form of an equation *slope – intercept form* .

Slope – Intercept Form is $y = mx+b$ – where m is the slope and b is the y – intercept.

Take a look at this graph and equation.

Graph the line $y = 3x+1$



Here we can calculate the slope of the line using the rise over the run and see that it is 3. The y – intercept is 1. Notice that we can find these values in our equation too.

When an equation is in slope – intercept form, we can spot the slope and the y – intercept by looking at the equation.

$y = mx + b$

Here m is the value of the slope and b is the value of the y – intercept.

For any equation written in the form $y = mx + b$, m is the slope and b is the y-intercept. For that reason, $y = mx + b$ is called the *slope-intercept form.* Using the properties of equations, you can write any equation in this form.

Because we can use slope – intercept form, we can rewrite equations in standard form into slope – intercept form. Then we can easily determine the slope and y – intercept of each equation.

Take a look here.

Write $4x + 2y = 6$ in slope – intercept form. Then determine the slope and the y – intercept by using the equation.

$$\begin{aligned}
4x + 2y &= 6 \\
4x + 2y - 2y &= 6 - 2y \\
4x &= 6 - 2y \\
4x - 6 &= -2y \\
\frac{4x - 6}{-2} &= y \\
y &= -2x + 3
\end{aligned}$$

Now we can determine the slope and the y – intercept from the equation.

$$-2 = \text{slope}$$
$$3 = y - \text{intercept}$$

Think back to our work with functions. Remember how we could write a function in function form? Well take a look at function form compared with slope – intercept form.

Function form $= f(x) = 2x+1$

Slope – Intercept Form $= y = 2x+1$

Yes! The two are the same. These two equations are equivalent!

Determine the slope and the y-intercept in each equation.

Example A

$y = x+4$

Solution: slope = 1, y-intercept = 4

Example B

$2x+y = 10$

Solution: slope = -2, y-intercept = 10

Example C

$-3x+y = 9$

Solution: slope = 3, y-intercept = 9

Now let's go back to the dilemma at the beginning of the Concept.

$y = -2x-8$

Looking at this equation, you can see that the slope is -2 **and the y-intercept is** 8**.**

Guided Practice

Here is one for you to try on your own.

Write this equation in slope-intercept form and then determine the slope and the y-intercept.

$$\begin{aligned}
-18x+6y &= 12\\
-18x+6y &= 12\\
-18x+6y+18x &= 18x+12\\
6y &= 18x+12\\
6y &= 18x+12\\
\frac{18x+12}{6} &= y\\
y &= 3x+2
\end{aligned}$$

Given this equation, the slope is 3 and the y-intercept is 2.

Practice

Directions: Look at each equation and identify the slope and the y – intercept by looking at each equation. There are two answers for each problem.

1. $y = 2x + 4$
2. $y = 3x - 2$
3. $y = 4x + 3$
4. $y = 5x - 1$
5. $y = \frac{1}{2}x + 2$
6. $y = -2x + 4$
7. $y = -3x - 1$
8. $y = \frac{-1}{3}x + 5$

Directions: Use what you have learned to write each in slope – intercept form and then answer each question.

9. $2x + 4y = 12$
10. Write this equation in slope – intercept form.
11. What is the slope?
12. What is the y – intercept?
13. $6x + 3y = 24$
14. Write this equation in slope – intercept form.
15. What is the slope?
16. What is the y – intercept?
17. $5x + 5y = 15$
18. Write this equation in slope – intercept form.
19. What is the slope?
20. What is the y – intercept?

3.4 Point-Slope Form

Guidance

Equations can be written in many forms. The previous Concepts taught you how to write equations of lines in slope-intercept form. This Concept will provide a second way to write an equation of a line: **point-slope form**.

The equation of the line between any two points (x_1, y_1) and (x_2, y_2) can be written in the following form: $y - y_1 = m(x - x_1)$.

To write an equation in point-slope form, you need two things:

1. The slope of the line
2. A point on the line

Example A

Write an equation for a line containing (9, 3) and (4, 5).

Solution: Begin by finding the slope.

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{4-9} = -\frac{2}{5}$$

Instead of trying to find b (the $y-$intercept), you will use the point-slope formula.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= \frac{-2}{5}(x - 9) \end{aligned}$$

It doesn't matter which point you use.

You could also use the other ordered pair to write the equation:

$$y - 5 = \frac{-2}{5}(x - 4)$$

These equations may look completely different, but by solving each one for y, you can compare the slope-intercept form to check your answer.

$$
\begin{aligned}
y-3 &= \frac{-2}{5}(x-9) \Rightarrow y = \frac{-2}{5}x+\frac{18}{5}+3 \\
y &= \frac{-2}{5}x+\frac{33}{5} \\
y-5 &= \frac{-2}{5}(x-4) \\
y &= \frac{-2}{5}x+\frac{8}{5}+5 \\
y &= \frac{-2}{5}x+\frac{33}{5}
\end{aligned}
$$

This process is called **rewriting in slope-intercept form.**

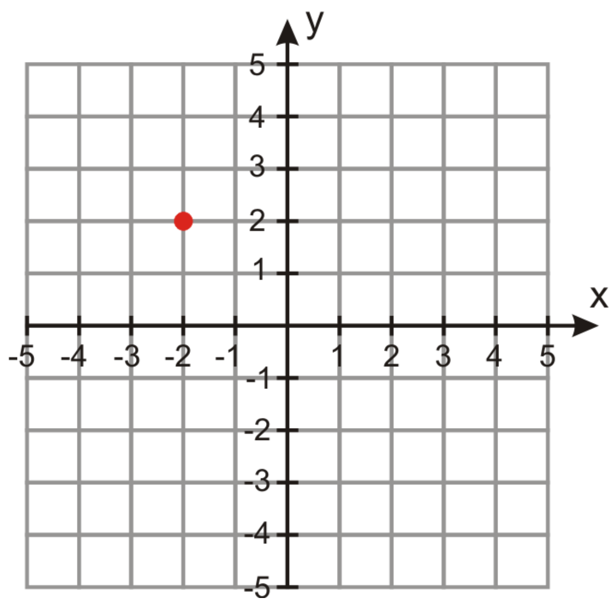
Graphing Equations Using Point-Slope Form

If you are given an equation in point-slope form, it is not necessary to re-write it in slope-intercept form in order to graph it. The point-slope form of the equation gives you enough information so you can graph the line.

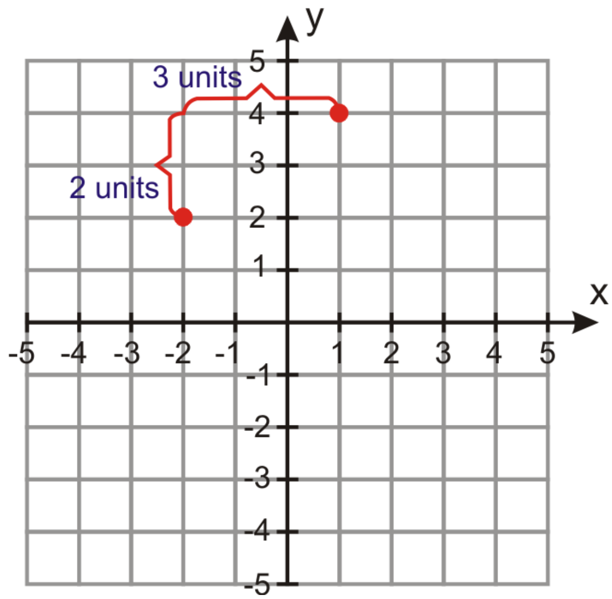
Example B

Make a graph of the line given by the equation $y-2=\frac{2}{3}(x+2)$.

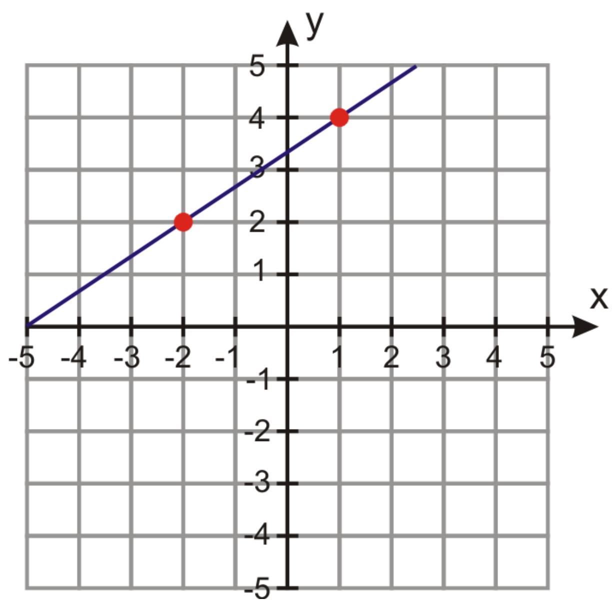
Solution: Begin by rewriting the equation to make it point-slope form: $y-2=\frac{2}{3}(x-(-2))$ Now we see that point (–2, 2) is on the line and that the slope $=\frac{2}{3}$. First plot point (–2, 2) on the graph.



A slope of $\frac{2}{3}$ tells you that from your point you should move 2 units up and 3 units to the right and draw another point.



Now draw a line through the two points and extend the line in both directions.



Writing a Linear Function in Point-Slope Form

Remember from the previous Concept that $f(x)$ and y are used interchangeably. Therefore, to write a function in point-slope form, you replace $y - y_1$ with $f(x) - y_1$.

Example C

Write the equation of the linear function in point-slope form.

$m = 9.8$ and $f(5.5) = 12.5$

Solution: This function has a slope of 9.8 and contains the ordered pair (5.5, 12.5). Substituting the appropriate values into point-slope form, we get the following:

$$y - 12.5 = 9.8(x - 5.5)$$

Replacing $y - y_1$ with $f(x) - y_1$, the equation in point-slope form is:

$$\begin{aligned} f(x) - 12.5 &= 9.8(x - 5.5) \\ f(x) - 12.5 &= 9.8x - 53.9 \\ f(x) &= 9.8x - 41.4 \end{aligned}$$

where the last equation is in slope-intercept form.

Guided Practice

Rewrite $y - 5 = 3(x - 2)$ in slope-intercept form.

Solution: Use the Distributive Property to simplify the right side of the equation:

$$y - 5 = 3x - 6$$

Solve for y:

$$\begin{aligned} y - 5 + 5 &= 3x - 6 + 5 \\ y &= 3x - 1 \end{aligned}$$

Practice

1. What is the equation for a line containing the points (x_1, y_1) and (x_2, y_2) in point-slope form?
2. In what ways is it easier to use point-slope form rather than slope-intercept form?

In 3-13, write the equation for the line in point-slope form.

3. The slope is $\frac{1}{3}$; the $y-$intercept is –4.
4. The slope is $-\frac{1}{10}$ and contains the point (10, 2).
5. The slope is –75 and contains the point (0, 125).
6. The slope is 10 and contains the point (8, –2).
7. The line contains the points (–2, 3) and (–1, –2).
8. The line contains the points (0, 0) and (1, 2).
9. The line contains the points (10, 12) and (5, 25).
10. The line contains the points (2, 3) and (0, 3).
11. The line has a slope of $\frac{3}{5}$ and a $y-$intercept of –3.
12. The line has a slope of –6 and a $y-$intercept of 0.5.
13. The line contains the points (–4, –2) and (8, 12).

In 14-17, write each equation in slope-intercept form.

14. $y - 2 = 3(x - 1)$
15. $y + 4 = \frac{-2}{3}(x + 6)$

16. $0 = x + 5$
17. $y = \frac{1}{4}(x - 24)$

In 18–25, write the equation of the linear function in point-slope form.

18. $m = -\frac{1}{5}$ and $f(0) = 7$
19. $m = -12$ and $f(-2) = 5$
20. $f(-7) = 5$ and $f(3) = -4$
21. $f(6) = 0$ and $f(0) = 6$
22. $m = 3$ and $f(2) = -9$
23. $m = -\frac{9}{5}$ and $f(0) = 32$
24. $m = 25$ and $f(0) = 250$
25. $f(32) = 0$ and $f(77) = 25$

3.5 Standard Form

Guidance

Slope-intercept form is one way to write the equation of a line. Another way is called standard form. Standard form looks like $Ax+By=C$, where A,B, and C are all real numbers.

Example A

Find the equation of a line, in standard form, where the slope is $\frac{3}{4}$ and passes through (4, -1).

Solution: To find the equation in standard form, you need to determine what A,B, and C are. Let's start this example by finding the equation in slope-intercept form.

$$\begin{aligned}-1 &= \frac{3}{4}(4)+b\\ -1 &= 3+b\\ -4 &= b\end{aligned}$$

In slope-intercept form, the equation is $y=\frac{3}{4}x-4$.

To change this to standard form we need to subtract the $x-$term from both sides of the equation.

$$-\frac{3}{4}x+y=-4$$

Example B

The equation of a line is $5x-2y=12$. What are the slope and $y-$intercept?

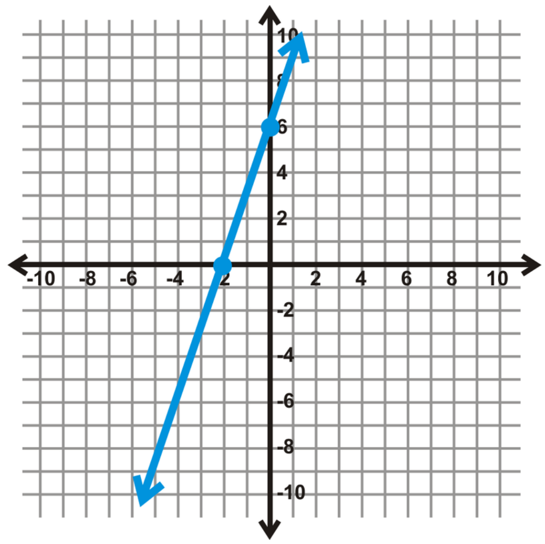
Solution: To find the slope and $y-$intercept of a line in standard form, we need to switch it to slope-intercept form. This means, we need to solve the equation for y.

$$\begin{aligned}5x-2y &= 12\\ -2y &= -5x+12\\ y &= \frac{5}{2}x-6\end{aligned}$$

From this, the slope is $\frac{5}{2}$ and the $y-$intercept is (0, -6).

Example C

Find the equation of the line below, in standard form.



Solution: Here, we are given the intercepts. The slope triangle is drawn by the axes, $\frac{-6}{-2} = 3$. And, the $y-$intercept is (0, 6). The equation of the line, in slope-intercept form, is $y = 3x + 6$. To change the equation to standard form, subtract the $x-$term to move it over to the other side.

$$-3x + y = 6 \ or \ 3x - y = -6$$

Example D

The equation of a line is $6x - 5y = 45$. What are the intercepts?

Solution: For the $x-$intercept, the $y-$value is zero. Plug in zero for y and solve for x.

$$\begin{aligned} 6x - 5y &= 45 \\ 6x - 5(0) &= 45 \\ 6x &= 45 \\ x &= \frac{45}{6} \ or \ \frac{15}{2} \end{aligned}$$

The $x-$intercept is $\left(\frac{15}{2}, 0\right)$.

For the $y-$intercept, the $x-$value is zero. Plug in zero for x and solve for y.

$$\begin{aligned} 6x - 5y &= 45 \\ 6(0) - 5y &= 45 \\ 5y &= 45 \\ y &= 9 \end{aligned}$$

The $y-$intercept is (0, 9).

Guided Practice

1. Find the equation of the line, in standard form that passes through (8, -1) and (-4, 2).

2. Change $2x+3y=9$ to slope-intercept form.

3. What are the intercepts of $3x-4y=-24$?

Answers

1. Like with Example A, we need to first find the equation of this line in $y-$intercept form and then change it to standard form. First, find the slope.

$$\frac{2-(-1)}{-4-8}=\frac{3}{-12}=-\frac{1}{4}$$

Find the $y-$intercept using slope-intercept form.

$$\begin{aligned}2&=-\frac{1}{4}(-4)+b\\2&=1+b\\1&=b\end{aligned}$$

The equation of the line is $y=-\frac{1}{4}x+1$.

To change this equation into standard form, add the $x-$term to both sides.

$$\frac{1}{4}x+y=1$$

2. To change $2x+3y=9$ into slope-intercept form, solve for y.

$$\begin{aligned}2x+3y&=9\\3y&=-2x+9\\y&=-\frac{2}{3}x+3\end{aligned}$$

3. Copy Example D to find the intercepts of $3x-4y=-24$. First, plug in zero for y and solve for x.

$$\begin{aligned}3x-4(0)&=-24\\3x&=-24\\x&=-8\end{aligned}$$

$x-$intercept is (-8, 0)

Now, start over and plug in zero for x and solve for y.

$$
\begin{aligned}
3(0) - 4y &= -24 \\
-4y &= -24 \\
y &= 6
\end{aligned}
$$

$y-$intercept is (6, 0)

Practice

Change the following equations into standard form.

1. $y = -\frac{2}{3}x + 4$
2. $y = x - 5$
3. $y = \frac{1}{5}x - 1$

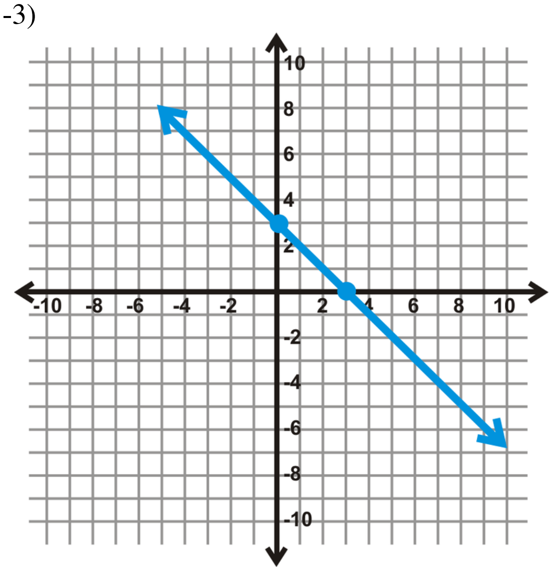
Change the following equations into slope-intercept form.

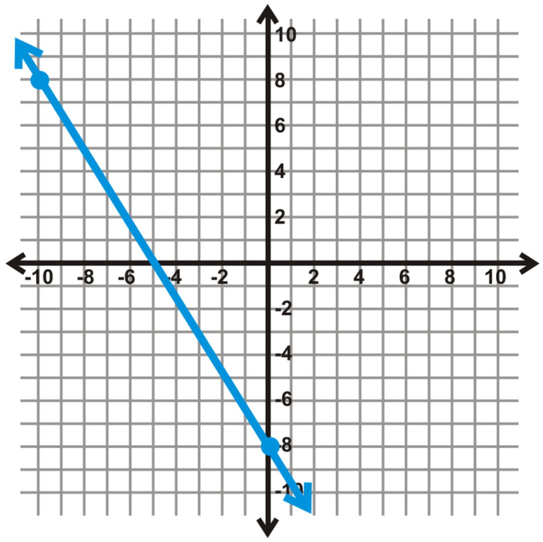
4. $4x + 5y = 20$
5. $x - 2y = 9$
6. $2x - 3y = 15$

Find the x and $y-$intercepts of the following equations.

7. $3x + 4y = 12$
8. $6x - y = 8$
9. $3x + 8y = -16$

Find the equation of the lines below, in standard form.

10. slope = 2 and passes through (3, -5)
11. slope $= -\frac{1}{2}$ and passes through (6, -3).
12. passes through (5, -7) and (-1, 2)
13. passes through (-5, -5) and (5, -3)
14. 

15. 

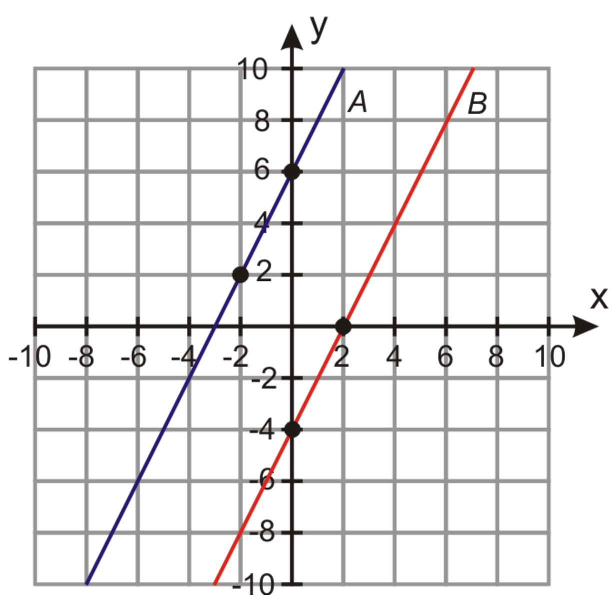
16. Change $Ax + By = C$ into slope-intercept form.
17. From #16, what are the slope and $y-$intercept equal to (in terms of A, B, and/or C)?
18. Using #16 and #17, find one possible combination of A, B, and C for $y = \frac{1}{2}x - 4$. Write your answer in standard form.
19. The measure of a road's slope is called the *grade*. The grade of a road is measured in a percentage, for how many vertical feet the road rises or declines over 100 feet. For example, a road with a grade incline of 5% means that for every 100 horizontal feet the road rises 5 vertical feet. What is the slope of a road with a grade decline of 8%?
20. The population of a small town in northern California gradually increases by about 50 people a year. In 2010, the population was 8500 people. Write an equation for the population of this city and find its estimated population in 2017.

3.6 Parallel and Perpendicular Lines

Here you'll learn how to use slopes to determine whether two lines are parallel or perpendicular.

Guidance

In this section you will learn how **parallel lines** and **perpendicular lines** are related to each other on the coordinate plane. Let's start by looking at a graph of two parallel lines.



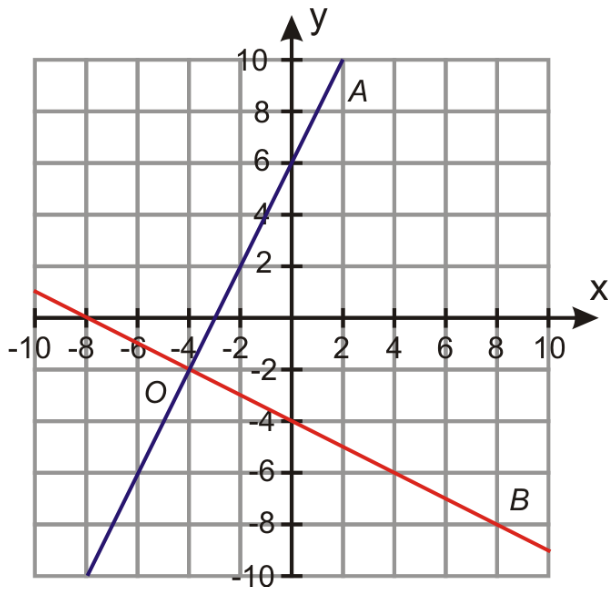
We can clearly see that the two lines have different $y-$intercepts: 6 and –4.

How about the slopes of the lines? The slope of line A is $\frac{6-2}{0-(-2)} = \frac{4}{2} = 2$, and the slope of line B is $\frac{0-(-4)}{2-0} = \frac{4}{2} = 2$. The slopes are the same.

Is that significant? Yes. By definition, parallel lines never meet. That means that when one of them slopes up by a certain amount, the other one has to slope up by the same amount so the lines will stay the same distance apart. If you look at the graph above, you can see that for any $x-$value you pick, the $y-$values of lines A and B are the same vertical distance apart—which means that both lines go up by the same vertical distance every time they go across by the same horizontal distance. In order to stay parallel, their slopes must stay the same.

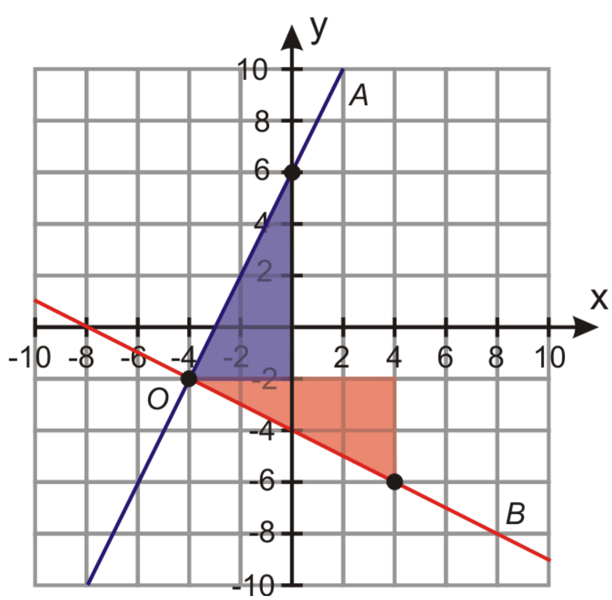
All parallel lines have the same slopes and different $y-$intercepts.

Now let's look at a graph of two perpendicular lines.



We can't really say anything about the $y-$intercepts. In this example, the $y-$intercepts are different, but if we moved the lines four units to the right, they would both intercept the $y-$axis at (0, -2). So perpendicular lines can have the same or different $y-$intercepts.

What about the relationship between the slopes of the two lines?



To find the slope of line A, we pick two points on the line and draw the blue (upper) right triangle. The legs of the triangle represent the rise and the run. We can see that the slope is $\frac{8}{4}$, or 2.

To find the slope of line B, we pick two points on the line and draw the red (lower) right triangle. Notice that the two triangles are identical, only rotated by 90°. Where line A goes 8 units up and 4 units right, line B goes 8 units right and 4 units down. Its slope is $-\frac{4}{8}$, or $-\frac{1}{2}$.

This is always true for perpendicular lines; where one line goes a units up and b units right, the other line will go a units right and b units down, so the slope of one line will be $\frac{a}{b}$ and the slope of the other line will be $-\frac{b}{a}$.

The slopes of **perpendicular lines** are always negative reciprocals of each other.

Determine When Lines are Parallel or Perpendicular

You can find whether lines are parallel or perpendicular by comparing the slopes of the lines. If you are given points on the lines, you can find their slopes using the formula. If you are given the equations of the lines, re-write each equation in a form that makes it easy to read the slope, such as the slope-intercept form.

Example A

Determine whether the lines are parallel or perpendicular or neither. One line passes through the points (2, 11) and (-1, 2); another line passes through the points (0, -4) and (-2, -10).

Solution

Find the slope of each line and compare them.

$m_1 = \frac{2-11}{-1-2} = \frac{-9}{-3} = 3 \quad \text{and} \quad m_2 = \frac{-10-(-4)}{-2-0} = \frac{-6}{-2} = 3$

The slopes are equal, so **the lines are parallel.**

Example B

Determine whether the lines are parallel or perpendicular or neither. One line passes through the points (-2, -7) and (1, 5); another line passes through the points (4, 1) and (-8, 4).

Solution:

$m_1 = \frac{5-(-7)}{1-(-2)} = \frac{12}{3} = 4 \quad \text{and} \quad m_2 = \frac{4-1}{-8-4} = \frac{3}{-12} = -\frac{1}{4}$

The slopes are negative reciprocals of each other, so **the lines are perpendicular.**

Example C

Determine whether the lines are parallel or perpendicular or neither. One line passes through the points (3, 1) and (-2, -2); another line passes through the points (5, 5) and (4, -6).

Solution:

$m_1 = \frac{-2-1}{-2-3} = \frac{-3}{-5} = \frac{3}{5} \quad \text{and} \quad m_2 = \frac{-6-5}{4-5} = \frac{-13}{-1} = 13$

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular.**

Vocabulary

- **All parallel lines** have the same slopes and different $y-$intercepts.
- The slopes of **perpendicular lines** are always negative reciprocals of each other.

Guided Practice

Determine whether the lines are parallel or perpendicular or neither:

a) $3x+4y=2$ and $8x-6y=5$

b) $2x=y-10$ and $y=-2x+5$

c) $7y+1=7x$ and $x+5=y$

Solution

Write each equation in slope-intercept form:

a) line 1: $3x+4y=2 \Rightarrow 4y=-3x+2 \Rightarrow y=-\frac{3}{4}x+\frac{1}{2} \Rightarrow$ slope $=-\frac{3}{4}$

line 2: $8x-6y=5 \Rightarrow 8x-5=6y \Rightarrow y=\frac{8}{6}x-\frac{5}{6} \Rightarrow y=\frac{4}{3}x-\frac{5}{6} \Rightarrow$ slope $=\frac{4}{3}$

The slopes are negative reciprocals of each other, so **the lines are perpendicular.**

b) line 1: $2x=y-10 \Rightarrow y=2x+10 \Rightarrow$ slope $=2$

line 2: $y=-2x+5 \Rightarrow$ slope $=-2$

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular.**

c) line 1: $7y+1=7x \Rightarrow 7y=7x-1 \Rightarrow y=x-\frac{1}{7} \Rightarrow$ slope $=1$

line 2: $x+5=y \Rightarrow y=x+5 \Rightarrow$ slope $=1$

The slopes are the same, so **the lines are parallel.**

Practice

For 1-10, determine whether the lines are parallel, perpendicular or neither.

1. One line passes through the points (-1, 4) and (2, 6); another line passes through the points (2, -3) and (8, 1).
2. One line passes through the points (4, -3) and (-8, 0); another line passes through the points (-1, -1) and (-2, 6).
3. One line passes through the points (-3, 14) and (1, -2); another line passes through the points (0, -3) and (-2, 5).
4. One line passes through the points (3, 3) and (-6, -3); another line passes through the points (2, -8) and (-6, 4).
5. One line passes through the points (2, 8) and (6, 0); another line has the equation $x-2y=5$.
6. One line passes through the points (-5, 3) and (2, -1); another line has the equation $2x+3y=6$.
7. Both lines pass through the point (2, 8); one line also passes through (3, 5), and the other line has slope 3.
8. Line 1: $4y+x=8$ Line 2: $12y+3x=1$
9. Line 1: $5y+3x=1$ Line 2: $6y+10x=-3$
10. Line 1: $2y-3x+5=0$ Line 2: $y+6x=-3$
11. Lines A, B, C, D, and E all pass through the point (3, 6). Line A also passes through (7, 12); line B passes through (8, 4); line C passes through (-1, -3); line D passes through (1, 1); and line Epasses through (6, 12).
    a. Are any of these lines perpendicular? If so, which ones? If not, why not?
    b. Are any of these lines parallel? If so, which ones? If not, why not?

3.7 Transformations of Linear Functions: Translations and Reflections

Guidance

A transformation is a change in the position, size, or shape of a figure or graph.

A linear function is a function, meaning we have an input and an output, than can be written in the form f(x) = mx + b.

Its graph is a line.

If we are transforming linear functions, we can say we are changing the linear function either the way it looks in the graph or equation.

There are four ways we can transform the linear function:

A. Horizontal Shift (Just remember...the x changes)

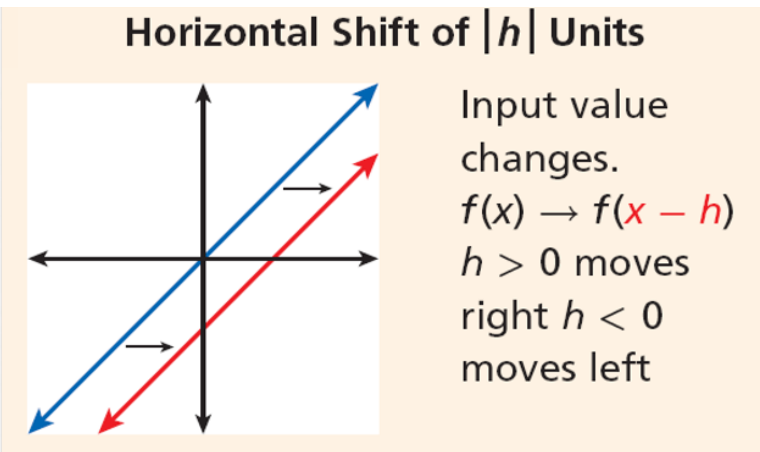


FIGURE 3.3

B. Vertical Shift (Just remember...the y changes)

C. Reflection Across the y-axis (Just remember...y is the mirror so the one that changes is the x)

D. Reflection Across the x-axis (Just remember...x is the mirror so the one that changes is the y)

Guided Practice

Example 1:

Let *g(x)* be the indicated transformation of *f(x)*. Write the rule for *g(x)*.

f(x) = 3*x* +2; *g(x)* is a horizontal shift 3 units to the right.

Solution:

g(x) = *f*(*x* -3) —>subtract 3 from the input

g(x) = 3(*x* -3) +2 —>evaluate *f* at (*x*-3)

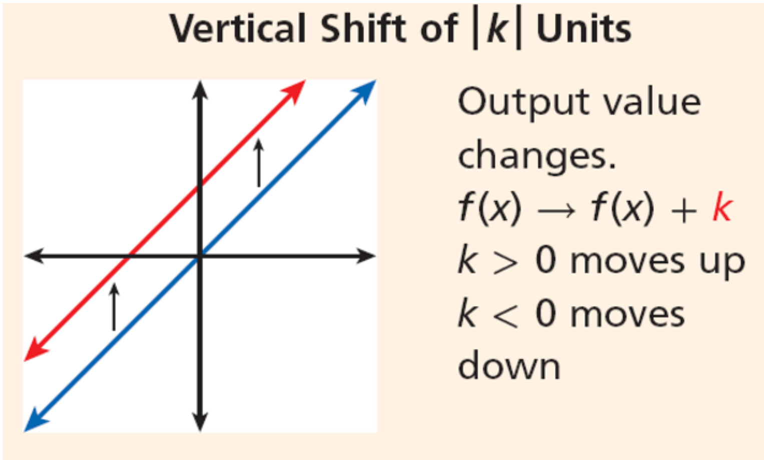


FIGURE 3.4

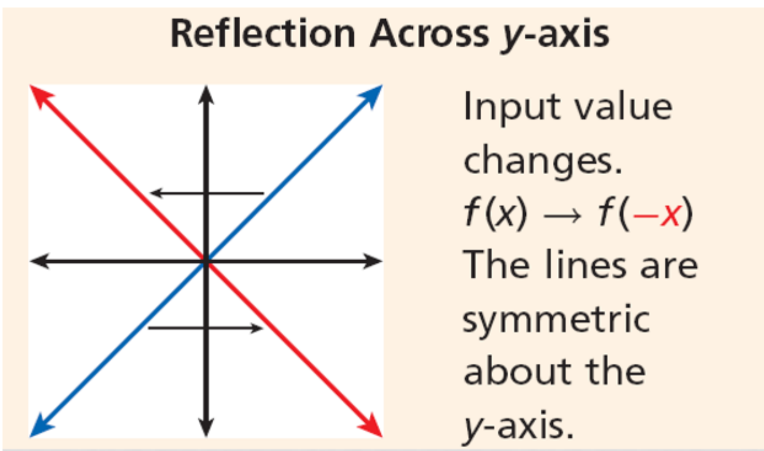


FIGURE 3.5

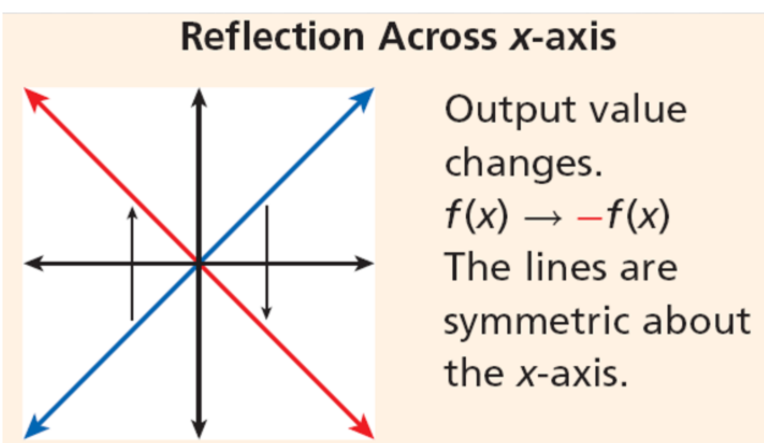


FIGURE 3.6

$g(x)$ = 3x - 9 + 2 —>simplify

$g(x)$ = 3x -7

Example 2:

Let $g(x)$ be the indicated transformation of $f(x)$. Write the rule for $g(x)$.

$f(x) = x$ +2; $g(x)$ is reflected about the y-axis.

Solution:

$g(x) = f(-x)$ —>change the input of f

$g(x)$ = (-x) + 2 —>Simplify

$g(x)$ = -x + 2

Example: 3:

Let $g(x)$ be the indicated transformation of $f(x)$. Write the rule for $g(x)$.

$f(x)$ = 2x - 6; $g(x)$ is a vertical shift (vertical translation) 3 units down followed by reflection across the x-axis.

Solution:

First, let's take care of the vertical translation

$g(x) = f(x)$ - 3 —>

$g(x)$ = (2x - 6) -3 —>Substitute

$g(x)$ = 2x - 9 —>Simplify

Then we continue with the reflection across the x-axis

$g(x) = -f(x)$

$g(x)$ = - (2x - 9)

$g(x)$ = -2x + 9

Guidance Continued

In addition, stretches and compressions change the slope of a linear function. If a line becomes steeper, the function has been stretched vertically or compressed horizontally.

If the line becomes flatter, the function has been compressed vertically or stretched horizontally. **Example 4:**

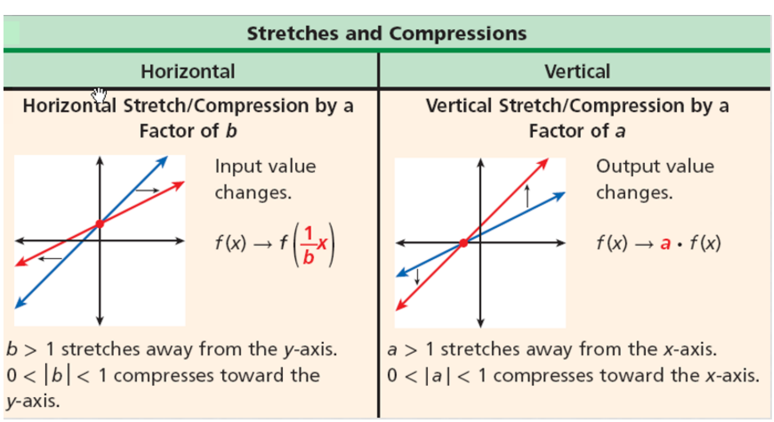


FIGURE 3.7

Let *g(x)* be a vertical compression of *f(x)* = 3*x* + 2 by a factor of 4 . Write the rule for *g(x)* and graph the function. Solution: Vertically compressing *f(x)* by a factor of replaces each *f(x)* with a · *f(x)* where a = 4 . *g(x)* = a∗f(x)=4∗f*(x)* *g(x)* = 4∗(3*x*+2) —>substitute *g(x)* = 12*x*+8 —>simplify **Example 5:**

Let *g(x)* be a horizontal compression of *f(x)* = 6*x* - 5by a factor of 1/3 followed by a vertical translation 4 units up . Lets *h(x)* be the horizontal compression and *g(x)* the vertical translation. Write the rule for *g(x)* and graph the function.

f(x) = *h*(1/*b x*)

f(x) = *h*(1/(1/3) *x*)=*h*(3*x*)

h(x) = 6(3*x*)-5=18*x*-5 Now lets take care of the translation *g(x)* = *h*(*x*)+4 *g(x)* = (18*x*-5)+4 —>substitute *g(x)* = 18*x*-1 —>simplify

Practice

1. Let g(x) be the indicated transformation of f(x).Write the rule for g(x). 2. f(x)=6x+2; g(x) is a vertical shift (vertical translation) 3 units down. 3. Let g(x) be the indicated transformation of f(x).Write the rule for g(x). 4. f(x)=6x+2; g(x) is a reflection across the x-axis 5. Let g(x) be a horizontal compression of f(x) = 5x - 2 by a factor of 1/3 . Write the rule for g(x) and graph the function.

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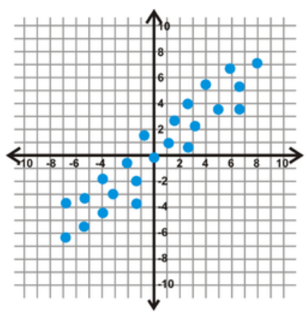
3.8 Scatter Plots and Trend Lines: Correlation and Finding Line of Best Fit

Here you'll learn how to find a linear equation that best fits a set of data or points.

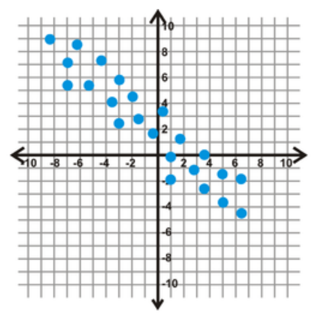
Guidance

A **scatter plot** is a set of points that represent data. We plot these points and try to find equations that best approximate their relationship. Because data is not always perfect, not every point will always fit on the line of best fit. The **line of best fit** is the line that is the closest to all the data points. It can be used to approximate data within the set or beyond the set. Scatter plots almost always represent a real-life situation.

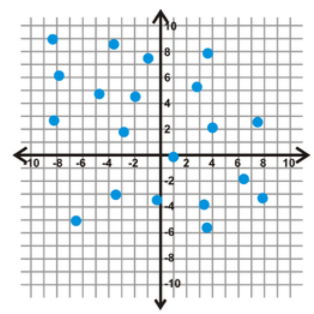
Scatter plots can have **positive correlation** if the x and y values tend to increase together. They can have **negative correlation** if y tends to decrease as x tends to increase. And, if the points have no sort of linear pattern, then the data would have relatively **no correlation.** Think of the type of correlations referring to the slope of the line that would best fit that data.



positive correlation



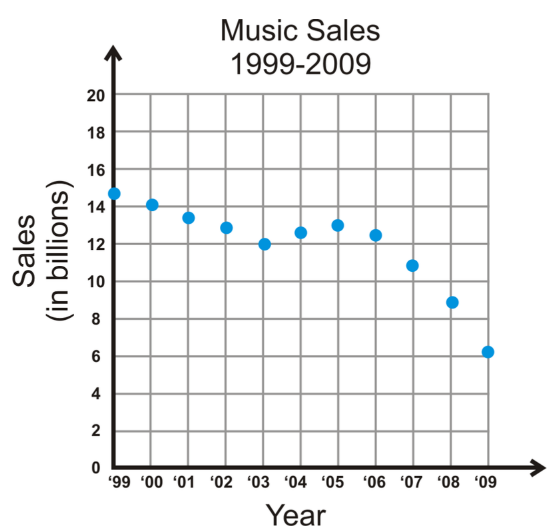
negative correlation



no correlation

Example A

Describe the type of correlation shown in the scatterplot. Explain your answer.



Solution: This is a negative correlation. As the years get larger, the sales go down. This could be because in the boom of online/digital and pirated music.

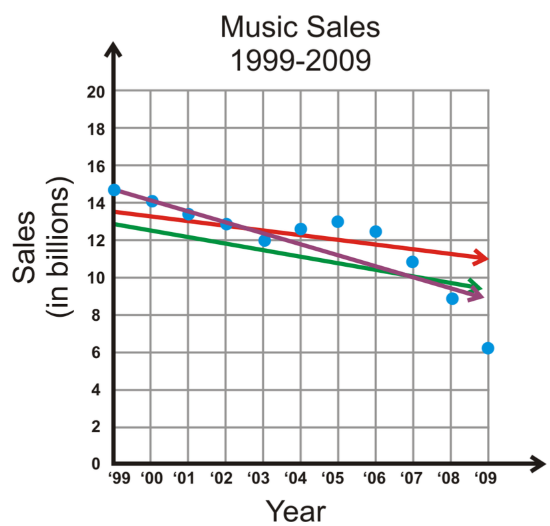
Example B

Find the linear equation of best fit for the data set above.

Solution: First, it can be very difficult to determine the "best" equation for a set of points. In general, you can use these steps to help you.

1. Draw the scatter plot on a graph.
2. Sketch the line that appears to most closely follow the data. Try to have the same number of points above and below the line.
3. Choose two points on the line and estimate their coordinates. These points do not have to be part of the original data set.
4. Find the equation of the line that passes through the two points from Step 3.

Let's use these steps on the graph above. We already have the scatter plot drawn, so let's sketch a couple lines to find the one that best fits the data.



From the lines in the graph, it looks like the purple line might be the best choice. The red line looks good from 2006-2009, but in the beginning, all the data is above it. The green line is well below all the early data as well. Only the purple line cuts through the first few data points, and then splits the last few years. Remember, it is very important to have the same number of points above and below the line.

Using the purple line, we need to find two points on it. The second point, crosses the grid perfectly at (2000, 14). Be careful! Our graph starts at 1999, so that would be considered zero. Therefore, (2000, 14) is actually (1, 14). The line also crosses perfectly at (2007, 10) or (8, 10). Now, let's find the slope and $y-$intercept.

$$m = \frac{14-10}{1-8} = -\frac{4}{7}$$

$$\begin{aligned}
y &= -\frac{4}{7}x + b \\
14 &= -\frac{4}{7}(1) + b \\
14 &= -0.57 + b \\
14.57 &= b
\end{aligned}$$

The equation of best fit is $y = -\frac{4}{7}x + 14.57$.

However, the equation above assumes that x starts with zero. In actuality, we started with 1999, so our final equation is $y = -\frac{4}{7}(x - 1999) + 14.57$.

Example C

Using the line of best fit above, what would you expect music sales to be in 2010?

Solution: In this example, we are using the line of best fit to predict data. Plug in 2010 for x and solve for y.

$$y = -\frac{4}{7}(2010 - 1999) + 14.57$$
$$y = -\frac{4}{7}(11) + 14.57$$
$$y = 8.3$$

It is estimated that music industry will make \$8.3 billion in music sales in 2010.

Guided Practice

Use the table below to answer the following questions.

Sleep Requirements, 0-3 years

TABLE 3.1:

Age, x	1	3	6	9	12	18	24	36
Sleep, y	16	15	14.25	14	13.75	13.5	13	12

The age is measured in months and sleep is measured in hours. Source: www.babycenter.com

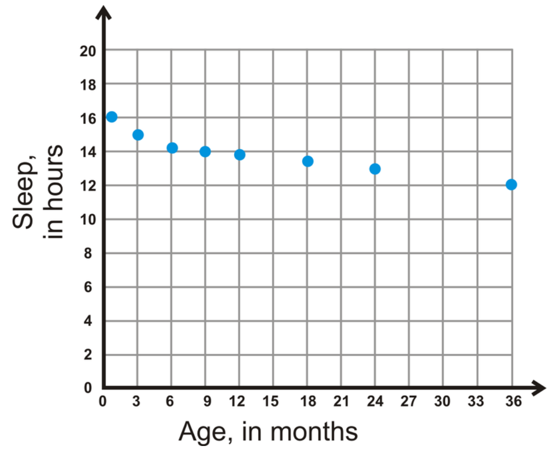
1. Draw a scatterplot with age across the $x-$axis and sleep along the $y-$axis. Count by 3's for the $x-$values and by 2's for the $y-$values.

2. Using the steps from Example B, find the line of best fit.

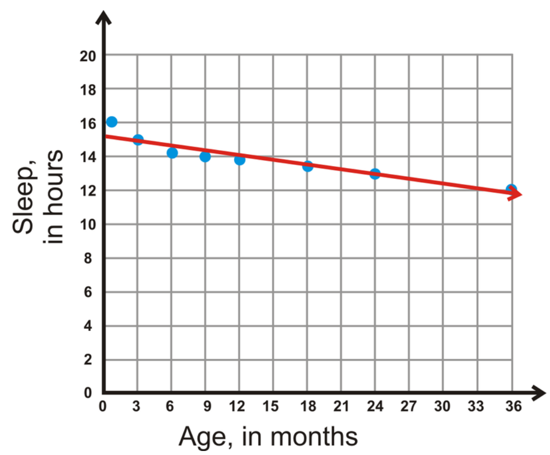
3. Determine the amount of sleep needed for a $2\frac{1}{2}$ year old and a 5 year old.

Answers

1. Here is the scatterplot.



2. Two points that seem to be on the red line are (3, 15) and (24, 13).



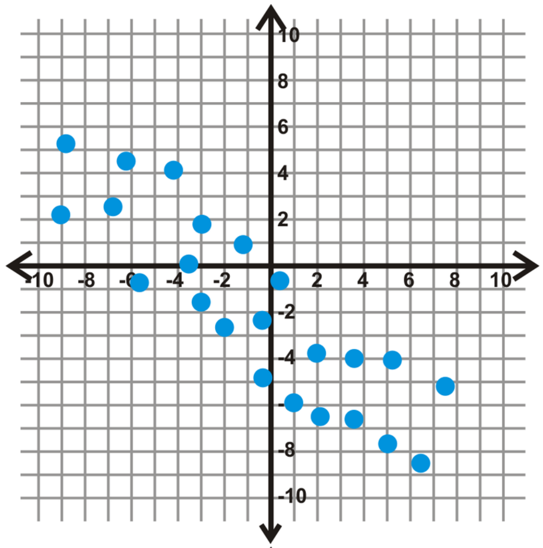
$$\begin{aligned}
m &= \frac{15-13}{3-24} = -\frac{2}{21} \\
15 &= -\frac{2}{21}(3) + b \\
15 &= -0.29 + b \\
15.29 &= b
\end{aligned}$$

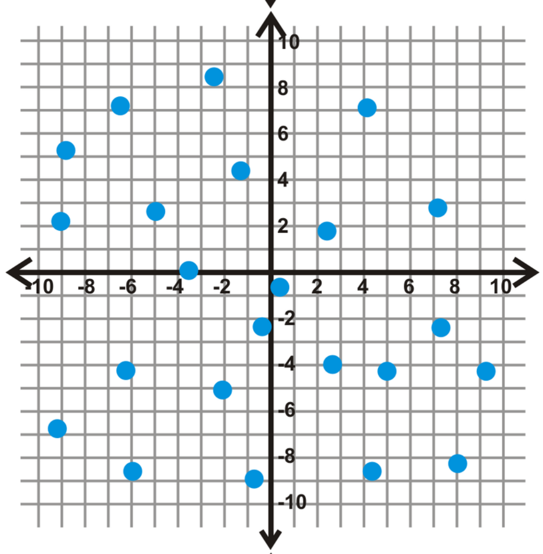
The equation of the line is $y = -\frac{2}{21}x + 15.29$.

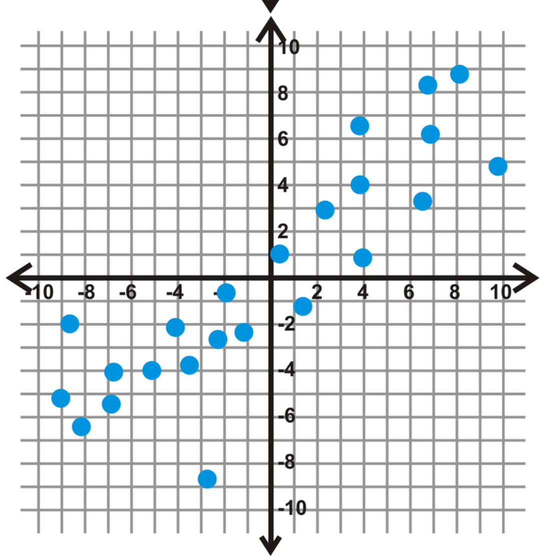
3. First, you need to change the age to months so that it corresponds with the units used in the graph. For a 2.5 year-old, 30 months, s/he should sleep $y = -\frac{2}{21}(30) + 15.29 \approx 12.4$ hours. For a 5-year-old, 60 months, s/he should sleep $y = -\frac{2}{21}(60) + 15.29 \approx 9.6$ hours.

Practice

Determine if the scatter plots below have positive, negative, or no correlation.

1. 

2. 

3. 

Plot each scatter plot and then determine the line of best fit.

4.

TABLE 3.2:

x	1	2	3	5	7	8
y	1	3	4	3	6	7

5.

TABLE 3.3:

x	10	9	7	6	5	2
y	5	6	4	3	3	2

Use the data below to answer questions 6-8.

The price of Apple stock from Oct 2009 - Sept 2011 source: *Yahoo! Finance*

TABLE 3.4:

10/09	**11/09**	**12/09**	**1/10**	**2/10**	**3/10**	**4/10**	**5/10**	**6/10**	**7/10**	**8/10**	**9/10**
$181	$189	$198	$214	$195	$208	$236	$249	$266	$248	$261	$258
10/10	**11/10**	**12/10**	**1/11**	**2/11**	**3/11**	**4/11**	**5/11**	**6/11**	**7/11**	**8/11**	**9/11**
$282	$309	$316	$331	$345	$352	$344	$349	$346	$349	$389	$379

6. Draw the scatter plot for the table above. Make the $x-$axis the month and the $y-$axis the price.
7. Find the linear equation of best fit.
8. According to your equation, what would be the predicted price of the stock in January 2012?

Use the data below to answer questions 9-11.

Total Number of Home Runs Hit in Major League Baseball, 2000-2010

TABLE 3.5:

2000	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**
5693	5458	5059	5207	5451	5017	5386	4957	4878	4655	4613

9. Draw the scatter plot for the table above. Make the $x-$axis the year and the $y-$axis the number of home runs.
10. Find the linear equation of best fit.
11. According to your equation, how many total home runs should be hit in 2011?

3.9 Scatter Plots and Trend Lines: Comparing and Contrasting Association and Causation

Guidance

Causation is caused when a change in one quantity causes change in another.

Casual relationships always have a correlation. However, two sets that have a correlation may not have a casual relationship.

In the following examples, we will identify if there is likely to be a correlation. If there is a correlation, we will identify if the example reflects a causal relationship or an associated relationship.

Example A:

The number of loaves of bread and the amount of flour used.

Solution: There is a positive correlation and also a causal relationship. As the number of loaves of bread baked increases, the amount of flour used increases. As the number of loaves baked increased, it caused the amount of flour used to increase as well.

Example B:

The number of mailboxes and the number of firefighters in a city.

Solution: There is likely to be a positive correlation because both the number of mailboxes and the number of firefighters tend to increase as the population of the city increases. However, installing more mailboxes will NOT cause the number of firefighters to increase. Therefore, there is no causal relationship. There is only association amongst the number of mailboxes and the number of firefighters in the city.

Practice

Indicate whether there is a correlation in each situation. If there is a correlation, indicate whether the correlation reflect a causal relationship or an association.

1. The price of hamburger meat at a grocery store and the amount of hamburger meat sold.
2. The shoe size and salary of a teacher.
3. The person's height and the number of letters in the person's name.
4. The amount you study for a test and the score you receive.

Vocabulary:

Rate of Change and Slope: Ratio, Rate of change, Slope

Direct Variation: Literal Equation, Constant of Variation for Direct Variation, Direct Variation

Slope-Intercept Form: Linear Equation, Linear Parent Function, Parent Function, Slope-Intercept Form, y-Interecept

Point-Slope Form: Linear Equation, Slope-Intercept Form, Point-Slope Form

Standard Form: y-Intercept, Standard Form of a Linear Equation, x-Intercept, Zero of a Function

Parallel and Perpendicular Lines: Opposite Reciprocals, Parallel Lines, Perpendicular Lines

Transformations of Linear Functions: Reflection, Transformation, Translation

Scatter Plots and Trend Lines: Causation, Correlation, Coefficient, Extrapolation, Interpolation, Line of Best Fit, Negative Correlation, No Correlation, Positive Correlation, Scatter Plot, Trend Line

CHAPTER 4 Systems of Equations and Inequalities

Chapter Outline

Solving Systems by Graphing: A.2i, A.3f, A.5c

Solving Systems by Substitution: A.2i, A.3g, A.5c

Solving Systems by Elimination: A.2i, A.5c

Application of Systems of Equations: A.2i, A.5c

Linear Inequalities: A.2h, A.3d

Systems of Linear Inequalities: A.2h, A.3d, A.3h

4.1 Solving Systems by Graphing

Here you'll learn how to determine whether an ordered pair is a solution to a system of equations. You'll also learn how to solve a system of equations by graphing. Finally, you'll solve word problems involving systems of equations.

Guidance

In this Concept, first we'll discover methods to determine if an ordered pair is a solution to a system of two equations. Then we'll learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, we'll look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

A linear system of equations is a set of equations that must be solved together to find the one solution that fits them both.

Consider this system of equations:

$$y = x + 2$$
$$y = -2x + 1$$

Since the two lines are in a system, we deal with them together by graphing them on the same coordinate axes. We can use any method to graph them; let's do it by making a table of values for each line.

Line 1: $y = x + 2$

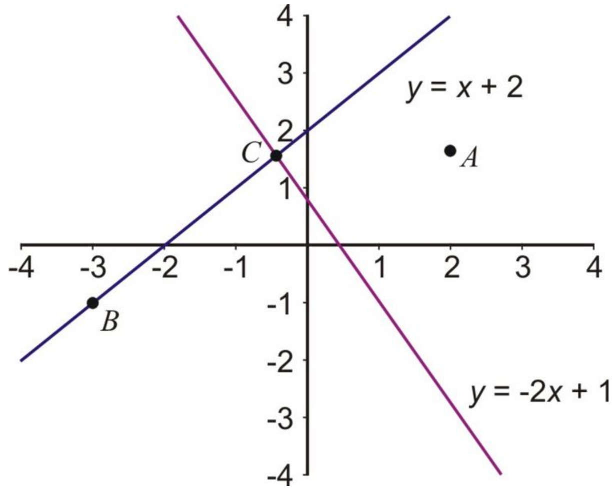
TABLE 4.1:

x	y
0	2
1	3

Line 2: $y = -2x + 1$

TABLE 4.2:

x	y
0	1
1	-1



We already know that any point that lies on a line is a solution to the equation for that line. That means that any point that lies on *both* lines in a system is a solution to both equations.

So in this system:

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point B is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In fact, point C is the only solution to the system, because it is the only point that lies on both lines. For a system of equations, the geometrical solution is the intersection of the two lines in the system. The algebraic solution is the ordered pair that solves both equations—in other words, the coordinates of that intersection point.

You can confirm the solution by plugging it into the system of equations, and checking that the solution works in each equation.

Example A

Determine which of the points (1, 3), (0, 2), or (2, 7) is a solution to the following system of equations:

$$y = 4x - 1$$
$$y = 2x + 3$$

Solution

To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.

Point (1, 3):

$$\begin{aligned} y &= 4x - 1 \\ 3 &\stackrel{?}{=} 4(1) - 1 \\ 3 &= 3 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 3 &\stackrel{?}{=} 2(1) + 3 \\ 3 &\neq 5 \text{ solution does not check} \end{aligned}$$

Point (1, 3) is on the line $y = 4x - 1$, but it is not on the line $y = 2x + 3$, so it is not a solution to the system.

Point (0, 2):

$$\begin{aligned} y &= 4x - 1 \\ 2 &\stackrel{?}{=} 4(0) - 1 \\ 2 &\neq -1 \text{ solution does not check} \end{aligned}$$

Point (0, 2) is not on the line $y = 4x - 1$, so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.

Point (2, 7):

$$\begin{aligned} y &= 4x - 1 \\ 7 &\stackrel{?}{=} 4(2) - 1 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 7 &\stackrel{?}{=} 2(2) + 3 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

Point (2, 7) is a solution to the system since it lies on both lines.

The solution to the system is the point (2, 7).

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point, (if there is one) that lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

This method most often offers only approximate solutions, so it's not sufficient when you need an exact answer. However, graphing the system of equations can be a good way to get a sense of what's really going on in the problem you're trying to solve, especially when it's a real-world problem.

Example B

Solve the following system of equations by graphing:

$$y = 3x - 5$$
$$y = -2x + 5$$

Solution

Graph both lines on the same coordinate axis using any method you like.

In this case, let's make a table of values for each line.

Line 1: $y = 3x - 5$

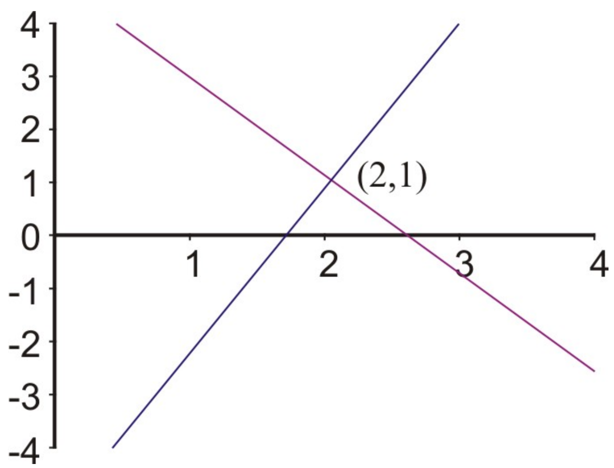
TABLE 4.3:

x	y
1	-2
2	1

Line 2: $y = -2x + 5$

TABLE 4.4:

x	y
1	3
2	1



The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point (2, 1). So **the solution is** $x = 2, y = 1$ **or (2, 1).**

Solving a System of Equations Using a Graphing Calculator

As an alternative to graphing by hand, you can use a graphing calculator to find or check solutions to a system of equations.

Example C

Solve the following system of equations using a graphing calculator.

$$\begin{aligned} x-3y&=4 \\ 2x+5y&=8 \end{aligned}$$

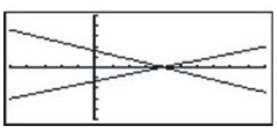
To input the equations into the calculator, you need to rewrite them in slope-intercept form (that is, $y=mx+b$ form).

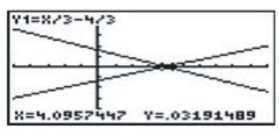
$$\begin{aligned} x-3y=4 \qquad & \qquad y=\frac{1}{3}x-\frac{4}{3} \\ & \qquad \Rightarrow \\ 2x+5y=8 \qquad & \qquad y=-\frac{2}{5}x+\frac{8}{5} \end{aligned}$$

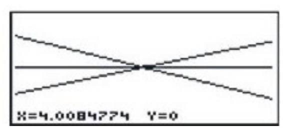
Press the **[y=]** button on the graphing calculator and enter the two functions as:

$$\begin{aligned} Y_1 &= \frac{x}{3}-\frac{4}{3} \\ Y_2 &= \frac{-2x}{5}+\frac{8}{5} \end{aligned}$$

Now press **[GRAPH]**. Here's what the graph should look like on a TI-83 family graphing calculator with the window set to $-5 \le x \le 10$ and $-5 \le y \le 5$.







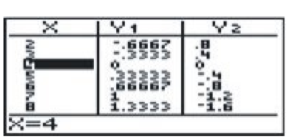
There are a few different ways to find the intersection point.

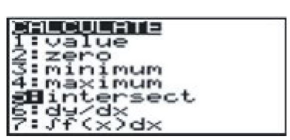
Option 1: Use **[TRACE]** and move the cursor with the arrows until it is on top of the intersection point. The values of the coordinate point will be shown on the bottom of the screen. The second screen above shows the values to be $X=4.0957447$ and $Y=0.03191489$.

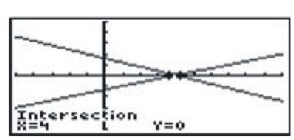
Use the **[ZOOM]** function to zoom into the intersection point and find a more accurate result. The third screen above shows the system of equations after zooming in several times. A more accurate solution appears to be $X=4$ and $Y=0$.

Option 2 Look at the table of values by pressing **[2nd] [GRAPH]**. The first screen below shows a table of values for this system of equations. Scroll down until the $Y-$values for the two functions are the same. In this case this occurs at $X=4$ and $Y=0$.

(Use the **[TBLSET]** function to change the starting value for your table of values so that it is close to the intersection point and you don't have to scroll too long. You can also improve the accuracy of the solution by setting the value of Δ Table smaller.)







Option 3 Using the **[2nd] [TRACE]** function gives the second screen shown above.

Scroll down and select "intersect."

The calculator will display the graph with the question **[FIRSTCURVE]**? Move the cursor along the first curve until it is close to the intersection and press **[ENTER]**.

The calculator now shows **[SECONDCURVE]**?

Move the cursor to the second line (if necessary) and press **[ENTER]**.

The calculator displays **[GUESS]**?

Press **[ENTER]** and the calculator displays the solution at the bottom of the screen (see the third screen above).

The point of intersection is $X = 4$ and $Y = 0$. Note that with this method, the calculator works out the intersection point for you, which is generally more accurate than your own visual estimate.

Solve Real-World Problems Using Graphs of Linear Systems

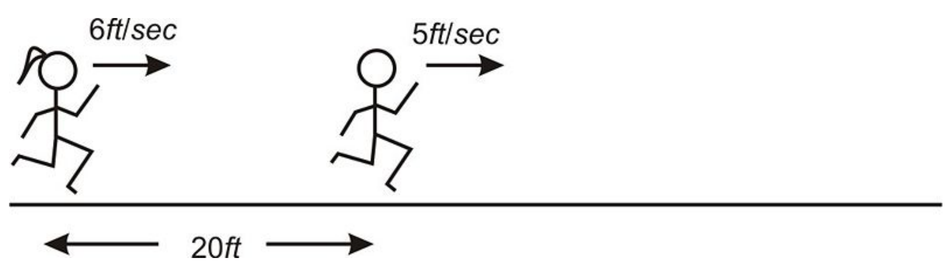
Consider the following problem:

Example D

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

Solution:

Let's start by drawing a sketch. Here's what the race looks like when Nadia starts running; we'll call this time $t = 0$.



Now let's define two variables in this problem:

$t =$ the time from when Nadia starts running

$d =$ the distance of the runners from the starting point.

Since there are two runners, we need to write equations for each of them. That will be the *system of equations* for this problem.

For each equation, we use the formula: distance $=$ speed $\times$ time

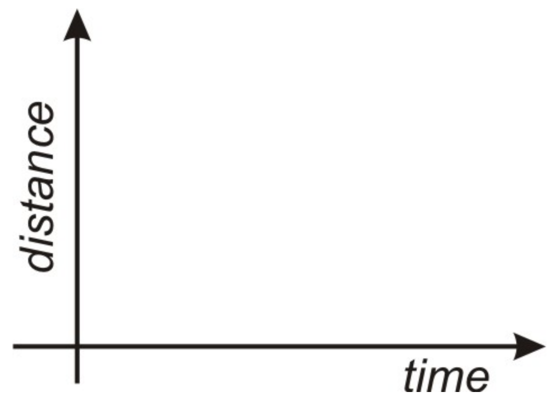
Nadia's equation: $d = 6t$

Peter's equation: $d = 5t + 20$

(Remember that Peter was already 20 feet from the starting point when Nadia started running.)

Let's graph these two equations on the same coordinate axes.

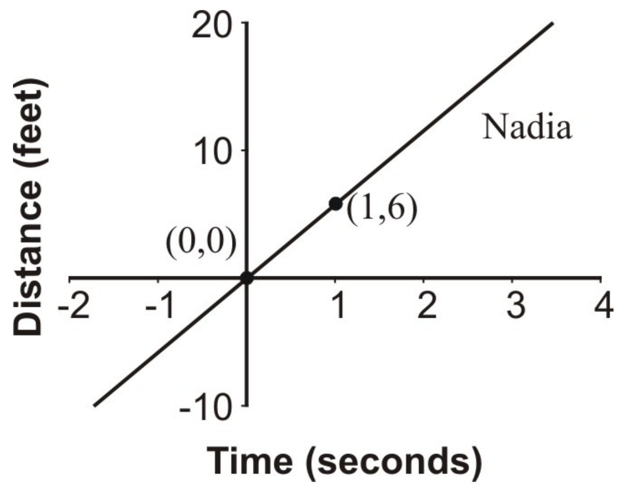
Time should be on the horizontal axis since it is the independent variable. Distance should be on the vertical axis since it is the dependent variable.



We can use any method for graphing the lines, but in this case we'll use the *slope–intercept* method since it makes more sense physically.

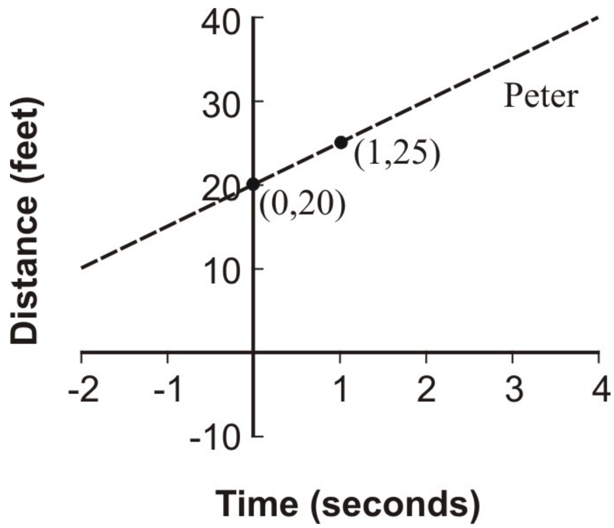
To graph the line that describes Nadia's run, start by graphing the $y-$intercept: (0, 0). (If you don't see that this is the $y-$intercept, try plugging in the test-value of $x = 0$.)

The slope tells us that Nadia runs 6 feet every one second, so another point on the line is (1, 6). Connecting these points gives us Nadia's line:

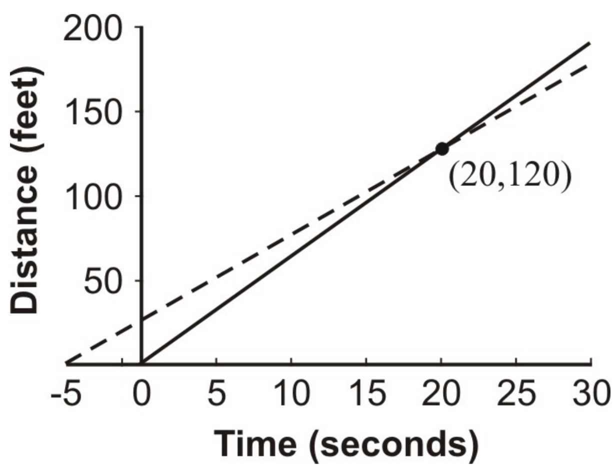


To graph the line that describes Peter's run, again start with the $y-$intercept. In this case this is the point (0, 20).

The slope tells us that Peter runs 5 feet every one second, so another point on the line is (1, 25). Connecting these points gives us Peter's line:



In order to find when and where Nadia and Peter meet, we'll graph both lines on the same graph and extend the lines until they cross. The crossing point is the solution to this problem.



The graph shows that Nadia and Peter meet **20 seconds after Nadia starts running, and 120 feet from the starting point.**

Vocabulary

- A **linear system of equations** is a set of equations that must be solved together to find the one solution that fits them both.

Guided Practice

Solve the following system of equations by graphing:

$$2x+3y=6$$
$$4x-y=-2$$

Solution

Since the equations are in standard form, this time we'll graph them by finding the $x-$ and $y-$intercepts of each of the lines.

Line 1: $2x+3y=6$

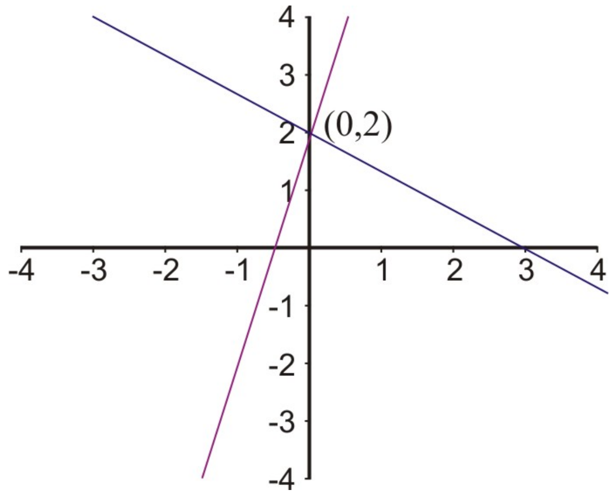
$x-$intercept: set $y=0 \Rightarrow 2x=6 \Rightarrow x=3$ so the intercept is (3, 0)

$y-$intercept: set $x=0 \Rightarrow 3y=6 \Rightarrow y=2$ so the intercept is (0, 2)

Line 2: $-4x+y=2$

$x-$intercept: set $y=0 \Rightarrow -4x=2 \Rightarrow x=-\frac{1}{2}$ so the intercept is $\left(-\frac{1}{2},0\right)$

$y-$intercept: set $x=0 \Rightarrow y=2$ so the intercept is (0, 2)



The graph shows that the lines intersect at (0, 2). Therefore, **the solution to the system of equations is** $x=0, y=2$.

Practice

Determine which ordered pair satisfies the system of linear equations.

1.
$$\begin{aligned} y &= 3x-2 \\ y &= -x \end{aligned}$$
   a. (1, 4)
   b. (2, 9)
   c. (1/2, -1/2)
2.
$$\begin{aligned} y &= 2x-3 \\ y &= x+5 \end{aligned}$$
   a. (8, 13)
   b. (-7, 6)
   c. (0, 4)
3.
$$\begin{aligned} 2x+y &= 8 \\ 5x+2y &= 10 \end{aligned}$$

a. (-9, 1)
b. (-6, 20)
c. (14, 2)

4.

$$\begin{aligned} 3x+2y &= 6 \\ y &= \frac{1}{2}x-3 \end{aligned}$$

a. (3, -3/2)
b. (-4, 3)
c. (1/2, 4)

5.

$$\begin{aligned} 2x-y &= 10 \\ 3x+y &= -5 \end{aligned}$$

a. (4, -2)
b. (1, -8)
c. (-2, 5)

Solve the following systems using the graphing method.

6.

$$\begin{aligned} y &= x+3 \\ y &= -x+3 \end{aligned}$$

7.

$$\begin{aligned} y &= 3x-6 \\ y &= -x+6 \end{aligned}$$

8.

$$\begin{aligned} 2x &= 4 \\ y &= -3 \end{aligned}$$

9.

$$\begin{aligned} y &= -x+5 \\ -x+y &= 1 \end{aligned}$$

10.

$$\begin{aligned} x+2y &= 8 \\ 5x+2y &= 0 \end{aligned}$$

11.

$$\begin{aligned} 3x+2y &= 12 \\ 4x-y &= 5 \end{aligned}$$

12.

$$5x+2y=-4$$
$$x-y=2$$

13.

$$2x+4=3y$$
$$x-2y+4=0$$

14.

$$y=\frac{1}{2}x-3$$
$$2x-5y=5$$

15.

$$y=4$$
$$x=8-3y$$

16. Try to solve the following system using the graphing method:

$$y=\frac{3}{5}x+5$$
$$y=-2x-\frac{1}{2}$$

.

a. What does it look like the $x-$coordinate of the solution should be?
b. Does that coordinate really give the same $y-$value when you plug it into both equations?
c. Why is it difficult to find the real solution to this system?

17. Try to solve the following system using the graphing method:

$$y=4x+8$$
$$y=5x+1$$

. Use a grid with $x-$values and $y-$values ranging from -10 to 10.

a. Do these lines appear to intersect?
b. Based on their equations, are they parallel?
c. What would we have to do to find their intersection point?

18. Try to solve the following system using the graphing method:

$$y=\frac{1}{2}x+4$$
$$y=\frac{4}{9}x+\frac{9}{2}$$

. Use the same grid as before.

a. Can you tell exactly where the lines cross?
b. What would we have to do to make it clearer?

Solve the following problems by using the graphing method.

19. Mary's car has broken down and it will cost her \$1200 to get it fixed—or, for \$4500, she can buy a new, more efficient car instead. Her present car uses about \$2000 worth of gas per year, while gas for the new car would cost about \$1500 per year. After how many years would the total cost of fixing the car equal the total cost of replacing it?
20. A tortoise and hare decide to race 30 feet. The hare, being much faster, decides to give the tortoise a 20 foot head start. The tortoise runs at 0.5 feet/sec and the hare runs at 5.5 feet per second. How long until the hare catches the tortoise?

4.2 Solving Systems Using Substitution

Here you'll learn how to use subtitution to solve systems of linear equations in two variables. You'll then solve real-world problems involving such systems.

Guidance

In this lesson, we'll learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Let's look again at the problem about Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

In that example we came up with two equations:

Nadia's equation: $d = 6t$

Peter's equation: $d = 5t + 20$

Each equation produced its own line on a graph, and to solve the system we found the point at which the lines intersected—the point where the values for d and t satisfied **both** relationships. When the values for d and t are equal, that means that Peter and Nadia are at the same place at the same time.

But there's a faster way than graphing to solve this system of equations. Since we want the value of d to be the same in both equations, we could just set the two right-hand sides of the equations equal to each other to solve for t. That is, if $d = 6t$ and $d = 5t + 20$, and the two d's are equal to each other, then by the transitive property we have $6t = 5t + 20$. We can solve this for t:

$$\begin{aligned} 6t &= 5t + 20 && \textit{subtract } 5t \textit{ from both sides}: \\ t &= 20 && \textit{substitute this value for } t \textit{ into Nadia's equation}: \\ d &= 6 \cdot 20 = 120 \end{aligned}$$

Even if the equations weren't so obvious, we could use simple algebraic manipulation to find an expression for one variable in terms of the other. If we rearrange Peter's equation to isolate t:

$$\begin{aligned} d &= 5t + 20 && \textit{subtract } 20 \textit{ from both sides}: \\ d - 20 &= 5t && \textit{divide by } 5: \\ \frac{d - 20}{5} &= t \end{aligned}$$

We can now ***substitute*** this expression for t into Nadia's equation $(d = 6t)$ to solve:

$$\begin{aligned}
d &= 6\left(\frac{d-20}{5}\right) && \textit{multiply both sides by } 5: \\
5d &= 6(d-20) && \textit{distribute the } 6: \\
5d &= 6d-120 && \textit{subtract } 6d \textit{ from both sides}: \\
-d &= -120 && \textit{divide by } -1: \\
d &= 120 && \textit{substitute value for } d \textit{ into our expression for } t: \\
t &= \frac{120-20}{5} = \frac{100}{5} = 20
\end{aligned}$$

So we find that Nadia and Peter meet 20 seconds after they start racing, at a distance of 120 feet away.

The method we just used is called the **Substitution Method.** In this lesson you'll learn several techniques for isolating variables in a system of equations, and for using those expressions to solve systems of equations that describe situations like this one.

Example A

Let's look at an example where the equations are written in **standard form.**

Solve the system

$$\begin{aligned}
2x+3y &= 6 \\
-4x+y &= 2
\end{aligned}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. So the easiest way to start is to use this equation to solve for y.

Solve the second equation for y:

$$\begin{aligned}
-4x+y &= 2 && \textit{add } 4x \textit{ to both sides}: \\
y &= 2+4x
\end{aligned}$$

Substitute this expression into the first equation:

$$\begin{aligned}
2x+3(2+4x) &= 6 && \textit{distribute the } 3: \\
2x+6+12x &= 6 && \textit{collect like terms}: \\
14x+6 &= 6 && \textit{subtract } 6 \textit{ from both sides}: \\
14x &= 0 && \textit{and hence}: \\
x &= 0
\end{aligned}$$

Substitute back into our expression for y:

$$y = 2+4\cdot 0 = 2$$

As you can see, we end up with the same solution $(x = 0, y = 2)$ that we found when we graphed these functions back in Lesson 7.1. So long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, let's look at a more complicated example. Here, the values of x and y we end up with aren't whole numbers, so they would be difficult to read off a graph!

Example B

Solve the system

$$\begin{aligned} 2x + 3y &= 3 \\ 2x - 3y &= -1 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x:

$$\begin{aligned} 2x + 3y &= 3 && \text{subtract } 3y \text{ from both sides :} \\ 2x &= 3 - 3y && \text{divide both sides by } 2 : \\ x &= \frac{1}{2}(3 - 3y) \end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned} \cancel{2} \cdot \frac{1}{\cancel{2}}(3 - 3y) - 3y &= -1 && \text{cancel the fraction and re-write terms :} \\ 3 - 3y - 3y &= -1 && \text{collect like terms :} \\ 3 - 6y &= -1 && \text{subtract } 3 \text{ from both sides :} \\ -6y &= -4 && \text{divide by } -6 : \\ y &= \frac{2}{3} \end{aligned}$$

Substitute into the expression we got for x:

$$\begin{aligned} x &= \frac{1}{2}\left(3 - \cancel{3}\left(\frac{2}{\cancel{3}}\right)\right) \\ x &= \frac{1}{2} \end{aligned}$$

So our solution is $x = \frac{1}{2}, y = \frac{2}{3}$. You can see how the graphical solution $\left(\frac{1}{2}, \frac{2}{3}\right)$ might have been difficult to read accurately off a graph!

Solving Real-World Problems Using Linear Systems

Simultaneous equations can help us solve many real-world problems. We may be considering a purchase—for example, trying to decide whether it's cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but aren't sure if you would really save any money by buying a new CD every month in that way. Or you might be considering two different phone contracts. Let's look at an example of that now.

Example C

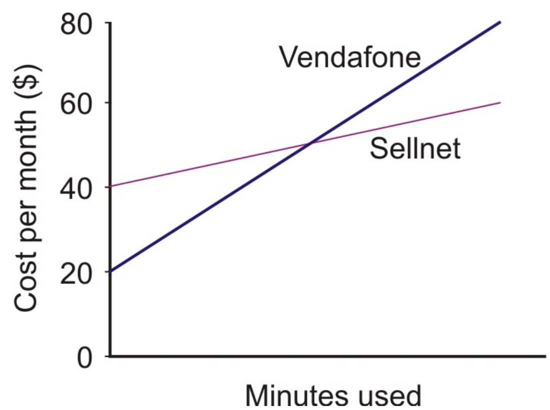
Anne is trying to choose between two phone plans. The first plan, with Vendafone, costs $20 per month, with calls costing an additional 25 cents per minute. The second company, Sellnet, charges $40 per month, but calls cost only 8 cents per minute. Which should she choose?

You should see that Anne's choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the *number of minutes* is the independent variable, it will be our x. Cost is *dependent* on minutes – the *cost per month* is the dependent variable and will be assigned y.

For Vendafone: $y = 0.25x + 20$

For Sellnet: $y = 0.08x + 40$

By writing the equations in slope-intercept form $(y = mx + b)$, you can sketch a graph to visualize the situation:



The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone line's slope). In order to help Anne decide which to choose, we'll find where the two lines cross, by solving the two equations as a system.

Since equation 1 gives us an expression for $y(0.25x + 20)$, we can substitute this expression directly into equation 2:

$$\begin{aligned}
0.25x + 20 &= 0.08x + 40 && \textit{subtract } 20 \textit{ from both sides}: \\
0.25x &= 0.08x + 20 && \textit{subtract } 0.08x \textit{ from both sides}: \\
0.17x &= 20 && \textit{divide both sides by } 0.17: \\
x &= 117.65 \textit{ minutes} && \textit{rounded to } 2 \textit{ decimal places.}
\end{aligned}$$

So if Anne uses 117.65 minutes a month (although she can't really do *exactly* that, because phone plans only count whole numbers of minutes), the phone plans will cost the same. Now we need to look at the graph to see which plan is better if she uses more minutes than that, and which plan is better if she uses fewer. You can see that the Vendafone plan costs more when she uses more minutes, and the Sellnet plan costs more with fewer minutes.

So, **if Anne will use 117 minutes or less every month she should choose *Vendafone.* If she plans on using 118 or more minutes she should choose *Sellnet.***

Vocabulary

- Solving linear systems **by substitution** means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.

Guided Practice

Solve the system

$$\begin{aligned}8x+10y &= 2\\ 4x-15y &= -19\end{aligned}$$

Solution:

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x:

$$\begin{aligned}8x+10y &= 2 && \textit{subtract } 10y \textit{ from both sides}:\\ 8x &= 2-10y && \textit{divide both sides by } 8:\\ x &= \frac{1}{8}(2-10y)\end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned}4\cdot\frac{1}{8}(2-10y)-15y &= -19 && \textit{simplify the fraction}:\\ \frac{1}{2}(2-10y)-15y &= -19 && \textit{distribute the fraction and re-write terms}:\\ 1-5y-15y &= -19 && \textit{collect like terms}:\\ 1-20y &= -19 && \textit{subtract } 1 \textit{ from both sides}:\\ -20y &= -20 && \textit{divide by } -20:\\ y &= 1\end{aligned}$$

Substitute into the expression we got for x:

$$\begin{aligned}x &= \frac{1}{8}(2-10y) && \textit{Substitute the y-value into the x equation}:\\ x &= \frac{1}{8}(2-10(1)) && \textit{Simplify}:\\ x &= \frac{1}{8}(2-10)\\ x &= \frac{1}{8}(-8)\\ x &= -1\end{aligned}$$

So our solution is $x=-1, y=1$.

Practice

1. Solve the system:

$$\begin{aligned}x+2y &= 9\\ 3x+5y &= 20\end{aligned}$$

2. Solve the system:

$$\begin{aligned} x-3y &= 10 \\ 2x+y &= 13 \end{aligned}$$

3. Solve the system:

$$\begin{aligned} 2x+0.5y &= -10 \\ x-y &= -10 \end{aligned}$$

4. Solve the system:

$$\begin{aligned} 2x+0.5y &= 3 \\ x+2y &= 8.5 \end{aligned}$$

5. Solve the system:

$$\begin{aligned} 3x+5y &= -1 \\ x+2y &= -1 \end{aligned}$$

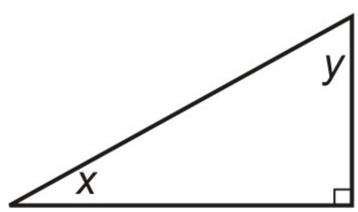
6. Solve the system:

$$\begin{aligned} 3x+5y &= -3 \\ x+2y &= -\frac{4}{3} \end{aligned}$$

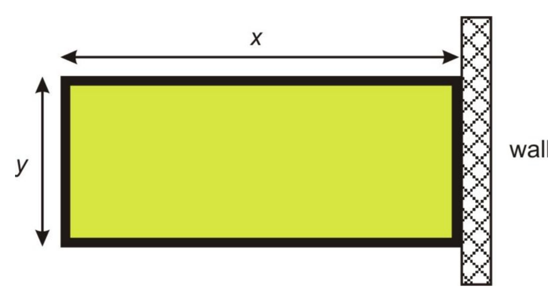
7. Solve the system:

$$\begin{aligned} x-y &= -\frac{12}{5} \\ 2x+5y &= -2 \end{aligned}$$

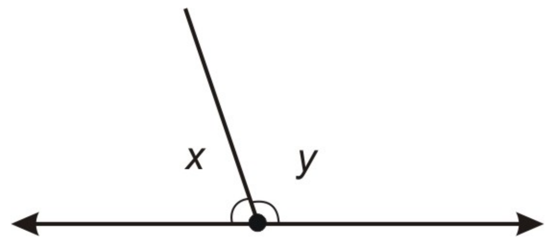
8. Of the two non-right angles in a right angled triangle, one measures twice as many degrees as the other. What are the angles?



9. The sum of two numbers is 70. They differ by 11. What are the numbers?
10. A number plus half of another number equals 6; twice the first number minus three times the second number equals 4. What are the numbers?
11. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



12. A ray cuts a line forming two angles. The difference between the two angles is 18°. What does each angle measure?



13. Jason is five years older than Becky, and the sum of their ages is 23. What are their ages?

4.3 Solving Systems by Elimination

Here you will review how to solve a system of two equations and two unknowns using the elimination method.

Guidance

There are many ways to solve a system that you have learned in the past including substitution and graphical intersection. Here you will focus on solving using elimination because the knowledge and skills used will transfer directly into using matrices.

When solving a system, the first thing to do is to count the number of variables that are missing and the number of equations. The number of variables needs to be the same or fewer than the number of equations. Two equations and two variables can be solved, but one equation with two variables cannot.

Get into the habit of always writing systems in standard form: $Ax+By=C$. This will help variables line up, avoid +/- errors and lay the groundwork for using matrices. Once two equations with two variables are in standard form, decide which variable you want to eliminate, scale each equation as necessary by multiplying through by constants and then add the equations together. This procedure should reduce both the number of equations and the number of variables leaving one equation and one variable. Solve and substitute to determine the value of the second variable.

Example A

Solve the following system of equations: $5x+12y=72$ and $3x-2y=18$.

Solution: Here is a system of two equations and two variables in standard form. Notice that there is an x column and a y column on the left hand side and a constant column on the right hand side when you rewrite the equations as shown. Also notice that if you add the system as written no variable will be eliminated.

Equation 1: $5x+12y=72$

Equation 2: $3x-2y=18$

Strategically choose to eliminate y by scaling the second equation by 6 so that the coefficient of y will match at 12 and -12.

$$\begin{aligned} 5x+12y&=72\\ 18x-12y&=108 \end{aligned}$$

Add the two equations:

$$\begin{aligned} 23x&=180\\ x&=\frac{180}{23} \end{aligned}$$

The value for x could be substituted into either of the original equations and the result could be solved for y; however, since the value is a fraction it will be easier to repeat the elimination process in order to solve for x. This time you will take the first two equations and eliminate x by making the coefficients of x to be 15 and -15. Scale the first equation by a factor of 3 and scale the second equation by a factor of -5.

Equation 1: $15x+36y=216$

Equation 2: $-15x+10y=-90$

Adding the two equations:

$$\begin{aligned} 0x+46y &= 126 \\ y &= \frac{126}{46} = \frac{63}{23} \end{aligned}$$

The point $\left(\frac{180}{23}, \frac{63}{23}\right)$ is where these two lines intersect.

Example B

Solve the following system of equations:

$$\begin{aligned} 6x-7y &= 8 \\ 15x-14y &= 21 \end{aligned}$$

Solution: Scaling the first equation by -2 will allow the y term to be eliminated when the equations are summed.

$$\begin{aligned} -12x+14y &= -16 \\ 15x-14y &= 21 \end{aligned}$$

The sum is:

$$\begin{aligned} 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

You can substitute x into the first equation to solve for y.

$$\begin{aligned} 6 \cdot \frac{5}{3} - 7y &= 8 \\ 10-7y &= 8 \\ -7y &= -2 \\ y &= \frac{2}{7} \end{aligned}$$

The point $\left(\frac{5}{3}, \frac{2}{7}\right)$ is where these two lines intersect.

Example C

Solve the following system of equations:

$$\begin{aligned} 5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\ \frac{1}{x} + \frac{1}{y} &= 4 \end{aligned}$$

Solution: The strategy of elimination still applies. You can eliminate the $\frac{1}{y}$ term if the second equation is scaled by a factor of -2.

$$\begin{aligned} 5\cdot\frac{1}{x}+2\cdot\frac{1}{y} &= 11 \\ -2\cdot\frac{1}{x}-2\cdot\frac{1}{y} &= -8 \end{aligned}$$

Add the equations together and solve for x.

$$\begin{aligned} -3\cdot\frac{1}{x}+0\cdot\frac{1}{y} &= 3 \\ -3\cdot\frac{1}{x} &= 3 \\ \frac{1}{x} &= -1 \\ x &= -1 \end{aligned}$$

Substitute into the second equation and solve for y.

$$\begin{aligned} \frac{1}{-1}+\frac{1}{y} &= 4 \\ -1+\frac{1}{y} &= 4 \\ \frac{1}{y} &= 5 \\ y &= \frac{1}{5} \end{aligned}$$

The point $\left(-1,\frac{1}{5}\right)$ is the point of intersection between these two curves.

Concept Problem Revisited

Plan A costs $40 per month plus $0.10 for each minute of talk time.

Plan B costs $25 per month plus $0.50 for each minute of talk time.

If you want to find out when the two plans cost the same, you can represent each plan with an equation and solve the system of equations. Let y represent cost and x represent number of minutes.

$$\begin{aligned} y &= 0.10x+40 \\ y &= 0.50x+25 \end{aligned}$$

First you put these equations in standard form.

$$\begin{aligned} x-10y &= -400 \\ x-2y &= -50 \end{aligned}$$

Then you scale the second equation by -1 and add the equations together and solve for y.

$$\begin{aligned} -8y &= -350 \\ y &= 43.75 \end{aligned}$$

To solve for x, you can scale the second equation by -5, add the equations together and solve for x.

$$\begin{aligned} -4x &= -150 \\ x &= 37.5 \end{aligned}$$

The equivalent costs of plan A and plan B will occur at 37.5 minutes of talk time with a cost of \$43.75.

Vocabulary

A ***system of equations*** is two or more equations.

Standard form for the equation of a line is $Ax + By = C$.

To ***scale an equation*** means to multiply every term by a constant.

Guided Practice

1. Solve the following system using elimination:

$$\begin{aligned} 20x + 6y &= -32 \\ 18x - 14y &= 10 \end{aligned}$$

2. Solve the following system using elimination:

$$\begin{aligned} 5x - y &= 22 \\ -2x + 7y &= 19 \end{aligned}$$

3. Solve the following system using elimination:

$$\begin{aligned} 11 \cdot \frac{1}{x} - 5 \cdot \frac{1}{y} &= -38 \\ 9 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= -25 \end{aligned}$$

Answers:

1. Start by scaling both of the equations by $\frac{1}{2}$. Then notice that you have $3y$ and $-7y$. Rescale the first equation by 7 and the second equation by 3 to make the coefficients of y at 21 and -21. There are a number of possible ways to eliminate y.

$$\begin{aligned} 70x+21y &= -112 \\ 27x-21y &= 15 \end{aligned}$$

Add, solve for $x=-1$, substitute and solve for y.

Final Answer: (-1, -2)

2. Start by scaling the first equation by 7 and notice that the y coefficient will immediately be eliminated when the equations are summed.

$$\begin{aligned} 35x-7y &= 154 \\ -2x+7y &= 19 \end{aligned}$$

Add, solve for $x=\frac{173}{33}$. Instead of substituting, practice eliminating x by scaling the first equation by 2 and the second equation by 5.

$$\begin{aligned} 10x-2y &= 44 \\ -10x+35y &= 95 \end{aligned}$$

Add, solve for y.

Final Answer: $\left(\frac{173}{33},\frac{139}{33}\right)$

3. To eliminate $\frac{1}{y}$, scale the first equation by 2 and the second equation by 5.

To eliminate $\frac{1}{x}$, scale the first equation by -9 and the second equation by 11.

Final Answer: $\left(-\frac{1}{3},1\right)$

Practice

Solve each system of equations using the elimination method.

1. $x+y=-4; -x+2y=13$
2. $\frac{3}{2}x-\frac{1}{2}y=\frac{1}{2}; -4x+2y=4$
3. $6x+15y=1; 2x-y=19$
4. $x-\frac{2y}{3}=\frac{-2}{3}; 5x-2y=10$
5. $-9x-24y=-243; \frac{1}{2}x+y=\frac{21}{2}$
6. $5x+\frac{28}{3}y=\frac{176}{3}; y+x=10$
7. $2x-3y=50; 7x+8y=-10$
8. $2x+3y=1; 2y=-3x+14$
9. $2x+\frac{3}{5}y=3; \frac{3}{2}x-y=-5$
10. $5x=9-2y; 3y=2x-3$
11. How do you know if a system of equations has no solution?
12. If a system of equations has no solution, what does this imply about the relationship of the curves on the graph?

13. Give an example of a system of two equations with two unknowns with an infinite number of solutions. Explain how you know the system has an infinite number of solutions.

14. Solve:

$$12 \cdot \frac{1}{x} - 18 \cdot \frac{1}{y} = 4$$
$$8 \cdot \frac{1}{x} + 9 \cdot \frac{1}{y} = 5$$

15. Solve:

$$14 \cdot \frac{1}{x} - 5 \cdot \frac{1}{y} = -3$$
$$7 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} = 3$$

4.4 Applications of Systems of Equations

Here you will solve systems of linear equations using the most efficient method.

Guidance

Any of the methods (graphing, substitution, linear combination) learned in this unit can be used to solve a linear system of equations. Sometimes, however it is more efficient to use one method over another based on how the equations are presented. For example

- If both equations are presented in slope intercept form $(y = mx + b)$, then either graphing or substitution would be most efficient.
- If one equation is given in slope intercept form or solved for x, then substitution might be easiest.
- If both equations are given in standard form $(Ax + By = C)$, then linear combinations is usually most efficient.

Example A

Solve the following system:

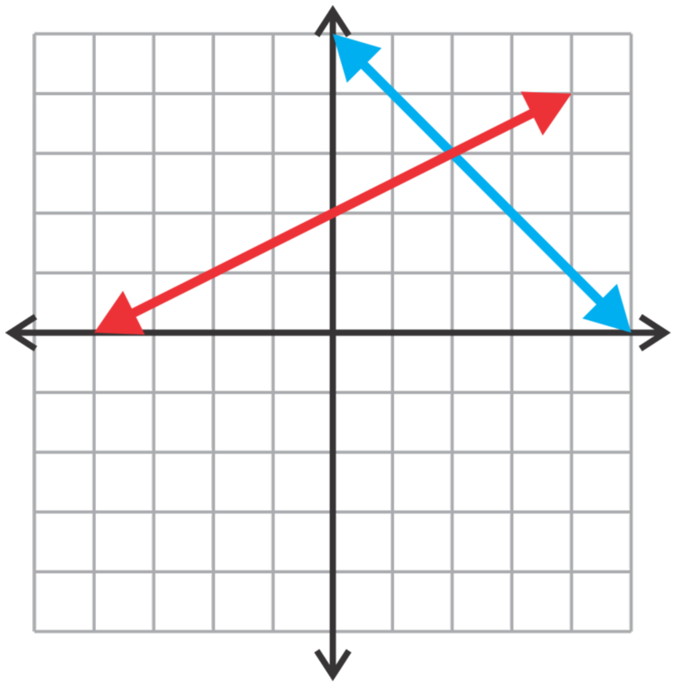
$$y = -x + 5$$
$$y = \frac{1}{2}x + 2$$

Solution: Since both equations are in slope intercept form we could easily graph these lines. The question is whether or not the intersection of the two lines will lie on the "grid" (whole numbers). If not, it is very difficult to determine an answer from a graph. One way to get around this difficulty is to use technology to graph the lines and find their intersection.

The first equation has a $y-$intercept of 5 and slope of -1. It is shown here graphed in **blue**.

The second equation has a $y-$intercept of 2 and a slope of $\frac{1}{2}$. It is shown here graphed in **red**.

The two lines clearly intersect at (2, 3).



Alternate Method: Substitution may be the preferred method for students who would rather solve equations algebraically. Since both of these equations are equal to y, we can let the right hand sides be equal to each other and solve for x:

$$\begin{aligned}
-x+5 &= \frac{1}{2}x+2 \\
2\left(-x+5 = \frac{1}{2}x+2\right) & \leftarrow \text{Multiplying the equation by 2 eliminates the fraction.} \\
-2x+10 &= x+4 \\
6 &= 3x \\
x &= 2
\end{aligned}$$

Now solve for y:

$$\begin{aligned}
y &= -(2)+5 \\
y &= 3
\end{aligned}$$

Solution: (2, 3)

Check your answer:

$$\begin{aligned}
3 &= -2+5 \\
3 &= \frac{1}{2}2+2 \rightarrow 1+2
\end{aligned}$$

Example B

Solve the system:

$$\begin{aligned} 15x+y&=24 \\ y&=-4x+2 \end{aligned}$$

Solution: This time one of our equations is already solved for y. It is easiest here to use this expression to substitute into the other equation and solve:

$$\begin{aligned} 15x+(-4x+2)&=24 \\ 15x-4x+2&=24 \\ 11x&=22 \\ x&=2 \end{aligned}$$

Now solve for y:

$$\begin{aligned} y&=-4(2)+2 \\ y&=-8+2 \\ y&=-6 \end{aligned}$$

Solution: (2, -6)

Check your answer:

$$\begin{aligned} 15(2)+(-6)=30-6&=24 \\ -6=-4(2)+2=-8+2&=-6 \end{aligned}$$

Example C

Solve the system:

$$\begin{aligned} -6x+11y&=86 \\ 9x-13y&=-115 \end{aligned}$$

Solution: Both equations in this example are in standard form so the easiest method to use here is linear combinations. Since the LCM of 6 and 9 is 18, we will multiply the first equation by 3 and the second equation by 2 to eliminate x first:

$$\begin{aligned} 3(-6x+11y=86) \quad &\Rightarrow \quad -\cancel{18x}+33y=258 \\ 2(9x-13y=-115) \quad & \qquad\quad \underline{\cancel{18x}-26y=-230} \\ & \qquad\qquad\qquad 7y=28 \\ & \qquad\qquad\qquad\; y=4 \end{aligned}$$

Now solve for x:

$$\begin{aligned}-6x+11(4)&=86\\-6x+44&=86\\-6x&=42\\x&=-7\end{aligned}$$

Solution: (-7, 4)

Check your answer:

$$\begin{aligned}-6(-7)+11(4)&=42+44=86\\9(-7)-13(4)&=-63-52=-115\end{aligned}$$

Guided Practice

Solve the following systems using the most efficient method:

1.

$$\begin{aligned}y&=-3x+2\\y&=2x-3\end{aligned}$$

2.

$$\begin{aligned}4x+5y&=-5\\x&=2y-11\end{aligned}$$

3.

$$\begin{aligned}4x-5y&=-24\\-15x+7y&=-4\end{aligned}$$

Answers

1. This one could be solved by graphing, graphing with technology or substitution. This time we will use substitution. Since both equations are solved for y, we can set them equal and solve for x:

$$\begin{aligned}-3x+2&=2x-3\\5&=5x\\x&=1\end{aligned}$$

Now solve for y:

$$\begin{aligned}y&=-3(1)+2\\y&=-3+2\\y&=-1\end{aligned}$$

Solution: (1, -1)

2. Since the second equation here is solved for x, it makes sense to use substitution:

$$\begin{aligned}4(2y-11)+5y&=-5\\8y-44+5y&=-5\\13y&=39\\y&=3\end{aligned}$$

Now solve for x:

$$\begin{aligned}x&=2(3)-11\\x&=6-11\\x&=-5\end{aligned}$$

Solution: (-5, 3)

3. This time, both equations are in standard form so it makes the most sense to use linear combinations. We can eliminate y by multiplying the first equation by 7 and the second equation by 5:

$$\begin{array}{lcr}7(4x-5y=-24) & \Rightarrow & 28x-\cancel{35y}=-168\\5(-15x+7y=-4) & & \underline{-75x+\cancel{35y}=-20}\\ & & -47x=-188\\ & & x=4\end{array}$$

Now find y:

$$\begin{aligned}4(4)-5y&=-24\\16-5y&=-24\\-5y&=-40\\y&=8\end{aligned}$$

Solution: (4, 8)

Practice

Solve the following systems using linear combinations.

1.

$$\begin{aligned}5x-2y&=-1\\8x+4y&=56\end{aligned}$$

2.

$$\begin{aligned} 3x+y &= -16 \\ -4x-y &= 21 \end{aligned}$$

3.

$$\begin{aligned} 7x+2y &= 4 \\ y &= -4x+1 \end{aligned}$$

4.

$$\begin{aligned} 6x+5y &= 25 \\ x &= 2y+24 \end{aligned}$$

5.

$$\begin{aligned} -8x+10y &= -1 \\ 2x-6y &= 2 \end{aligned}$$

6.

$$\begin{aligned} 3x+y &= 18 \\ -7x+3y &= -10 \end{aligned}$$

7.

$$\begin{aligned} 2x+15y &= -3 \\ -3x-5y &= -6 \end{aligned}$$

8.

$$\begin{aligned} 15x-y &= 19 \\ 13x+2y &= 48 \end{aligned}$$

9.

$$\begin{aligned} x &= -9y - 2 \\ -2x - 15y &= 6 \end{aligned}$$

10.

$$\begin{aligned} 3x - 4y &= 1 \\ -2x + 3y &= 1 \end{aligned}$$

11.

$$\begin{aligned} x - y &= 2 \\ 3x - 2y &= -7 \end{aligned}$$

12.

$$\begin{aligned} 3x + 12y &= -18 \\ y &= -\frac{1}{4}x - \frac{3}{2} \end{aligned}$$

13.

$$\begin{aligned} -2x - 8y &= -2 \\ x &= \frac{1}{2}y + 10 \end{aligned}$$

14.

$$\begin{aligned} 14x + y &= 3 \\ -21x - 3y &= -3 \end{aligned}$$

15.

$$\begin{aligned} y &= \frac{4}{5}x + 7 \\ 8x - 10y &= 2 \end{aligned}$$

Solve the following word problem by creating and solving a system of linear equations.

16. Jack and James each buy some small fish for their new aquariums. Jack buys 10 clownfish and 7 goldfish for \$28.25. James buys 5 clownfish and 6 goldfish for \$17.25. How much does each type of fish cost?
17. The sum of two numbers is 35. The larger number is one less than three times the smaller number. What are the two numbers?
18. Rachel offers to go to the coffee shop to buy cappuccinos and lattes for her coworkers. She buys a total of nine drinks for \$35.75. If cappuccinos cost \$3.75 each and the lattes cost \$4.25 each, how many of each drink did she buy?

4.5 Linear Inequalities: Identifying Solutions

Here you'll learn how to solve inequalities by isolating the variable on one side of the inequality sign. You'll also learn how to graph their solution set.

Guidance

To solve an inequality we must isolate the variable on one side of the inequality sign. To isolate the variable, we use the same basic techniques used in solving equations.

We can solve some inequalities by adding or subtracting a constant from one side of the inequality.

Example A

Solve the inequality and graph the solution set.

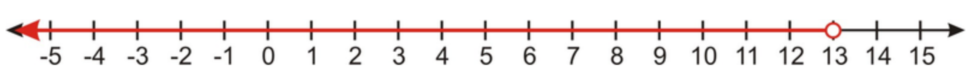
$x-3<10$

Solution

Starting inequality: $x-3<10$

Add **3** to both sides of the inequality: $x-3+3<10+3$

Simplify: $x<13$



Example B

Solve the inequality and graph the solution set.

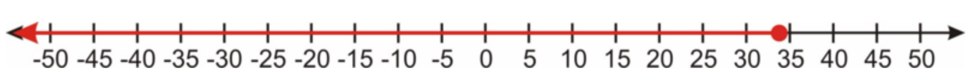
$x-20 \leq 14$

Solution:

Starting inequality: $x-20 \leq 14$

Add **20** to both sides of the inequality: $x-20+20 \leq 14+20$

Simplify: $x \leq 34$



Solving Inequalities Using Multiplication and Division

We can also solve inequalities by multiplying or dividing both sides by a constant. For example, to solve the inequality $5x<3$, we would divide both sides by 5 to get $x<\frac{3}{5}$.

However, something different happens when we multiply or divide by a negative number. We know, for example, that 5 is greater than 3. But if we multiply both sides of the inequality $5>3$ by -2, we get $-10>-6$. And we know that's not true; -10 is less than -6.

This happens whenever we multiply or divide an inequality by a negative number, and so we have to flip the sign around to make the inequality true. For example, to multiply $2 < 4$ by -3, first we multiply the 2 and the 4 each by -3, and then we change the <sign to a >sign, so we end up with $-6 > -12$.

The same principle applies when the inequality contains variables.

Example C

Solve the inequality.

$4x < 24$

Solution:

Original problem: $4x < 24$

Divide both sides by 4: $\frac{4x}{4} < \frac{24}{4}$

Simplify: $x < 6$

Example D

Solve the inequality.

$-5x \leq 21$

Solution:

Original problem: $-5x \leq 21$

Divide both sides by -5 : $\frac{-5x}{-5} \geq \frac{21}{-5}$ ***Flip the inequality sign.***

Simplify: $x \geq -\frac{21}{5}$

Guided Practice

Solve each inequality.

a) $x+8 \leq -7$

b) $x+4 > 13$

c) $\frac{x}{25} < \frac{3}{2}$

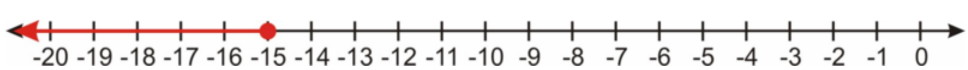
d) $\frac{x}{-7} \geq 9$

Solutions:

a) Starting inequality: $x+8 \leq -7$

Subtract **8** from both sides of the inequality: $x+8-8 \leq -7-8$

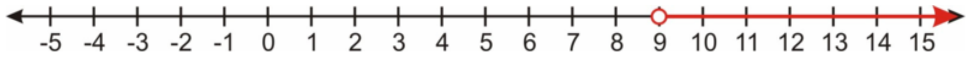
Simplify: $x \leq -15$



b) Starting inequality: $x+4 > 13$

Subtract **4** from both sides of the inequality: $x+4-4 > 13-4$

Simplify: $x > 9$



c) Original problem: $\frac{x}{25} < \frac{3}{2}$

Multiply both sides by 25: $25 \cdot \frac{x}{25} < \frac{3}{2} \cdot 25$

Simplify: $x < \frac{75}{2}$ or $x < 37.5$

d) Original problem: $\frac{x}{-7} \geq 9$

Multiply both sides by -7: $-7 \cdot \frac{x}{-7} \leq 9 \cdot (-7)$ ***Flip the inequality sign.***

Simplify: $x \leq -63$

Practice

For 1-8, solve each inequality and graph the solution on the number line.

1. $x - 5 < 35$
2. $x + 15 \geq -60$
3. $x - 2 \leq 1$
4. $x - 8 > -20$
5. $x + 11 > 13$
6. $x + 65 < 100$
7. $x - 32 \leq 0$
8. $x + 68 \geq 75$

For 9-12, solve each inequality. Write the solution as an inequality and graph it.

9. $3x \leq 6$
10. $\frac{x}{5} > -\frac{3}{10}$
11. $-10x > 250$
12. $\frac{x}{-7} \geq -5$

4.6 Linear Inequalities: Graphing an Inequality in Two Variables

Here you'll learn how to graph linear inequalities in two variables of the form $y > mx+b$ or $y < mx+b$. You'll also solve real-world problems involving such inequalities.

Guidance

The general procedure for graphing inequalities in two variables is as follows:

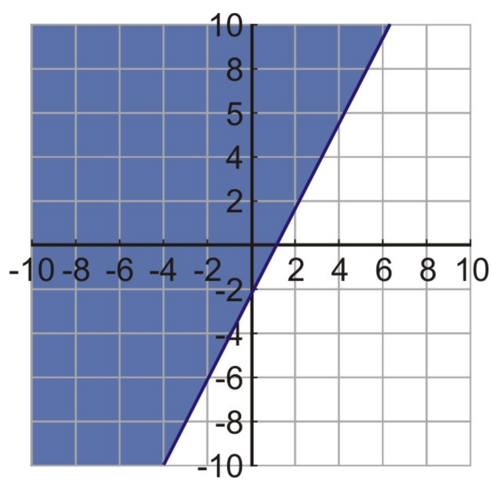
1. Re-write the inequality in slope-intercept form: $y = mx+b$. Writing the inequality in this form lets you know the direction of the inequality.
2. Graph the line of the equation $y = mx+b$ using your favorite method (plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
3. Shade the half plane above the line if the inequality is "greater than." Shade the half plane under the line if the inequality is "less than."

Example A

Graph the inequality $y \geq 2x-3$.

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x-3$; then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.



Example B

Graph the inequality $5x-2y > 4$.

Solution

First we need to rewrite the inequality in slope-intercept form:

$$-2y > -5x + 4$$
$$y < \frac{5}{2}x - 2$$

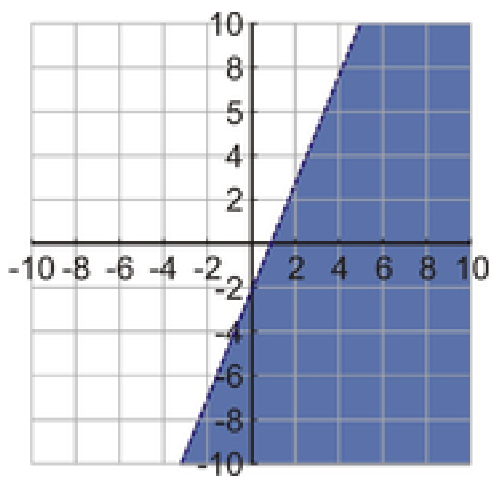
Notice that the inequality sign changed direction because we divided by a negative number.

To graph the equation, we can make a table of values:

TABLE 4.5:

x	y
-2	$\frac{5}{2}(-2) - 2 = -7$
0	$\frac{5}{2}(0) - 2 = -2$
2	$\frac{5}{2}(2) - 2 = 3$

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example C

A retailer sells two types of coffee beans. One type costs \$9 per pound and the other type costs \$7 per pound. Find all the possible amounts of the two different coffee beans that can be mixed together to get a quantity of coffee beans costing \$8.50 or less.

Solution

Let $x =$ weight of \$9 per pound coffee beans in pounds.

Let $y =$ weight of \$7 per pound coffee beans in pounds.

The cost of a pound of coffee blend is given by $9x + 7y$.

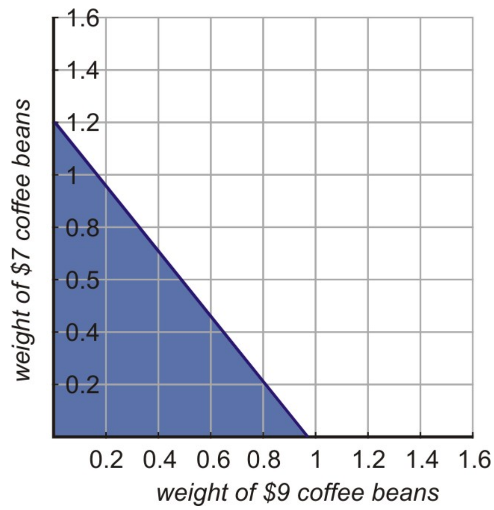
We are looking for the mixtures that cost \$8.50 or less. We write the inequality $9x + 7y \leq 8.50$.

Since this inequality is in standard form, it's easiest to graph it by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $7y = 8.50$ or $y = \frac{8.50}{7} \approx 1.21$. When $y = 0$, we have $9x = 8.50$ or $x = \frac{8.50}{9} \approx 0.94$. We can then graph the line that includes those two points.

Now we have to figure out which side of the line to shade. In $y-$intercept form, we shade the area **below** the line when the inequality is "less than." But in standard form that's not always true. We could convert the inequality to $y-$intercept form to find out which side to shade, but there is another way that can be easier.

The other method, which works for any linear inequality in any form, is to plug a random point into the inequality and see if it makes the inequality true. Any point that's not on the line will do; the point (0, 0) is usually the most convenient.

In this case, plugging in 0 for x and y would give us $9(0)+7(0) \leq 8.50$, which is true. That means we should shade the half of the plane that includes (0, 0). If plugging in (0, 0) gave us a false inequality, that would mean that the solution set is the part of the plane that does *not* contain (0, 0).



Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always nonnegative, so points outside the first quadrant don't represent real-world solutions to this problem.

Vocabulary

- For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.
- The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

$>$The solution set is the half plane above the line.

$\geq$ The solution set is the half plane above the line and also all the points on the line.

$<$The solution set is the half plane below the line.

$\leq$ The solution set is the half plane below the line and also all the points on the line.

Guided Practice

Julius has a job as an appliance salesman. He earns a commission of \$60 for each washing machine he sells and \$130 for each refrigerator he sells. How many washing machines and refrigerators must Julius sell in order to make \$1000 or more in commissions?

Solution

Let $x =$ number of washing machines Julius sells.

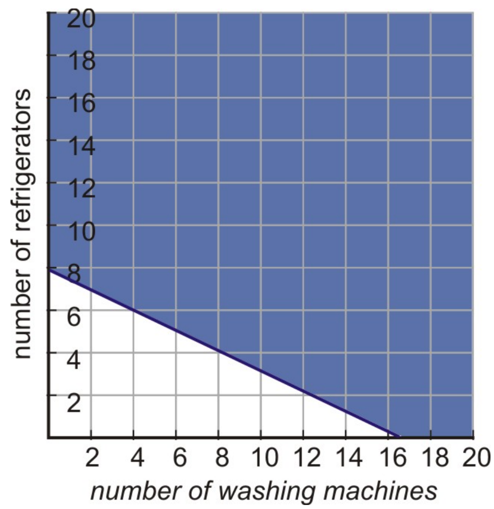
Let $y =$ number of refrigerators Julius sells.

The total commission is $60x + 130y$.

We're looking for a total commission of \$1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Once again, we can do this most easily by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $130y = 1000$, or $y = \frac{1000}{30} \approx 7.69$. When $y = 0$, we have $60x = 1000$, or $x = \frac{1000}{60} \approx 16.67$.

We draw a solid line connecting those points, and shade above the line because the inequality is "greater than." We can check this by plugging in the point (0, 0): selling 0 washing machines and 0 refrigerators would give Julius a commission of \$0, which is *not* greater than or equal to \$1000, so the point (0, 0) is *not* part of the solution; instead, we want to shade the side of the line that does *not* include it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius's commission should be non-negative.

Practice

Graph the following inequalities on the coordinate plane.

1. $y \leq 4x + 3$
2. $y > -\frac{x}{2} - 6$
3. $3x - 4y \geq 12$
4. $x + 7y < 5$
5. $6x + 5y > 1$
6. $y + 5 \leq -4x + 10$
7. $x - \frac{1}{2}y \geq 5$
8. $6x + y < 20$
9. $30x + 5y < 100$
10. Remember what you learned in the last chapter about families of lines.
    a. What do the graphs of $y > x + 2$ and $y < x + 5$ have in common?
    b. What do you think the graph of $x + 2 < y < x + 5$ would look like?
11. How would the answer to problem 6 change if you subtracted 2 from the right-hand side of the inequality?

12. How would the answer to problem 7 change if you added 12 to the right-hand side?
13. How would the answer to problem 8 change if you flipped the inequality sign?
14. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and nighttime minutes could you use in one week if you wanted to pay less than $20?
15. Suppose you are graphing the inequality $y > 5x$.
    a. Why can't you plug in the point (0, 0) to tell you which side of the line to shade?
    b. What happens if you do plug it in?
    c. Try plugging in the point (0, 1) instead. Now which side of the line should you shade?
16. A theater wants to take in at least $2000 for a certain matinee. Children's tickets cost $5 each and adult tickets cost $10 each.
    a. If x represents the number of adult tickets sold and y represents the number of children's tickets, write an inequality describing the number of tickets that will allow the theater to meet their minimum take.
    b. If 100 children's tickets and 100 adult tickets have already been sold, what inequality describes how many *more* tickets of both types the theater needs to sell?
    c. If the theater has only 300 seats (so only 100 are still available), what inequality describes the *maximum* number of additional tickets of both types the theater can sell?

4.7 Systems of Linear Inequalities: Graphing and Writing

Here you'll learn how to graph and solve a system of two or more linear inequalities. You'll also determine if such systems are consistent or inconsistent.

Guidance

In the last chapter you learned how to graph a linear inequality in two variables. To do that, you graphed the equation of the straight line on the coordinate plane. The line was solid for $\leq$ or $\geq$ signs (where the equals sign is included), and the line was dashed for $<$or $>$signs (where the equals sign is not included). Then you shaded above the line (if the inequality began with $y >$ or $y \geq$) or below the line (if it began with $y <$ or $y \leq$).

In this section, we'll see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes.

Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example A

Solve the following system:

$$\begin{aligned} 2x+3y &\leq 18 \\ x-4y &\leq 12 \end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

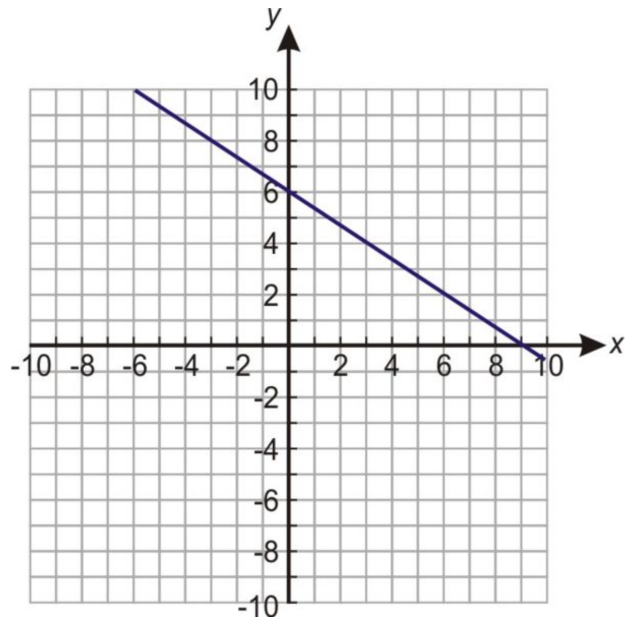
First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

$$\begin{aligned} 3y &\leq -2x+18 & \qquad y &\leq -\frac{2}{3}x+6 \\ && \Rightarrow & \\ -4y &\leq -x+12 & \qquad y &\geq \frac{x}{4}-3 \end{aligned}$$

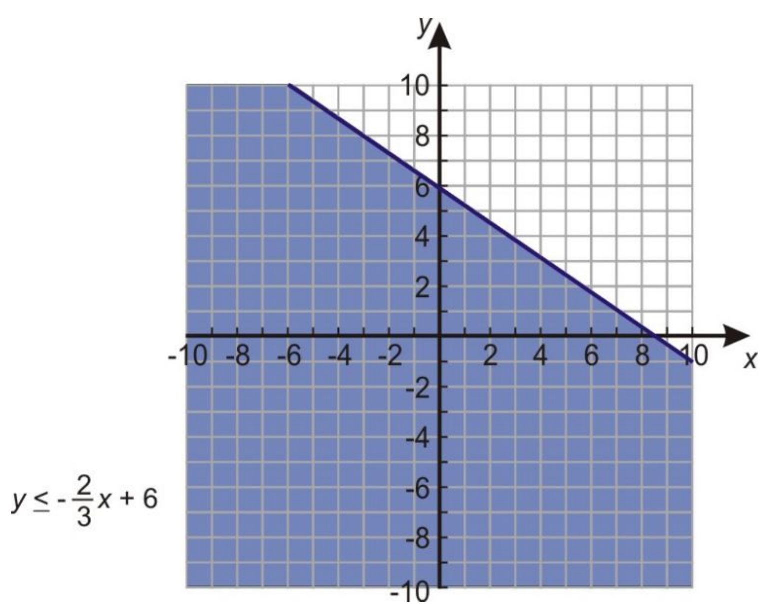
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

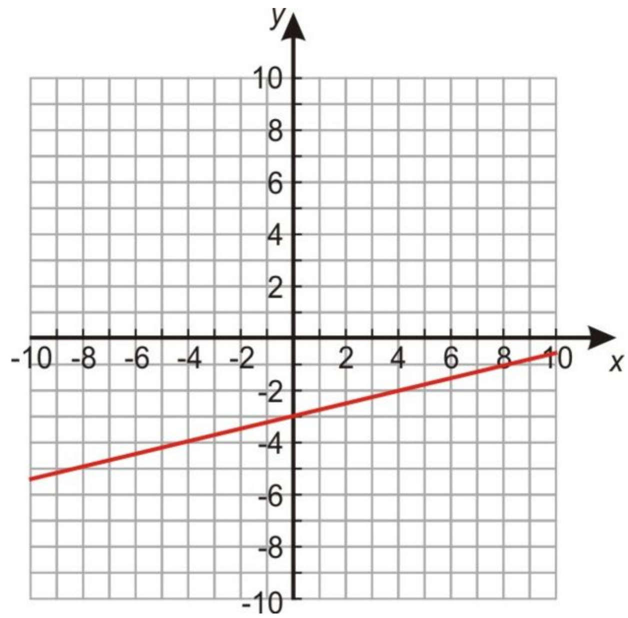
Here's the graph of the first inequality:



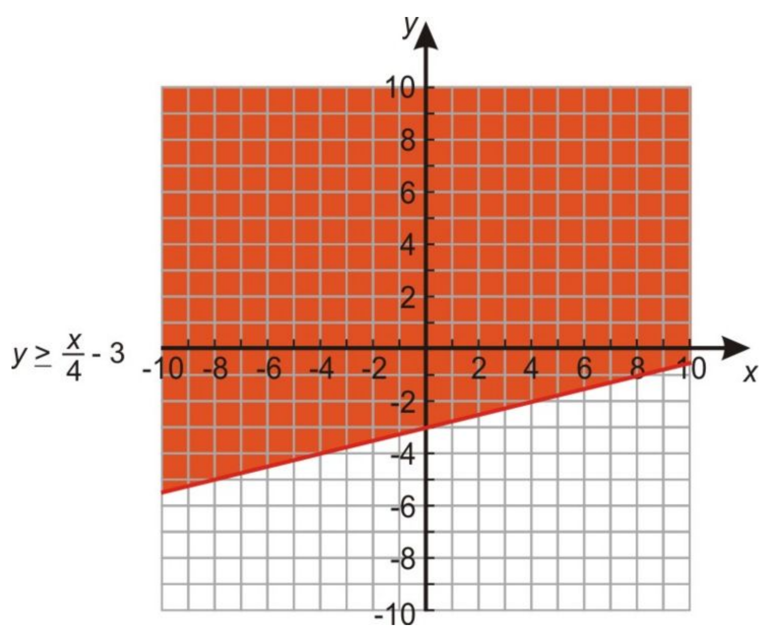
The line is solid because the equals sign is included in the inequality. Since the inequality is **less** than or equal to, we shade **below** the line.



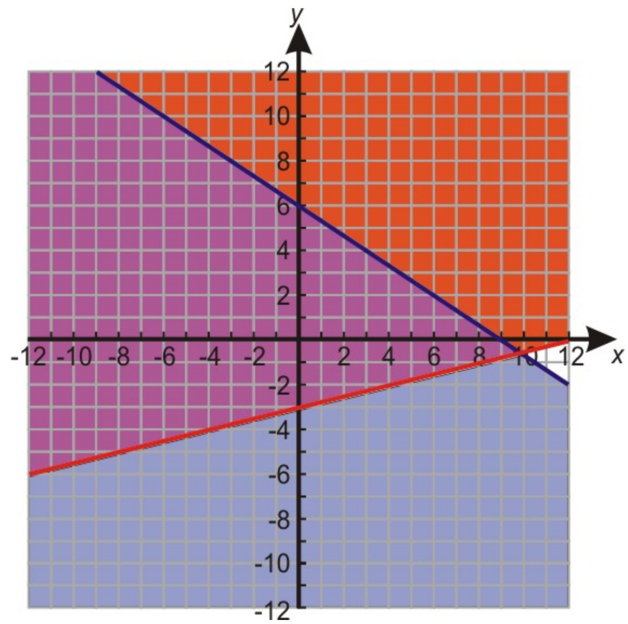
And here's the graph of the second inequality:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because y is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

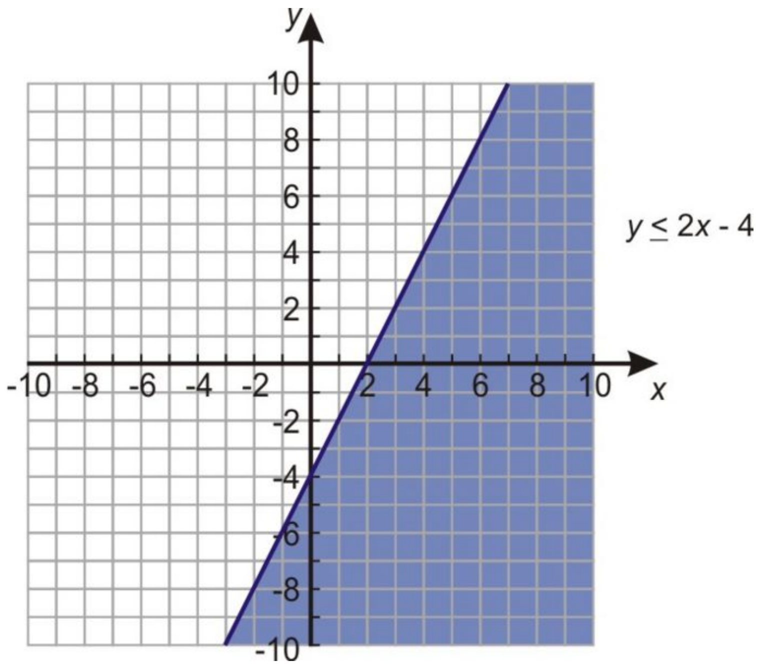
Example B

There are also situations where a system of inequalities has no solution. For example, let's solve this system.

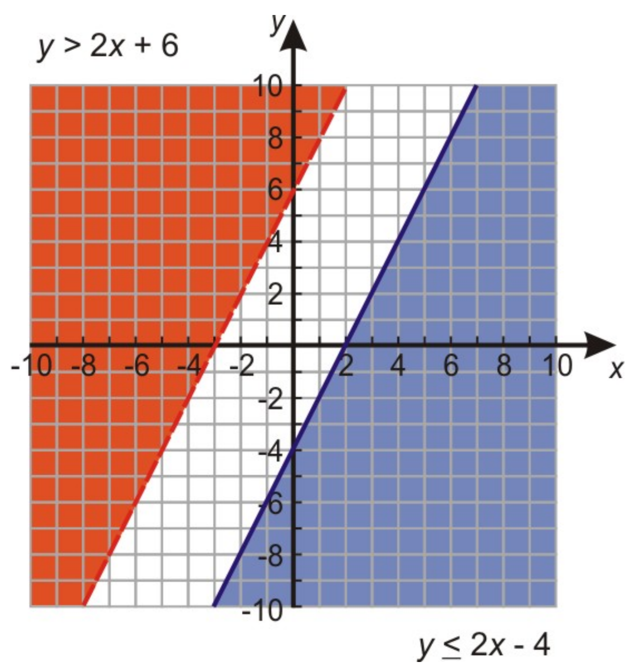
$$\begin{aligned} y &\leq 2x - 4 \\ y &> 2x + 6 \end{aligned}$$

Solution

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because y is greater than.



It doesn't look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won't ever overlap since the lines won't ever cross. So this system of inequalities has no solution.

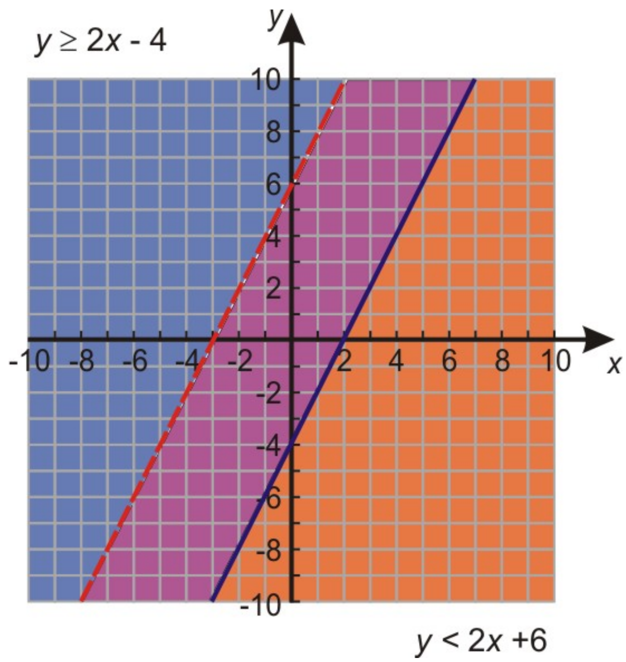
But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

To graph the system

$$y \geq 2x - 4$$
$$y < 2x + 6$$

,

we draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region—one that continues infinitely in at least one direction. But if we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded**—a finite region with three or more sides.

Let's look at a simple example.

Example C

Find the solution to the following system of inequalities.

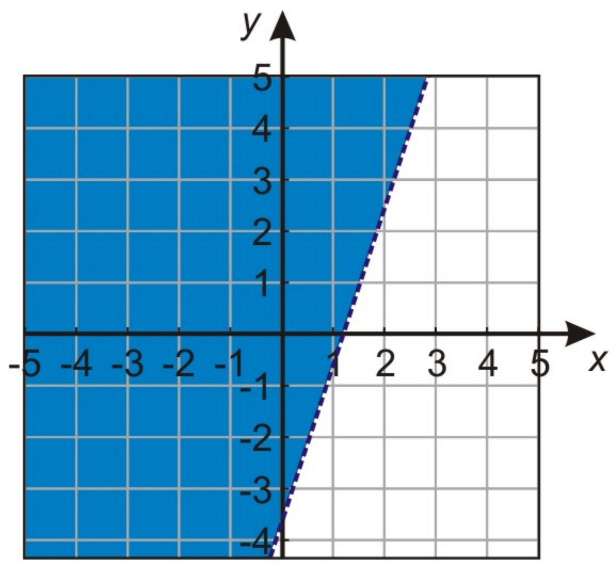
$$3x - y < 4$$
$$4y + 9x < 8$$
$$x \geq 0$$
$$y \geq 0$$

Solution

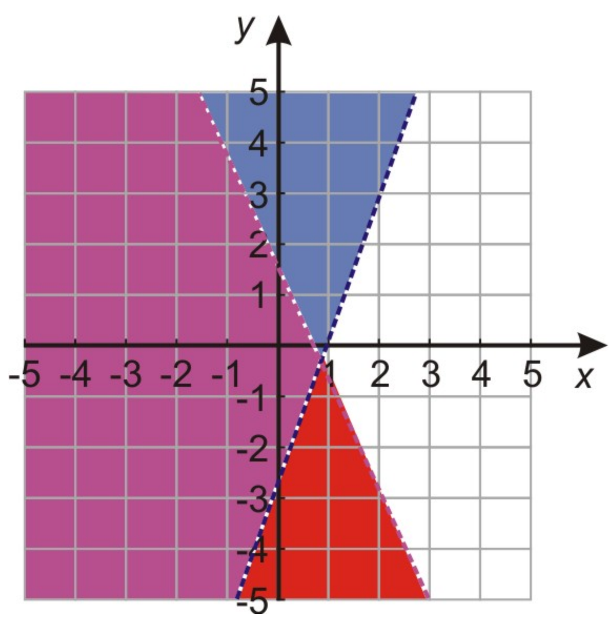
Let's start by writing our inequalities in slope-intercept form.

$$y > 3x - 4$$
$$y < -\frac{9}{4}x + 2$$
$$x \geq 0$$
$$y \geq 0$$

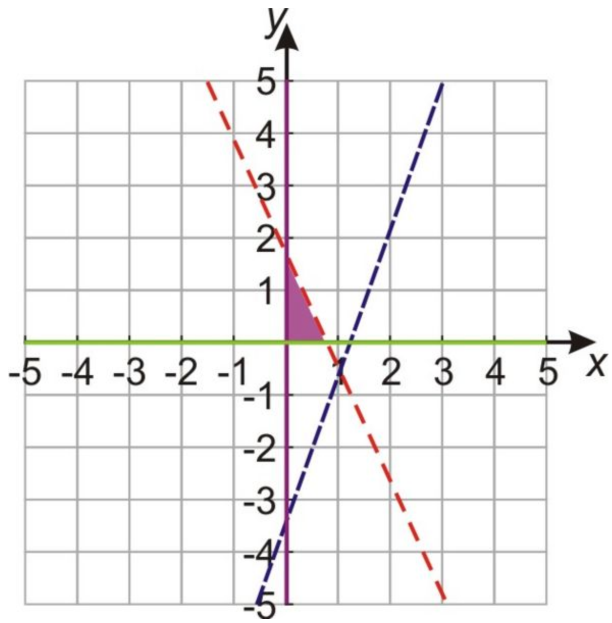
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$:



Next we graph $y < -\frac{9}{4}x + 2$:



Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.



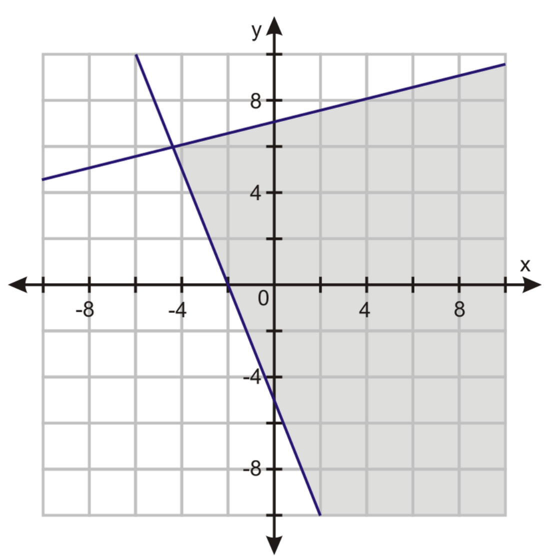
The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x - 4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

But that wasn't obvious until we actually drew the graph!

Guided Practice

Write the system of inequalities shown below.



Solution:

There are two boundary lines, so there are two inequalities. Write each one in slope-intercept form.

$$y \leq \frac{1}{4}x + 7$$
$$y \geq -\frac{5}{2}x - 5$$

Practice

1. Consider the system

$$y < 3x - 5$$
$$y > 3x - 5$$

Is it consistent or inconsistent? Why?

2. Consider the system

$$y \leq 2x + 3$$
$$y \geq 2x + 3$$

Is it consistent or inconsistent? Why?

3. Consider the system

$$y \leq -x + 1$$
$$y > -x + 1$$

Is it consistent or inconsistent? Why?

4. In example 3 in this lesson, we solved a system of four inequalities and saw that one of the inequalities, $y > 3x - 4$, didn't affect the solution set of the system.
   a. What would happen if we changed that inequality to $y < 3x - 4$?
   b. What's another inequality that we could add to the original system without changing it? Show how by sketching a graph of that inequality along with the rest of the system.
   c. What's another inequality that we could add to the original system to make it inconsistent? Show how by sketching a graph of that inequality along with the rest of the system.
5. Recall the compound inequalities in one variable that we worked with back in chapter 6. Compound inequalities with "and" are simply systems like the ones we are working with here, except with one variable instead of two.
   a. Graph the inequality $x > 3$ in two dimensions. What's another inequality that could be combined with it to make an inconsistent system?
   b. Graph the inequality $x \leq 4$ on a number line. What two-dimensional system would have a graph that looks just like this one?

Find the solution region of the following systems of inequalities.

6.

$$x - y < -6$$
$$2y \geq 3x + 17$$

7.

$$\begin{aligned} 4y-5x &< 8 \\ -5x &\geq 16-8y \end{aligned}$$

8.

$$\begin{aligned} 5x-y &\geq 5 \\ 2y-x &\geq -10 \end{aligned}$$

9.

$$\begin{aligned} 5x+2y &\geq -25 \\ 3x-2y &\leq 17 \\ x-6y &\geq 27 \end{aligned}$$

10.

$$\begin{aligned} 2x-3y &\leq 21 \\ x+4y &\leq 6 \\ 3x+y &\geq -4 \end{aligned}$$

11.

$$\begin{aligned} 12x-7y &< 120 \\ 7x-8y &\geq 36 \\ 5x+y &\geq 12 \end{aligned}$$

Vocabulary:

Solving Systems by Graphing: Consistent System, Dependent System, Inconsistent System, Independent System, Solution of a System, System of Linear Equations

Solving Systems by Substitution: Solution of system of Linear Equations, System of Linear Equations, Substitution

Solving Systems by Elimination: Identity, Properties of Equality, Elimination Method

Application of Systems of Equations: Elimination Method, Substitution Method, System of Linear Equations

Linear Inequalities: Slope, Linear Inequality, Solution of a Inequality

Systems of Linear Inequalities: Linear Inequality, Solution of a System of Linear Inequalities, System of Linear Inequalities

CHAPTER 5 Exponents and Radicals

Chapter Outline

Zero and Negative Exponents: A.11b

Multiplying Powers with the Same Base: A.11b

More Multiplication Properties of Exponents: A.11b

Division Properties of Exponents: A.11b

Rational Exponents and Radicals: A.11b

Simplifying Radicals: A.11a

The Pythagorean Theorem: A.8a, A.11a

5.1 Zero and Negative Exponents: Simplifying and Evaluating

Here you'll evaluate and use negative and zero exponents.

Guidance

In this concept, we will introduce negative and zero exponents. First, let's address a zero in the exponent through an investigation.

Investigation: Zero Exponents

1. Evaluate $\frac{5^6}{5^6}$ by using the Quotient of Powers property.

$\frac{5^6}{5^6} = 5^{6-6} = 5^0$

2. What is a number divided by itself? Apply this to #1.

$\frac{5^6}{5^6} = 1$

3. Fill in the blanks. $\frac{a^m}{a^m} = a^{m-m} = a^{-} = \_$

$a^0 = 1$

Investigation: Negative Exponents

1. Expand $\frac{3^2}{3^7}$ and cancel out the common 3's and write your answer with positive exponents.

$\frac{3^2}{3^7} = \frac{\cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{1}{3^5}$

2. Evaluate $\frac{3^2}{3^7}$ by using the Quotient of Powers property.

$\frac{3^2}{3^7} = 3^{2-7} = 3^{-5}$

3. Are the answers from #1 and #2 equal? Write them as a single statement.

$\frac{1}{3^5} = 3^{-5}$

4. Fill in the blanks. $\frac{1}{a^m} = a-$ and $\frac{1}{a^{-m}} = a-$

$\frac{1}{a^m} = a^{-m}$ and $\frac{1}{a^{-m}} = a^m$

From the two investigations above, we have learned two very important properties of exponents. First, anything to the zero power is one. Second, negative exponents indicate placement. If an exponent is negative, it needs to be moved from where it is to the numerator or denominator. We will investigate this property further in the Problem Set.

Example A

Simplify the following expressions. Your answer should only have positive exponents.

(a) $\frac{5^2}{5^5}$

(b) $\frac{x^7yz^{12}}{x^{12}yz^7}$

(c) $\frac{a^4b^0}{a^8b}$

Solution: Use the two properties from above. An easy way to think about where the "leftover" exponents should go, is to look at the fraction and determine which exponent is greater. For example, in b, there are more x's in the denominator, so the leftover should go there.

(a) $\frac{5^2}{5^5} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

(b) $\frac{x^7yz^{12}}{x^{12}yz^7} = \frac{y^{1-1}z^{12-7}}{x^{12-7}} = \frac{y^0z^5}{x^5} = \frac{z^5}{x^5}$

(c) $\frac{a^4b^0}{a^8b} = a^{4-8}b^{0-1} = a^{-4}b^{-1} = \frac{1}{a^4b}$

Alternate Method: Part c

$\frac{a^4b^0}{a^8b} = \frac{1}{a^{8-4}b} = \frac{1}{a^4b}$

Example B

Simplify the expressions. Your answer should only have positive exponents.

(a) $\frac{xy^5}{8y^{-3}}$

(b) $\frac{27g^{-7}h^0}{18g}$

Solution: In these expressions, you will need to move the negative exponent to the numerator or denominator and then change it to a positive exponent to evaluate. Also, simplify any numerical fractions.

(a) $\frac{xy^5}{8y^{-3}} = \frac{xy^5y^3}{8} = \frac{xy^{5+3}}{8} = \frac{xy^8}{8}$

(b) $\frac{27g^{-7}h^0}{18g} = \frac{3}{2g^1g^7} = \frac{3}{2g^{1+7}} = \frac{3}{2g^8}$

Example C

Multiply the two fractions together and simplify. Your answer should only have positive exponents.

$$\frac{4x^{-2}y^5}{20x^8} \cdot \frac{-5x^6y}{15y^{-9}}$$

Solution: The easiest way to approach this problem is to multiply the two fractions together first and then simplify.

$$\frac{4x^{-2}y^5}{20x^8} \cdot \frac{-5x^6y}{15y^{-9}} = -\frac{20x^{-2+6}y^{5+1}}{300x^8y^{-9}} = -\frac{x^{-2+6-8}y^{5+1+9}}{15} = -\frac{x^{-4}y^{15}}{15} = -\frac{y^{15}}{15x^4}$$

Guided Practice

Simplify the expressions.

1. $\frac{8^6}{8^9}$

2. $\frac{3x^{10}y^2}{21x^7y^{-4}}$

3. $\frac{2a^8b^{-4}}{16a^{-5}} \cdot \frac{4^3a^{-3}b^0}{a^4b^7}$

Answers

1. $\frac{8^6}{8^9} = 8^{6-9} = \frac{1}{8^3} = \frac{1}{512}$
2. $\frac{3x^{10}y^2}{21x^7y^{-4}} = \frac{x^{10-7}y^{2-(-4)}}{7} = \frac{x^3y^6}{7}$
3. $\frac{2a^8b^{-4}}{16a^{-5}} \cdot \frac{4^3a^{-3}b^0}{a^4b^7} = \frac{128a^{8-3}b^{-4}}{16a^{-5+4}b^7} = \frac{8a^{5+1}}{b^{7+4}} = \frac{8a^6}{b^{11}}$

Practice

Simplify the following expressions. Answers cannot have negative exponents.

1. $\frac{8^2}{8^4}$
2. $\frac{x^6}{x^{15}}$
3. $\frac{7^{-3}}{7^{-2}}$
4. $\frac{y^{-9}}{y^{10}}$
5. $\frac{x^0y^5}{xy^7}$
6. $\frac{a^{-1}b^8}{a^5b^7}$
7. $\frac{14c^{10}d^{-4}}{21c^6d^{-3}}$
8. $\frac{8g^0h}{30g^{-9}h^2}$
9. $\frac{5x^4}{10y^{-2}} \cdot \frac{y^7x}{x^{-1}y}$
10. $\frac{g^9h^5}{6gh^{12}} \cdot \frac{18h^3}{g^8}$
11. $\frac{4a^{10}b^7}{12a^{-6}} \cdot \frac{9a^{-5}b^4}{20a^{11}b^{-8}}$
12. $\frac{-g^8h}{6g^{-8}} \cdot \frac{9g^{15}h^9}{-h^{11}}$
13. Rewrite the following exponential pattern with positive exponents: $5^{-4}, 5^{-3}, 5^{-2}, 5^{-1}, 5^0, 5^1, 5^2, 5^3, 5^4$.
14. Evaluate each term in the pattern from #13.
15. Fill in the blanks.

As the numbers increase, you ______________ the previous term by 5.

As the numbers decrease, you _____________ the previous term by 5.

5.2 Multiplying Powers with the Same Base: Powers with the Same Base

Here you'll learn how to write repeated multiplication in exponential form. You'll also learn how to multiply and simplify exponential expressions.

Guidance

Back in chapter 1, we briefly covered expressions involving exponents, like 3^5 or x^3. In these expressions, the number on the bottom is called the **base** and the number on top is the **power** or **exponent**. The whole expression is equal to the base multiplied by itself a number of times equal to the exponent; in other words, the exponent tells us how many copies of the base number to multiply together.

Example A

Write in exponential form.

a) $2 \cdot 2$

b) $(-3)(-3)(-3)$

c) $y \cdot y \cdot y \cdot y \cdot y$

d) $(3a)(3a)(3a)(3a)$

Solution

a) $2 \cdot 2 = 2^2$ because we have 2 factors of 2

b) $(-3)(-3)(-3) = (-3)^3$ because we have 3 factors of (-3)

c) $y \cdot y \cdot y \cdot y \cdot y = y^5$ because we have 5 factors of y

d) $(3a)(3a)(3a)(3a) = (3a)^4$ because we have 4 factors of $3a$

When the base is a variable, it's convenient to leave the expression in exponential form; if we didn't write x^7, we'd have to write $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ instead. But when the base is a number, we can simplify the expression further than that; for example, 2^7 equals $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, but we can multiply all those 2's to get 128.

Let's simplify the expressions from Example A.

Example B

Simplify.

a) 2^2

b) $(-3)^3$

c) y^5

d) $(3a)^4$

Solution

a) $2^2 = 2 \cdot 2 = 4$

b) $(-3)^3 = (-3)(-3)(-3) = -27$

c) y^5 is already simplified

d) $(3a)^4 = (3a)(3a)(3a)(3a) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot a \cdot a \cdot a \cdot a = 81a^4$

Be careful when taking powers of negative numbers. Remember these rules:

$$(negative\ number) \cdot (positive\ number) = negative\ number$$
$$(negative\ number) \cdot (negative\ number) = positive\ number$$

So **even powers of negative numbers** are always positive. Since there are an even number of factors, we pair up the negative numbers and all the negatives cancel out.

$$\begin{aligned}(-2)^6 &= (-2)(-2)(-2)(-2)(-2)(-2)\\ &= \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4}\\ &= +64\end{aligned}$$

And **odd powers of negative numbers** are always negative. Since there are an odd number of factors, we can still pair up negative numbers to get positive numbers, but there will always be one negative factor left over, so the answer is negative:

$$\begin{aligned}(-2)^5 &= (-2)(-2)(-2)(-2)(-2)\\ &= \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)}_{-2}\\ &= -32\end{aligned}$$

Use the Product of Powers Property

So what happens when we multiply one power of x by another? Let's see what happens when we multiply x ***to the power of 5*** by x ***cubed***. To illustrate better, we'll use the full factored form for each:

$$\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{x^5} \cdot \underbrace{(x \cdot x \cdot x)}_{x^3} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^8}$$

So $x^5 \times x^3 = x^8$. You may already see the pattern to multiplying powers, but let's confirm it with another example. We'll multiply x ***squared*** by x ***to the power of 4***:

$$\underbrace{(x \cdot x)}_{x^2} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^6}$$

So $x^2 \times x^4 = x^6$. Look carefully at the powers and how many factors there are in each calculation. 5 x's times 3 x's equals $(5+3) = 8$ x's. 2 x's times 4 x's equals $(2+4) = 6$ x's.

You should see that when we take the product of two powers of x, the number of x's in the answer is the total number of x's in all the terms you are multiplying. In other words, the exponent in the answer is the sum of the exponents in the product.

Product Rule for Exponents: $x^n \cdot x^m = x^{(n+m)}$

There are some easy mistakes you can make with this rule, however. Let's see how to avoid them.

Example C

Multiply $2^2 \cdot 2^3$.

Solution

$2^2 \cdot 2^3 = 2^5 = 32$

Note that when you use the product rule you **don't multiply the bases**. In other words, you must avoid the common error of writing $2^2 \cdot 2^3 = 4^5$. You can see this is true if you multiply out each expression: 4 times 8 is definitely 32, not 1024.

Example D

Multiply $2^2 \cdot 3^3$.

Solution

$2^2 \cdot 3^3 = 4 \cdot 27 = 108$

In this case, we can't actually use the product rule at all, because it only applies to terms that have the *same base*. In a case like this, where the bases are different, we just have to multiply out the numbers by hand—the answer is *not* 2^5 or 3^5 or 6^5 or anything simple like that.

Vocabulary

- An ***exponent*** is a power of a number that shows how many times that number is multiplied by itself. An example would be 2^3. You would multiply 2 by itself 3 times: $2 \times 2 \times 2$. The number 2 is the ***base*** and the number 3 is the ***exponent.*** The value 2^3 is called the ***power.***

- **Product Rule for Exponents:** $x^n \cdot x^m = x^{(n+m)}$

Guided Practice

Simplify the following exponents:

a. $(-2)^5$

b. $(10x)^2$

Solutions:

a. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$

b. $(10x)^2 = 10^2 \cdot x^2 = 100x^2$

Practice

Write in exponential notation:

1. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
2. $3x \cdot 3x \cdot 3x$
3. $(-2a)(-2a)(-2a)(-2a)$
4. $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$
5. $2 \cdot x \cdot y \cdot 2 \cdot 2 \cdot y \cdot x$

Find each number.

6. 5^4
7. $(-2)^6$
8. $(0.1)^5$
9. $(-0.6)^3$
10. $(1.2)^2 + 5^3$
11. $3^2 \cdot (0.2)^3$

Multiply and simplify:

12. $6^3 \cdot 6^6$
13. $2^2 \cdot 2^4 \cdot 2^6$
14. $3^2 \cdot 4^3$
15. $x^2 \cdot x^4$
16. $(-2y^4)(-3y)$
17. $(4a^2)(-3a)(-5a^4)$

5.3 Multiplying Powers with the Same Base: Scientific Notation

Here you'll learn how to write very large and very small numbers so that they are easier to work with and evaluate.

Guidance

Consider the number six hundred and forty three thousand, two hundred and ninety seven. We write it as 643,297 and each digit's position has a "value" assigned to it. You may have seen a table like this before:

hundred-thousands	ten-thousands	thousands	hundreds	tens	units
6	4	3	2	9	7

We've seen that when we write an exponent above a number, it means that we have to multiply a certain number of copies of that number together. We've also seen that a zero exponent always gives us 1, and negative exponents give us fractional answers.

Look carefully at the table above. Do you notice that all the column headings are powers of ten? Here they are listed:

$$\begin{aligned}100,000 &= 10^5\\10,000 &= 10^4\\1,000 &= 10^3\\100 &= 10^2\\10 &= 10^1\end{aligned}$$

Even the "units" column is really just a power of ten. ***Unit*** means 1, and 1 is 10^0.

If we divide 643,297 by 100,000 we get 6.43297; if we multiply 6.43297 by 100,000 we get 643, 297. But we have just seen that 100,000 is the same as 10^5, so if we multiply 6.43297 by 10^5 we should also get 643,297. In other words,

$$643,297 = 6.43297 \times 10^5$$

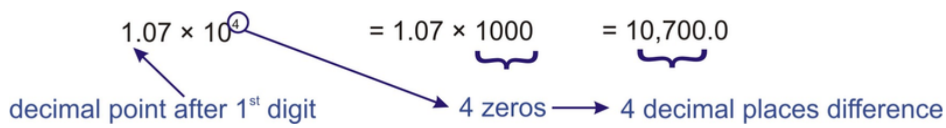
Writing Numbers in Scientific Notation

In scientific notation, numbers are always written in the form $a \times 10^b$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal). This notation is especially useful for numbers that are either very small or very large.

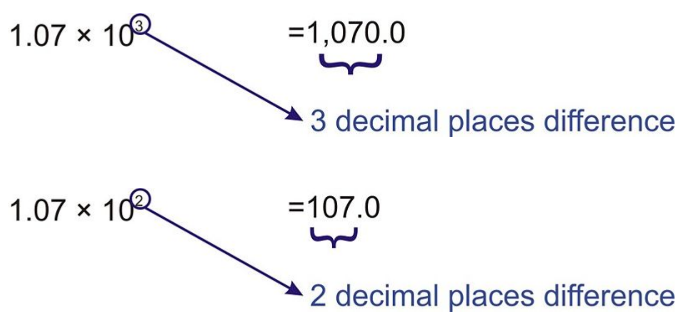
Here's a set of examples:

$$
\begin{aligned}
1.07 \times 10^4 &= 10,700 \\
1.07 \times 10^3 &= 1,070 \\
1.07 \times 10^2 &= 107 \\
1.07 \times 10^1 &= 10.7 \\
1.07 \times 10^0 &= 1.07 \\
1.07 \times 10^{-1} &= 0.107 \\
1.07 \times 10^{-2} &= 0.0107 \\
1.07 \times 10^{-3} &= 0.00107 \\
1.07 \times 10^{-4} &= 0.000107
\end{aligned}
$$

Look at the first example and notice where the decimal point is in both expressions.



So the exponent on the ten acts to move the decimal point over to the right. An exponent of 4 moves it 4 places and an exponent of 3 would move it 3 places.



This makes sense because each time you multiply by 10, you move the decimal point one place to the right. 1.07 times 10 is 10.7, times 10 again is 107.0, and so on.

Similarly, if you look at the later examples in the table, you can see that a negative exponent on the 10 means the decimal point moves that many places to the left. This is because multiplying by 10^{-1} is the same as multiplying by $\frac{1}{10}$, which is like dividing by 10. So instead of moving the decimal point one place to the right for every multiple of 10, we move it one place to the left for every multiple of $\frac{1}{10}$.

That's how to convert numbers from scientific notation to standard form. When we're converting numbers *to* scientific notation, however, we have to apply the whole process backwards. First we move the decimal point until it's immediately after the first nonzero digit; then we count how many places we moved it. If we moved it to the *left,* the exponent on the 10 is positive; if we moved it to the *right,* it's negative.

So, for example, to write 0.000032 in scientific notation, we'd first move the decimal five places to the right to get 3.2; then, since we moved it right, the exponent on the 10 should be *negative* five, so the number in scientific notation is 3.2×10^{-5}.

You can double-check whether you've got the right direction by comparing the number in scientific notation with the number in standard form, and thinking "Does this represent a *big* number or a *small* number?" A positive exponent on the 10 represents a number bigger than 10 and a negative exponent represents a number smaller than 10, and you can easily tell if the number in standard form is bigger or smaller than 10 just by looking at it.

Example A

Write the following numbers in scientific notation.

a) 63

b) 9,654

c) 653,937,000

d) 0.003

e) 0.000056

f) 0.00005007

Solution

a) $63 = 6.3 \times 10 = 6.3 \times 10^1$

b) $9,654 = 9.654 \times 1,000 = 9.654 \times 10^3$

c) $653,937,000 = 6.53937000 \times 100,000,000 = 6.53937 \times 10^8$

d) $0.003 = 3 \times \frac{1}{1000} = 3 \times 10^{-3}$

e) $0.000056 = 5.6 \times \frac{1}{100,000} = 5.6 \times 10^{-5}$

f) $0.00005007 = 5.007 \times \frac{1}{100,000} = 5.007 \times 10^{-5}$

Example B

Evaluate the following expressions and write your answer in scientific notation.

a) $(3.2 \times 10^6) \cdot (8.7 \times 10^{11})$

b) $(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19})$

c) $(1.7 \times 10^6) \cdot (2.7 \times 10^{-11})$

Solution

The key to evaluating expressions involving scientific notation is to group the powers of 10 together and deal with them separately.

a) $(3.2 \times 10^6)(8.7 \times 10^{11}) = \underbrace{3.2 \times 8.7}_{27.84} \times \underbrace{10^6 \times 10^{11}}_{10^{17}} = 27.84 \times 10^{17}$. But 27.84×10^{17} isn't in proper scientific notation, because it has more than one digit before the decimal point. We need to move the decimal point one more place to the left and add 1 to the exponent, which gives us 2.784×10^{18}.

b)

$$\begin{aligned}(5.2 \times 10^{-4})(3.8 \times 10^{-19}) &= \underbrace{5.2 \times 3.8}_{19.76} \times \underbrace{10^{-4} \times 10^{-19}}_{10^{-23}} \\ &= 19.76 \times 10^{-23} \\ &= 1.976 \times 10^{-22}\end{aligned}$$

c) $(1.7 \times 10^6)(2.7 \times 10^{-11}) = \underbrace{1.7 \times 2.7}_{4.59} \times \underbrace{10^6 \times 10^{-11}}_{10^{-5}} = 4.59 \times 10^{-5}$

When we use scientific notation in the real world, we often round off our calculations. Since we're often dealing with very big or very small numbers, it can be easier to round off so that we don't have to keep track of as many digits—and scientific notation helps us with that by saving us from writing out all the extra zeros. For example, if we round off 4,227, 457,903 to 4,200,000,000, we can then write it in scientific notation as simply 4.2×10^9.

When rounding, we often talk of **significant figures** or **significant digits**. Significant figures include

- all nonzero digits
- all zeros that come *before* a nonzero digit and *after* either a decimal point or another nonzero digit

For example, the number 4000 has one significant digit; the zeros don't count because there's no nonzero digit after them. But the number 4000.5 has five significant digits: the 4, the 5, and all the zeros in between. And the number 0.003 has three significant digits: the 3 and the two zeros that come between the 3 and the decimal point.

Example C

Evaluate the following expressions. Round to 3 significant figures and write your answer in scientific notation.

a) $(3.2 \times 10^6) \div (8.7 \times 10^{11})$

b) $(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19})$

Solution

It's easier if we convert to fractions and THEN separate out the powers of 10.

a)

$$\begin{aligned}
(3.2 \times 10^6) \div (8.7 \times 10^{11}) &= \frac{3.2 \times 10^6}{8.7 \times 10^{11}} && - separate\ out\ the\ powers\ of\ 10: \\
&= \frac{3.2}{8.7} \times \frac{10^6}{10^{11}} && - evaluate\ each\ fraction\ (round\ to\ 3\ s.f.): \\
&= 0.368 \times 10^{(6-11)} \\
&= 0.368 \times 10^{-5} && - remember\ how\ to\ write\ scientific\ notation! \\
&= 3.68 \times 10^{-6}
\end{aligned}$$

b)

$$\begin{aligned}
(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19}) &= \frac{5.2 \times 10^{-4}}{3.8 \times 10^{-19}} && - separate\ the\ powers\ of\ 10: \\
&= \frac{5.2}{3.8} \times \frac{10^{-4}}{10^{-19}} && - evaluate\ each\ fraction\ (round\ to\ 3\ s.f.) \\
&= 1.37 \times 10^{((-4)-(-19))} \\
&= 1.37 \times 10^{15}
\end{aligned}$$

Vocabulary

- In **scientific notation**, numbers are always written in the form $a \times 10^b$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal).

Guided Practice

Evaluate the following expression. Round to 3 significant figures and write your answer in scientific notation.

$(1.7 \times 10^6) \div (2.7 \times 10^{-11})$

Solution:

$$\begin{aligned}
(1.7 \times 10^6) \div (2.7 \times 10^{-11}) &= \frac{1.7 \times 10^6}{2.7 \times 10^{-11}} && - \textit{next we separate the powers of } 10: \\
&= \frac{1.7}{2.7} \times \frac{10^6}{10^{-11}} && - \textit{evaluate each fraction (round to 3 s.f.)} \\
&= 0.630 \times 10^{(6-(-11))} \\
&= 0.630 \times 10^{17} \\
&= 6.30 \times 10^{16}
\end{aligned}$$

Note that we have to leave in the final zero to indicate that the result has been rounded.

Practice

Write the numerical value of the following.

1. 3.102×10^2
2. 7.4×10^4
3. 1.75×10^{-3}
4. 2.9×10^{-5}
5. 9.99×10^{-9}

Write the following numbers in scientific notation.

6. 120,000
7. 1,765,244
8. 12
9. 0.00281
10. 0.000000027

How many significant digits are in each of the following?

11. 38553000
12. 2754000.23
13. 0.0000222
14. 0.0002000079

Round each of the following to two significant digits.

15. 3.0132
16. 82.9913

5.4 More Multiplication Properties of Exponents

Here you'll discover and use the power properties of exponents.

Guidance

The last set of properties to explore are the power properties. Let's investigate what happens when a power is raised to another power.

Investigation: Power of a Power Property

1. Rewrite $(2^3)^5$ as 2^3 five times.

$(2^3)^5 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3$

2. Expand each 2^3. How many 2's are there?

$(2^3)^5 = \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} = 2^{15}$

3. What is the *product* of the powers?

$3 \cdot 5 = 15$

4. Fill in the blank. $(a^m)^n = a^{\_ \cdot \_}$

$(a^m)^n = a^{mn}$

The other two exponent properties are a form of the distributive property.

Power of a Product Property: $(ab)^m = a^m b^m$

Power of a Quotient Property: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example A

Simplify the following.

(a) $(3^4)^2$

(b) $(x^2y)^5$

Solution: Use the new properties from above.

(a) $(3^4)^2 = 3^{4 \cdot 2} = 3^8 = 6561$

(b) $(x^2y)^5 = x^{2 \cdot 5}y^5 = x^{10}y^5$

Example B

Simplify $\left(\frac{3a^{-6}}{2^2a^2}\right)^4$ without negative exponents.

Solution: This example uses the Negative Exponent Property from the previous concept. Distribute the 4^{th} power first and then move the negative power of a from the numerator to the denominator.

$$\left(\frac{3a^{-6}}{2^2a^2}\right)^4 = \frac{3^4a^{-6\cdot 4}}{2^{2\cdot 4}a^{2\cdot 4}} = \frac{81a^{-24}}{2^8a^8} = \frac{81}{256a^{8+24}} = \frac{81}{256a^{32}}$$

Example C

Simplify $\frac{4x^{-3}y^4z^6}{12x^2y} \div \left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2$ without negative exponents.

Solution: This example is definitely as complicated as these types of problems get. Here, all the properties of exponents will be used. Remember that dividing by a fraction is the same as multiplying by its reciprocal.

$$\begin{aligned}\frac{4x^{-3}y^4z^6}{12x^2y} \div \left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2 &= \frac{4x^{-3}y^4z^6}{12x^2y} \cdot \frac{225x^6z^{-4}}{25x^2y^{-2}} \\ &= \frac{y^3z^6}{3x^5} \cdot \frac{9x^4y^2}{z^4} \\ &= \frac{3x^4y^5z^6}{x^5z^4} \\ &= \frac{3y^5z^2}{x}\end{aligned}$$

Intro Problem Revisit To find the number of bacteria remaining, we use the exponential expression $1000(\frac{1}{2})^n$ where n is the number of four-hour periods.

There are 6 four-hour periods in 24 hours, so we set n equal to 6 and solve.

$1000(\frac{1}{2})^6$

Applying the Power of a Quotient Property, we get:

$1000(\frac{1^6}{2^6}) = \frac{1000\cdot 1}{2^6} = \frac{1000}{64} = 15.625$

Therefore, there are 15.625 bacteria remaining after 24 hours.

Guided Practice

Simplify the following expressions without negative exponents.

1. $\left(\frac{5a^3}{b^4}\right)^7$
2. $(2x^5)^{-3}(3x^9)^2$
3. $\frac{(5x^2y^{-1})^3}{10y^6} \cdot \left(\frac{16x^8y^5}{4x^7}\right)^{-1}$

Answers

1. Distribute the 7 to every power within the parenthesis.

$$\left(\frac{5a^3}{b^4}\right)^7 = \frac{5^7a^{21}}{b^{28}} = \frac{78,125a^{21}}{b^{28}}$$

2. Distribute the -3 and 2 to their respective parenthesis and then use the properties of negative exponents, quotient and product properties to simplify.

$$(2x^5)^{-3}(3x^9)^2 = 2^{-3}x^{-15}3^2x^{18} = \frac{9x^3}{8}$$

3. Distribute the exponents that are outside the parenthesis and use the other properties of exponents to simplify. Anytime a fraction is raised to the -1 power, it is equal to the reciprocal of that fraction to the first power.

$$\begin{aligned}\frac{\left(5x^2y^{-1}\right)^3}{10y^6} \cdot \left(\frac{16x^8y^5}{4x^7}\right)^{-1} &= \frac{5^3x^{-6}y^{-3}}{10y^6} \cdot \frac{4x^7}{16x^8y^5}\\ &= \frac{500xy^{-3}}{160x^8y^{11}}\\ &= \frac{25}{8x^7y^{14}}\end{aligned}$$

Practice

Simplify the following expressions without negative exponents.

1. $(2^5)^3$
2. $(3x)^4$
3. $\left(\frac{4}{5}\right)^2$
4. $(6x^3)^3$
5. $\left(\frac{2a^3}{b^5}\right)^7$
6. $(4x^8)^{-2}$
7. $\left(\frac{1}{7^2h^9}\right)^{-1}$
8. $\left(\frac{2x^4y^2}{5x^{-3}y^5}\right)^3$
9. $\left(\frac{9m^5n^{-7}}{27m^6n^5}\right)^{-4}$
10. $\frac{(4x)^2(5y)^{-3}}{(2x^3y^5)^2}$
11. $(5r^6)^4\left(\frac{1}{3}r^{-2}\right)^5$
12. $(4t^{-1}s)^3(2^{-1}ts^{-2})^{-3}$
13. $\frac{6a^2b^4}{18a^{-3}b^4} \cdot \left(\frac{8b^{12}}{40a^{-8}b^5}\right)^2$
14. $\frac{2(x^4y^4)^0}{2^4x^3y^5z} \div \frac{8z^{10}}{32x^{-2}y^5}$
15. $\frac{5g^6}{15g^0h^{-1}} \cdot \left(\frac{h}{9g^{15}j^7}\right)^{-3}$
16. **Challenge** $\frac{a^7b^{10}}{4a^{-5}b^{-2}} \cdot \left[\frac{(6ab^{12})^2}{12a^9b^{-3}}\right]^2 \div (3a^5b^{-4})^3$
17. Rewrite 4^3 as a power of 2.
18. Rewrite 9^2 as a power of 3.
19. Solve the equation for x. $3^2 \cdot 3^x = 3^8$
20. Solve the equation for x. $(2^x)^4 = 4^8$

5.5 Division Properties of Exponents: Dividing Expressions and Scientific Notation

Here you'll learn how to simplify a fraction with exponential expressions in both its numerator and denominator that is raised to another secondary power.

Guidance

When we raise a whole quotient to a power, another special rule applies. Here is an example:

$$\begin{aligned}\left(\frac{x^3}{y^2}\right)^4 &= \left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\\ &= \frac{(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)}{(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)}\\ &= \frac{x^{12}}{y^8}\end{aligned}$$

Notice that the exponent outside the parentheses is multiplied by the exponent in the numerator and the exponent in the denominator, separately. This is called the power of a quotient rule:

Power Rule for Quotients: $\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n\cdot p}}{y^{m\cdot p}}$

Let's apply these new rules to a few examples.

Example A

Simplify the following expressions.

a) $\frac{4^5}{4^2}$

b) $\frac{5^3}{5^7}$

c) $\left(\frac{3^4}{5^2}\right)^2$

Solution

Since there are just numbers and no variables, we can evaluate the expressions and get rid of the exponents completely.

a) We can use the quotient rule first and then evaluate the result: $\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$

OR we can evaluate each part separately and then divide: $\frac{4^5}{4^2} = \frac{1024}{16} = 64$

b) Use the quotient rule first and then evaluate the result: $\frac{5^3}{5^7} = \frac{1}{5^4} = \frac{1}{625}$

OR evaluate each part separately and then reduce: $\frac{5^3}{5^7} = \frac{125}{78125} = \frac{1}{625}$

Notice that it makes more sense to apply the quotient rule first for examples (a) and (b). Applying the exponent rules to simplify the expression *before* plugging in actual numbers means that we end up with smaller, easier numbers to work with.

c) Use the power rule for quotients first and then evaluate the result: $\left(\frac{3^4}{5^2}\right)^2 = \frac{3^8}{5^4} = \frac{6561}{625}$

OR evaluate inside the parentheses first and then apply the exponent: $\left(\frac{3^4}{5^2}\right)^2 = \left(\frac{81}{25}\right)^2 = \frac{6561}{625}$

Example B

Simplify the following expressions:

a) $\frac{x^{12}}{x^5}$

b) $\left(\frac{x^4}{x}\right)^5$

Solution

a) Use the quotient rule: $\frac{x^{12}}{x^5} = x^{12-5} = x^7$

b) Use the power rule for quotients and then the quotient rule: $\left(\frac{x^4}{x}\right)^5 = \frac{x^{20}}{x^5} = x^{15}$

OR use the quotient rule inside the parentheses first, then apply the power rule: $\left(\frac{x^4}{x}\right)^5 = (x^3)^5 = x^{15}$

Example C

Simplify the following expressions.

a) $\frac{6x^2y^3}{2xy^2}$

b) $\left(\frac{2a^3b^3}{8a^7b}\right)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number or each variable separately.

a) Group like terms together: $\frac{6x^2y^3}{2xy^2} = \frac{6}{2} \cdot \frac{x^2}{x} \cdot \frac{y^3}{y^2}$

Then reduce the numbers and apply the quotient rule on each fraction to get $3xy$.

b) Apply the quotient rule inside the parentheses first: $\left(\frac{2a^3b^3}{8a^7b}\right)^2 = \left(\frac{b^2}{4a^4}\right)^2$

Then apply the power rule for quotients: $\left(\frac{b^2}{4a^4}\right)^2 = \frac{b^4}{16a^8}$

Vocabulary

- **Quotient of Powers Property:** For all real numbers x,

$\frac{x^n}{x^m} = x^{n-m}$.

- **Power of a Quotient Property:**

$\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n \cdot p}}{y^{m \cdot p}}$

Guided Practice

Simplify the following expressions.

a) $(x^2)^2 \cdot \frac{x^6}{x^4}$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Solution

In problems where we need to apply several rules together, we must keep the order of operations in mind.

a) We apply the power rule first on the first term:

$$(x^2)^2 \cdot \frac{x^6}{x^4} = x^4 \cdot \frac{x^6}{x^4}$$

Then apply the quotient rule to simplify the fraction:

$$x^4 \cdot \frac{x^6}{x^4} = x^4 \cdot x^2$$

And finally simplify with the product rule:

$$x^4 \cdot x^2 = x^6$$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Simplify inside the parentheses by reducing the numbers:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$$

Then apply the power rule to the first fraction:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}} = \frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}}$$

Group like terms together:

$$\frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}} = 64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}}$$

And apply the quotient rule to each fraction:

$$64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}} = \frac{64}{a^{10}b^{13}}$$

Practice

Evaluate the following expressions.

1. $\left(\frac{3}{8}\right)^2$
2. $\left(\frac{2^2}{3^3}\right)^3$
3. $\left(\frac{2^3 \cdot 4^2}{2^4}\right)^2$

Simplify the following expressions.

4. $\left(\frac{a^3b^4}{a^2b}\right)^3$
5. $\left(\frac{18a^4}{15a^{10}}\right)^4$
6. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
7. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
8. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$
9. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$ for $a = 2, b = 1$, and $c = 3$
10. $\left(\frac{3x^2y}{2z}\right)^3 \cdot \frac{z^2}{x}$ for $x = 1, y = 2$, and $z = -1$
11. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 2, y = -3$
12. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 0, y = 6$
13. If $a = 2$ and $b = 3$, simplify $\frac{(a^2b)(bc)^3}{a^3c^2}$ as much as possible.

5.6 Rational Exponents and Radicals

Here you'll learn about rational exponents and relate them to n^{th} roots.

Guidance

Now that you are familiar with nth roots, we will convert them into exponents. Let's look at the square root and see if we can use the properties of exponents to determine what exponential number it is equivalent to.

Investigation: Writing the Square Root as an Exponent

1. Evaluate $\left(\sqrt{x}\right)^2$. What happens?

- The $\sqrt{\ }$ and the ^{2} cancel each other out, $\left(\sqrt{x}^2\right) = x$.

2. Recall that when a power is raised to another power, we multiply the exponents. Therefore, we can rewrite the exponents and root as an equation, $n \cdot 2 = 1$. Solve for n.

- $$\begin{aligned} \frac{n \cdot \cancel{2}}{\cancel{2}} &= \frac{1}{2} \\ n &= \frac{1}{2} \end{aligned}$$

3. From #2, we can conclude that $\sqrt{\ } = \frac{1}{2}$.

- $\left(\sqrt{x}\right)^2 = \left(x^{\frac{1}{2}}\right)^2 = x^{\left(\frac{1}{2}\right) \cdot 2} = x^1 = x$

From this investigation, we see that $\sqrt{x} = x^{\frac{1}{2}}$. We can extend this idea to the other roots as well; $\sqrt[3]{x} = x^{\frac{1}{3}} = \sqrt[4]{x} = x^{\frac{1}{4}}, \ldots \sqrt[n]{x} = x^{\frac{1}{n}}$.

Example A

Find $256^{\frac{1}{4}}$.

Solution: Rewrite this expression in terms of roots. A number to the one-fourth power is the same as the fourth root.

$256^{\frac{1}{4}} = \sqrt[4]{256} = \sqrt[4]{4^4} = 4$

Therefore, $256^{\frac{1}{4}} = 4$.

Example B

Find $49^{\frac{3}{2}}$.

Solution: This problem is the same as the ones in the previous concept. However, now, the root is written in the exponent. Rewrite the problem.

$49^{\frac{3}{2}} = \left(49^3\right)^{\frac{1}{2}} = \sqrt{49^3}$ or $\left(\sqrt{49}\right)^3$

From the previous concept, we know that it is easier to evaluate the second option above. $\left(\sqrt{49}\right)^3 = 7^3 = 343$.

The Rational Exponent Theorem: For any real number a, root n, and exponent m, the following is always true: $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

Example C

Find $5^{\frac{2}{3}}$ using a calculator. Round your answer to the nearest hundredth.

Solution: To type this into a calculator, the keystrokes would probably look like: $5^{\frac{2}{3}}$. The "^" symbol is used to indicate a power. Anything in parenthesis after the "^" would be in the exponent. Evaluating this, we have 2.924017738..., or just 2.92.

Other calculators might have a x^y button. This button has the same purpose as the ^ and would be used in the exact same way.

Intro Problem Revisit Substitute 27 for p and solve.

$d = 27^{\frac{2}{3}}$

Rewrite the problem.

$27^{\frac{2}{3}} = \left(27^2\right)^{\frac{1}{3}} = \sqrt[3]{27^2}$ or $\sqrt[3]{27}^2$

$\left(\sqrt[3]{27}\right)^2 = 3^2 = 9.$

Therefore, the planet's distance from the sun is 9 astronomical units.

Guided Practice

1. Rewrite $\sqrt[7]{12}$ using rational exponents. Then, use a calculator to find the answer.

2. Rewrite $845^{\frac{4}{9}}$ using roots. Then, use a calculator to find the answer.

Evaluate without a calculator.

3. $125^{\frac{4}{3}}$

4. $256^{\frac{5}{8}}$

5. $\sqrt{81^{\frac{1}{2}}}$

Answers

1. Using rational exponents, the 7^{th} root becomes the $\frac{1}{7}$ power;$12^{\frac{1}{7}} = 1.426$.

2. Using roots, the 9 in the denominator of the exponent is the root; $\sqrt[9]{845^4} = 19.99$. To enter this into a calculator, you can use the rational exponents. If you have a TI-83 or 84, press **MATH** and select **5**: $\sqrt[x]{\ }$. On the screen, you should type 9 $\sqrt[x]{\ }$ 845^4 to get the correct answer. You can also enter 845^$\left(\frac{4}{9}\right)$ and get the exact same answer

3. $125^{\frac{4}{3}} = \left(\sqrt[3]{125}\right)^4 = 5^4 = 625$

4. $256^{\frac{5}{8}} = \left(\sqrt[8]{256}\right)^5 = 2^5 = 32$

5. $\sqrt{81^{\frac{1}{2}}} = \sqrt{\sqrt{81}} = \sqrt{9} = 3$

Practice

Write the following expressions using rational exponents and then evaluate using a calculator. Answers should be rounded to the nearest hundredth.

1. $\sqrt[5]{45}$
2. $\sqrt[9]{140}$
3. $\sqrt[8]{50^3}$

Write the following expressions using roots and then evaluate using a calculator. Answers should be rounded to the nearest hundredth.

4. $72^{\frac{5}{3}}$
5. $95^{\frac{2}{3}}$
6. $125^{\frac{3}{4}}$

Evaluate the following without a calculator.

7. $64^{\frac{2}{3}}$
8. $27^{\frac{4}{3}}$
9. $16^{\frac{5}{4}}$
10. $\sqrt{25^3}$
11. $\sqrt[2]{9^5}$
12. $\sqrt[5]{32^2}$

For the following problems, rewrite the expressions with rational exponents and then simplify the exponent and evaluate without a calculator.

13. $\sqrt[4]{\left(\frac{2}{3}\right)^8}$
14. $\sqrt[3]{\frac{7}{2}^6}$
15. $\sqrt{(16)^{\frac{1}{2}}}^6$

5.7 Simplifying Radicals

Here you'll learn how to simplify expressions containing radicals.

Guidance

In algebra, you learned how to simplify radicals. Let's review it here. Some key points to remember:

1. One way to simplify a radical is to factor out the perfect squares (see Example A).
2. When adding radicals, you can only combine radicals with the same number underneath it. For example, $2\sqrt{5}+3\sqrt{6}$ cannot be combined, because 5 and 6 are not the same number (see Example B).
3. To multiply two radicals, multiply what is under the radicals and what is in front (see Example B).
4. To divide radicals, you need to simplify the denominator, which means multiplying the top and bottom of the fraction by the radical in the denominator (see Example C).

Example A

Simplify the radicals.

a) $\sqrt{50}$

b) $\sqrt{27}$

c) $\sqrt{272}$

Answers: For each radical, find the square number(s) that are factors.

a) $\sqrt{50} = \sqrt{25\cdot 2} = 5\sqrt{2}$

b) $\sqrt{27} = \sqrt{9\cdot 3} = 3\sqrt{3}$

c) $\sqrt{272} = \sqrt{16\cdot 17} = 4\sqrt{17}$

Example B

Simplify the radicals.

a) $2\sqrt{10}+\sqrt{160}$

b) $5\sqrt{6}\cdot 4\sqrt{18}$

c) $\sqrt{8}\cdot 12\sqrt{2}$

d) $\left(5\sqrt{2}\right)^2$

Answers:

a) Simplify $\sqrt{160}$ before adding: $2\sqrt{10}+\sqrt{160} = 2\sqrt{10}+\sqrt{16\cdot 10} = 2\sqrt{10}+4\sqrt{10} = 6\sqrt{10}$

b) $5\sqrt{6}\cdot 4\sqrt{18} = 5\cdot 4\sqrt{6\cdot 18} = 20\sqrt{108} = 20\sqrt{36\cdot 3} = 20\cdot 6\sqrt{3} = 120\sqrt{3}$

c) $\sqrt{8}\cdot 12\sqrt{2} = 12\sqrt{8\cdot 2} = 12\sqrt{16} = 12\cdot 4 = 48$

d) $\left(5\sqrt{2}\right)^2 = 5^2\left(\sqrt{2}\right)^2 = 25\cdot 2 = 50 \rightarrow$ the $\sqrt{\ }$ and the ^{2} cancel each other out

Example C

Divide and simplify the radicals.

a) $4\sqrt{6} \div \sqrt{3}$

b) $\frac{\sqrt{30}}{\sqrt{8}}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}}$

Answers: Rewrite all division problems like a fraction.

a)

$$4\sqrt{6} \div \sqrt{3} = \frac{4\sqrt{6}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{18}}{\sqrt{9}} = \frac{4\sqrt{9 \cdot 2}}{3} = \frac{4 \cdot 3\sqrt{2}}{3} = 4\sqrt{2}$$

like multiplying by 1, $\frac{\sqrt{3}}{\sqrt{3}}$ does not change the value of the fraction

b) $\frac{\sqrt{30}}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{240}}{\sqrt{64}} = \frac{\sqrt{16 \cdot 15}}{8} = \frac{4\sqrt{15}}{8} = \frac{\sqrt{15}}{2}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{8\sqrt{14}}{6 \cdot 7} = \frac{4\sqrt{14}}{3 \cdot 7} = \frac{4\sqrt{14}}{21}$

Notice, we do not really "divide" radicals, but get them out of the denominator of a fraction.

Guided Practice

Simplify the radicals.

1. $\sqrt{75}$
2. $2\sqrt{5} + 3\sqrt{80}$
3. $\frac{\sqrt{45}}{\sqrt{2}}$

Answers:

1. $\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}$
2. $2\sqrt{5} + 3\sqrt{80} = 2\sqrt{5} + 3(\sqrt{16 \cdot 5}) = 2\sqrt{5} + (3 \cdot 4)\sqrt{5} = 14\sqrt{5}$
3. $\frac{\sqrt{45}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{90}}{\sqrt{4}} = \frac{\sqrt{9 \cdot 10}}{2} = \frac{3\sqrt{10}}{2}$

Practice

Simplify the radicals.

1. $\sqrt{48}$
2. $2\sqrt{5} + \sqrt{20}$
3. $\sqrt{24}$
4. $\left(6\sqrt{3}\right)^2$
5. $8\sqrt{8} \cdot \sqrt{10}$
6. $\left(2\sqrt{30}\right)^2$
7. $\sqrt{320}$

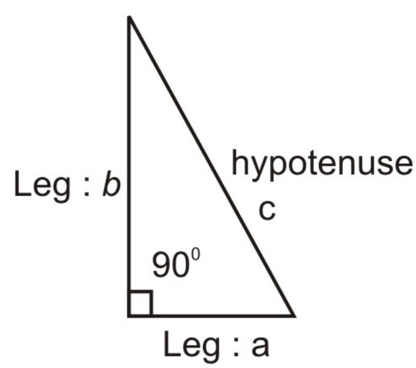
8. $\frac{4\sqrt{5}}{\sqrt{6}}$
9. $\frac{12}{\sqrt{10}}$
10. $\frac{21\sqrt{5}}{9\sqrt{15}}$

5.8 The Pythagorean Theorem

rHere you'll learn how to solve problems by using the Pythagorean Theorem and the converse of this theorem.

Guidance

One of the most important theorems in mathematics and science is Pythagorean's Theorem. Simply put, it states, "The sum of the square of each leg of a right triangle is equal to the square of the hypotenuse."



Let's review basic right triangle anatomy.

The two segments forming the right angle (90°) are called the *legs* of the right triangle. The segment opposite the right angle is called the *hypotenuse*.

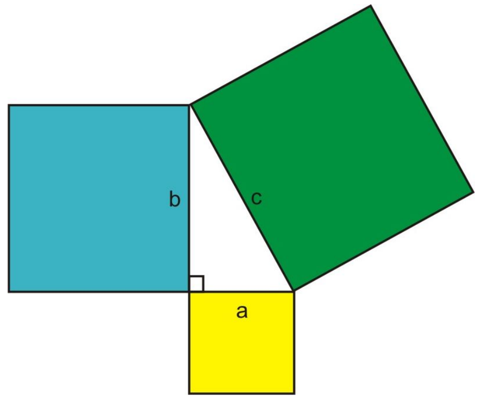
The Pythagorean Theorem states, $(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$:

$$a^2 + b^2 = c^2$$

Or, to find the hypotenuse, $c = \sqrt{a^2 + b^2}$.

Notice this relationship is only true for **right triangles**. In later courses, you will learn how to determine relationships with non-right triangles.

Although we usually refer to the Pythagorean Theorem when determining side lengths of a right triangle, the theorem originally made a statement about areas. If we build squares on each side of a right triangle, the Pythagorean Theorem says that the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares formed by the legs of the triangle.



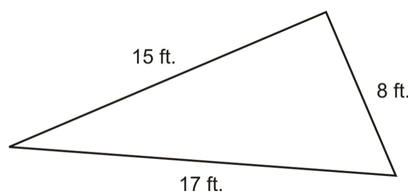
The Converse of Pythagorean's Theorem

The **Converse of the Pythagorean Theorem** is also true. That is, if the lengths of three sides of a triangle make the equation $a^2 + b^2 = c^2$ true, then they represent the sides of a right triangle.

With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any of the triangle's angle measurements.

Example A

Does the triangle below contain a right angle?



Solution: This triangle does not have any right angle marks or measured angles, so you cannot assume you know whether the triangle is acute, right, or obtuse just by looking at it. Take a moment to analyze the side lengths and see how they are related. Two of the sides, 15 and 17, are relatively close in length. The third side, 8, is about half the length of the two longer sides.

To see if the triangle might be right, try substituting the side lengths into the Pythagorean Theorem to see if they make the equation true. *The hypotenuse is always the longest side,* so 17 should be substituted for c. The other two values can represent a and b and the order is not important.

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
8^2 + 15^2 &= 17^2 \\
64 + 225 &= 289 \\
289 &= 289
\end{aligned}$$

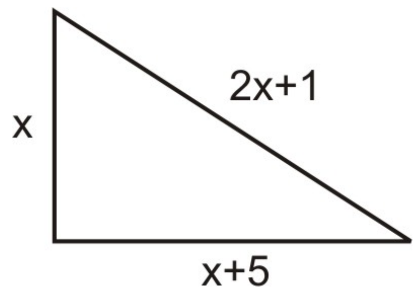
Since both sides of the equation are equal, these values satisfy the Pythagorean Theorem. Therefore, the triangle described in the problem is a right triangle.

Example B

One leg of a right triangle is 5 more than the other leg. The hypotenuse is one more than twice the size of the short leg. Find the dimensions of the triangle.

Solution:

Let $x =$ length of the short leg. Then, $x+5 =$ length of the long leg and $2x+1 =$ length of the hypotenuse.



The sides of the triangle must satisfy the Pythagorean Theorem.

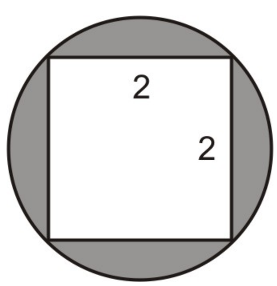
	$x^2+(x+5)^2 = (2x+1)$
Eliminate the parentheses.	$x^2+x^2+10x+25 = 4x^2+4x+1$
Move all terms to the right hand side of the equation.	$0 = 2x^2-6x-24$
Divide all terms by 2.	$0 = x^2-3x-12$
Solve using the quadratic formula.	$x = \frac{3 \pm \sqrt{9+48}}{2} = \frac{3 \pm \sqrt{57}}{2}$
	$x \approx 5.27$ or $x \approx -2.27$

The negative solution does not make sense in the context of this problem. So, use $x = 5.27$ and we get $\text{short}-\text{leg} = 5.27,\ \text{long}-\text{leg} = 10.27$ and $\text{hypotenuse} = 11.54$.

Real-World Right Triangles

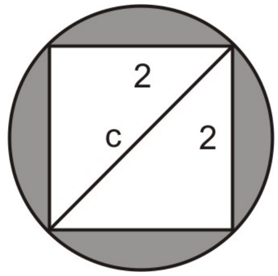
Example C

Find the area of the shaded region in the following diagram.



Solution:

Draw the diagonal of the square on the figure.



Notice that the diagonal of the square is also the diameter of the circle.

Define variables. Let $c =$ diameter of the circle.

$$2^2 + 2^2 = c^2$$
$$4 + 4 = c^2$$

Write the formula and solve. $c^2 = 8 \Rightarrow c = \sqrt{8} \Rightarrow c = 2\sqrt{2}$

The diameter of the circle is $2\sqrt{2}$. Therefore, the radius is $r = \sqrt{2}$.

The area of a circle is $A = \pi r^2 = \pi \left(\sqrt{2}\right)^2 = 2\pi$.

The area of the shaded region is, therefore, $2\pi - 4 \approx 2.28$.

Guided Practice

Determine whether or not a triangle with sides of lengths 5, 6 and 8 forms a right triangle.

Solution:

Use the Converse of the Pythagorean Theorem:

Start with the Pythagorean equation.	$a^2 + b^2 = c^2$
Substitute in the values of the sides.	$5^2 + 6^2 \stackrel{?}{=} 8^2$
Simplify.	$25 + 36 \stackrel{?}{=} 64$
Check.	$61 \neq 64$

Since these lengths of sides do not satisfy the equation of the Pythagorean Theorem, the triangle is not a right triangle.

Practice

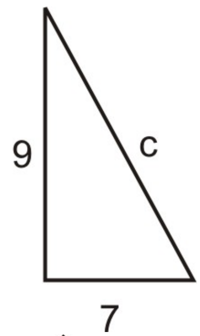
Verify that each triangle is a right triangle.

1. $a = 12,\ b = 9, c = 15$
2. $a = 6,\ b = 6,\ c = 6\sqrt{2}$

3. $a = 8,\ b = 8\sqrt{3},\ c = 16$

Find the missing length of each right triangle.

4. $a = 12,\ b = 16,\ c = ?$
5. $a = ?,\ b = 20,\ c = 30$
6. $a = 4,\ b = ?,\ c = 11$



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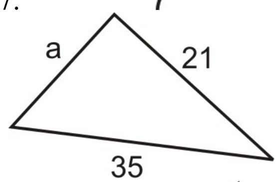

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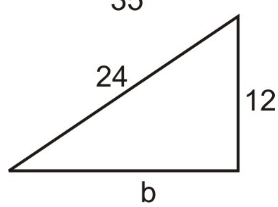
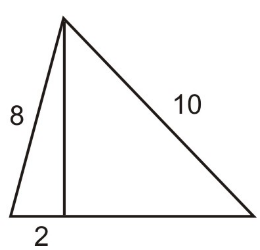


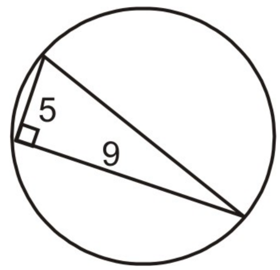
9.

10. One leg of a right triangle is 4 feet less than the hypotenuse. The other leg is 12 feet. Find the lengths of the three sides of the triangle.
11. One leg of a right triangle is 3 more than twice the length of the other. The hypotenuse is 3 times the length of the short leg. Find the lengths of the three legs of the triangle.
12. A regulation baseball diamond is a square with 90 feet between bases. How far is second base from home plate?
13. Emanuel has a cardboard box that measures 20 $cm \times 10\ cm \times 8\ cm$(length $\times$ width $\times$ height). What is the length of the diagonal from a bottom corner to the opposite top corner?
14. Samuel places a ladder against his house. The base of the ladder is 6 feet from the house and the ladder is 10 feet long. How high above the ground does the ladder touch the wall of the house?
15. Find the area of the triangle by using the formula $A = \frac{1}{2}\text{base} \times \text{height}$.



16. Instead of walking along the two sides of a rectangular field, Mario decided to cut across the diagonal. He saves a distance that is half of the long side of the field. Find the length of the long side of the field given that the short side is 123 feet.
17. Marcus sails due north and Sandra sails due east from the same starting point. In two hours, Marcus's boat is 35 miles from the starting point and Sandra's boat is 28 miles from the starting point. How far are the boats from each other?

18. Determine the area of the circle.



19. In a right triangle, one leg is twice as long as the other and the perimeter is 28. What are the measures of the sides of the triangle?
20. Maria has a rectangular cookie sheet that measures 10 *inches* $\times$ 14 *inches*. Find the length of the diagonal of the cookie sheet.
21. Mike is loading a moving van by walking up a ramp. The ramp is 10 feet long and the bed of the van is 2.5 feet above the ground. How far does the ramp extend past the back of the van?

e

Vocabulary:

Zero and Negative Exponents: Identity, Reciprocal

Multiplying Powers with the Same Base: Base, Power

More Multiplication Properties of Exponents: Exponent, Scientific Notation

Division Properties of Exponents: Simplified Form, Simplify

Rational Exponents and Radicals: Radical, Radicand, Index

Simplifying Radicals: Index, Radical, Rationalize the Denominator

The Pythagorean Theorem: Hypotenuse, Leg, Pythagorean Theorem

CHAPTER 6 Sequences

Chapter Outline

Arithmetic and Geometric Sequences: A.12b, A.12d

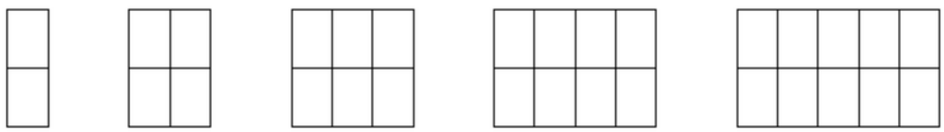
Arithmetic Sequences and Recursive Form: A.12b, A.12c, A.12d

Geometric Sequences in Recursive Form: A.12c, A.12d

6.1 Arithmetic and Geometric Sequences: Arithmetic Sequences

Guidance

Look at this sequence.



You probably saw a pattern right away. If there were another set of boxes, you'd probably guess at how many there would be.

If you saw this same pattern in terms of numbers, it would look like this:

$$2, 4, 6, 8, 10$$

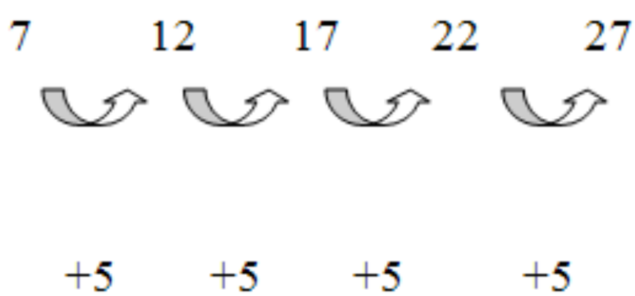
This set of numbers is called a ***sequence*** **; it is a series of numbers that follow a pattern.**

If there was another set of boxes, you'd probably guess there would be 12, right? Just like if you added another number to the sequence, you'd write 12. You noticed that there was a difference of 2 between each two numbers, or *terms*, in the sequence.

When we have a sequence with a fixed number between each of the terms, we call this sequence an *arithmetic sequence* .

Take a look at this one.

What is the common difference between each of the terms in the sequence?



The difference is 5 between each number.

This is an arithmetic sequence. You can see that you have to be a bit of a detective to figure out the number patterns.

Finding the difference between two terms in a sequence is one way to look at sequences. We have used tables of values for several types of equations and we have used those tables of values to create graphs. Graphs are helpful because they are visual representations of the same numbers. When values rise, we can see them rise on a graph. Let's use the same ideas, then, to graph arithmetic sequences.

Take a look at this one.

Graph the sequence 2, 5, 8, 11, 14, 17,...

First convert it into a table of values with independent values being the term number and the dependent values being the actual term.

1^{st} term

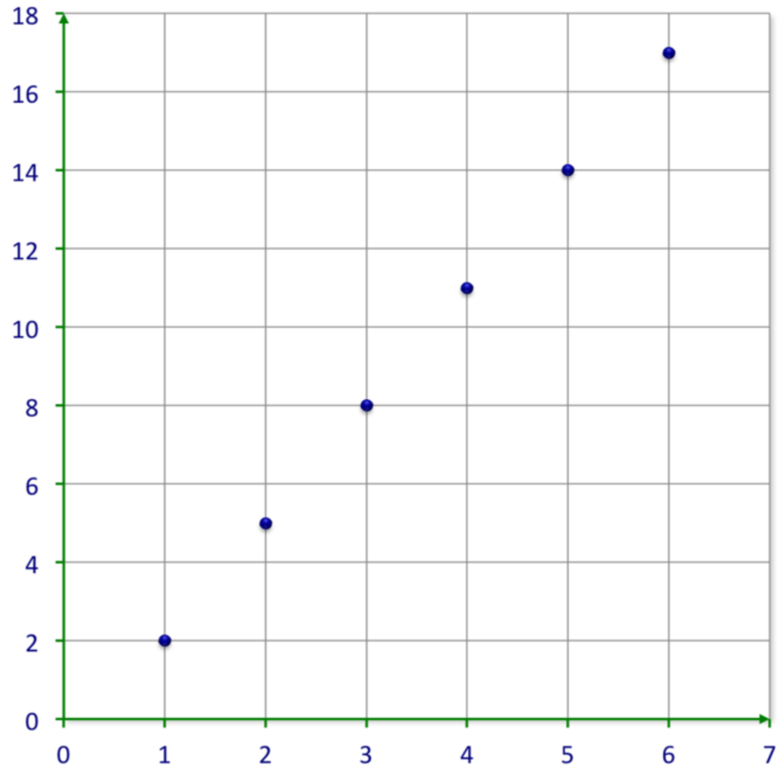
2^{nd} term

x	y
1	2
2	5
3	8
4	11
5	14
6	17

value of 1^{st} term

value of 2^{nd} term

Use this table to create a graph.



You can see the pattern clearly in the graph. That is one of the wonderful things about graphing arithmetic sequences.

In the graph that we created in the example, each term was expressed as a single point. This is called ***discrete data* —only the exact points are shown.** This type of data is usually involves things that are counted in whole numbers like people or boxes. Depending on what type of situation you are graphing, you may choose to connect the points with a line. The line demonstrates that there are data points *between* the points that we have graphed. This is called ***continuous data* and usually involves things like temperature or length that can change fractionally.**

So, we can graph sequences and classify them as either discrete or continuous data. Yet another possibility is continuing a sequence in either direction by adding terms that follow the same pattern.

Identify the pattern in the following sequences.

Example A

$3, 7, 11, 15$

Solution: Add four

Example B

$18, 8, -2$

Solution: Subtract 10

Example C

$81, 86, 91, 96$

Solution: Add five

Now let's go back to the dilemma from the beginning of the Concept.

$34, 38, 42, 46, 54$

If we look for the difference between the values, you will see that each value has 8 added to it to equal the next value.

The pattern is add 8.

This is our answer.

Guided Practice

Here is one for you to try on your own.

What is the common difference in the following sequence?

$-15, -13, -11, -9...$

Solution

You can see that positive two is added to the first value to find the second value. This happens for each of the values in the sequence.

Practice

Directions: Write the common difference for each sequence. If there is not a pattern, indicate this in your answer.

1. -9, -7, -5, -3, -1
2. 5.05, 5.1, 5.15, 5.2, 5.25
3. 3, 6, 10, 15, 21, 28
4. 17, 14, 11, 8, 5, 2
5. 10, 9, 8, 7, 6
6. 3, 5, 7, 9, 11
7. 3, 9, 27
8. 4, 8, 16, 32
9. 2, 3, 5, 9
10. 5, 11, 23, 47
11. 16, 8, 4, 2
12. 5, 10, 15, 20
13. 3, 6, 9, 12

Directions: Solve this problem by using what you know about arithmetic sequences.

An ant colony invades the caramels in a candy store. The first day they eat a $\frac{1}{4}$ of a caramel, the second day $\frac{1}{2}$ of a caramel, the third day $\frac{3}{4}$.

14. What is the difference between each day?
15. How many do you think they'll eat on the fourth, fifth, and sixth days?

Ari

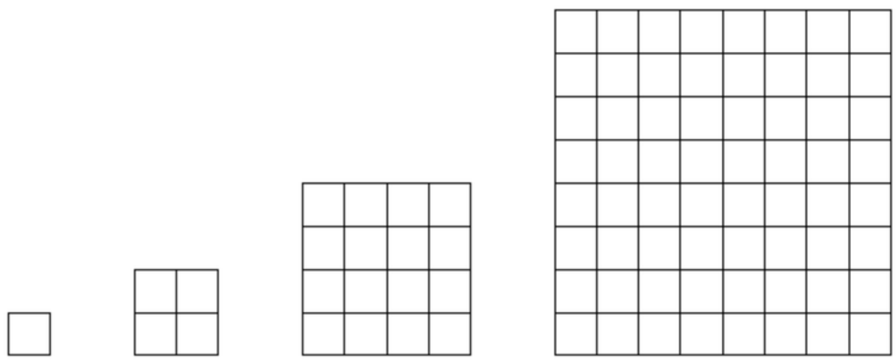
6.2 Arithmetic and Geometric Sequences: Geometric Sequences

Guidance

Have you ever seen a number sequence? There are several different types of sequences that follow patterns.

An arithmetic sequence has a fixed sum or difference between each term.

Now look at the boxes below, and you will see another type of sequence.



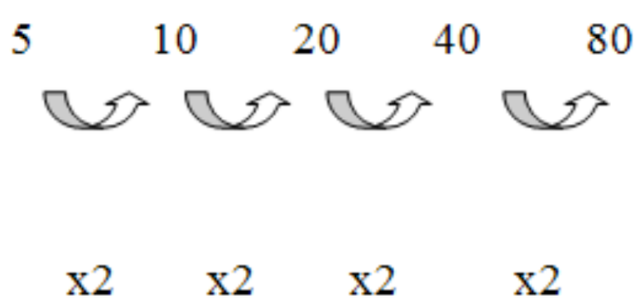
Can you see a pattern? The boxes increase each time. Using numbers, the sequence could be written 1, 4, 16, 64. You might even guess at what would come next. Is there a common difference between them? Not really. There is a difference of 3 between the first two terms, 12 between the second and third terms, and 48 between the third and fourth terms. If you guessed that 256 would follow it's because you figured out the pattern. You noticed that to get to the next term, you have to **multiply by 4** instead of by adding a certain number.

This is a *geometric sequence* ; it's a sequence in which the terms are found by multiplying by a fixed number called the *common ratio* . In the situation above, the common ratio is 4.

Once you know the common ratio, then you can figure out the next step in the pattern.

Take a look at this one.

What is the common ratio between each of the terms in the sequence?



The ratio is 2 between each number.

You can see how knowing the common ratio helped us with our problem solving.

Consider the following sequence:

$$8, 24, 72, 216, \dots$$

Doesn't your brain want to find the next number? You've probably figured out that the common ratio here is 3. So the next term in the sequence would be $216 \cdot 3$ or 648. You would continue the same process to find the term that follows. Or, you could divide by 3 to find the previous term.

Just as we did with arithmetic sequences, it can be useful to graph geometric sequences. We'll use the same method as before—create a table of values and then use a coordinate plane to plot the points.

Take a look at this situation.

The amount of memory that computer chips can hold in the same amount of space doubles every year. In 1992, they could hold 1MB. Chart the next 15 years in a table of values and show the amount of memory on the same size chip in 2007.

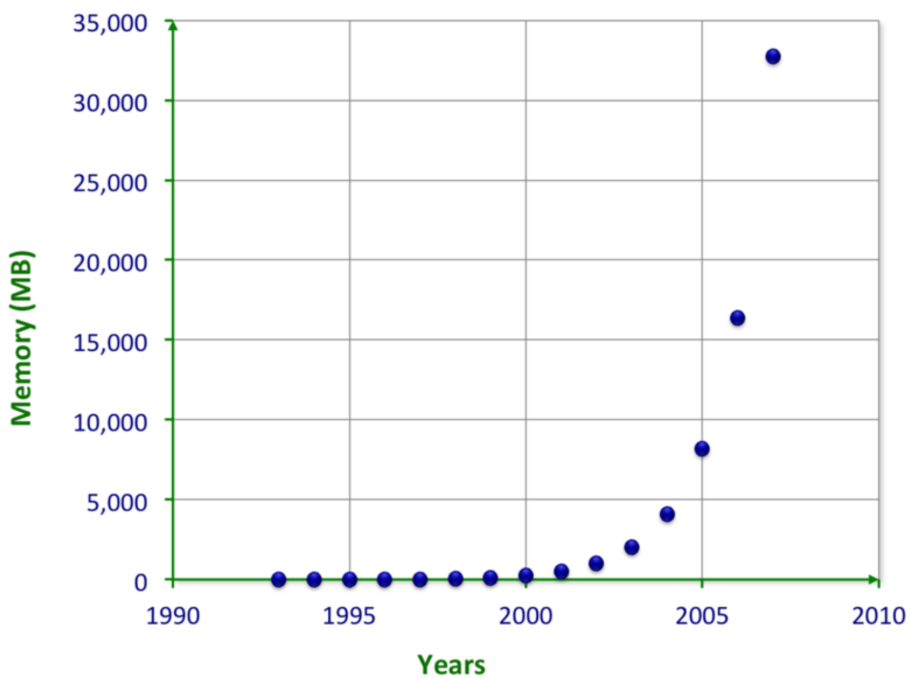


TABLE 6.1:

Year	Memory (MB)
1992	1
1993	2
1994	4
1995	8
1996	16
1997	32
1998	64
1999	128

TABLE 6.1: (continued)

Year	**Memory (MB)**
2000	256
2001	512
2002	1024
2003	2048
2004	4096
2005	8192
2006	16384
2007	32768

Find the common ratio for each sequence.

Example A

$2, 4, 8, 16$

Solution: 2 **is the common ratio.**

Example B

$1, 7, 49, 343$

Solution: 7 **is the common ratio.**

Example C

$400, 100, 25$

Solution: $\frac{1}{4}$ **is the common ratio.**

Now let's go back to the dilemma from the beginning of the Concept.

We can write a number pattern.

2, 4, 8, 16, 32, 64, 128, 256, 512, 1024

1024 aliens after 10 splits!

This is the answer to our problem.

Guided Practice

Here is one for you to try on your own.

Find the common ratio in the sequence.

$\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

Solution

Each of the values was divided in half. Therefore, the common ratio is $\frac{1}{2}$.

This is the answer.

Practice

Directions: Find the common ratio between each term.

1. $-4, 20, -100, 500, -2500$
2. $60, 15, \frac{15}{4}, \frac{15}{16}$
3. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2$
4. $3, 6, 12, 24$
5. $4, 2, 1, .5, .25$
6. $12, 24, 48$
7. $\frac{1}{2}, 1, 2$
8. $100, 50, 25, 12.5$

Directions: Identify the following sequences as an arithmetic sequence, a geometric sequence, or neither. For arithmetic sequences, find the common difference. For geometric sequences, find the common ratio.

9. 1, 4, 7, 10, 13
10. 180, 60, 20, $6\frac{2}{3}$
11. 102, 94, 86, 78
12. 19, 27, 35, 43, 50
13. 5, -50, 500, -5000, 50000
14. $\frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$
15. $99, 33, 11$

6.3 Arithmetic and Geometric Sequences: Explicit Formulas

Guidance

When we represent a sequence with a formula that lets us find any term in the sequence without knowing any other terms, we are representing the sequence explicitly.

Given a recursive definition of an arithmetic or geometric sequence, you can always find an explicit formula, or an equation to represent the n^{th} term of the sequence. Consider for example the sequence of odd numbers we started with: 1,3,5,7,...

We can find an explicit formula for the n^{th} term of the sequence if we analyze a few terms:

$$a_1 = 1$$
$$a_2 = a_1 + 2 = 1 + 2 = 3$$
$$a_3 = a_2 + 2 = 1 + 2 + 2 = 5$$
$$a_4 = a_3 + 2 = 1 + 2 + 2 + 2 = 7$$
$$a_5 = a_4 + 2 = 1 + 2 + 2 + 2 + 2 = 9$$
$$a_6 = a_5 + 2 = 1 + 2 + 2 + 2 + 2 + 2 = 11$$

Note that every term is made up of a 1, and a set of 2's. How many 2's are in each term?

TABLE 6.2:

a_1	$= 1$
a_2	$= 1 + 2 = 3$
a_3	$= 1 + 2 \times 2 = 5$
a_4	$= 1 + 3 \times 2 = 7$
a_5	$= 1 + 4 \times 2 = 9$
a_6	$= 1 + 5 \times 2 = 11$

The n^{th} term has (n - 1) 2's. For example, $a_{99} = 1 + 98 \times 2 = 197$. We can therefore represent the sequence as $a_n = 1 + 2(n - 1)$. We can simplify this expression:

TABLE 6.3:

a_n	$= 1 + 2(n - 1)$
a_n	$= 1 + 2n - 2$
a_n	$= 2n - 1$

In general, we can represent an arithmetic sequence in this way, as long as we know the first term and the common difference, d. Notice that in the previous example, the first term was 1, and the common difference, d, was 2. The n^{th} term is therefore the first term, plus $d(n - 1)$:

TABLE 6.4:

a_n	$=a_1 + d(n - 1)$

You can use this general equation to find an explicit formula for any term in an arithmetic sequence.

Example A

Find an explicit formula for the nth term of the sequence 3, 7, 11, 15... and use the equation to find the 50^{th} term in the sequence.

Solution:

$a_n = 4n - 1$, and $a_{50} = 199$

The first term of the sequence is 3, and the common difference is 4.

TABLE 6.5:

a_n a_n a_n a_n	$= a_1 + d(n - 1)$ $= 3 + 4(n - 1)$ $= 3 + 4n - 4$ $= 4n - 1$
a_{50}	$= 4(50) - 1 = 200 - 1 = 199$

We can also find an explicit formula for a geometric sequence. Consider again the sequence in example 2a:

TABLE 6.6:

$t_1 = 3$ $t_n = 2 \times t_{n-1}$	$\rightarrow$	$t_2 = 2t_1 = 2 \times 3 = 6$ $t_3 = 2t_2 = 2 \times 6 = 12$ $t_4 = 2t_3 = 2 \times 12 = 24$ $t_5 = 2t_4 = 2 \times 24 = 48$

Notice that every term is the first term, multiplied by a power of 2. This is because 2 is the common ratio for the sequence.

TABLE 6.7:

t_1	$= 3$
t_2	$= 2 \times 3 = 6$
t_3	$= 2 \times 2 \times 6 = 2^2 \times 6 = 12$
t_4	$= 2 \times 2 \times 2 \times 6 = 2^3 \times 6 = 24$
t_5	$= 2 \times 2 \times 2 \times 2 \times 6 = 2^4 \times 6 = 48$

The power of 2 in the n^{th} term is (n-1). Therefore the n^{th} term in this sequence can be defined as: $t_n = 3(2^{n-1})$. In general, we can define the n^{th} term of a geometric sequence in terms of its first term and its common ratio, r:

TABLE 6.8:

t_n	$= t_1(\mathrm{r}^{n-1})$

You can use this general equation to find an explicit formula for any term in a geometric sequence.

Example B

Find an explicit formula for the n^{th} term of the sequence 5, 15, 45, 135... and use the equation to find the 10^{th} term in the sequence.

Solution:

$a_n = 5 \times 3^{n-1}$, and $a_{10} = 98{,}415$

The first term in the sequence is 5, and $r = 3$.

TABLE 6.9:

a_n	$= a^1 \times r^{n-1}$
a_n	$= 5 \times 3^{n-1}$
a_{10}	$= 5 \times 3^{10-1}$
a_{10}	$= 5 \times 3^9 = 5 \times 19{,}683 = 98{,}415$

Again, it is always possible to write an explicit formula for terms of an arithmetic or geometric sequence. However, you can also write an explicit formula for other sequences, as long as you can identify a pattern. To do this, you must remember that a sequence is a function, which means there is a relationship between the input and the output. That is, you must identify a pattern between the term and its **index**, or the term's "place" in the sequence.

Example C

Write an explicit formula for the nth term of the sequence 1, (1/2), (1/3), (1/4)...

Solution:

$a_n = (1/n)$

Initially you may see a pattern in the fractions, but you may also wonder about the first term. If you write 1 as (1/1), then it should become clear that the n^{th} term is ($1/n$).

Guided Practice

1) Write an explicit formula for the sequence: 2, 9, 16... and use the formula to find the value of the 20^{th} term.

2) Write an explicit formula for the sequence: 5, 10, 20... and use the formula to find the value of the 9_{th} term.

3) Write an explicit formula for the sequence: (1/2), (1/4), (1/8) and use the formula to find the value of the 7_{th} term.

4) Identify all sequences in the previous three problems that are geometric. What is the common ratio in each sequence?

5) The membership of an online dating service increases at an average rate of 8% per year. In the first year, there are 500 members.

a. How many members are there in the second year?
b. How many members are there in the eighteenth year?

Answers

1) For the sequence: 2, 9, 16...

$a_n = 7n - 5$

$\therefore a_{20} = 7(20) - 5$
$a_{20} = 135$

2) For the sequence: 5, 10, 20...

$a_n = 5 \cdot 2^{n-1}$

$\therefore a_9 = 5 \cdot 2^8$
$a_9 = 5 \cdot 256$
$a_9 = 1280$

3) For the sequence: (1/2), (1/4), (1/8)...

$a_n = \frac{1}{2^n}$

$\therefore a_7 = \frac{1}{2^7}$
$a_7 = \frac{1}{128}$

4) The sequence in question 1 is arithmetic.

The sequence is question 2 is geometric, and has $r = 2$.

The sequence in question 3 is geometric and has $r = 1/2$.

5) a. 540 members

b. Approximately 1,998 members

Practice

Name the sequence as arithmetic, geometric, or neither.

1. $-21, -6, 18, -3, 20, -2$
2. $0, \frac{-1}{5}, \frac{-2}{5}, \frac{-3}{5}, \frac{-4}{5}, -1$
3. $1, 3, 9, 27, 81, 243$
4. $2, 9, -2, 1, 18, 2$

Write the first 5 terms of the arithmetic sequence(explicit).

5. $a_n = -8 - 9(n-1)$
6. $a_n = 6 - \frac{2}{3}(n-1)$
7. $a_n = 8 + \frac{1}{3}(n-1)$

Solve the following:

8. What are the first five terms of the sequence? $a_n = a_{n-1} - \frac{10}{3}; a_1 = -6$
9. Given the sequence, write a recursive function to generate it: $2, -4, -10, -16, -22, -28$
10. Write the equation of a_n without using recursion: $a_n = a_{n-1} - \frac{3}{2}; a_1 = 10$
11. Write as a recursion: $a_n = 6 - \frac{5}{3}(n-1)$
12. Write the equation of a_n without using recursion: $a_n = a_{n-1} + 8; a_1 = 3$
13. What are the first five terms of the sequence? $a_n = a_{n-1} - 1; a_1 = -5$

Write an explicit formula for the n_{th} term of the arithmetic sequence.

14. $-7, \frac{-13}{3}, \frac{-5}{3}, 1, \frac{11}{3}, \frac{19}{3}$
15. $6, -4, -14, -24, -34, -44$
16. $9, 16, 23, 30, 37, 44$
17. In a particular arithmetic sequence, the second term is 4 and the fifth term is 13. Write an explicit formula for this sequence.

Write the first 5 terms of the geometric sequence.

18. $a_n = 5(-3)^{(n-1)}$
19. $a_n = -6(\frac{-10}{3}^{(n-1)})$

Write the explicit formula for the n_{th} term of the geometric sequence.

20. $-8, 16, -32, 64, -128, 256$
21. $9, \frac{27}{2}, \frac{81}{4}, \frac{243}{8}, \frac{729}{16}, \frac{2187}{32}$

Convert the explicit formula to a recursive formula.

22. $a_n = 9(\frac{-4}{3})^{(n-1)}$
23. $a_n = -6(-4)^{(n-1)}$
24. $a_n = -5(5)^{(n-1)}$

6.4 Arithmetic and Geometric Sequences: Recursive Formulas

Guidance

A sequence is an ordered list of objects. The simplest way to represent a sequence is by listing some of its terms.

The sequence of odd, positive integers is shown here:

TABLE 6.10:

1, 3, 5, 7 ...

In this lesson you will learn to represent a sequence ***recursively***, which means that you need to know the previous term in order to find the next term in the sequence.

Consider the sequence shown above. What is the next term?

As long as you are familiar with the odd integers (i.e., you can count in 2's) you can figure out that the next term is 9. If we want to describe this sequence in general, we can do so by stating what the first term is, and then by stating the relationship between successive terms. When we represent a sequence by describing the relationship between its successive terms, we are representing the sequence **recursively**.

The terms in a sequence are often denoted with a variable and a subscript. All of the terms in a given sequence are written with the same variable, and increasing subscripts. So we might list terms in a sequence as a_1, a_2, a_3, a_4, a_5 ...

We can use this notation to represent the example above. This sequence is defined as follows:

TABLE 6.11:

$a_1 = 1$ $a_n = a_{n-1} + 2$

At first glance this notation may seem confusing. What is important to keep in mind is that the subscript of a term represents its "place in line." So a_n just means the nth term in the sequence. The term a_{n-1} just means the term before a_n. In the sequence of odd numbers above, $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$, $a_5 = 9$ and so on. If, for example, we wanted to find a_{10}, we would need to find the 9th term in the sequence first. To find the 9th term we need to find the 8th term, and so on, back to a term that we know.

Example A

For the sequence of odd numbers, list a_6, a_7, a_8, a_9, and a_{10}.

Solution: Each term is two more than the previous term.

$$a_6 = a_5 + 2 = 9 + 2 = 11$$
$$a_7 = a_6 + 2 = 11 + 2 = 13$$
$$a_8 = a_7 + 2 = 13 + 2 = 15$$
$$a_9 = a_8 + 2 = 15 + 2 = 17$$
$$a_{10} = a_9 + 2 = 17 + 2 = 19$$

The sequence of odd numbers is ***linear***, and it is referred to as an ***arithmetic sequence***. Every arithmetic sequence has a **common difference**, or a constant difference between each term. (The common difference is analogous to the slope of a line.) The sequence of odd numbers has a common difference of 2 because for all n, a_n - a_{n-1} = 2.

Finding terms in this sequence is relatively straightforward, as the pattern is familiar. However, this would clearly be tedious if you needed to find the 100^{th} term.

Example B

Find the 5^{th} term for the sequence:

TABLE 6.12:

$t_1 = 3$ $t_n = 2t_{n-1}$

Solution:

$t_5 = 48$

TABLE 6.13:

$t_1 = 3$ $t_n = 2 \times t_{n-1}$	$\rightarrow$	$t_2 = 2t_1 = 2 \times 3 = 6$ $t_3 = 2t_2 = 2 \times 6 = 12$ $t_4 = 2t_3 = 2 \times 12 = 24$ $t_5 = 2t_4 = 2 \times 24 = 48$

This example is a **geometric sequence**. Every geometric sequence has a **common ratio**, which is 2 in this example, because for all n, $\frac{t_n}{t_{n-1}} = 2$. The terms of a geometric sequence follow an exponential pattern.

Example C

Find the 4^{th} term for the sequence:

TABLE 6.14:

$b_1 = 2$ $b_n = (b_{n-1})^2 + 1$

Solution:

$b_4 = 677$

TABLE 6.15:

$b_1 = 2$ $b_n = (b_{n-1})^2 + 1$	$\rightarrow$	$b_2 = (b_1)^2 + 1 = 2^2 + 1 = 4 + 1 = 5$ $b_3 = (b_2)^2 + 1 = 5^2 + 1 = 25 + 1 = 26$ $b_4 = (b_3)^2 + 1 = 26^2 + 1 = 676 + 1 = 677$

This sequence is neither arithmetic nor geometric, though its values follow a cubic pattern. As you can see from just a few terms here, the terms in a sequence can grow quickly.

For any of these sequences, as noted above, determining more than a few values by hand can be time consuming. In another lesson, we will introduce ***explicit formulas***, which can be used to define a sequence in a way that makes finding the n^{th} term faster.

Guided Practice

Write the next 5 terms of each sequence.

1) Given:$a_1 = 2$ and $a_n = 3a_{n-1} + 3$.

2) Given: $a_1 = -4 a_2 = -4$ and $a_n = 2a_{n-1} + a_{n-2}$

3) Write a recursive formula that fits the following sequence: 1, 5, 9, 13, 17

4) Given the following sequence, write a recursive formula, then find the next three numbers in the sequence: 3, -4, -1, -5, -6, -11, -17

Answers

1) Given:$a_1 = 2$ and $a_n = 3a_{n-1} + 3$.

$$\begin{aligned}
&a_2 = 3(2) + 3 = 9\\
&a_3 = 3(9) + 3 = 30\\
&a_4 = 3(30) + 3 = 93\\
&a_5 = 3(93) + 3 = 282\\
&a_6 = 3(282) + 3 = 849
\end{aligned}$$

So our answer is: 9, 30, 93, 282 and 849

2) Given: $a_1 = -4 a_2 = -4$ and $a_n = 2a_{n-1} + a_{n-2}$

$$\begin{aligned}
&a_2 = 2(-4) + (-4) = -12\\
&a_3 = 2(-12) + (-4) = -28\\
&a_4 = 2(-28) + (-12) = -68\\
&a_5 = 2(-68) + (-28) = -164\\
&a_6 = 2(-164) + (-68) = -396
\end{aligned}$$

So our answer is: -12, -28, -68, -164, and -396.

3) In this problem we deduct that each term differs by the same amount. Once, we identify the difference of each term, +4 each time in this case, then we know that the sequence requires that we add the same amount (4) to each term.

We write that as: $a_n = a_{n-1} + 4$

4) Getting to the next term is not going to be as easy as the previous example. We need to examine the number sequence more closely to solve this problem.

In this sequence, the Fibonacci Series was applied. What this means is that the two previous terms were added together to get the next term in the sequence.

It is written as: $a_n = a_{n-1} + a_{n-2}$

Now that we know the formula, we can find the next three numbers in the sequence:

$$\begin{aligned}
&a_8 = (-11) + (-17) = -28\\
&a_9 = (-17) + (-28) = -45
\end{aligned}$$

$a_{10} = (-28) + (-45) = -73$

So the next three numbers in the sequence are: -28, -45, and -73.

Practice

1. A sequence in which you know the previous term in order to find the next term is:
2. Why is the sequence of odd numbers linear?
3. Which type of sequence has a common difference?
4. A sequence that uses the same multiple to get from one term to another is:
5. Find the value of a_6 , given the sequence defined as: $a_1 = 4$ $a_n = 5a_{n-1}$
6. Find the value of a_5, given the sequence defined as: $a_1 = 32$ $a_n = (1/2)a_{n-1}$
7. Find the value of a_{n-1} , given the sequence defined as: $a_1 = 1$ $a_n = 3a_{n-1}$-n

Using the given recursive formulas, identify the next 5 terms in the sequences that follow:

8. $a_1 = -2a_2 = 1$ and $a_n = 3a_{n-1} - 5a_{n-2}$
9. $a_1 = -2$ and $a_n = 3a_{n-1}$
10. $a_1 = 3a_2 = -2$ and $a_n = -5a_{n-1} + a_{n-2}$
11. $a_1 = 1$ and $a_n = 4a_{n-1}$
12. $a_1 = -4a_2 = 1$ and $a_n = -a_{n-1} + a_{n-2}$

Given the following sequence of numbers find the recursive formula.

13. 1, 5, 9, 13, 17
14. -1, 3, 2, 5, 7, 12, 19
15. -4, 16, -64, 256, -1024

Given the following sequence of numbers find the recursive formula and the next three numbers in the sequence.

16. 1, -1, 1, -1, 1
17. -5, -1, -6, -7, -13, -20, -33
18. 1, - 3, 9, -27, 81
19. -3, -4, -7, -11, -18, -29, -47
20. -1, -5, -9, -13, -17
21. Write the next three terms of the sequence: $a_n = (-1)^n \cdot 5a_{n-1}$
22. Given the formula: $a_n = 4n - 1$, is the number 27 a term in the sequence of numbers?
23. Given the formula: $a_n = 4n - 1$ is the number 97 a term in the sequence of numbers?

6.5 Arithmetic Sequences in Recursive Form

Guidance

Arithmetic Sequence – Recursive Form

A recursive formula is a function rule that relates each term of a sequence after the first to the ones before it.

Example: 7, 11, 15, 19, …

The common difference of this arithmetic sequence is 4.

7 + 4 = 11

11 + 4 = 15

15 + 4 = 19

You can use the common difference to write the recursive formula.

FIGURE 6.1

A general recursive definition for an arithmetic sequence with a common difference, d, has TWO parts:

A (1) = first term	the starting value
A (n) = *A (n-1)* + *d*, for *n* must be > or = 2	Recursive formula

FIGURE 6.2

Example:

Let *n* = the term number in the sequence

Let *A (n)* = the value of the nth term of the sequence

Value of term 1 = *A* (1) = 7

Value of term 2 = *A* (2) + 4 = 11

Value of term 3 = *A* (3) + 4 = 15

Value of term 4 = *A* (4) + 4 = 19

Value of term n = *A* (*n* – 1) + 4

- The recursive formula for the arithmetic sequence above is

  A (n) = A (n-1) +4

  A (n) = value of previous term + common difference

- The recursive formula together with the starting value *A* (1) = 7 defines the sequence

FIGURE 6.3

Practice

Indicate whether each sequence is arithmetic. Explain your answer. If the sequence is arithmetic, write the recursive formula.

1. -5, 5, -5, 5, ...
2. 1, 8, 27, 64, ...
3. 0.3, 0.9, 1.5, 2.1, ...
4. -3, -7, -11, -15, ...
5. 2, 8, 32, 128, ...
6. -10, -5, 0, 5, ...

6.6 Geometric Sequence in Recursive Form

Guidance

Geometric Sequences – Recursive From

In a geometric sequence, the ratio of any term to its preceding term is a constant value.

A geometric sequence with a starting value of a and common ratio r is a sequence form

a, ar, ar²,...

Example:

Geometric sequence of 4, 20, 200, 500, ...

The common ratio of the terms of geometric sequence is 5

Value term 1 = *A* (1) = 4

Value term 2 = *A* (2) = *A* (1) * 5 = 20

Value term 3 = *A* (3) = *A* (2) * 5 = 100

Value term 4 = *A* (4) = *A* (3) * 5 = 500

FIGURE 6.4

A general recursive formula definition for a geometric sequence with common ratio, *r*, has TWO parts:

A (1) = *a* starting value

A (n) = *A (n-1)* * *r*, for *n* > or = 2 Recursive formula

FIGURE 6.5

Value term n = A (n -1) * 5

Value of the previous term times 5

- The recursive formula for the geometric sequence example is
- A (n) = A (n-1) * 5
- The recursive formula together with the starting value A (1) = 4 defines the sequence

FIGURE 6.6

Practice

Identify if each sequence is a a geometric sequence. If it is, find the common ratio and write the recursive formula for each geometric sequence.

1. 200, -100, 50, -25, ...
2. -3, -1, 1, 3, ...
3. 98, 14, 2, 2/7, ...
4. 3, -9, 27, -81, ...
5. 20, 15, 10, 5, ...
6. 5, 10, 20, 40, ...

Vocabulary:

Arithmetic and Geometric Sequences: Arithmetic Sequence, Common Difference, Common Ratio, Explicit, Formula, Geometric Sequence, Sequence, Term of a Sequence

Arithmetic Sequences and Recursive Form: Arithmetic Sequence, Explicit Formula, Recursive Definition, Recursive Formula

Geometric Sequences in Recursive Form: Common Ratio, Geometric Sequence, Recursive Formula

CHAPTER 7 Polynomials and Factorization

Chapter Outline

Adding and Subtracting Polynomials: A.10a

Multiplying and Factoring Polynomials: A.10a, A.10d

Multiplying Binomials: A.10b, A.10d

Multiplying Special Cases: A.10b

Factoring $x^2 + bx + c$: A.10e

Factoring $ax^2 + bx + c$: A.10d, A.10e

Factoring Special Cases: A.10e, A.10f

Factoring by Grouping: A.10b

Simplifying Rational Expressions: A.10d, A.10e, A.10f

Dividing Polynomials: A.10d

7.1 Adding and Subtracting Polynomials: Classifying and Degree

Guidance

You can add and subtract polynomials by lining up terms and then adding or subtracting each part separately.

Before we begin to add or subtract polynomials, we first need to learn the vocabulary and then how to find the degree and classify polynomials.

Vocabulary

Monomial – is an expression that is a number, a variable, or a product of a number and one or more variables. Consequently, a monomial has no variable in its denominator. It has one term. (mono implies one).

$13, \quad 3x, \quad -57, \quad x^2, \quad 4y^2, \quad -2xy, \quad \text{or } 520x^2y^2$

FIGURE 7.1

Binomial – is the sum of two monomials. It has two unlike terms (bi implies two).

$3x+1, \quad x^2-4x, \quad 2x+y, \quad \text{or} \quad y-y^2$

FIGURE 7.2

Trinomial – is the sum of three monomials. It has three unlike terms. (tri implies three).

$x^2+2x+1, \quad 3x^2-4x+10, \quad 2x+3y+2$

FIGURE 7.3

Polynomial – is a monomial or the sum (+) or difference (-) of one or more terms. (poly implies many). •Polynomials are in simplest form when they contain no like terms. x2 + 2x + 1 + $3x^2$ – 4x when simplified becomes $4x^2$ – 2x + 1 •Polynomials are generally written in descending order. Descending: $4x^2$ – 2x + 1 (exponents of variables decrease from left to right)

$x^2+2x, \quad 3x^3+x^2+5x+6, \quad 4x+6y+8$

FIGURE 7.4

The **degree of a monomial** is the sum of the exponents of its variables. For a non-zero constant, the degree is zero. Zero has no degree.

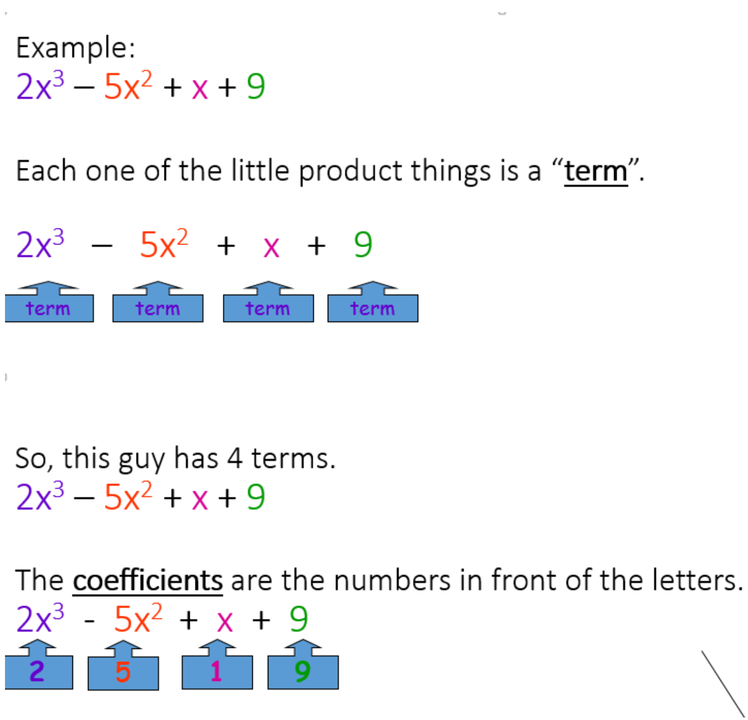


FIGURE 7.5

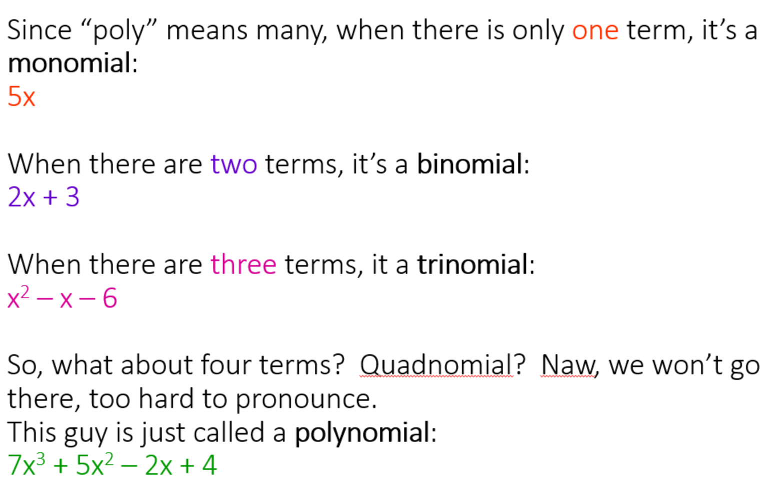


FIGURE 7.6

Practice

1. 6x2 + 7 – 9x4
2. 3y – 4 – y3
3. 8 + 7v – 11v

So, there's one word to remember to classify:
degree

Here's how you find the degree of a polynomial:
Look at each term,
whoever has the most letters wins!
$3x^2 - 8x^4 + x^5$

This guy has 5 letters…
The degree is 5.

This is a 7^{th} degree polynomial:
$6mn^2 + m^3n^4 + 8$

This guy has 7 letters…
The degree is 7

FIGURE 7.7

$$3x^4 + 5x^2 - 7x + 1$$

The polynomial above is in <u>standard form</u>. <u>Standard form of a polynomial</u> - means that the degrees of its monomial terms decrease from left to right.

FIGURE 7.8

Polynomial	Degree	Name using Degree	Number of Terms	Name using number of terms
$7x + 4$	1	Linear	2	Binomial
$3x^2 + 2x + 1$	2	Quadratic	3	Trinomial
$4x^3$	3	Cubic	1	Monomial
$9x^4 + 11x$	4	Fourth degree	2	Binomial
5	0	Constant	1	monomial

Once you simplify a polynomial by combining like terms, you can name the polynomial based on degree or number of monomials it contains.

FIGURE 7.9

add

7.2 Adding and Subtracting Polynomials: Add and Subtract

Guidance

A **polynomial** is an expression with multiple variable terms, such that the exponents are greater than or equal to zero. All quadratic and linear equations are polynomials. Equations with negative exponents, square roots, or variables in the denominator are not polynomials.

Polynomials	Not Polynomials
$2x^2 + 6x - 9$	$10x^{-1} + 6x^2$
$-x^3 + 9$	$\sqrt{x} - 2$
$4x^4 + 5x^3 - 8x^2 + 12x + 24$	$\frac{3}{x} + 5$

Now that we have established what a polynomial is, there are a few important parts. Just like with a quadratic, a polynomial can have a **constant**, which is a number without a variable. The **degree** of a polynomial is the largest exponent. For example, all quadratic equations have a degree of 2. Lastly, the **leading coefficient** is the coefficient in front of the variable with the degree. In the polynomial $4x^4 + 5x^3 - 8x^2 + 12x + 24$ above, the degree is 4 and the leading coefficient is also 4. Make sure that when finding the degree and leading coefficient you have the polynomial in standard form. **Standard form** lists all the variables in order, from greatest to least.

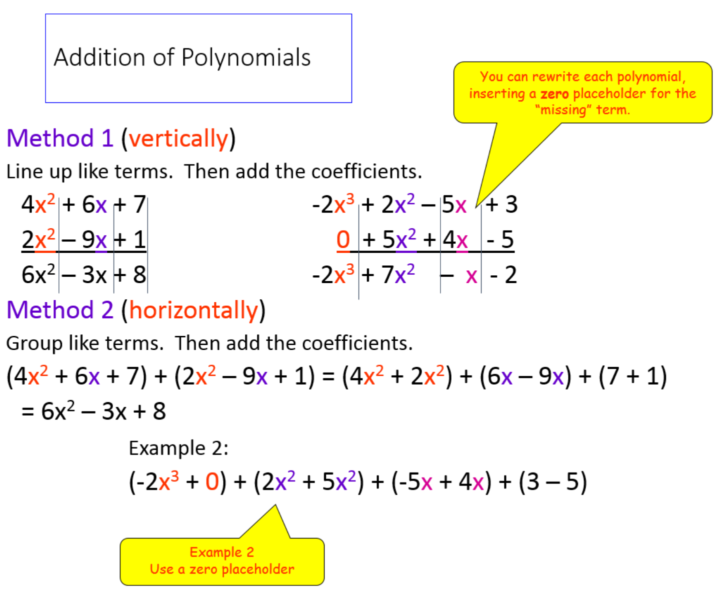


FIGURE 7.10

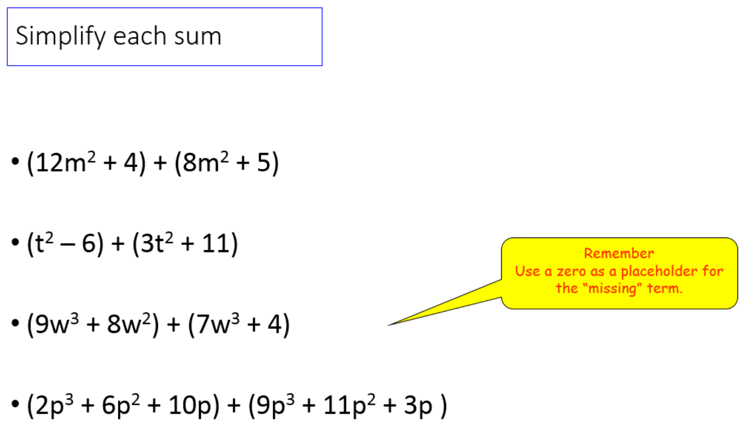


FIGURE 7.11

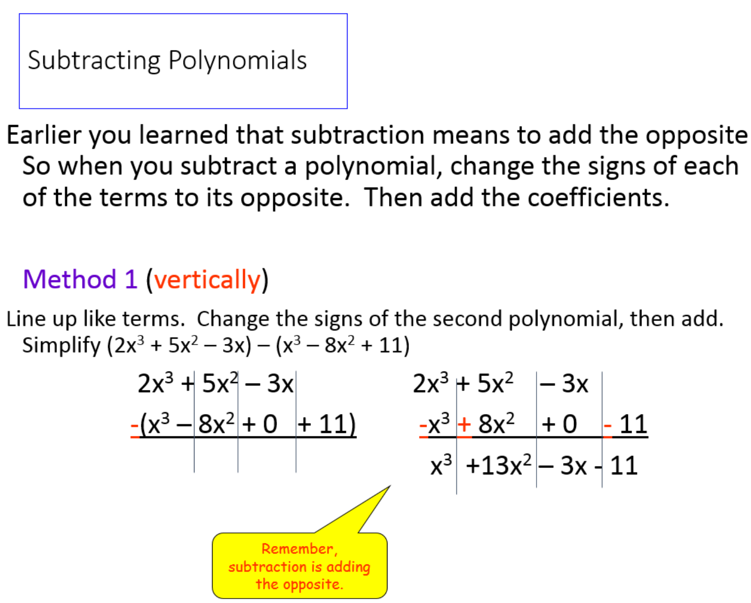


FIGURE 7.12

Guided Practice

1. Is $\sqrt{2x^3 - 5x} + 6$ a polynomial? Why or why not?

2. Find the leading coefficient and degree of $6x^2 - 3x^5 + 16x^4 + 10x - 24$.

Add or subtract.

3. $(9x^2 + 4x^3 - 15x + 22) + (6x^3 - 4x^2 + 8x - 14)$

4. $(7x^3 + 20x - 3) - (x^3 - 2x^2 + 14x - 18)$

Answers

1. No, this is not a polynomial because x is under a square root in the equation.

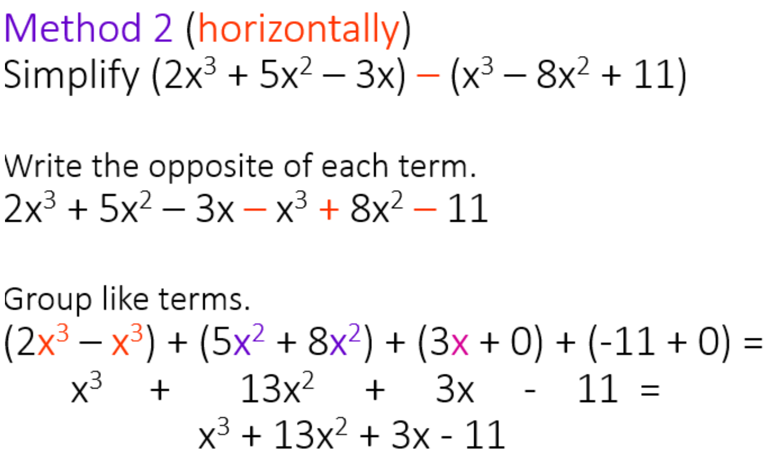


FIGURE 7.13

2. In standard form, this polynomial is $-3x^5+16x^4+6x^2+10x-24$. Therefore, the degree is 5 and the leading coefficient is -3.

3. $(9x^2+4x^3-15x+22)+(6x^3-4x^2+8x-14)=10x^3+5x^2-7x+8$

4. $(7x^3+20x-3)-(x^3-2x^2+14x-18)=6x^3+2x^2+6x+15$

Practice

Determine if the following expressions are polynomials. If not, state why. If so, write in standard form and find the degree and leading coefficient.

1. $\frac{1}{x^2}+x+5$
2. $x^3+8x^4-15x+14x^2-20$
3. x^3+8
4. $5x^{-2}+9x^{-1}+16$
5. $x^2\sqrt{2}-x\sqrt{6}+10$
6. $\frac{x^4+8x^2+12}{3}$
7. $\frac{x^2-4}{x}$
8. $-6x^3+7x^5-10x^6+19x^2-3x+41$

Add or subtract the following polynomials.

9. $(x^3+8x^2-15x+11)+(3x^3-5x^2-4x+9)$
10. $(-2x^4+x^3+12x^2+6x-18)-(4x^4-7x^3+14x^2+18x-25)$
11. $(10x^3-x^2+6x+3)+(x^4-3x^3+8x^2-9x+16)$
12. $(7x^3-2x^2+4x-5)-(6x^4+10x^3+x^2+4x-1)$
13. $(15x^2+x-27)+(3x^3-12x+16)$
14. $(2x^5-3x^4+21x^2+11x-32)-(x^4-3x^3-9x^2+14x-15)$
15. $(8x^3-13x^2+24)-(x^3+4x^2-2x+17)+(5x^2+18x-19)$

7.3 Multiplying and Factoring Polynomials

Guidance

Factor the following polynomial: $12x^4 + 6x^3 + 3x^2$.

Step 1: Identify the GCF of the polynomial.

If you look at just the factors of the numbers you can see the following:

$$\begin{array}{ccc} & 12x^4+6x^3+3x^2 & \\ \swarrow & \downarrow & \searrow \\ 1\times 12 & 1\times 6 & 1\times 3 \\ 2\times 6 & 2\times 3 & \\ 3\times 4 & & \end{array}$$

Looking at the factors for each of the numbers, you can see that 12, 6, and 3 can all be divided by 3.

Also notice that each of the terms has an x^2 in common.

$$\begin{aligned} 12x^4 &= 12\cdot x\cdot x\cdot x\cdot x \\ 6x^3 &= 6\cdot x\cdot x\cdot x \\ 3x^2 &= 3\cdot x\cdot x \end{aligned}$$

So the GCF for this polynomial is $3x^2$

Step 2: Divide out the GCF from each term of the polynomial.

$$\begin{array}{llll} 12x^4+6x^3+3x^2 = 3x^2(4x^2+2x+1) & & & \\ & \swarrow & \downarrow & \searrow \\ & \text{Remember the} & \text{Remember the} & \text{Remember} \\ & \text{factors of } 12 & \text{factors of } 6 & \text{the factors of} \\ & \text{included } 3\times 4 & \text{included } 2\times 3 & 3 \text{ are } 1\times 3 \\ & \text{and } x^2\cdot x^2 = x^4 & \text{and } x^2\cdot x = x^3 & \text{and } x^2\cdot 1 = x^2 \end{array}$$

Example A

Factor the following binomial: $5a + 15$

Step 1: Identify the GCF.

$$\begin{array}{ccc} & 5a+15 & \\ \swarrow & & \searrow \\ 1\times 5 & & 1\times 15 \\ & & 3\times 5 \end{array}$$

Looking at the factors for each of the numbers, you can see that 5 and 15 can both be divided by 5.

So the GCF for this binomial is 5.

Step 2: Divide out the GCF from each term of the binomial.

$5a+15=5(a+3)$

Example B

Factor the following polynomial: $4x^2+8x-2$

Step 1: Identify the GCF.

$$\begin{array}{ccc} & 4x^2+8x-2 & \\ \swarrow & \downarrow & \searrow \\ 1\times 4 & 1\times 8 & 1\times 2 \\ 2\times 2 & 2\times 4 & \end{array}$$

Looking at the factors for each of the numbers, you can see that 4, 8, and 2 can all be divided by 2.

So the GCF for this polynomial is 2.

Step 2: Divide out the GCF from each term of the polynomial.

$4x^2+8x-2=2(2x^2+4x-1)$

Example C

Factor the following polynomial: $3x^5-9x^3-6x^2$

Step 1: Identify the GCF.

$$\begin{array}{ccc} & 3x^5-9x^3-6x^2 & \\ \swarrow & \downarrow & \searrow \\ 1\times 3 & 1\times 9 & 1\times 6 \\ & 3\times 3 & 2\times 3 \end{array}$$

Looking at the factors for each of the numbers, you can see that 3, 9, and 6 can all be divided by 3.

Also notice that each of the terms has an x^2 in common.

$$\begin{aligned} 3x^5 &= 3\cdot x\cdot x\cdot x\cdot x\cdot x \\ -9x^3 &= -9\cdot x\cdot x\cdot x \\ -6x^2 &= -6\cdot x\cdot x \end{aligned}$$

So the GCF for this polynomial is $3x^2$.

Step 2: Divide out the GCF from each term of the polynomial.

$3x^5 - 9x^3 - 6x^2 = 3x^2(x^3 - 3x - 2)$

Vocabulary

Common Factor

> ***Common factors*** are numbers (numerical coefficients) or letters (literal coefficients) that are factors in all terms of the polynomials.

Greatest Common Factor

> The ***Greatest Common Factor*** (or ***GCF***) is the largest monomial that is a factor of (or divides into evenly) each of the terms of the polynomial.

Guided Practice

1. Find the common factors of the following: $a^2(b+7) - 6(b+7)$
2. Factor the following polynomial: $5k^6 + 15k^4 + 10k^3 + 25k^2$
3. Factor the following polynomial: $27x^3y + 18x^2y^2 + 9xy^3$

Answers

1. $a^2(b+7) - 6(b+7)$

Step 1: Identify the GCF.

$$a^2(b+7) - 6(b+7)$$

This problem is a little different in that if you look at the expression you notice that $(b+7)$ is common in both terms. Therefore $(b+7)$ is the common factor.

So the GCF for this expression is $(b+7)$.

Step 2: Divide out the GCF from each term of the expression.

$$a^2(b+7) - 6(b+7) = (a^2 - 6)(b+7)$$

2. $5k^6 + 15k^4 + 10k^3 + 25k^2$

Step 1: Identify the GCF.

$$\begin{array}{cccc} \multicolumn{4}{c}{5k^6 + 15k^4 + 10k^3 + 25k^2} \\ \swarrow & \downarrow & \downarrow & \searrow \\ 1 \times 5 & 1 \times 15 & 1 \times 10 & 1 \times 25 \\ & 3 \times 5 & 2 \times 5 & 5 \times 5 \end{array}$$

Looking at the factors for each of the numbers, you can see that 5, 15, 10, and 25 can all be divided by 5.

Also notice that each of the terms has an k^2 in common.

$$\begin{aligned}5k^6 &= 5 \cdot k \cdot k \cdot k \cdot k \cdot k \cdot k\\ 15k^4 &= 15 \cdot k \cdot k \cdot k \cdot k\\ 10k^3 &= 10 \cdot k \cdot k \cdot k\\ 25k^2 &= 25 \cdot k \cdot k\end{aligned}$$

So the GCF for this polynomial is $5k^2$.

Step 2: Divide out the GCF out of each term of the polynomial.

$$5k^6 + 15k^4 + 10k^3 + 25k^2 = 5k^2(k^4 + 3k^2 + 2k + 5)$$

3. $27x^3y + 18x^2y^2 + 9xy^3$

Step 1: Identify the GCF.

$$\begin{array}{ccc} & 27x^3y + 18x^2y^2 + 9xy^3 & \\ \swarrow & \downarrow & \searrow \\ 1 \times 27 & 1 \times 18 & 1 \times 9 \\ 3 \times 9 & 2 \times 9 & 3 \times 3 \\ & 3 \times 6 & \end{array}$$

Looking at the factors for each of the numbers, you can see that 27, 18, and 9 can all be divided by 9.

Also notice that each of the terms has an xy in common.

$$\begin{aligned}27x^3y &= 27 \cdot x \cdot x \cdot x \cdot y\\ 18x^2y^2 &= 18 \cdot x \cdot x \cdot y \cdot y\\ 9xy^3 &= 9 \cdot x \cdot y \cdot y \cdot y\end{aligned}$$

So the GCF for this polynomial is $9xy$.

Step 2: Divide out the GCF out of each term of the polynomial

$27x^3y + 18x^2y^2 + 9xy^3 = 9xy(3x^2 + 2xy + y^2)$

Practice

Find the common factors of the following:

1. $2x(x-5) + 7(x-5)$
2. $4x(x-3) + 5(x-3)$
3. $3x^2(e+4) - 5(e+4)$
4. $8x^2(c-3) - 7(c-3)$
5. $ax(x-b) + c(x-b)$

Factor the following polynomial:

1. $7x^2 + 14$
2. $9c^2 + 3$
3. $8a^2 + 4a$
4. $16x^2 + 24y^2$
5. $2x^2 - 12x + 8$

Factor the following polynomial:

1. $32w^2x + 16xy + 8x^2$
2. $12abc + 6bcd + 24acd$
3. $20x^2y - 10x^2y^2 + 25x^2y$
4. $12a^2b - 18ab^2 - 24a^2b^2$
5. $4s^3t^2 - 16s^2t^3 + 12st^2 - 24st^3$

7.4 Multiplying Binomials

Guidance

Multiplying together polynomials is very similar to multiplying together factors. You can FOIL or we will also present an alternative method. When multiplying together polynomials, you will need to use the properties of exponents, primarily the Product Property $(a^m \cdot a^n = a^{m+n})$ and combine like terms.

Example A

Find the product of $(x^2-5)(x^3+2x-9)$.

Solution: Using the FOIL method, you need be careful. First, take the x^2 in the first polynomial and multiply it by every term in the second polynomial.

$$(x^2-5)(x^3+2x-9) = x^5+2x^3-9x^2$$

Now, multiply the -5 and multiply it by every term in the second polynomial.

$$(x^2-5)(x^3+2x-9) = x^5+2x^3-9x^2-5x^3-10x+45$$

Lastly, combine any like terms. In this example, only the x^3 terms can be combined.

$$(x^2-5)(x^3+2x-9) = x^5\boxed{+2x^3}-9x^2\boxed{-5x^3}-10x+45$$

$$= x^5-3x^3-9x^2-10x+45$$

Example B

Multiply $(x^2+4x-7)(x^3-8x^2+6x-11)$.

Solution: In this example, we will use the "box" method. Align the two polynomials along the top and left side of a rectangle and make a row or column for each term. Write the polynomial with more terms along the top of the rectangle.

	x^3	$-8x^2$	$+6x$	-11
x^2				
$4x$				
-7				

Multiply each term together and fill in the corresponding spot.

	x^3	$-8x^2$	$+6x$	-11
x^2	x^5	$-8x^4$	$6x^3$	$-11x^2$
$4x$	$4x^4$	$-32x^3$	$24x^2$	$-44x$
-7	$-7x^3$	$56x^2$	$-42x$	77

Finally, combine like terms. The final answer is $x^5 - 4x^4 - 33x^3 + 69x^2 - 86x + 77$. This method presents an alternative way to organize the terms. Use whichever method you are more comfortable with. Keep in mind, no matter which method you use, you will multiply every term in the first polynomial by every term in the second.

Example C

Find the product of $(x-5)(2x+3)(x^2+4)$.

Solution: In this example we have three binomials. When multiplying three polynomials, start by multiplying the first two binomials together.

$$\begin{aligned}(x-5)(2x+3) &= 2x^2 + 3x - 10x - 15 \\ &= 2x^2 - 7x - 15\end{aligned}$$

Now, multiply the answer by the last binomial.

$$\begin{aligned}(2x^2 - 7x - 15)(x^2+4) &= 2x^4 + 8x^2 - 7x^3 - 28x - 15x^2 - 60 \\ &= 2x^4 - 7x^3 - 7x^2 - 28x - 60\end{aligned}$$

Intro Problem Revisit Recall that the area of a rectangle is $A = lw$, where l is the length and w is the width. Therefore, we need to multiply.

$$\begin{aligned}A &= (x^3 + 5x^2 - 1)(x^2 + 3) \\ &= x^5 + 3x^3 + 5x^4 + 15x^2 - x^2 - 3\end{aligned}$$

.

Now combine like terms and simplify. Be sure to write your answer in standard form

$$\begin{aligned} &x^5+3x^3+5x^4+(15x^2-x^2)-3 \\ &=x^5+3x^3+5x^4+14x^2-3 \\ &=x^5+5x^4+3x^3+14x^2-3 \end{aligned}$$

Therefore, the area of the garden plot is $x^5+5x^4+3x^3+14x^2-3$.

Guided Practice

Find the product of the polynomials.

1. $-2x^2(3x^3-4x^2+12x-9)$
2. $(4x^2-6x+11)(-3x^3+x^2+8x-10)$
3. $(x^2-1)(3x-4)(3x+4)$
4. $(2x-7)^2$

Answers

1. Use the distributive property to multiply $-2x^2$ by the polynomial.

$$-2x^2(3x^3-4x^2+12x-9)=-6x^5+8x^4-24x^3+18x^2$$

2. Multiply each term in the first polynomial by each one in the second polynomial.

$$\begin{aligned} (4x^2-6x+11)(-3x^3+x^2+8x-10) &= -12x^5+4x^4+32x^3-40x^2 \\ &\quad +18x^4-6x^3-48x^2+60x \\ &\quad -33x^3+11x^2+88x-110 \\ &= -12x^5+22x^4-7x^3-77x^2+148x-110 \end{aligned}$$

3. Multiply the first two binomials together.

$$(x^2-1)(3x-4)=3x^3-4x^2-3x+4$$

Multiply this product by the last binomial.

$$\begin{aligned} (3x^3-4x^2-3x+4)(3x+4) &= 9x^4+12x^3-12x^3-16x^2-9x^2-12x+12x-16 \\ &= 9x^4-25x^2-16 \end{aligned}$$

4. The square indicates that there are two binomials. Expand this and multiply.

$$
\begin{aligned}
(2x-7)^2 &= (2x-7)(2x-7)\\
&= 4x^2-14x-14x+49\\
&= 4x^2-28x+49
\end{aligned}
$$

Practice

Find the product.

1. $5x(x^2-6x+8)$
2. $-x^2(8x^3-11x+20)$
3. $7x^3(3x^3-x^2+16x+10)$
4. $(x^2+4)(x-5)$
5. $(3x^2-4)(2x-7)$
6. $(9-x^2)(x+2)$
7. $(x^2+1)(x^2-2x-1)$
8. $(5x-1)(x^3+8x-12)$
9. $(x^2-6x-7)(3x^2-7x+15)$
10. $(x-1)(2x-5)(x+8)$
11. $(2x^2+5)(x^2-2)(x+4)$
12. $(5x-12)^2$
13. $-x^4(2x+11)(3x^2-1)$
14. $(4x+9)^2$
15. $(4x^3-x^2-3)(2x^2-x+6)$
16. $(2x^3-6x^2+x+7)(5x^2+2x-4)$
17. $(x^3+x^2-4x+15)(x^2-5x-6)$

7.5 Multiplying Special Cases

Here you'll learn how to find two special polynomial products: 1) the square of a binomial and 2) two binomials where the sum and difference formula can be applied. You'll also learn how to apply special products of polynomials to solve real-world problems.

Guidance

We saw that when we multiply two binomials we need to make sure to multiply each term in the first binomial with each term in the second binomial. Let's look at another example.

Multiply two linear binomials (binomials whose degree is 1):

$$(2x+3)(x+4)$$

When we multiply, we obtain a quadratic polynomial (one with degree 2) with four terms:

$$2x^2+8x+3x+12$$

The middle terms are like terms and we can combine them. We simplify and get $2x^2+11x+12$. This is a quadratic, or second-degree, **trinomial** (polynomial with three terms).

You can see that every time we multiply two linear binomials with one variable, we will obtain a quadratic polynomial. In this section we'll talk about some special products of binomials.

Find the Square of a Binomial

One special binomial product is the **square of a binomial**. Consider the product $(x+4)(x+4)$.

Since we are multiplying the same expression by itself, that means we are squaring the expression. $(x+4)(x+4)$ is the same as $(x+4)^2$.

When we multiply it out, we get $x^2+4x+4x+16$, which simplifies to $x^2+8x+16$.

Notice that the two middle terms—the ones we added together to get $8x$—were the same. Is this a coincidence? In order to find that out, let's square a general linear binomial.

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Sure enough, the middle terms are the same. How about if the expression we square is a difference instead of a sum?

$$\begin{aligned}(a-b)^2 = (a-b)(a-b) &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

It looks like the middle two terms are the same in general whenever we square a binomial. The general pattern is: to square a binomial, take the square of the first term, add or subtract twice the product of the terms, and add the square of the second term. You should remember these formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$
and
$$(a-b)^2 = a^2 - 2ab + b^2$$

Remember! Raising a polynomial to a power means that we multiply the polynomial by itself however many times the exponent indicates. For instance, $(a+b)^2 = (a+b)(a+b)$. **Don't make the common mistake of thinking that** $(a+b)^2 = a^2+b^2$**!** To see why that's not true, try substituting numbers for a and b into the equation (for example, $a = 4$ and $b = 3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

We can apply the formulas for squaring binomials to any number of problems.

Example A

Square each binomial and simplify.

a) $(x+10)^2$

b) $(2x-3)^2$

c) $(x^2+4)^2$

Solution

Let's use the square of a binomial formula to multiply each expression.

a) $(x+10)^2$

If we let $a = x$ and $b = 10$, then our formula $(a+b)^2 = a^2+2ab+b^2$ becomes $(x+10)^2 = x^2+2(x)(10)+10^2$, which simplifies to $x^2+20x+100$.

b) $(2x-3)^2$

If we let $a = 2x$ and $b = 3$, then our formula $(a-b)^2 = a^2-2ab+b^2$ becomes $(2x-3)^2 = (2x^2)-2(2x)(3)+(3)^2$, which simplifies to $4x^2-12x+9$.

c) $(x^2+4)^2$

If we let $a = x^2$ and $b = 4$, then

$$\begin{aligned}(x^2+4)^2 &= (x^2)^2+2(x^2)(4)+(4)^2\\ &= x^4+8x^2+16\end{aligned}$$

Find the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$\begin{aligned}(x+4)(x-4) &= x^2-4x+4x-16\\ &= x^2-16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they *cancel out* when we collect like terms. This is not a coincidence. This always happens when we multiply a sum and difference of the same terms. In general,

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula: $(a+b)(a-b) = a^2 - b^2$

Let's apply this formula to a few examples.

Example B

Multiply the following binomials and simplify.

a) $(x+3)(x-3)$

b) $(5x+9)(5x-9)$

c) $(2x^3+7)(2x^3-7)$

Solution

a) Let $a = x$ and $b = 3$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2 - b^2 \\ (x+3)(x-3) &= x^2 - 3^2 \\ &= x^2 - 9\end{aligned}$$

b) Let $a = 5x$ and $b = 9$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2 - b^2 \\ (5x+9)(5x-9) &= (5x)^2 - 9^2 \\ &= 25x^2 - 81\end{aligned}$$

c) Let $a = 2x^3$ and $b = 7$, then:

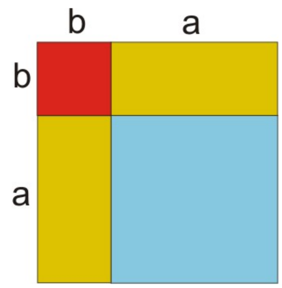
$$\begin{aligned}(2x^3+7)(2x^3-7) &= (2x^3)^2 - (7)^2 \\ &= 4x^6 - 49\end{aligned}$$

Solve Real-World Problems Using Special Products of Polynomials

Now let's see how special products of polynomials apply to geometry problems and to mental arithmetic.

Example C

Find the area of the following square:



Solution

The length of each side is $(a+b)$, so the area is $(a+b)(a+b)$.

Notice that this gives a visual explanation of the square of a binomial. The blue square has area a^2, the red square has area b^2, and each rectangle has area ab, so added all together, the area $(a+b)(a+b)$ is equal to $a^2+2ab+b^2$.

The next example shows how you can use the special products to do fast mental calculations.

Example D

Use the difference of squares and the binomial square formulas to find the products of the following numbers without using a calculator.

a) 43×57

b) 45^2

c) 481×319

Solution

The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

a) Rewrite 43 as $(50-7)$ and 57 as $(50+7)$.

Then $43 \times 57 = (50-7)(50+7) = (50)^2-(7)^2 = 2500-49 = 2451$

b) $45^2 = (40+5)^2 = (40)^2+2(40)(5)+(5)^2 = 1600+400+25 = 2025$

c) Rewrite 481 as $(400+81)$ and 319 as $(400-81)$.

Then $481 \times 319 = (400+81)(400-81) = (400)^2-(81)^2$

$(400)^2$ is easy - it equals 160000.

$(81)^2$ is not easy to do mentally, so let's rewrite 81 as $80+1$.

$(81)^2 = (80+1)^2 = (80)^2+2(80)(1)+(1)^2 = 6400+160+1 = 6561$

Then $481 \times 319 = (400)^2-(81)^2 = 160000-6561 = 153439$

Vocabulary

- **Square of a binomial:** $(a+b)^2 = a^2+2ab+b^2$, *and* $(a-b)^2 = a^2-2ab+b^2$
- **Sum and difference formula:** $(a+b)(a-b) = a^2-b^2$

Guided Practice

1. *Square the binomial and simplify:* $(5x-2y)^2$.

2. *Multiply* $(4x+5y)(4x-5y)$ *and simplify.*

3. *Use the difference of squares and the binomial square formulas to find the product of* 112×88 *without using a calculator.*

Solutions:

1. $(5x-2y)^2$

If we let $a = 5x$ and $b = 2y$, then

$$\begin{aligned} (5x-2y)^2 &= (5x)^2 - 2(5x)(2y) + (2y)^2 \\ &= 25x^2 - 20xy + 4y^2 \end{aligned}$$

2. Let $a = 4x$ and $b = 5y$, then:

$$\begin{aligned} (4x+5y)(4x-5y) &= (4x)^2 - (5y)^2 \\ &= 16x^2 - 25y^2 \end{aligned}$$

3. The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

Rewrite 112 as $(100+12)$ and 88 as $(100-12)$.

Then

$$\begin{aligned} 112 \times 88 &= (100+12)(100-12) \\ &= (100)^2 - (12)^2 \\ &= 10000 - 144 \\ &= 9856 \end{aligned}$$

Practice

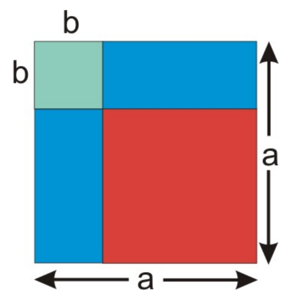
Use the special product rule for squaring binomials to multiply these expressions.

1. $(x+9)^2$
2. $(3x-7)^2$
3. $(5x-y)^2$
4. $(2x^3-3)^2$
5. $(4x^2+y^2)^2$
6. $(8x-3)^2$
7. $(2x+5)(5+2x)$
8. $(xy-y)^2$

Use the special product of a sum and difference to multiply these expressions.

9. $(2x-1)(2x+1)$
10. $(x-12)(x+12)$
11. $(5a-2b)(5a+2b)$
12. $(ab-1)(ab+1)$
13. $(z^2+y)(z^2-y)$
14. $(2q^3+r^2)(2q^3-r^2)$
15. $(7s-t)(t+7s)$
16. $(x^2y+xy^2)(x^2y-xy^2)$

Find the area of the lower right square in the following figure.

17. 

Multiply the following numbers using special products.

18. 45×55
19. 56^2
20. 1002×998
21. 36×44
22. 10.5×9.5
23. 100.2×9.8
24. -95×-105
25. 2×-2

7.6 Factoring $x^2 + bx + c$

Guidance

If a trinomial of the form $x^2 + bx + c$ can be written as the product of two binomials, then:

- The coeffecient of the x-term in the trinomial is the sum of the constants in the binomials.
- The trinomial's constant term is the product of the constants in the binomials.

Example:

What is the factored form of $x^2+12x+32$

To write the factored form, you are looking for two factors of 32 that have a sum of 12.

Make a table showing the factors of 32

TABLE 7.1:

Factors of 32	Sum of Factors
1 and 32	33
2 and 16	18
4 and 8	12

Solution: $x^2+12x+32=(x+4)(x+8)$

Some factorable trinomials in the form of x^2+bx+cwill have negative coeffecients. The rules for factoring are the same as when the x-term and the constant are positive.

- The coeffecient of the x-term of the trinomial is the sum of the constants in the binomials.
- The trinomial's constant term is the product of the constants in the binomials.

However, one or both constants in the binomial will be negative.

Example:

What is the factored form of $x^2-3x-40$

To write the factored form, you are looking for two factors of -40 that have a sum of -3. The negative constant will have a greater absolute value than the positive constant

Make a table showing the factors of -40

TABLE 7.2:

Factors of -40	Sum of factors
1 and -40	-39
2 and -20	-18
4 and -10	-6
5 and -8	-3

Solution: $x^2 - 3x - 40 = (x-8)(x+5)$

Practice

1. $x^2 + 10x + 9$
2. $x^2 + 15x + 50$
3. $x^2 + 10x + 21$
4. $x^2 + 16x + 48$
5. $x^2 - 11x + 24$
6. $x^2 - 13x + 42$
7. $x^2 - 14x + 33$
8. $x^2 - 9x + 20$
9. $x^2 + 5x - 14$
10. $x^2 + 6x - 27$
11. $x^2 + 7x - 78$
12. $x^2 + 4x - 32$
13. $x^2 - 12x - 45$
14. $x^2 - 5x - 50$
15. $x^2 - 3x - 40$
16. $x^2 - x - 56$
17. $-x^2 - 2x - 1$
18. $-x^2 - 5x + 24$
19. $-x^2 + 18x - 72$
20. $-x^2 + 25x - 150$
21. $x^2 + 21x + 108$
22. $-x^2 + 11x - 30$
23. $x^2 + 12x - 64$
24. $x^2 - 17x - 60$

7.7 Factoring $ax^2 + bx + c$

Guidance

We will learn how to factor $x^2 + bx + c$ using the X-Box Method. This is a guaranteed method for factoring quadratic equations—no guessing necessary! •Background knowledge needed: •Basic x-solve problems •General form of a quadratic equation •Dividing a polynomial by a monomial using the box method

Guided Practice

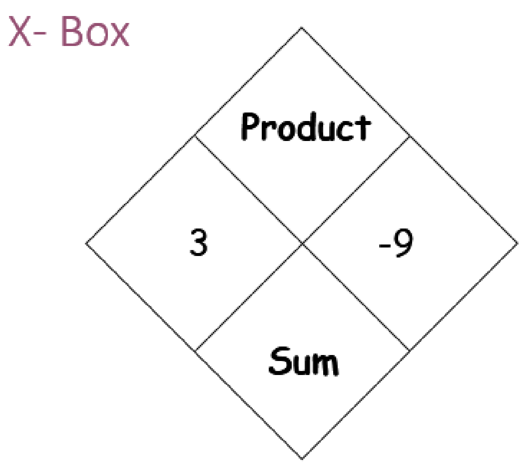


FIGURE 7.14

Practice

1. $2x^2 + 13x + 6$ 2. $3d^2 = 23d + 14$ 3. $3n^2 + 8n + 3$ 4. $4p^2 + 7p + 3$ 5. $6r^2 - 23r + 14$ 6. $8g^2 - 14g + 3$ 7. $3x^2 + 23x - 36$ 8. $4w^2 - 5w - 6$ 9. $4d^2 - 4d - 35$ 10. $12p^2 + 20p - 8$

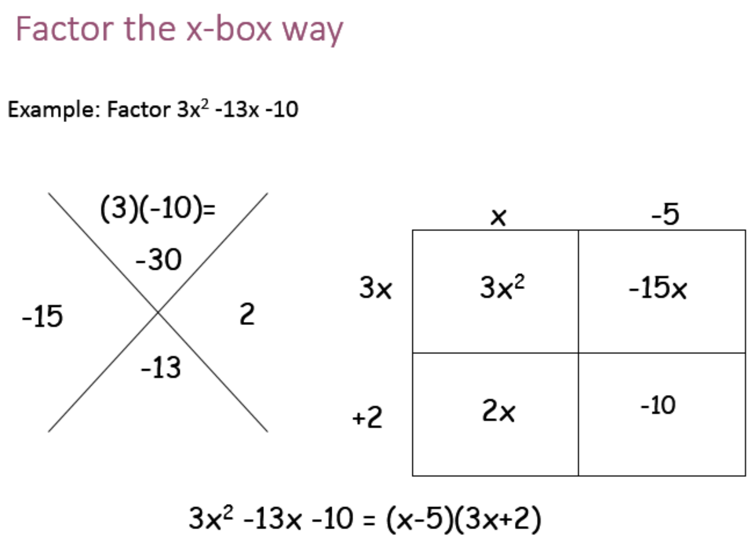


FIGURE 7.15

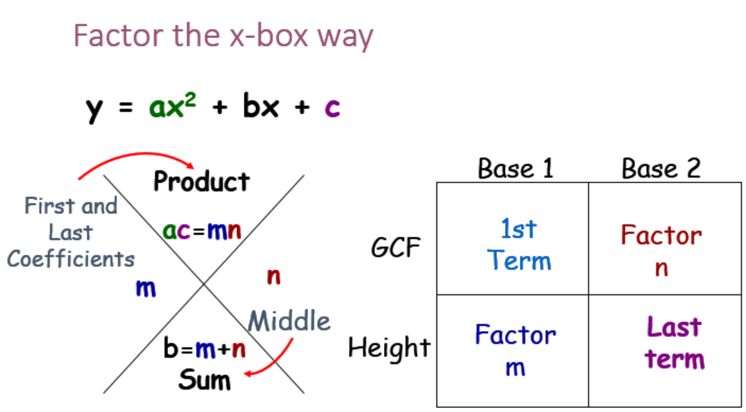


FIGURE 7.16

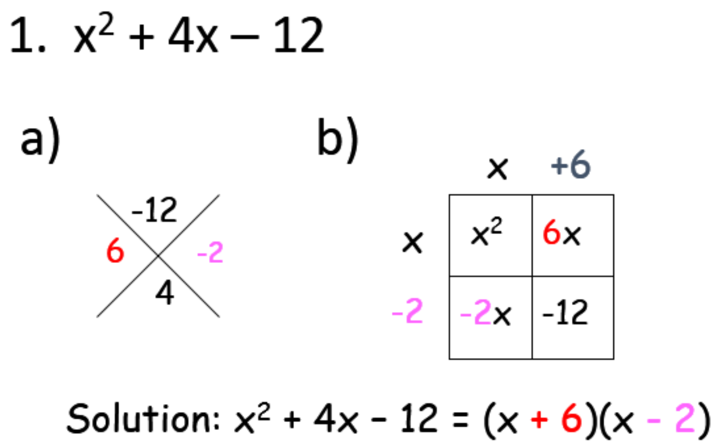


FIGURE 7.17

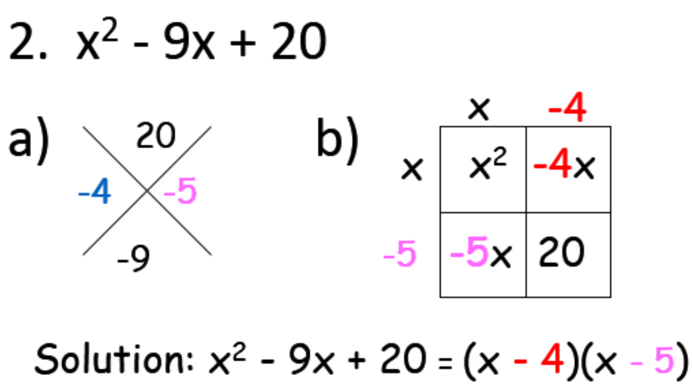


FIGURE 7.18

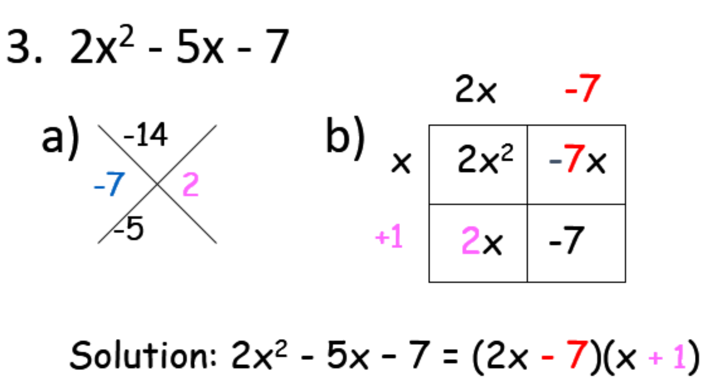


FIGURE 7.19

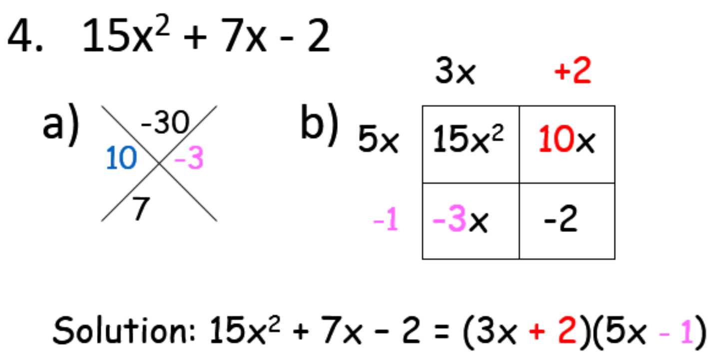

4. 15x² + 7x - 2
a)
-30
10
-3
7
b)
3x
+2
5x
15x²
10x
-1
-3x
-2
Solution: 15x² + 7x - 2 = (3x + 2)(5x - 1)

FIGURE 7.20

7.8 Factoring Special Cases

Guidance

When we learned how to multiply binomials, we talked about two special products: the Sum and Difference Formula and the Square of a Binomial Formula. In this Concept, we will learn how to recognize and factor these special products.

Factoring the Difference of Two Squares

We use the Sum and Difference Formula to factor a difference of two squares. A difference of two squares can be a quadratic polynomial in this form: $a^2 - b^2$. Both terms in the polynomial are perfect squares. In a case like this, the polynomial factors into the sum and difference of the square root of each term.

$$a^2 - b^2 = (a+b)(a-b)$$

In these problems, the key is figuring out what the a and b terms are. Let's do some examples of this type.

Example A

Factor the difference of squares.

(a) $x^2 - 9$

(b) $x^2y^2 - 1$

Solution:

(a) Rewrite $x^2 - 9$ as $x^2 - 3^2$. Now it is obvious that it is a difference of squares.

We substitute the values of a and b in the Sum and Difference Formula:

$$(x+3)(x-3)$$

The answer is $x^2 - 9 = (x+3)(x-3)$.

(b) Rewrite $x^2y^2 - 1$ as $(xy)^2 - 1^2$. This factors as $(xy+1)(xy-1)$.

Factoring Perfect Square Trinomials

A **perfect square trinomial** has the form:

$$a^2 + 2ab + b^2 \qquad \text{or} \qquad a^2 - 2ab + b^2$$

The **factored form** of a perfect square trinomial has the form:

$$(a+b)^2 \; if \; a^2 + 2(ab) + b^2$$

And

$$(a-b)^2 \; if \; a^2 - 2(ab) + b^2$$

In these problems, the key is figuring out what the a and b terms are. Let's do some examples of this type.

Example B

Factor $x^2+8x+16$.

Solution:

Check that the first term and the last term are perfect squares.

$$x^2+8x+16 \qquad \text{as} \qquad x^2+8x+4^2.$$

Check that the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite them.

$$x^2+8x+16 \qquad \text{as} \qquad x^2+2\cdot 4\cdot x+4^2$$

This means we can factor $x^2+8x+16$ as $(x+4)^2$.

Example C

Factor x^2-4x+4.

Solution:

Rewrite x^2-4x+4 as $x^2+2\cdot(-2)\cdot x+(-2)^2$.

We notice that this is a perfect square trinomial and we can factor it as $(x-2)^2$.

Solving Polynomial Equations Involving Special Products

We have learned how to factor quadratic polynomials that are helpful in solving polynomial equations like $ax^2+bx+c=0$. Remember that to solve polynomials in expanded form, we use the following steps:

Step 1: Rewrite the equation in standard form such that: Polynomial expression = 0.

Step 2: Factor the polynomial completely.

Step 3: Use the Zero Product Property to set **each factor equal to zero.**

Step 4: Solve each equation from step 3.

Step 5: Check your answers by substituting your solutions into the original equation.

Guided Practice

Solve the following polynomial equations.

$$x^2+7x+6=0$$

Solution: No need to rewrite because it is already in the correct form.

Factor: We write 6 as a product of the following numbers:

$$\begin{array}{lll} 6 = 6 \times 1 & \text{and} & 6+1=7 \\ x^2+7x+6=0 & \text{factors as} & (x+1)(x+6)=0 \end{array}$$

Set each factor equal to zero:

$$x+1=0 \qquad \text{or} \qquad x+6=0$$

Solve:

$$x=-1 \qquad \text{or} \qquad x=-6$$

Check: Substitute each solution back into the original equation.

$$(-1)^2+7(-1)+6=1+(-7)+6=0$$
$$(-6)^2+7(-6)+6=36+(-42)+6=0$$

Practice

Factor the following perfect square trinomials.

1. $x^2+8x+16$
2. $x^2-18x+81$
3. $-x^2+24x-144$
4. $x^2+14x+49$
5. $4x^2-4x+1$
6. $25x^2+60x+36$
7. $4x^2-12xy+9y^2$
8. x^4+22x^2+121

Factor the following differences of squares.

9. x^2-4
10. x^2-36
11. $-x^2+100$
12. x^2-400
13. $9x^2-4$
14. $25x^2-49$
15. $-36x^2+25$
16. $16x^2-81y^2$

Solve the following quadratic equations using factoring.

17. $x^2-11x+30=0$

18. $x^2+4x=21$
19. $x^2+49=14x$
20. $x^2-64=0$
21. $x^2-24x+144=0$
22. $4x^2-25=0$
23. $x^2+26x=-169$
24. $-x^2-16x-60=0$

7.9 Factoring by Grouping

Guidance

It may be possible to factor a polynomial containing four or more terms by factoring common monomials from groups of terms. This method is called **factoring by grouping**. The following example illustrates how this process works.

Example A

Factor $2x+2y+ax+ay$.

Solution: There isn't a common factor for all four terms in this example. However, there is a factor of 2 that is common to the first two terms and there is a factor of a that is common to the last two terms. Factor 2 from the first two terms and factor a from the last two terms.

$$2x+2y+ax+ay = 2(x+y)+a(x+y)$$

Now we notice that the binomial $(x+y)$ is common to both terms. We factor the common binomial and get:

$$(x+y)(2+a)$$

Our polynomial is now factored completely.

We know how to factor quadratic trinomials (ax^2+bx+c) where $a=1$ using methods we have previously learned. To factor a quadratic polynomial where $a \neq 1$, we follow the following steps.

1. We find the product ac.
2. We look for two numbers that multiply to give ac and add to give b.
3. We rewrite the middle term using the two numbers we just found.
4. We factor the expression by grouping.

Let's apply this method to the following examples.

Example B

Factor $3x^2+8x+4$ *by grouping*.

Solution: Follow the steps outlined above.

$$ac = 3 \cdot 4 = 12$$

The number 12 can be written as a product of two numbers in any of these ways:

$$\begin{array}{llll} 12 = 1 \times 12 & \textit{and} & 1+12=13 & \\ 12 = 2 \times 6 & \textit{and} & 2+6=8 & \text{This is the correct choice.} \end{array}$$

Rewrite the middle term as: $8x = 2x + 6x$, so the problem becomes the following.

$$3x^2 + 8x + 4 = 3x^2 + 2x + 6x + 4$$

Factor an x from the first two terms and 2 from the last two terms.

$$x(3x+2) + 2(3x+2)$$

Now factor the common binomial $(3x+2)$.

$$(3x+2)(x+2)$$

Our answer is $(3x+2)(x+2)$.

In this example, all the coefficients are positive. What happens if the b is negative?

Example C

Factor $6x^2 - 11x + 4$ *by grouping*.

Solution: $ac = 6 \cdot 4 = 24$

The number 24 can be written as a product of two numbers in any of these ways.

$$\begin{array}{llll} 24 = 1 \times 24 & \textit{and} & 1+24=25 & \\ 24 = (-1) \times (-24) & \textit{and} & (-1)+(-24)=-25 & \\ 24 = 2 \times 12 & \textit{and} & 2+12=14 & \\ 24 = (-2) \times (-12) & \textit{and} & (-2)+(-12)=-14 & \\ 24 = 3 \times 8 & \textit{and} & 3+8=11 & \\ 24 = (-3) \times (-8) & \textit{and} & (-3)+(-8)=-11 & \text{This is the correct choice.} \end{array}$$

Rewrite the middle term as $-11x = -3x - 8x$, so the problem becomes:

$$6x^2 - 11x + 4 = 6x^2 - 3x - 8x + 4$$

Factor by grouping. Factor a $3x$ from the first two terms and factor –4 from the last two terms.

$$3x(2x-1) - 4(2x-1)$$

Now factor the common binomial $(2x-1)$.

Our answer is $(2x-1)(3x-4)$.

Guided Practice

Factor $10x^2 - 43x + 28$.

Solution:

First, find $a \cdot c$: $10 \cdot 28 = 280$. Since $b = -43$, the factors of 280 need to add up to -43, so consider pairs of negative factors of 280. There will be a lot of pairs of factors, and you could list them all in order until you find the correct pair, as in Examples B and C. Or, use some number sense to help make the search shorter. Start with -1 as a factor:

$280 = -1 \cdot -280$ and $-1 + -280 = -281$

Since -281 is much more negative than -43, you need to have a pair of factors where one is not so negative. Try:

$280 = -7 \cdot -40$ and $-7 + -40 = -47$

This is close! Since it is still too negative, you need a factor that is less negative than -40, and one that is slightly more negative than -7. Try:

$280 = -8 \cdot -35$ and $-8 + -35 = -43$

This works! So, -8 and -35 are the factors needed. Rewrite the middle term as $-43x = -35x - 8x$ and factor by grouping:

$$10x^2 - 43x + 28 = 10x^2 - 35x + -8x + 28 = 5x(2x-7) - 4(2x-7) = (5x-4)(2x-7)$$

Practice

Factor by Grouping

1. $6x^2 - 9x + 10x - 15$
2. $5x^2 - 35x + x - 7$
3. $9x^2 - 9x - x + 1$
4. $4x^2 + 32x - 5x - 40$
5. $12x^3 - 14x^2 + 42x - 49$
6. $4x^2 + 25x - 21$
7. $24b^3 + 32b^2 - 3b - 4$
8. $2m^3 + 3m^2 + 4m + 6$
9. $6x^2 + 7x + 1$
10. $4x^2 + 8x - 5$
11. $5a^3 - 5a^2 + 7a - 7$
12. $3x^2 + 16x + 21$
13. $4xy + 32x + 20y + 160$
14. $10ab + 40a + 6b + 24$
15. $9mn + 12m + 3n + 4$
16. $4jk - 8j^2 + 5k - 10j$
17. $24ab + 64a - 21b - 56$

7.10 Simplifying Rational Expressions

Guidance

A simplified rational expression is one where the numerator and denominator have no common factors. In order to simplify an expression to **lowest terms**, we factor the numerator and denominator as much as we can and cancel common factors from the numerator and the denominator.

Simplify Rational Expressions

Example 1

Reduce each rational expression to simplest terms.

a) $\frac{4x-2}{2x^2+x-1}$

b) $\frac{x^2-2x+1}{8x-8}$

c) $\frac{x^2-4}{x^2-5x+6}$

Solution

Factor the numerator and denominator completely:	$\frac{2(2x-1)}{(2x-1)(x+1)}$
Cancel the common factor $(2x-1)$:	$\frac{2}{x+1}$

Factor the numerator and denominator completely:	$\frac{(x-1)(x-1)}{8(x-1)}$
Cancel the common factor $(x-1)$:	$\frac{x-1}{8}$

Factor the numerator and denominator completely:	$\frac{(x-2)(x+2)}{(x-2)(x-3)}$
Cancel the common factor$(x-2)$:	$\frac{x+2}{x-3}$

Find Excluded Values of Rational Expressions

Whenever there's a variable expression in the denominator of a fraction, we must remember that the denominator could be zero when the independent variable takes on certain values. Those values, corresponding to the vertical asymptotes of the function, are called **excluded** values. To find the excluded values, we simply set the denominator equal to zero and solve the resulting equation.

Example 2

Find the excluded values of the following expressions.

a) $\frac{x}{x+4}$

When we set the denominator equal to zero, we obtain: x + 4 = 0 —>x = - 4

So, - 4 is the excluded value.

b) $\frac{2x+1}{x^2-x-6}$

When we set the denominator equal to zero, we obtain: x^2+ x - 6 = 0

Solve by factoring (x - 3) (x + 2) = 0

x = 3 and x = -2

So, 3 and 2 are the excluded values.

Practice

Reduce each fraction to lowest terms

1. $\frac{4}{2x-8}$
2. $\frac{x^2+2x}{x}$
3. $\frac{9x+3}{12x+4}$
4. $\frac{6x^2+2x}{4x}$
5. $\frac{x-2}{x^2-4x+4}$
6. $\frac{x^2-9}{5x+15}$
7. $\frac{x^2+6x+8}{x^2+4x}$
8. $\frac{2x^2+10x}{x^2+10x+25}$
9. $\frac{x^2+6x+5}{x^2-x-2}$
10. $\frac{x^2-16}{x^2+2x-8}$
11. $\frac{3x^2+3x-18}{2x^2+5x-3}$
12. $\frac{x^3+x^2-20x}{6x^2+6x-120}$

Find the excluded values for each rational expression.

13. $\frac{2}{x}$
14. $\frac{4}{x+2}$
15. $\frac{2x-1}{(x-1)^2}$
16. $\frac{3x+1}{x^2-4}$
17. $\frac{x^2}{x^2+9}$
18. $\frac{2x^2+3x-1}{x^2-3x-28}$
19. $\frac{5x^3-4}{x^2+3x}$
20. $\frac{9}{x^3+11x^2+30x}$
21. $\frac{4x-1}{x^2+3x-5}$
22. $\frac{5x+11}{3x^2-2x-4}$
23. $\frac{x^2-1}{2x^2+x+3}$
24. $\frac{12}{x^2+6x+1}$

7.11 Dividing Polynomials: Monomials and Polynomials

Guidance

A **rational expression** is formed by taking the quotient of two polynomials.

Some examples of rational expressions are

$$\frac{2x}{x^2-1} \qquad \frac{4x^2-3x+4}{2x} \qquad \frac{9x^2+4x-5}{x^2+5x-1} \qquad \frac{2x^3}{2x+3}$$

Just as with rational numbers, the expression on the top is called the **numerator** and the expression on the bottom is called the **denominator**. In special cases we can simplify a rational expression by dividing the numerator by the denominator.

Divide a Polynomial by a Monomial

We'll start by dividing a polynomial by a monomial. To do this, we divide each term of the polynomial by the monomial. When the numerator has more than one term, the monomial on the bottom of the fraction serves as the **common** denominator to all the terms in the numerator.

Example A

Divide.

a) $\frac{8x^2-4x+16}{2}$

b) $\frac{3x^2+6x-1}{x}$

c) $\frac{-3x^2-18x+6}{9x}$

Solution

a) $\frac{8x^2-4x+16}{2} = \frac{8x^2}{2} - \frac{4x}{2} + \frac{16}{2} = 4x^2-2x+8$

b) $\frac{3x^3+6x-1}{x} = \frac{3x^3}{x} + \frac{6x}{x} - \frac{1}{x} = 3x^2+6-\frac{1}{x}$

c) $\frac{-3x^2-18x+6}{9x} = -\frac{3x^2}{9x} - \frac{18x}{9x} + \frac{6}{9x} = -\frac{x}{3} - 2 + \frac{2}{3x}$

A common error is to cancel the denominator with just one term in the numerator.

Consider the quotient $\frac{3x+4}{4}$.

Remember that the denominator of 4 is common to both the terms in the numerator. In other words we are dividing both of the terms in the numerator by the number 4.

The correct way to simplify is:

$$\frac{3x+4}{4} = \frac{3x}{4} + \frac{4}{4} = \frac{3x}{4} + 1$$

A common mistake is to cross out the number 4 from the numerator and the denominator, leaving just $3x$. This is incorrect, because the entire numerator needs to be divided by 4.

Example B

Divide $\frac{5x^3-10x^2+x-25}{-5x^2}$.

Solution

$$\frac{5x^3-10x^2+x-25}{-5x^2}=\frac{5x^3}{-5x^2}-\frac{10x^2}{-5x^2}+\frac{x}{-5x^2}-\frac{25}{-5x^2}$$

The negative sign in the denominator changes all the signs of the fractions:

$$-\frac{5x^3}{5x^2}+\frac{10x^2}{5x^2}-\frac{x}{5x^2}+\frac{25}{5x^2}=-x+2-\frac{1}{5x}+\frac{5}{x^2}$$

Divide a Polynomial by a Binomial

We divide polynomials using a method that's a lot like long division with numbers. We'll explain the method by doing an example.

Example C

Divide $\frac{x^2+4x+5}{x+3}$.

Solution

When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor**.

To start the division we rewrite the problem in the following form:

$$x+3\overline{)x^2+4x+5}$$

We start by dividing the first term in the dividend by the first term in the divisor: $\frac{x^2}{x}=x$.

We place the answer on the line above the x term:

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \end{array}$$

Next, we multiply the x term in the answer by the divisor, $x+3$, and place the result under the dividend, matching like terms. x times $(x+3)$ is x^2+3x, so we put that under the divisor:

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \\ x^2+3x \end{array}$$

Now we subtract x^2+3x from x^2+4x+5. It is useful to change the signs of the terms of x^2+3x to $-x^2-3x$ and add like terms vertically:

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x} \\ x \end{array}$$

Now, we bring down the 5, the next term in the dividend.

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x } \\ x+5 \end{array}$$

And now we go through that procedure once more. First we divide the first term of $x+5$ by the first term of the divisor. x divided by x is 1, so we place this answer on the line above the constant term of the dividend:

$$\begin{array}{r} x+1 \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x } \\ x+5 \end{array}$$

Multiply 1 by the divisor, $x+3$, and write the answer below $x+5$, matching like terms.

$$\begin{array}{r} x+1 \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x } \\ x+5 \\ x+3 \end{array}$$

Subtract $x+3$ from $x+5$ by changing the signs of $x+3$ to $-x-3$ and adding like terms:

$$\begin{array}{r} x+1 \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x } \\ x+5 \\ \underline{-x-3} \\ 2 \end{array}$$

Since there are no more terms from the dividend to bring down, we are done. The quotient is $x+1$ and the remainder is 2.

Remember that for a division with a remainder the answer is quotient $+\frac{\text{remainder}}{\text{divisor}}$. So the answer to this division problem is $\frac{x^2+4x+5}{x+3}=x+1+\frac{2}{x+3}$.

Check

To check the answer to a long division problem we use the fact that

$$(\text{divisor} \times \text{quotient}) + \text{remainder} = \text{dividend}$$

For the problem above, here's how we apply that fact to check our solution:

$$\begin{aligned}(x+3)(x+1)+2 &= x^2+4x+3+2 \\ &= x^2+4x+5\end{aligned}$$

The answer checks out.

Vocabulary

- A **rational expression** is formed by taking the quotient of two polynomials.

Guided Practice

Divide $\frac{x^2+8x+17}{x+4}$.

Solution

When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor**.

To start the division we rewrite the problem in the following form:

$$x+4\overline{)x^2+8x+17}$$

We start by dividing the first term in the dividend by the first term in the divisor: $\frac{x^2}{x} = x$.

We place the answer on the line above the x term:

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \end{array}$$

Next, we multiply the x term in the answer by the divisor, $x+4$, and place the result under the dividend, matching like terms. x times $(x+4)$ is x^2+4x, so we put that under the divisor:

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \\ x^2+4x \end{array}$$

Now we subtract x^2+4x from $x^2+8x+17$. It is useful to change the signs of the terms of x^2+4x to $-x^2-4x$ and add like terms vertically:

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ 4x \end{array}$$

Now, we bring down the 17, the next term in the dividend.

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ 4x+17 \end{array}$$

And now we go through that procedure once more. First we divide the first term of $4x+17$ by the first term of the divisor. $4x$ divided by x is 4, so we place this answer on the line above the constant term of the dividend:

$$\begin{array}{r} x\ +\ 4 \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ x+17 \end{array}$$

Multiply 4 by the divisor, $x+4$, and write the answer below $4x+16$, matching like terms.

$$\begin{array}{r} x\ +\ 4 \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ 4x+17 \\ 4x+16 \end{array}$$

Subtract $4x+16$ from $4x+17$ by changing the signs of $4x+16$ to $-4x-16$ and adding like terms:

$$\begin{array}{r} x\ +\ 4 \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ x+17 \\ \underline{-4x-16} \\ 1 \end{array}$$

Since there are no more terms from the dividend to bring down, we are done. The quotient is $x+4$ and the remainder is 1.

Remember that for a division with a remainder the answer is quotient $+\frac{\text{remainder}}{\text{divisor}}$. So the answer to this division problem is $\frac{x^2+8x+17}{x+4} = x+4+\frac{1}{x+4}$.

Check

To check the answer to a long division problem we use the fact that

$$(\text{divisor} \times \text{quotient}) + \text{remainder} = \text{dividend}$$

For the problem above, here's how we apply that fact to check our solution:

$$\begin{aligned}(x+4)(x+4)+1 &= x^2+8x+16+1\\ &= x^2+8x+17\end{aligned}$$

The answer checks out.

Practice

Divide the following polynomials:

1. $\frac{2x+4}{2}$
2. $\frac{x-4}{x}$
3. $\frac{5x-35}{5x}$
4. $\frac{x^2+2x-5}{x}$
5. $\frac{4x^2+12x-36}{-4x}$
6. $\frac{2x^2+10x+7}{2x^2}$
7. $\frac{x^3-x}{-2x^2}$
8. $\frac{5x^4-9}{3x}$
9. $\frac{x^3-12x^2+3x-4}{12x^2}$
10. $\frac{3-6x+x^3}{-9x^3}$
11. $\frac{x^2+3x+6}{x+1}$
12. $\frac{x^2-9x+6}{x-1}$
13. $\frac{x^2+5x+4}{x+4}$
14. $\frac{x^2-10x+25}{x-5}$
15. $\frac{x^2-20x+12}{x-3}$
16. $\frac{3x^2-x+5}{x-2}$
17. $\frac{9x^2+2x-8}{x+4}$
18. $\frac{3x^2-4}{3x+1}$
19. $\frac{5x^2+2x-9}{2x-1}$
20. $\frac{x^2-6x-12}{5x^4}$

Vocabulary:

Adding and Subtracting Polynomials: Binomial, Degree of Monomial, Degree of Polynomial, Monomial, Polynomial, Standard Form of a Polynomial, Trinomial

Multiplying and Factoring Polynomials: Coefficient, Exponent, Greatest Common Factor (GFC)

Multiplying Binomials: Distributive Property, Factor, Like Terms

Multiplying Special Cases: Area of a Square, Polynomial, Square of a Number

Factoring x^2 + bx + c: Monomial, Binomial, Trinomial

Factoring ax^2 + bx + c: Degree of Monomial, Degree of Polynomial, Factored Form

Factoring Special Cases: Standard Form of a Polynomial, Difference of Two Squares, Perfect Square Trinomial

Factoring by Grouping: Difference of Squares, Perfect Square Trinomial, Factor by Grouping

Simplifying Rational Expressions: Denominator, Numerator, Excluded Value, Rational Expression

Dividing Polynomials: Binomial, Coefficient, Polynomial

CHAPTER 8 Quadratic Equations and Functions

Chapter Outline

Quadratic Graphs and Properties: A.6a, A.7a

Quadratic Functions: A.7a, A.8b

Quadratic Function Transformations: A.7a, A.7c

Vertex Form of Quadratic Function: A.6a, A.6b, A.7a, A.12b

Solving Quadratic Equations: A.7a, A.8a

Factoring and Solving Quadratic Equations: A.8a

Writing Quadratic Functions: A.6c, A.7b

Completing the Square of Quadratic Functions: A.8a

The Quadratic Formula and the Discriminant: A.8a

8.1 Quadratic Graphs and Properties

Guidance

A quadratic function has a form

y = ax2 + bx + c

where a $\neq$ 0. The graph of a quadratic function is U-shaped and is called parabola. The lowest or highest point on the graph of a quadratic function is called the vertex.

A parabola that open upward has minimum or lower point.

A parabola that opens downward has maximum or highest point The graphs of y = x2 and y = -x2 are symmetric about the y-axis, called the axis of symmetry. The parabola opens up if a>0 and opens down if a<0. The parabola is wider than the graph of y = x2 if |a| <1 and narrower than the graph of y = x2 if |a| >1. The x-coordinate of the vertex is -b/2a. The axis of symmetry is the vertical line x = -b/2a. **Example:** What is the vertex of the graph below?

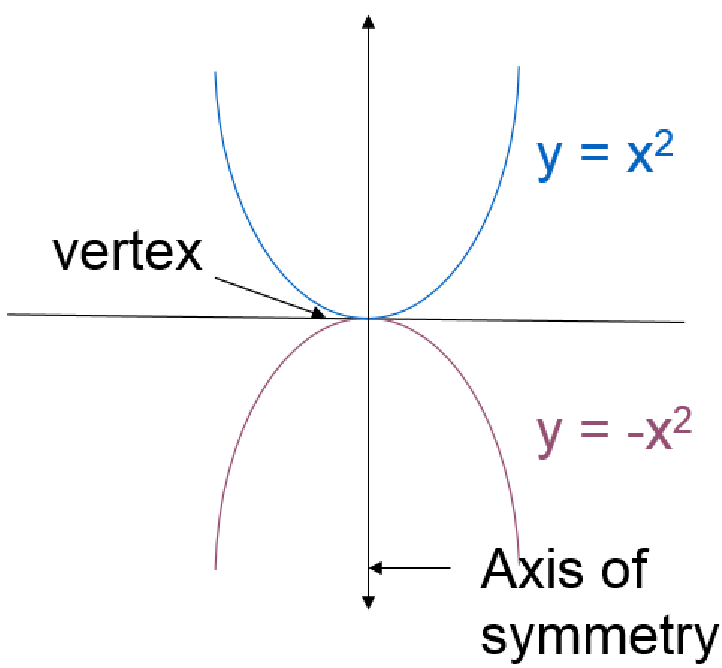


FIGURE 8.1

Indicate whether it is a minimum or maximum. The graph opens downward, so you are looking at the highest point. The vertex of the graph is (-3, 2) and it is a maximum. **Example:**

Any function in the form $y = ax^2 + bx + c$ where a cannot equal to 0 is called a quadratic function. The graph of a quadratic function is a parabola.

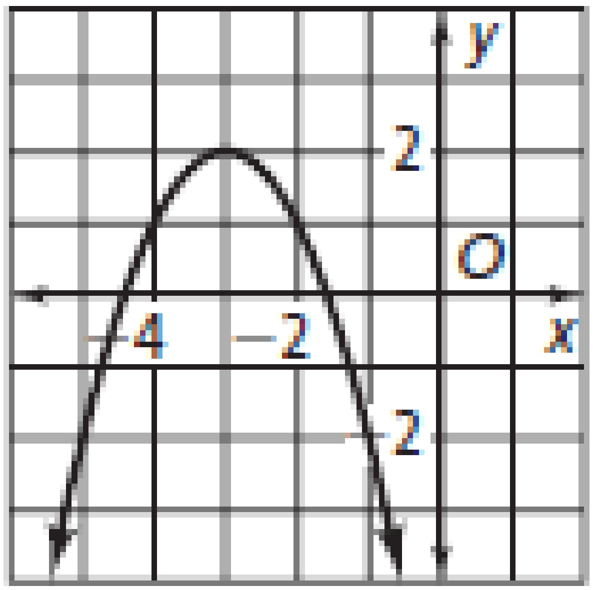


FIGURE 8.2

What is the graph of y =12x^2- 4 ?

This is a quadratic function where a = $\frac{1}{2}$, b = 0, and c = - 4
The graph will be a parabola. Use a table to fi nd some points on the graph. Then use what you know about parabolas to complete the graph.

Practice

Identify the vertex of each graph. Indicate whether it is a maximum or minimum.

1.

2.

3.

Graph each quadratic function.

4. $y = -x + 5$

5. $y = x^2 - 4$

6. $y = -x - 1$

Order quadratic functions from widest to narrowest

7.

8.

9.

10.

x	$y = \frac{1}{2}x^2 - 4$	(x, y)
-4	$y = \frac{1}{2}(-4)^2 - 4 = 4$	$(-4, 4)$
-2	$y = \frac{1}{2}(-2)^2 - 4 = -2$	$(-2, -2)$
0	$y = \frac{1}{2}(0)^2 - 4 = -4$	$(0, -4)$
2	$y = \frac{1}{2}(2)^2 - 4 = -2$	$(2, -2)$
4	$y = \frac{1}{2}(4)^2 - 4 = 4$	$(4, 4)$

FIGURE 8.3

Qua

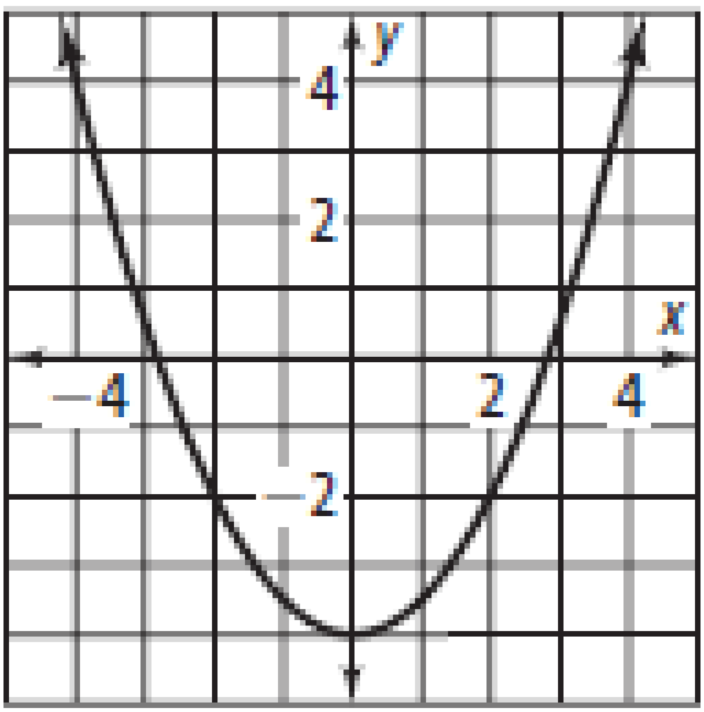


FIGURE 8.4

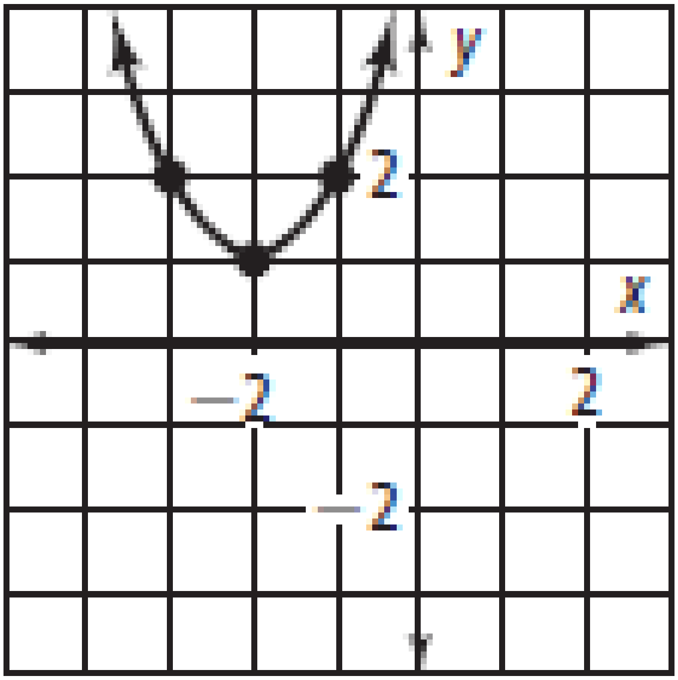


FIGURE 8.5

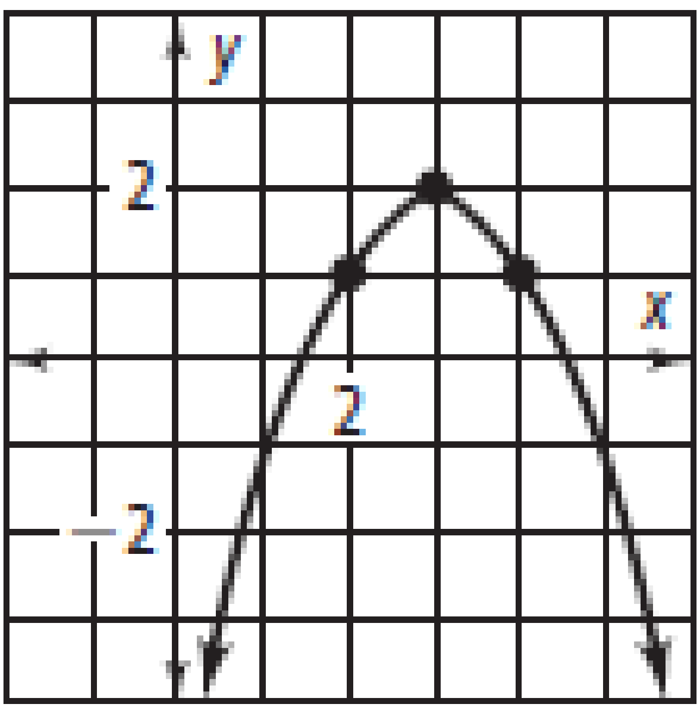


FIGURE 8.6

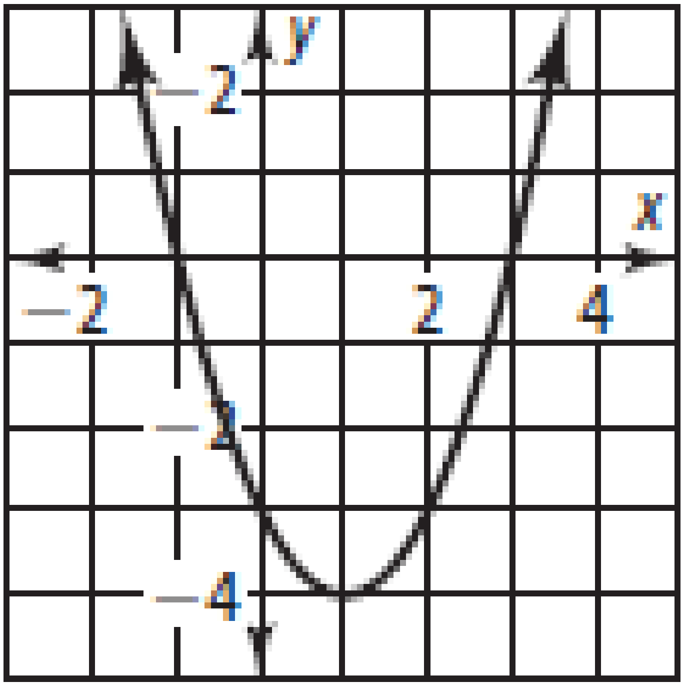


FIGURE 8.7

$$y = -3x^2, y = -5x^2, y = -1x^2$$

FIGURE 8.8

$$y = 4x^2, y = -2x^2, y = -6x^2$$

FIGURE 8.9

$$y = x^2, y = \frac{1}{3}x^2, y = 2x^2$$

FIGURE 8.10

$y = \frac{1}{6}x^2, y = \frac{1}{4}x^2, y = \frac{1}{2}x^2$

FIGURE 8.11

8.2 Quadratic Functions: Graphing ax2 + bx + c

Here you will explore different methods of graphing quadratic functions. You have likely been exposed to one or all of these methods in the past, so pay particular attention to any that remain confusing or difficult. Being skilled at graphing quadratic functions can save a lot of time on more complex problems, particularly when only a good approximate answer is needed.

Guidance

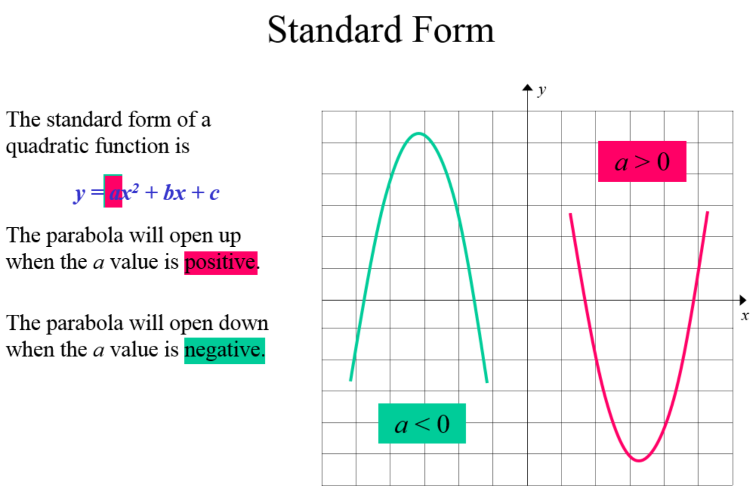


FIGURE 8.12

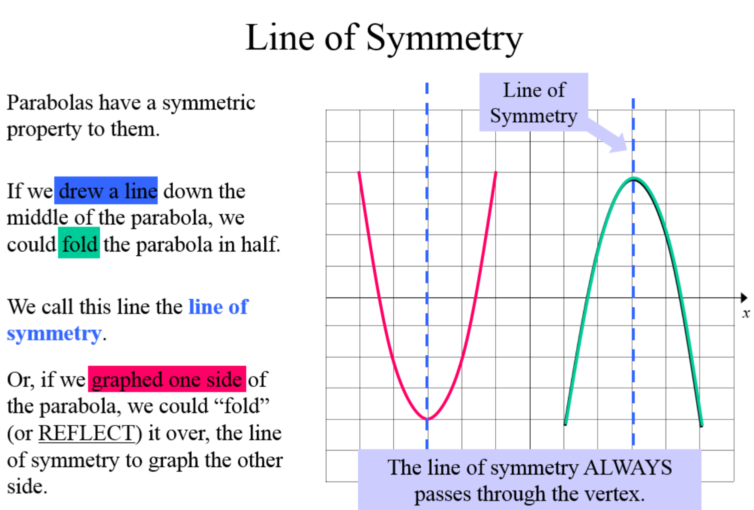


FIGURE 8.13

Finding the Line of Symmetry

When a quadratic function is in standard form

$y = ax^2 + bx + c$,

The equation of the line of symmetry is

$$x = \frac{-b}{2a}$$

This is best read as …

the opposite of b divided by the quantity of 2 times a.

For example…

Find the line of symmetry of

$y = 3x^2 - 18x + 7$

Using the formula…

$$x = \frac{18}{2(3)} = \frac{18}{6} = 3$$

Thus, the line of symmetry is $x = 3$.

FIGURE 8.14

Finding the Vertex

We know the line of symmetry always goes through the vertex.

Thus, the line of symmetry gives us the x – coordinate of the vertex.

To find the y – coordinate of the vertex, we need to plug the x – value into the original equation.

$y = -2x^2 + 8x - 3$

STEP 1: Find the line of symmetry

$$x = \frac{-b}{2a} = \frac{-8}{2(-2)} = \frac{-8}{-4} = 2$$

STEP 2: Plug the x – value into the original equation to find the y value.

$y = -2(2)^2 + 8(2) - 3$

$y = -2(4) + 8(2) - 3$

$y = -8 + 16 - 3$

$y = 5$

Therefore, the vertex is (2 , 5)

FIGURE 8.15

The standard form of a quadratic function is given by

$y = ax^2 + bx + c$

STEP 1: Find the line of symmetry

STEP 2: Find the vertex

STEP 3: Find two other points and reflect them across the line of symmetry. Then connect the five points with a smooth curve.

There are 3 steps to graphing a parabola in standard form.

USE the equation

using $x = \frac{-b}{2a}$ close to

the mmetry.

FIGURE 8.16

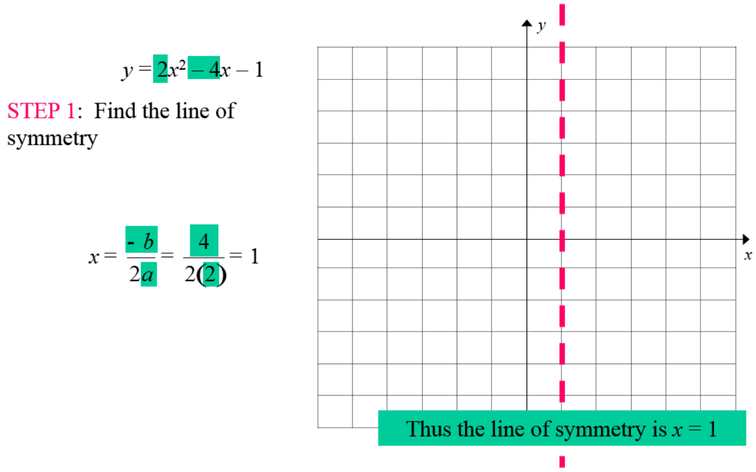


FIGURE 8.17

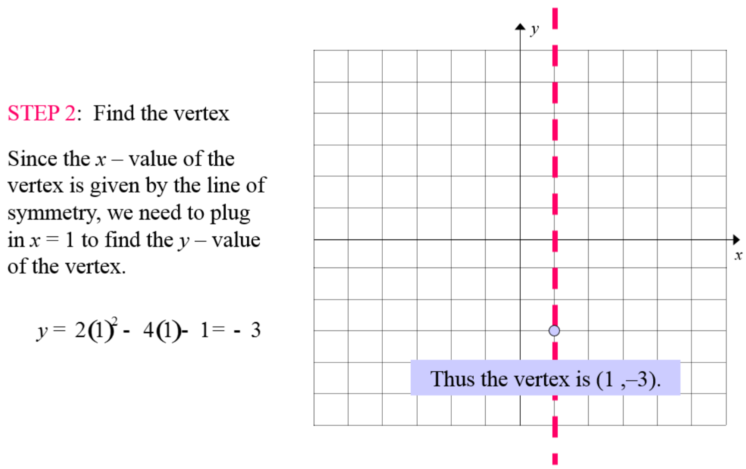


FIGURE 8.18

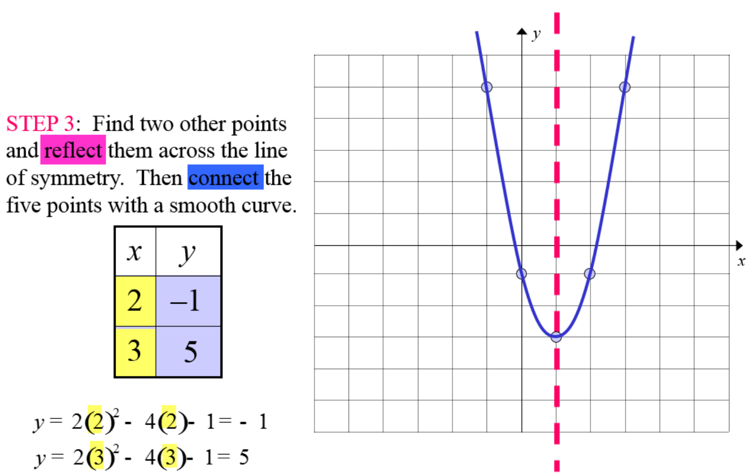


FIGURE 8.19

Practice

1. What is the U-Shaped graph of a quadratic function called?
2. Which direction does a parabola open if the leading coefficient (a) is positive?
3. For $y^2 = x$ If the coefficient of y is positive, which way does the parabola open?
4. What is the name of the lowest point of a parabola that opens up and the highest point of a parabola that opens down.
5. What is the name of the line passing through the vertex that divides the parabola into two symmetric parts?
6. Sketch the graph of $y = x^2 + 3$
7. Sketch the graph of $y = -x^2 + 4x - 4$
8. Sketch the graph of $y = 2x^2 + 8x$
9. Consider the following quadratic function: $y = -x^2 - 2x + 1$ a) Which direction does it open? b) What is the vertex? c) Is it stretched in any way?
10. Consider the quadratic functions: $y = 2x^2 y = 4x^2 y = 6x^2$ Which quadratic function would you expect to have the narrowest parabola? Explain your answer.

Sketch the graph of each function:

11. $y = -x^2$
12. $y = 3x^2 + 6x + 1$
13. $y = \frac{1}{2}x^2 + 2x + 4$
14. $y = (x-3)^2 + 4$
15. $y = -x^2 - 8x - 17$

8.3 Quadratic Functions: Using a Vertical Motion Model

Guidance

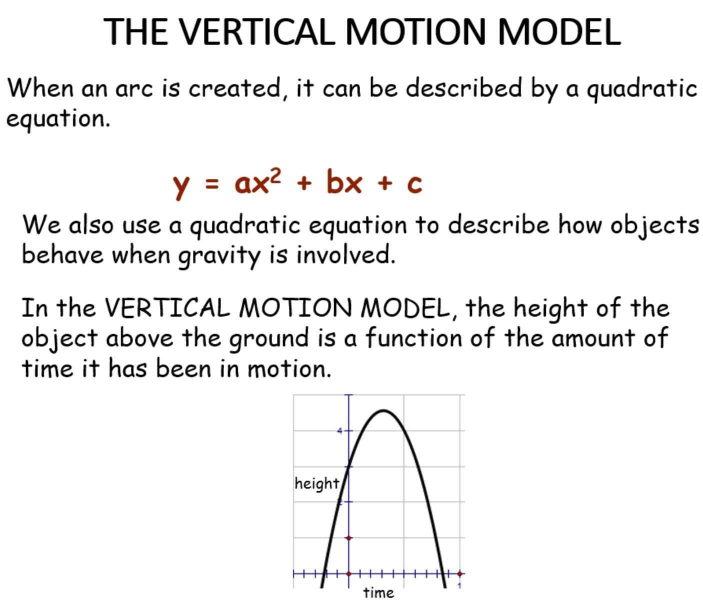


FIGURE 8.20

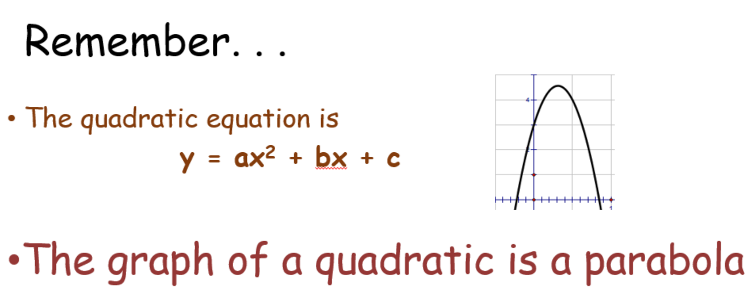


FIGURE 8.21

Practice

1. A startled armadillo jumps straight into the air with the initial velocity of 14 feet per second. After how many seconds does it land on the ground?

2. An athlete throws a discus from an initial height of 6 feet and with an initial vertical velocity of 46 feet per second. Write an equation that gives the height in feet of the discus as a function of function (in seconds) since it left the athlete's hand. After how many seconds does the discus hit the ground?

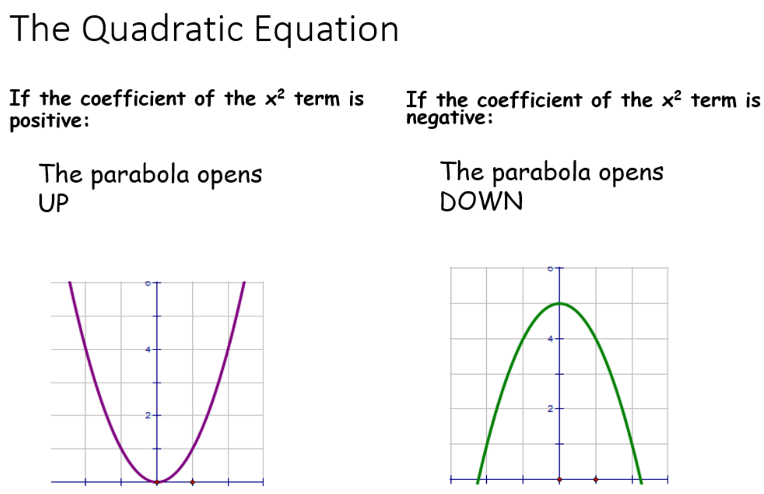


FIGURE 8.22

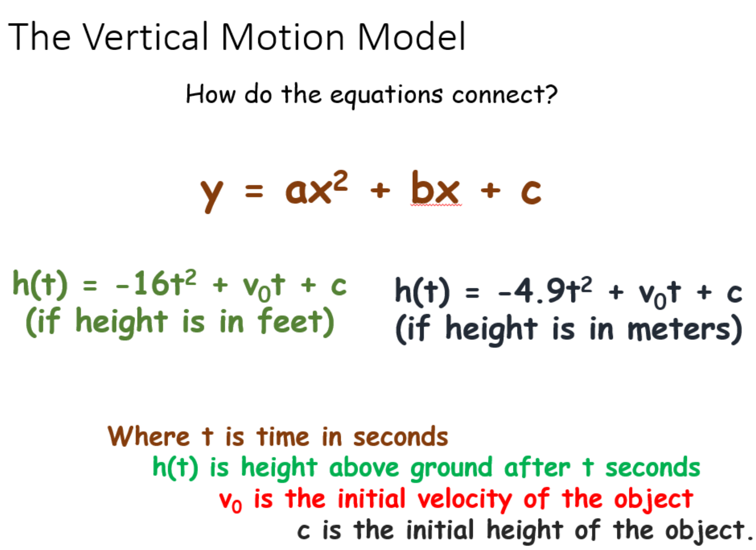


FIGURE 8.23

3. A grasshopper jumps straight up from the ground with an initial vertical velocity of 8 feet per second. After how many seconds will the grasshopper be 1 foot off of the ground?

4. A squirrel is 27 feet up in a tree and tosses a nut out of the tree with an initial vertical velocity of 6 feet per second. the squirrel climbs down the tree in 2 seconds. Does it reach the ground before the nut?

5. A kangaroo can jump with an initial vertical velocity of 18 feet per second.

a) When will the kangaroo land on the ground?

b) How high is the kangaroo after 0.2 seconds?

c) How high is the kangaroo after 0.8 seconds?

d) When does the kangaroo reach its maximum height? Approximately how high does it jump?

6. A baseball player releases a baseball at a height of 7 feet with an initial velocity of 54 feet per second. How long will it take the ball to reach the ground?

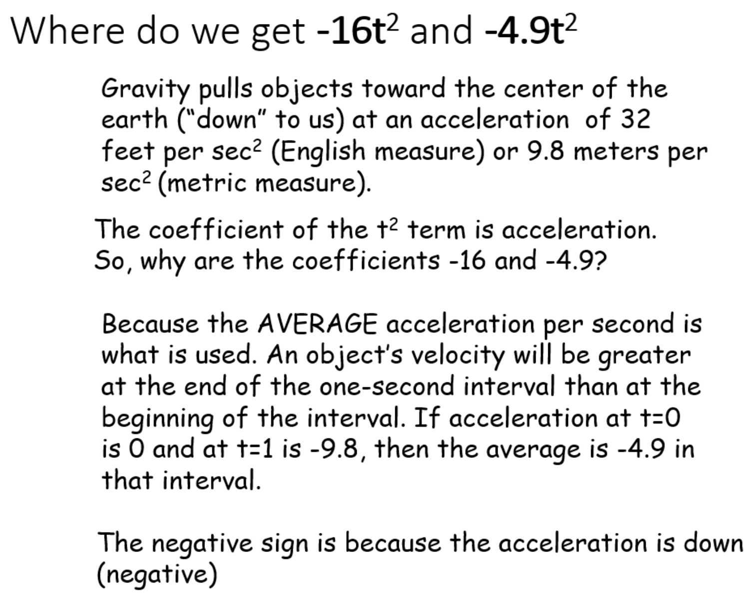


FIGURE 8.24

7. A miniature rocket is launched off a roof of 20 feet above the ground with an initial velocity of 22 feet per second. How much time will elapse before the rocket reaches the ground?

8. A cliff diver jumps from a ledge of 88 feet above the ocean with an initial upward velocity of 12 feet per second. How long will it take until the diver enters the water?

9. You throw a football form a height of 6 feet into the air with an initial vertical velocity of 12 feet per second. the football is caught at a height of 2 feet. After how many seconds is the football caught?

10. You hit a badminton birdie upward with a racket from a height of 2 feet with an initial velocity of 4 feet per second.

a) How high is the birdie at 0.1 seconds?

b) How high is the birdie at 0.25 seconds?

c) How long will it take the birdie to reach the ground?

11. A tarantula jumps into the air with an initial vertical velocity of 10 feet per second.

a) When will the tarantula land on the ground?

b) The tarantula reaches its maximum height after 0.3125 seconds. How high can it jump?

c) How does the time in part (b) relate to the time in part (a)?

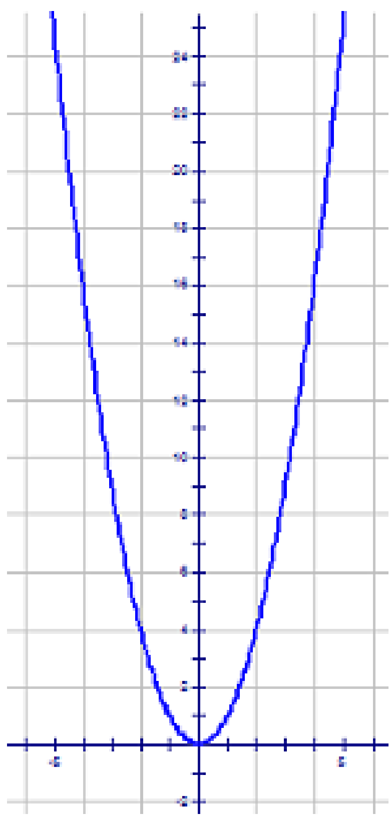
d) What relationship do you think there is between these coordinates in parts (a) and (b)?

Qu

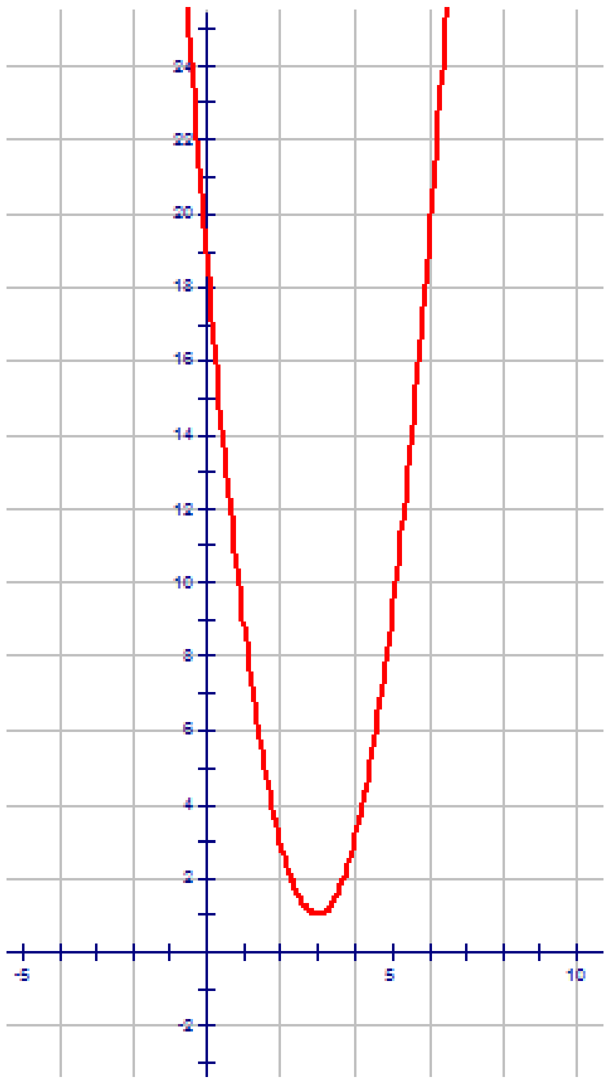
8.4 Quadratic Function Transformations

Guidance

This is the graph of $y = x^2$:



This is the graph of $y = x^2$ that has undergone transformations:



The vertex of the red parabola is (3, 1). The sides of the parabola open upward but they appear steeper and longer than those on the blue parabola.

As shown above, you can apply changes to the graph of $y = x^2$ to create a new parabola (still a 'U' shape) that no longer has its vertex at (0, 0) and no longer has y-values of 1, 4 and 9. These changes are known as transformations.

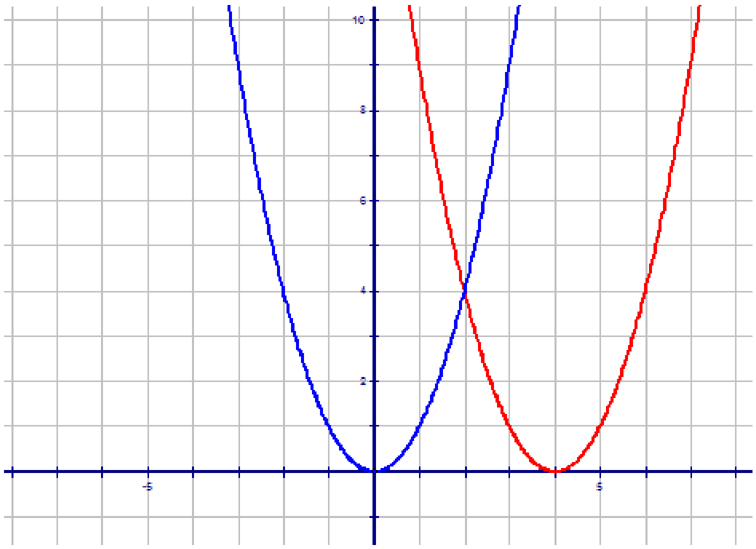
The vertex of (0, 0) will change if the parabola undergoes either a horizontal translation and/or a vertical translation. These transformations cause the parabola to slide left or right and up or down.

If the parabola undergoes a vertical stretch, the y-values of 1, 4 and 9 can increase if the stretch is a whole number. This will produce a parabola that will appear to be narrower than the original base graph. If the vertical stretch is a fraction less than 1, the values of 1, 4 and 9 will decrease. This will produce a parabola that will appear to be wider than the original base graph.

Finally, a parabola can undergo a vertical reflection that will cause it to open downwards as opposed to upwards. For example, $y = -x^2$ is a vertical reflection of $y = x^2$.

Example A

Look at the two parabolas below. Describe the transformation from the blue parabola to the red parabola. What is the coordinate of the vertex of the red parabola?

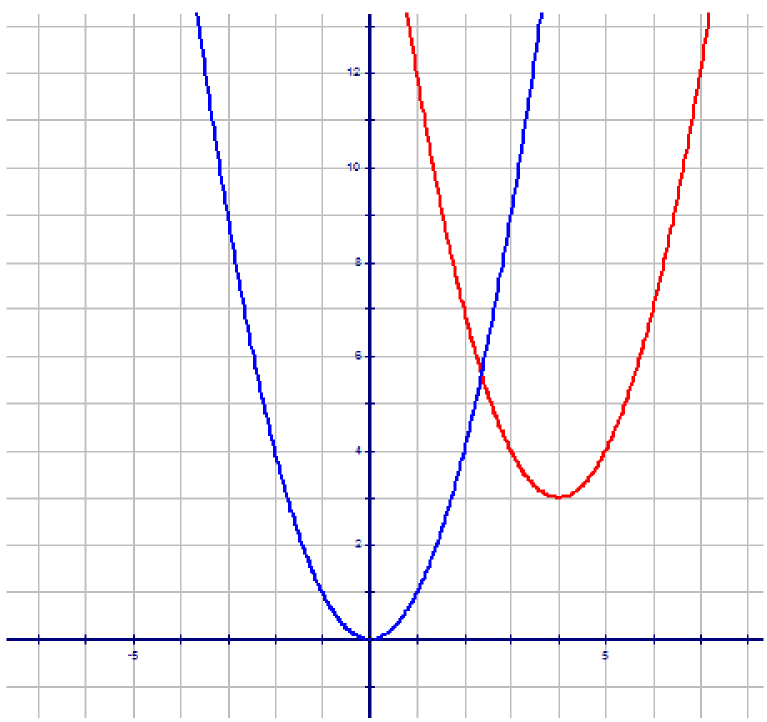


Solution:

The blue parabola is the graph of $y = x^2$. Its vertex is (0, 0). The red graph is the graph of $y = x^2$ that has been moved four units to the right. When the graph undergoes a slide of four units to the right, it has undergone a horizontal translation of +4. The vertex of the red graph is (4, 0). A horizontal translation changes the $x-$coordinate of the vertex of the graph of $y = x^2$.

Example B

Look at the two parabolas below. Describe the transformation from the blue parabola to the red parabola. What is the coordinate of the vertex of the red parabola?

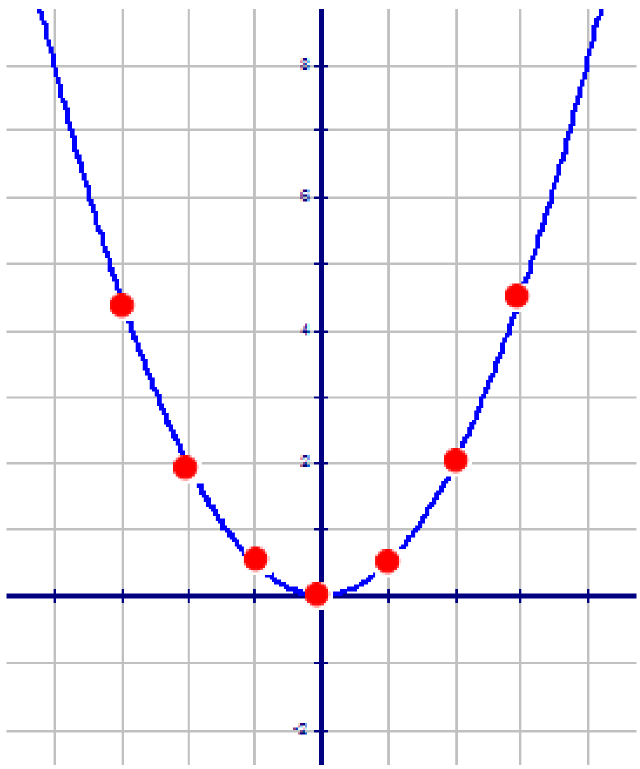


Solution:

The blue parabola is the graph of $y = x^2$. Its vertex is (0, 0). The red graph is the graph of $y = x^2$ that has been moved four units to the right and three units upward. When the graph undergoes a slide of four units to the right, it has undergone a horizontal translation of +4. When the graph undergoes a slide of three units upward, it has undergone a vertical translation of +3.The vertex of the red graph is (4, 3). A horizontal translation changes the $x-$coordinate of the vertex of the graph of $y = x^2$ while a vertical translation changes the $y-$coordinate of the vertex.

Example C

Look at the parabola below. How is this parabola different from $y = x^2$? What do you think the equation of this parabola is?

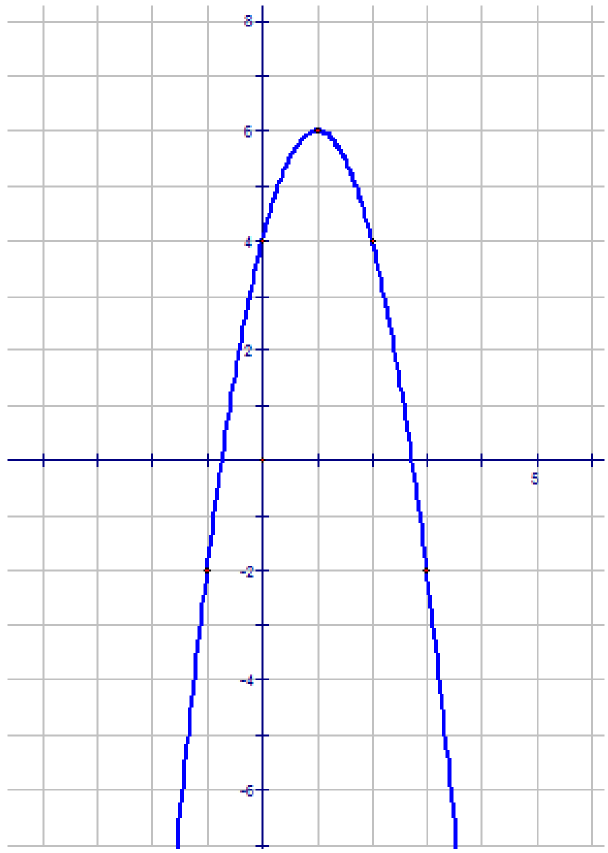


Solution:

This is the graph of $y = \frac{1}{2}x^2$. The points are plotted from the vertex as right and left one and up one-half, right and left 2 and up two, right and left three and up four and one-half. The original y-values of 1, 4 and 9 have been divided by two or multiplied by one-half. When the y-values are multiplied, the y-values either increase or decrease. This transformation is known as a vertical stretch.

Guided Practice

1. Identify the transformations of the base graph $y = x^2$.

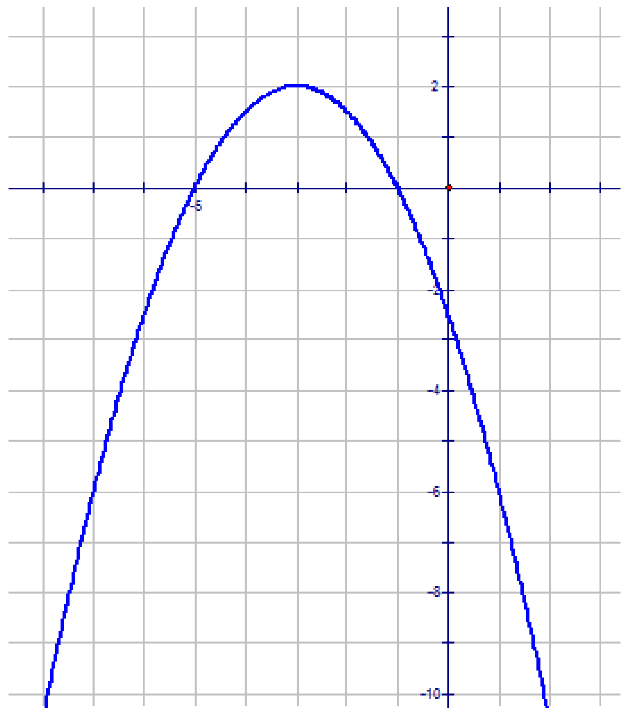


2. Draw the image graph of $y = x^2$ that has undergone a vertical reflection, a vertical stretch by a factor of $\frac{1}{2}$, a vertical translation up 2 units, and a horizontal translation left 3 units.

Answers:

1. The vertex is (1, 6). The base graph has undergone a horizontal translation of +1 and a vertical translation of +6. The parabola opens downward, so the graph is a vertical reflection. The points have been plotted such that the y-values of 1 and 4 are now 2 and 8. It is not unusual for a parabola to be plotted with five points rather than seven. The reason for this is the vertical stretch often multiplies the y-values such that they are difficult to graph on a Cartesian grid. If all the points are to be plotted, a different scale must be used for the y-axis.

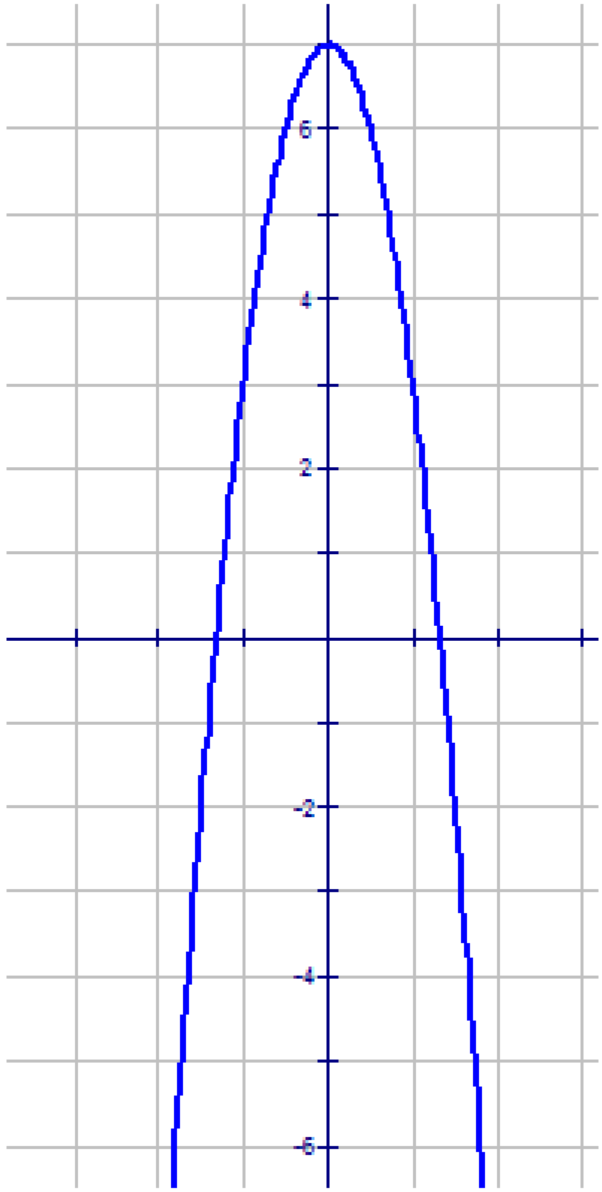
2. The vertex given by the horizontal and vertical translations and is (–3, 2). The y-values of 1, 4 and 9 must be multiplied by $\frac{1}{2}$ to create values of $\frac{1}{2}, 2$ and $4\frac{1}{2}$. The graph is a vertical reflection which means the graph opens downward and the y-values become negative.



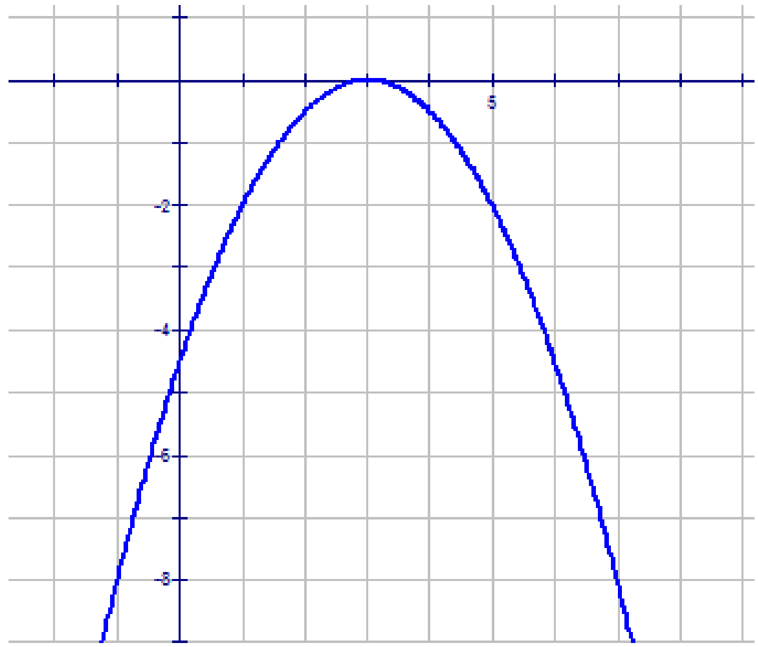
Practice

For each of the following graphs, list the transformations of $y = x^2$.

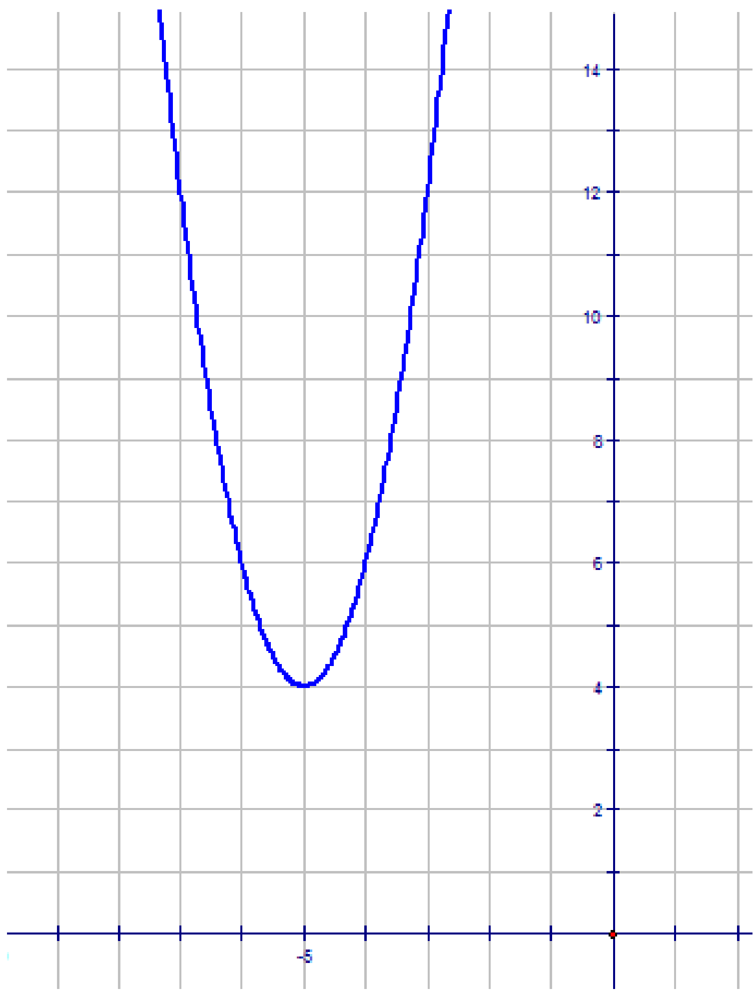
1.



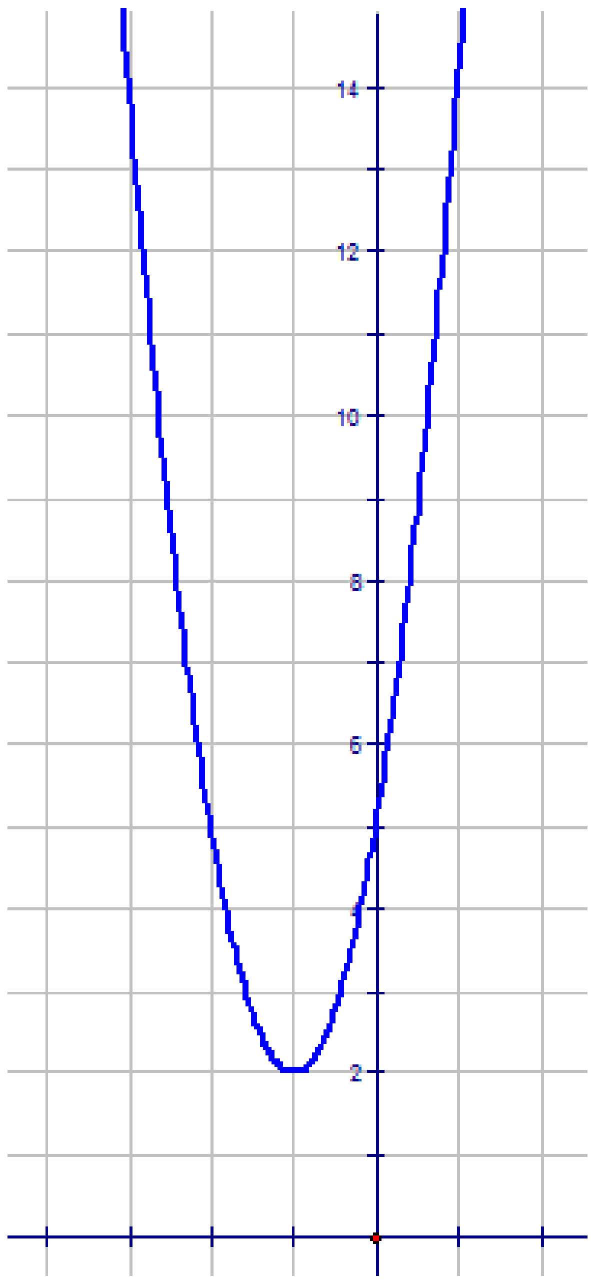
2.



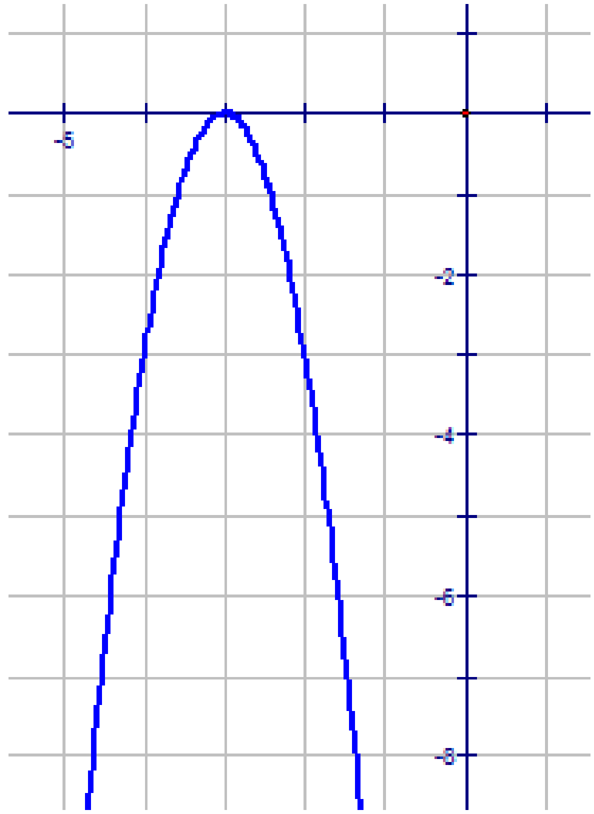
3.



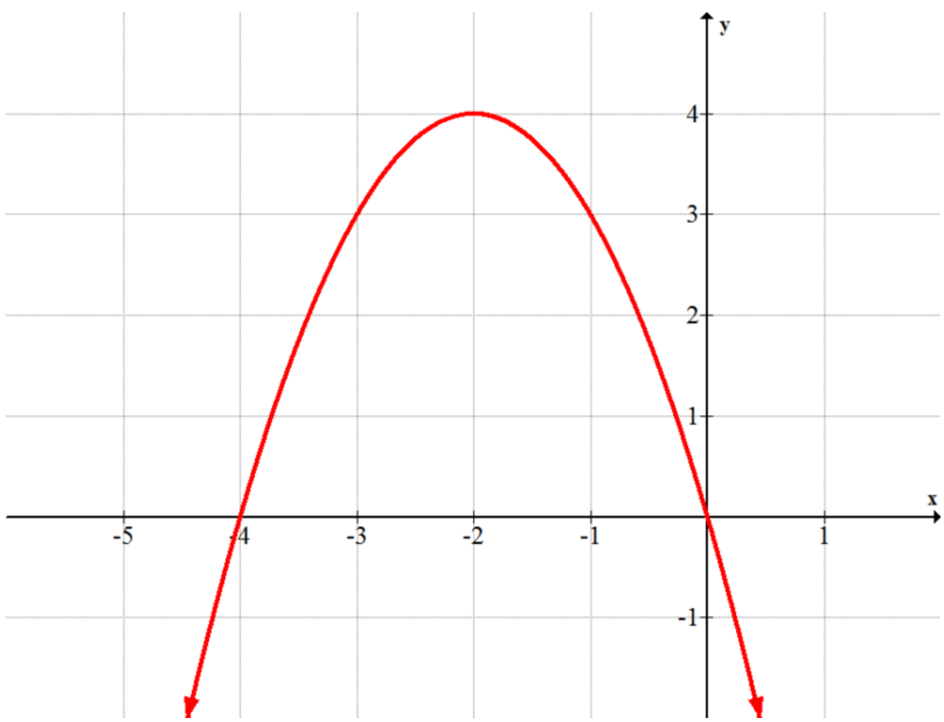
4.



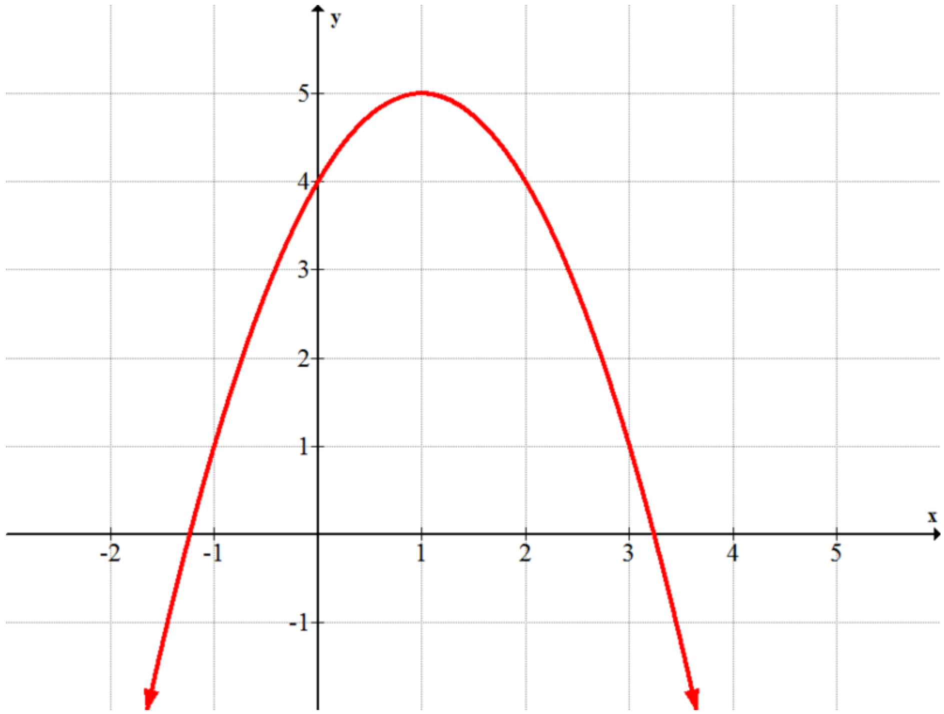
5.



6.



7.



y
5
4
3
2
1
-1
x
-2
-1
1
2
3
4
5

8.5 Vertex Form of Quadratic Function

Here you'll learn how to find the vertex, the x-intercepts, and the y-intercept of parabolas that are written in vertex form. You'll also learn how to graph such parabolas. Finally, you'll rewrite quadratic functions in vertex form.

Guidance

Probably one of the best applications of the method of completing the square is using it to rewrite a quadratic function in vertex form. The vertex form of a quadratic function is

$$y-k=a(x-h)^2$$

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at the point (h,k).

It is also simple to find the $x-$intercepts from the vertex form: just set $y=0$ and take the square root of both sides of the resulting equation.

To find the $y-$intercept, set $x=0$ and simplify.

Example A

Find the vertex, the $x-$ intercepts and the $y-$ intercept of the following parabolas:

a) $y-2=(x-1)^2$

b) $y+8=2(x-3)^2$

Solution

a) $y-2=(x-1)^2$

Vertex: $\underline{(1,2)}$

To find the $x-$intercepts,

$$\begin{array}{rlcl} \text{Set } y=0: & -2=(x-1)^2 & & \\ \text{Take the square root of both sides}: & \sqrt{-2}=x-1 & \text{and} & -\sqrt{-2}=x-1 \end{array}$$

The solutions are not real so there are **no $x-$ intercepts.**

To find the $y-$intercept,

$$\begin{array}{rl} \text{Set } x=0: & y-2=(-1)^2 \\ \text{Simplify}: & y-2=1 \Rightarrow \underline{y=3} \end{array}$$

b) $y+8=2(x-3)^2$

$$\begin{array}{ll} \text{Rewrite :} & y-(-8)=2(x-3)^2 \\ \text{Vertex :} & \underline{(3,-8)} \end{array}$$

To find the $x-$intercepts,

$$\begin{array}{rlll} \text{Set } y=0: & 8=2(x-3)^2 & & \\ \text{Divide both sides by 2 :} & 4=(x-3)^2 & & \\ \text{Take the square root of both sides :} & 2=x-3 & \text{and} & -2=x-3 \\ \text{Simplify :} & \underline{\underline{x=5}} & \text{and} & \underline{\underline{x=1}} \end{array}$$

To find the $y-$intercept,

$$\begin{array}{ll} \text{Set } x=0: & y+8=2(-3)^2 \\ \text{Simplify :} & y+8=18 \Rightarrow \underline{\underline{y=10}} \end{array}$$

To graph a parabola, we only need to know the following information:

- the vertex
- the $x-$intercepts
- the $y-$intercept
- whether the parabola turns up or down (remember that it turns up if $a > 0$ and down if $a < 0$)

Example B

Graph the parabola given by the function $y+1=(x+3)^2$.

Solution

$$\begin{array}{lll} \text{Rewrite :} & y-(-1)=(x-(-3))^2 & \\ \text{Vertex :} & \underline{(-3,-1)} & \text{vertex} : (-3,-1) \end{array}$$

To find the $x-$intercepts,

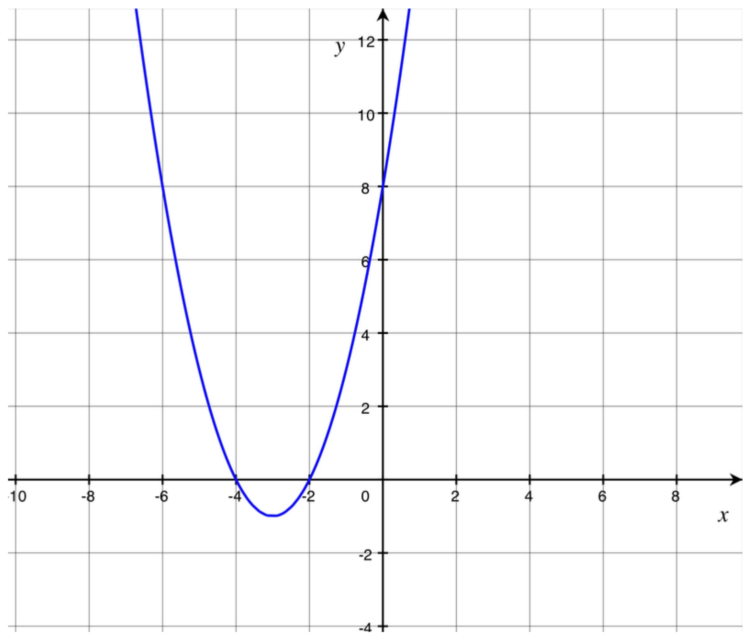
$$\begin{array}{llll} \text{Set } y=0: & 1=(x+3)^2 & & \\ \text{Take the square root of both sides :} & 1=x+3 & \text{and} & -1=x+3 \\ \text{Simplify :} & \underline{\underline{x=-2}} & \text{and} & \underline{\underline{x=-4}} \\ & x-\text{intercepts} : (-2,0) \text{ and } (-4,0) & & \end{array}$$

To find the $y-$intercept,

$$\begin{array}{lll} \text{Set } x=0: & y+1=(3)^2 & \\ \text{Simplify:} & \underline{\underline{y=8}} & y-\text{intercept} : (0,8) \end{array}$$

And since $a > 0$, the parabola **turns up.**

Graph all the points and connect them with a smooth curve:



Example C

Graph the parabola given by the function $y = -\frac{1}{2}(x-2)^2$.

Solution:

$$\begin{array}{lll} \text{Rewrite} & y-(0) = -\frac{1}{2}(x-2)^2 & \\ \text{Vertex:} & \underline{(2,0)} & \text{vertex:}(2,0) \end{array}$$

To find the $x-$intercepts,

$$\begin{array}{rll} \text{Set } y=0: & 0 = -\frac{1}{2}(x-2)^2 & \\ \text{Multiply both sides by } -2: & 0 = (x-2)^2 & \\ \text{Take the square root of both sides}: & 0 = x-2 & \\ \text{Simplify}: & \underline{\underline{x=2}} & x-\text{intercept:}(2,0) \end{array}$$

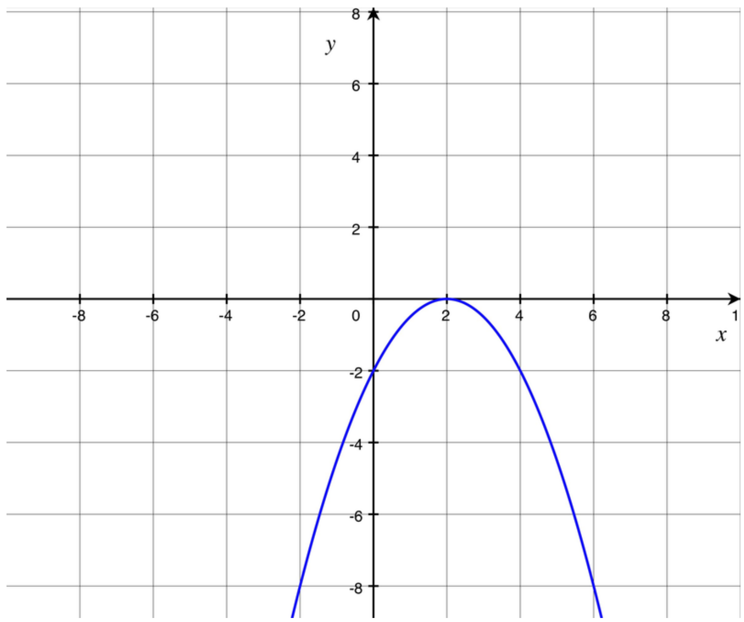
Note: there is only one $x-$intercept, indicating that the vertex is located at this point, (2, 0).

To find the $y-$intercept,

$$\begin{array}{lll} \text{Set } x=0: & y = -\frac{1}{2}(-2)^2 & \\ \text{Simplify:} & y = -\frac{1}{2}(4) \Rightarrow \underline{\underline{y=-2}} & y-\text{intercept:}(0,-2) \end{array}$$

Since $a < 0$, the parabola **turns down.**

Graph all the points and connect them with a smooth curve:



Vocabulary

- The vertex form of a quadratic function is

$$y-k=a(x-h)^2$$

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at the point (h,k).

- To find the $x-$intercepts from the vertex form: just set $y=0$ and take the square root of both sides of the resulting equation.

- To find the $y-$intercept, set $x=0$ and simplify.

Guided Practice

Graph the parabola given by the function $y=4(x+2)^2-1$.

Solution:

Rewrite	$y-(-1)=4(x+2)^2$	
Simplify	$y+1=4(x+2)^2$	
Vertex:	$\underline{(-2,-1)}$	vertex:$(-2,-1)$

To find the $x-$intercepts,

$$\begin{aligned}
\text{Set. } y=0: &\quad 0=4(x+2)^2-1 \\
\text{Subtract 1 from each side}: &\quad 1=4(x+2)^2 \\
\text{Divide both sides by 4}: &\quad \frac{1}{4}=(x+2)^2 \\
\text{Take the square root of both sides}: &\quad \frac{1}{2}=\pm(x+2) \\
\text{Separate}: &\quad \frac{1}{2}=-(x+2) \qquad\qquad \frac{1}{2}=x+2) \\
\text{Simplify}: &\quad \underline{\underline{x=-2.5}} \qquad\qquad \underline{\underline{x=-1.5}}
\end{aligned}$$

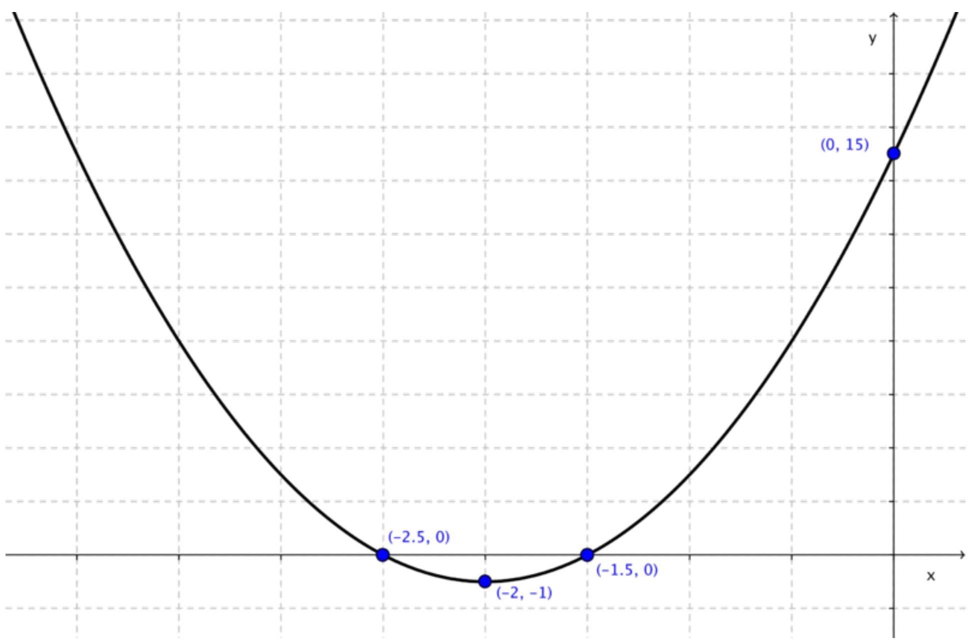
The $x-$intercepts are $(-2.5,0)$ and $(-1.5,0)$.

To find the $y-$intercept,

$$\begin{aligned}
\text{Set } x=0: &\quad y=4(0+2)^2-1 \\
\text{Simplify:} &\quad y=15 \Rightarrow \underline{\underline{y=15}} \qquad\qquad y-\text{intercept:}(0,15)
\end{aligned}$$

Since $a<0$, the parabola **turns up.**

Graph all the points and connect them with a smooth curve:



Practice

Rewrite each quadratic function in vertex form.

1. $y=x^2-6x$
2. $y+1=-2x^2-x$
3. $y=9x^2+3x-10$
4. $y=-32x^2+60x+10$

For each parabola, find the vertex; the $x-$ and $y-$intercepts; and if it turns up or down. Then graph the parabola.

5. $y-4=x^2+8x$
6. $y=-4x^2+20x-24$
7. $y=3x^2+15x$
8. $y+6=-x^2+x$
9. $x^2-10x+25=9$
10. $x^2+18x+81=1$
11. $4x^2-12x+9=16$
12. $x^2+14x+49=3$
13. $4x^2-20x+25=9$
14. $x^2+8x+16=25$

8.6 Solving Quadratic Equations: Solve by Graphing

Here you'll learn how to identify the number of solutions to a quadratic equation and how to find those solutions. You'll also learn how to use a graphing calculator to find the roots and the vertex of polynomials. Finally, you'll solve real-world problems by graphing quadratic functions.

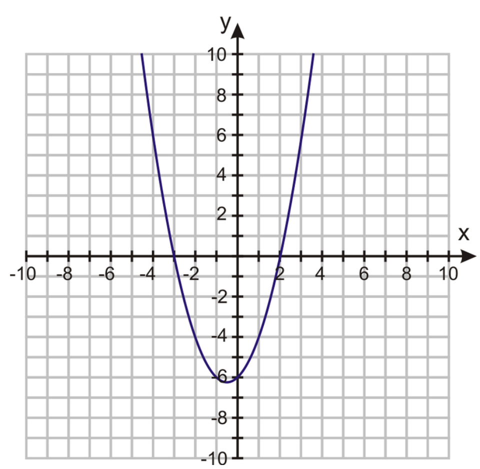
Guidance

Solving a quadratic equation means finding the $x-$values that will make the quadratic function equal zero; in other words, it means finding the points where the graph of the function crosses the $x-$axis. The solutions to a quadratic equation are also called the **roots** or **zeros** of the function, and in this section we'll learn how to find them by graphing the function.

Identify the Number of Solutions of a Quadratic Equation

Three different situations can occur when graphing a quadratic function:

Case 1: The parabola crosses the $x-$axis at two points. An example of this is $y = x^2 + x - 6$:

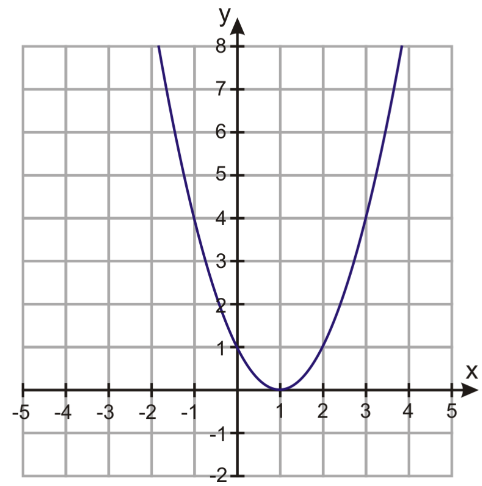


Looking at the graph, we see that the parabola crosses the $x-$axis at $x = -3$ and $x = 2$.

We can also find the solutions to the equation $x^2 + x - 6 = 0$ by setting $y = 0$. We solve the equation by factoring: $(x+3)(x-2) = 0$, so $x = -3$ or $x = 2$.

When the graph of a quadratic function crosses the $x-$axis at two points, we get **two distinct solutions** to the quadratic equation.

Case 2: The parabola touches the $x-$axis at one point. An example of this is $y = x^2 - 2x + 1$:

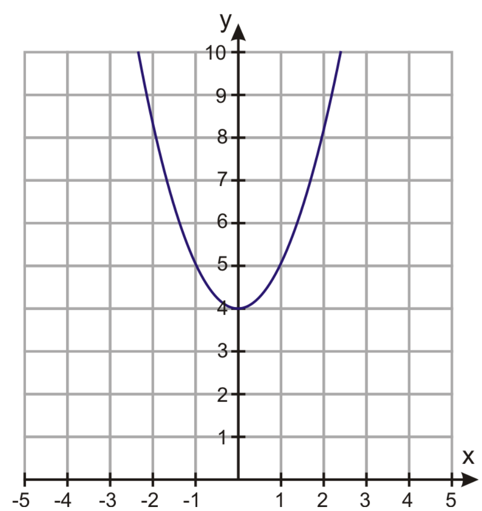


We can see that the graph touches the $x-$axis at $x = 1$.

We can also solve this equation by factoring. If we set $y = 0$ and factor, we obtain $(x-1)^2 = 0$, so $x = 1$. Since the quadratic function is a perfect square, we get only one solution for the equation—it's just the same solution repeated twice over.

When the graph of a quadratic function touches the $x-$axis at one point, the quadratic equation has one solution and the solution is called a **double root.**

Case 3: The parabola does not cross or touch the $x-$axis. An example of this is $y = x^2 + 4$:



If we set $y = 0$ we get $x^2 + 4 = 0$. This quadratic polynomial does not factor.

When the graph of a quadratic function does not cross or touch the $x-$axis, the quadratic equation has **no real solutions.**

Solve Quadratic Equations by Graphing

So far we've found the solutions to quadratic equations using factoring. However, in real life very few functions factor easily. As you just saw, graphing a function gives a lot of information about the solutions. We can find exact or approximate solutions to a quadratic equation by graphing the function associated with it.

Example A

Find the solutions to the following quadratic equations by graphing.

a) $-x^2+3=y$

b) $-x^2+x-3=y$

c) $y=-x^2+4x-4$

Solution

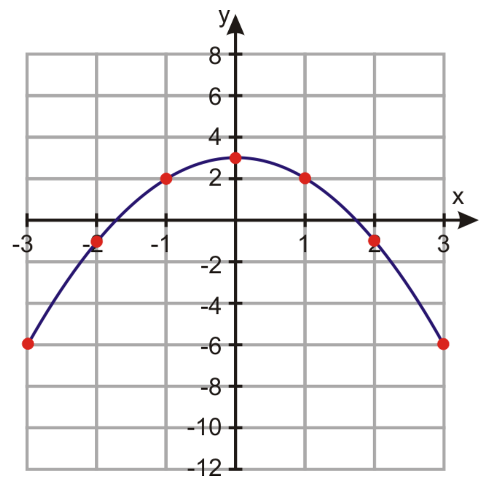
Since we can't factor any of these equations, we won't be able to graph them using intercept form (if we could, we wouldn't need to use the graphs to find the intercepts!) We'll just have to make a table of arbitrary values to graph each one.

a)

TABLE 8.1:

x	$y=-x^2+3$
-3	$y=-(-3)^2+3=-6$
-2	$y=-(-2)^2+3=-1$
-1	$y=-(-1)^2+3=2$
0	$y=-(0)^2+3=3$
1	$y=-(1)^2+3=2$
2	$y=-(2)^2+3=-1$
3	$y=-(3)^2+3=-6$

We plot the points and get the following graph:



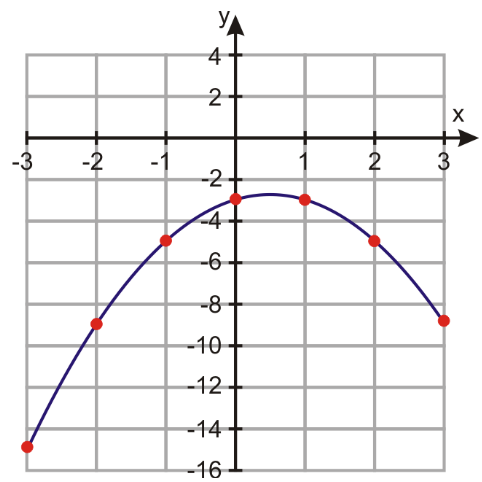
From the graph we can read that the $x-$intercepts are approximately $x=1.7$ **and** $x=-1.7$. These are the solutions to the equation.

b)

TABLE 8.2:

x	$y=-x^2+x-3$
-3	$y=-(-3)^2+(-3)-3=-15$
-2	$y=-(-2)^2+(-2)-3=-9$
-1	$y=-(-1)^2+(-1)-3=-5$
0	$y=-(0)^2+(0)-3=-3$
1	$y=-(1)^2+(1)-3=-3$
2	$y=-(2)^2+(2)-3=-5$
3	$y=-(3)^2+(3)-3=-9$

We plot the points and get the following graph:



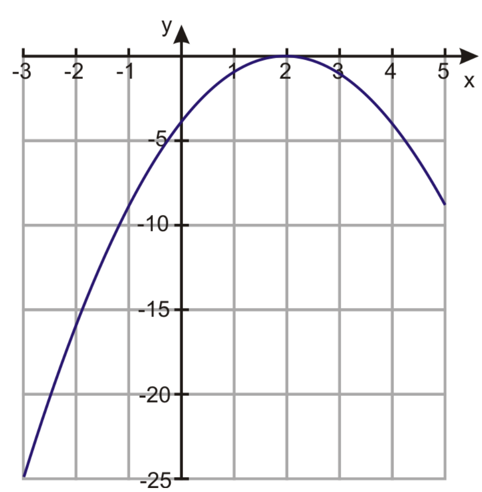
The graph curves up toward the $x-$axis and then back down without ever reaching it. This means that the graph never intercepts the $x-$axis, and so the corresponding equation has **no real solutions.**

c)

TABLE 8.3:

x	$y = -x^2 + 4x - 4$
-3	$y = -(-3)^2 + 4(-3) - 4 = -25$
-2	$y = -(-2)^2 + 4(-2) - 4 = -16$
-1	$y = -(-1)^2 + 4(-1) - 4 = -9$
0	$y = -(0)^2 + 4(0) - 4 = -4$
1	$y = -(1)^2 + 4(1) - 4 = -1$
2	$y = -(2)^2 + 4(2) - 4 = 0$
3	$y = -(3)^2 + 4(3) - 4 = -1$
4	$y = -(4)^2 + 4(4) - 4 = -4$
5	$y = -(5)^2 + 4(5) - 4 = -9$

Here is the graph of this function:



The graph just touches the $x-$axis at $x = 2$, so the function has a **double root** there. $x = 2$ is the only solution to the equation.

'Analyze Quadratic Functions Using a Graphing Calculator

A graphing calculator is very useful for graphing quadratic functions. Once the function is graphed, we can use the calculator to find important information such as the roots or the vertex of the function.

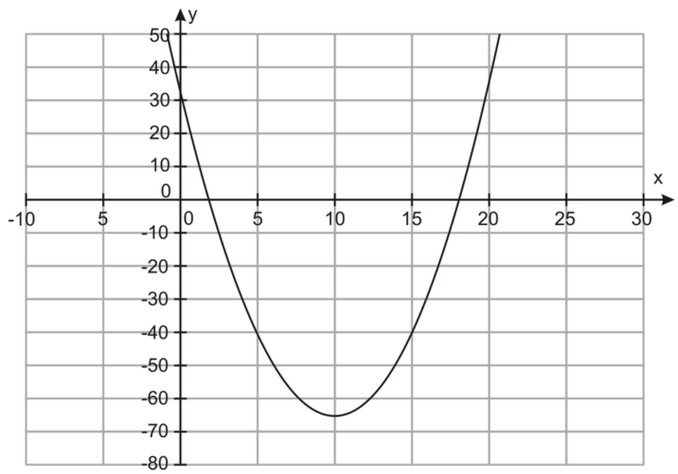
Example B

Use a graphing calculator to analyze the graph of $y = x^2 - 20x + 35$.

Solution

1. **Graph** the function.

Press the **[Y=]** button and enter "$x^2 - 20x + 35$" next to $[Y_1 =]$. Press the **[GRAPH]** button. This is the plot you should see:



If this is not what you see, press the **[WINDOW]** button to change the window size. For the graph shown here, the $x-$values should range from -10 to 30 and the $y-$values from -80 to 50.

2. Find the **roots.**

There are at least three ways to find the roots:

Use **[TRACE]** to scroll over the $x-$intercepts. The approximate value of the roots will be shown on the screen. You can improve your estimate by zooming in.

OR

Use **[TABLE]** and scroll through the values until you find values of y equal to zero. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** (i.e. 'calc' button) and use option 'zero'.

Move the cursor to the left of one of the roots and press **[ENTER]**.

Move the cursor to the right of the same root and press **[ENTER]**.

Move the cursor close to the root and press **[ENTER]**.

The screen will show the value of the root. Repeat the procedure for the other root.

Whichever technique you use, you should get about $x = 1.9$ **and** $x = 18$ for the two roots.

3. Find the **vertex.**

There are three ways to find the vertex:

Use **[TRACE]** to scroll over the highest or lowest point on the graph. The approximate value of the roots will be shown on the screen.

OR

Use **[TABLE]** and scroll through the values until you find values the lowest or highest value of y. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** and use the option 'maximum' if the vertex is a maximum or 'minimum' if the vertex is a minimum.

Move the cursor to the left of the vertex and press **[ENTER]**.

Move the cursor to the right of the vertex and press **[ENTER]**.

Move the cursor close to the vertex and press **[ENTER]**.

The screen will show the $x-$ and $y-$values of the vertex.

Whichever method you use, you should find that the vertex is at **(10, -65)**.

Solve Real-World Problems by Graphing Quadratic Functions

Here's a real-world problem we can solve using the graphing methods we've learned.

Example C

Andrew is an avid archer. He launches an arrow that takes a parabolic path. The equation of the height of the ball with respect to time is $y = -4.9t^2 + 48t$, *where* y *is the height of the arrow in meters and* t *is the time in seconds since Andrew shot the arrow. Find how long it takes the arrow to come back to the ground.*

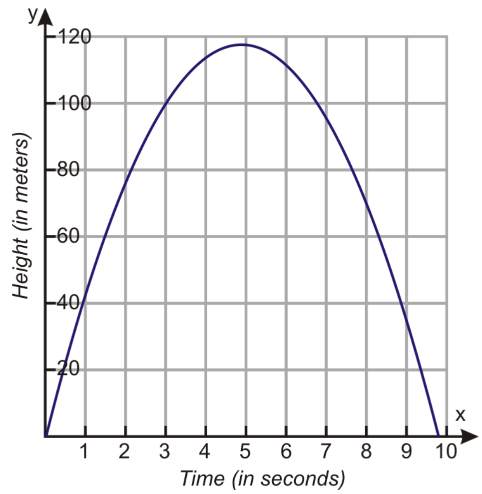
Solution

Let's graph the equation by making a table of values.

TABLE 8.4:

t	$y = -4.9t^2 + 48t$
0	$y = -4.9(0)^2 + 48(0) = 0$
1	$y = -4.9(1)^2 + 48(1) = 43.1$
2	$y = -4.9(2)^2 + 48(2) = 76.4$
3	$y = -4.9(3)^2 + 48(3) = 99.9$
4	$y = -4.9(4)^2 + 48(4) = 113.6$
5	$y = -4.9(5)^2 + 48(5) = 117.6$
6	$y = -4.9(6)^2 + 48(6) = 111.6$
7	$y = -4.9(7)^2 + 48(7) = 95.9$
8	$y = -4.9(8)^2 + 48(8) = 70.4$
9	$y = -4.9(9)^2 + 48(9) = 35.1$
10	$y = -4.9(10)^2 + 48(10) = -10$

Here's the graph of the function:



The roots of the function are approximately $x = 0$ sec and $x = 9.8$ sec. The first root tells us that the height of the arrow was 0 meters when Andrew first shot it. The second root says that it takes approximately **9.8 seconds** for the arrow to return to the ground.

Vocabulary

- The **solutions of a quadratic equation** are often called the ***roots*** or ***zeros.***

Guided Practice

Find the solutions to $2x^2 + 5x - 7 = 0$ *by graphing.*

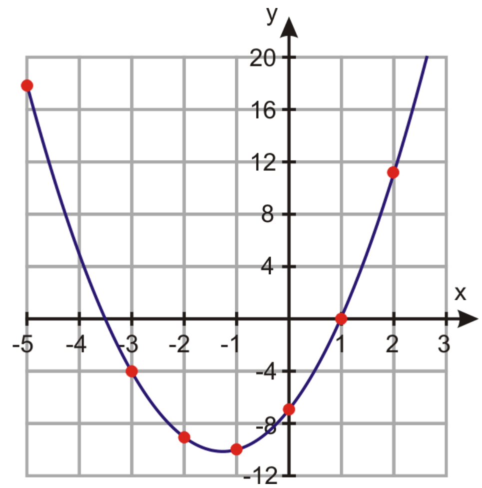
Solution

Since we can't factor this equation, we won't be able to graph it using intercept form (if we could, we wouldn't need to use the graphs to find the intercepts!) We'll just have to make a table of arbitrary values to graph the equation.

TABLE 8.5:

x	$y = 2x^2 + 5x - 7$
-5	$y = 2(-5)^2 + 5(-5) - 7 = 18$
-4	$y = 2(-4)^2 + 5(-4) - 7 = 5$
-3	$y = 2(-3)^2 + 5(-3) - 7 = -4$
-2	$y = 2(-2)^2 + 5(-2) - 7 = -9$
-1	$y = 2(-1)^2 + 5(-1) - 7 = -10$
0	$y = 2(0)^2 + 5(0) - 7 = -7$
1	$y = 2(1)^2 + 5(1) - 7 = 0$
2	$y = 2(2)^2 + 5(2) - 7 = 11$
3	$y = 2(3)^2 + 5(3) - 7 = 26$

We plot the points and get the following graph:



From the graph we can read that the $x-$intercepts are $x = 1$ **and** $x = -3.5$. These are the solutions to the equation.

Practice

For 1-6, find the solutions of the following equations by graphing.

1. $x^2+3x+6=0$
2. $-2x^2+x+4=0$
3. $x^2-9=0$
4. $x^2+6x+9=0$
5. $10x-3x^2=0$
6. $\frac{1}{2}x^2-2x+3=0$

For 7-12, find the roots of the following quadratic functions by graphing.

7. $y=-3x^2+4x-1$
8. $y=9-4x^2$
9. $y=x^2+7x+2$
10. $y=-x^2-10x-25$
11. $y=2x^2-3x$
12. $y=x^2-2x+5$

For 13-18, use your graphing calculator to find the roots and the vertex of each polynomial.

13. $y=x^2+12x+5$
14. $y=x^2+3x+6$
15. $y=-x^2-3x+9$
16. $y=-x^2+4x-12$
17. $y=2x^2-4x+8$
18. $y=-5x^2-3x+2$
19. Graph the equations $y=2x^2-4x+8$ and $y=x^2-2x+4$on the same screen. Find their roots and vertices.
    a. What is the same about the graphs? What is different?
    b. How are the two equations related to each other? (Hint: factor them.)
    c. What might be another equation with the same roots? Graph it and see.
20. Graph the equations $y=x^2-2x+2$ and $y=x^2-2x+4$on the same screen. Find their roots and vertices.

a. What is the same about the graphs? What is different?
b. How are the two equations related to each other?

21. Phillip throws a ball and it takes a parabolic path. The equation of the height of the ball with respect to time is $y = -16t^2 + 60t$, where y is the height in feet and t is the time in seconds. Find how long it takes the ball to come back to the ground.

8.7 Solving Quadratic Equations: Solving Using Square Roots

Here you'll use the properties of square roots to solve certain types of quadratic equations.

Guidance

Now that you are familiar with square roots, we will use them to solve quadratic equations. Keep in mind, that square roots cannot be used to solve every type of quadratic. In order to solve a quadratic equation by using square roots, an $x-$term *cannot* be present. Solving a quadratic equation by using square roots is very similar to solving a linear equation. In the end, you must isolate the x^2 or whatever is being squared.

Example A

Solve $2x^2-3=15$.

Solution: Start by isolating the x^2.

$$\begin{aligned}2x^2-3&=15\\2x^2&=18\\x^2&=9\end{aligned}$$

At this point, you can take the square root of both sides.

$$\begin{aligned}\sqrt{x^2}&=\pm\sqrt{9}\\x&=\pm 3\end{aligned}$$

Notice that x has two solutions; 3 or -3. When taking the square root, always put the $\pm$ (plus or minus sign) in front of the square root. This indicates that the positive or negative answer will be the solution.

Check:

$$\begin{array}{rcr}2(3)^2-3=15 & & 2(-3)^2-3=15\\2\cdot 9-3=15 & \textit{or} & 2\cdot 9-3=15\\18-3=15 & & 18-3=15\end{array}$$

Example B

Solve $\frac{x^2}{16}+3=27$.

Solution: Isolate x^2 and then take the square root.

$$\begin{aligned}\frac{x^2}{16}+3&=27\\ \frac{x^2}{16}&=24\\ x^2&=384\\ x&=\pm\sqrt{384}=\pm 8\sqrt{6}\end{aligned}$$

Example C

Solve $3(x-5)^2+7=43$.

Solution: In this example, x is not the only thing that is squared. Isolate the $(x-5)^2$ and then take the square root.

$$\begin{aligned}3(x-5)^2+7&=43\\ 3(x-5)^2&=36\\ (x-5)^2&=12\\ x-5&=\pm\sqrt{12}\ or\ \pm 2\sqrt{3}\end{aligned}$$

Now that the square root is gone, add 5 to both sides.

$$\begin{aligned}x-5&=\pm 2\sqrt{3}\\ x&=5\pm 2\sqrt{3}\end{aligned}$$

$x=5+2\sqrt{3}$ or $5-2\sqrt{3}$. We can estimate these solutions as decimals; 8.46 or 1.54. Remember, that the most accurate answer includes the radical numbers.

Guided Practice

Solve the following quadratic equations.

1. $\frac{2}{3}x^2-14=38$
2. $11+x^2=4x^2+5$
3. $(2x+1)^2-6=19$

Answers

1. Isolate x^2 and take the square root.

$$\begin{aligned}\frac{2}{3}x^2-14&=38\\ \frac{2}{3}x^2&=52\\ x^2&=78\\ x&=\pm\sqrt{78}\end{aligned}$$

2. Combine all like terms, then isolate x^2.

$$\begin{aligned}11+x^2 &= 4x^2+5\\ -3x^2 &= -6\\ x^2 &= 2\\ x &= \pm\sqrt{2}\end{aligned}$$

3. Isolate what is being squared, take the square root, and then isolate x.

$$\begin{aligned}(2x+1)^2-6 &= 19\\ (2x+1)^2 &= 25\\ 2x+1 &= \pm 5\\ 2x &= -1\pm 5\\ x &= \frac{-1\pm 5}{2} \rightarrow x = \frac{-1+5}{2} = 2 \text{ or } x = \frac{-1-5}{2} = -3\end{aligned}$$

Practice

Solve the following quadratic equations. Reduce answers as much as possible. No decimals.

1. $x^2 = 144$
2. $5x^2-4 = 16$
3. $8-10x^2 = -22$
4. $(x+2)^2 = 49$
5. $6(x-5)^2+1 = 19$
6. $\frac{3}{4}x^2-19 = 26$
7. $x^2-12 = 36-2x^2$
8. $9-\frac{x^2}{3} = -33$
9. $-4(x+7)^2 = -52$
10. $2(3x+4)^2-5 = 45$
11. $\frac{1}{3}(x-10)^2-8 = 16$
12. $\frac{(x-1)^2}{6}-\frac{8}{3} = \frac{7}{2}$

8.8 Factoring and Solving Quadratic Equations: Using the Zero-Product Property

Guidance

Recall that when solving an equation, you are trying to determine the values of the variable that make the equation true. For the equation $2x^2+10x+8=0$, $x=-1$ and $x=-4$ are both solutions. You can check this:

- $2(-1)^2+10(-1)+8=2(1)-10+8=0$
- $2(-4)^2+10(-4)+8=2(16)-40+8=32-40+8=0$

Here you will focus on solving quadratic equations. One of the methods for quadratic equations utilizes your factoring skills and a property called the **zero product property**.

If $a \cdot b=0$, what can you say about a or b? What you should realize is that either a or b have to be equal to 0, because that is the only way that their product will be 0. If both a and b were non-zero, then their product would have to be non-zero. This is the idea of the zero product property. The zero product property states that if the product of two quantities is zero, then one or both of the quantities must be zero.

The zero product property has to do with products being equal to zero. When you factor, you turn a quadratic expression into a product. If you have a quadratic expression equal to zero, you can factor it and then use the zero product property to solve. So, if you were given the equation $2x^2+5x-3=0$, first you would want to turn the quadratic expression into a product by factoring it:

$2x^2+5x-3=(x+3)(2x-1)$

You can rewrite the equation you are trying to solve as $(x+3)(2x-1)=0$.

Now, you have the product of two binomials equal to zero. This means at least one of those binomials must be equal to zero. So, you have two mini-equations that you can solve to find the values of x that cause each binomial to be equal to zero.

- $x+3=0$, which means $x=-3$ OR
- $2x-1=0$, which means $x=\frac{1}{2}$

The two solutions to the equation $2x^2+5x-3=0$ are $x=-3$ and $x=\frac{1}{2}$.

Keep in mind that you can only use the zero product property if your equation is set equal to zero! If you have an equation not set equal to zero, first rewrite it so that it is set equal to zero. Then factor and use the zero product property.

Example A

Solve for x: $x^2+5x+6=0$.

Solution: First, change x^2+5x+6 into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+5x+6=(x+3)(x+2)$

Next, rewrite the equation you are trying to solve:

$x^2+5x+6=0$ becomes $(x+3)(x+2)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+3=0$, which means that $x=-3$
- $x+2=0$, which means that $x=-2$

The solutions are $x=-3$ or $x=-2$.

Example B

Solve for x: $6x^2+x-15=0$.

In order to solve for x you need to factor the polynomial.

Solution: First, change $6x^2+x-15$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$6x^2+x-15=(3x+5)(2x-3)$

Next, rewrite the equation you are trying to solve:

$6x^2+x-15=0$ becomes $(3x+5)(2x-3)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $3x+5=0$, which means that $x=-\frac{5}{3}$
- $2x-3=0$, which means that $x=\frac{3}{2}$

The solutions are $x=-\frac{5}{3}$ or $x=\frac{3}{2}$.

Example C

Solve for x: $x^2+2x-35=0$.

Solution: First, change $x^2+2x-35$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+2x-35=(x+7)(x-5)$

Next, rewrite the equation you are trying to solve:

$x^2+2x-35=0$ becomes $(x+7)(x-5)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+7=0$, which means that $x=-7$
- $x-5=0$, which means that $x=5$

The solutions are $x=-7$ or $x=5$.

Guided Practice

1. Solve for the variable in the polynomial: $x^2+4x-21=0$

2. Solve for the variable in the polynomial: $20m^2+11m-4=0$

3. Solve for the variable in the polynomial: $2e^2+7e+6=0$

Answers:

1. $x^2+4x-21=(x-3)(x+7)$

$$\begin{array}{cc} \multicolumn{2}{c}{(x-3)(x+7)=0} \\ \swarrow & \searrow \\ (x-3)=0 & (x+7)=0 \\ x=3 & x=-7 \end{array}$$

2. $20m^2+11m-4=(4m-1)(5m+4)$

$$\begin{array}{cc} \multicolumn{2}{c}{(4m-1)(5m+4)=0} \\ \swarrow & \searrow \\ 4m-1=0 & 5m+4=0 \\ 4m=1 & 5m=-4 \\ m=\dfrac{1}{4} & m=\dfrac{-4}{5} \end{array}$$

3. $2e^2+7e+6=(2e+3)(e+2)$

$$\begin{array}{cc} \multicolumn{2}{c}{(2e+3)(e+2)=0} \\ \swarrow & \searrow \\ 2e+3=0 & e+2=0 \\ 2e=-3 & e=-2 \\ e=\dfrac{-3}{2} & \end{array}$$

Practice

Solve for the variable in each of the following equations.

1. $(x+1)(x-3)=0$
2. $(a+3)(a+5)=0$
3. $(x-5)(x+4)=0$
4. $(2t-4)(t+3)=0$
5. $(x-8)(3x-7)=0$
6. $x^2+x-12=0$
7. $b^2+2b-24=0$
8. $t^2+3t-18=0$
9. $w^2+3w-108=0$
10. $e^2-2e-99=0$
11. $6x^2-x-2=0$
12. $2d^2+14d-16=0$

13. $3s^2 + 20s + 12 = 0$
14. $18x^2 + 12x + 2 = 0$
15. $3j^2 - 17j + 10 = 0$

8.9 Writing Quadratic Functions: Expressions, Equations, and Functions

Guidance

You can use the solutions of a quadratic equation to write an equation for a related quadratic function.

Each linear factor of a quadratic expression corresponds to a solution of a related quadratic equation and a zero of a related quadratic function.

Quadratic Expression: $x^2 + 3x - 4$

Quadratic Equation: $x^2 + 3x - 4 = 0$

Quadratic Function: $f(x) = x^2 + 3x - 4$

If a real number r is a solution of a quadratic equation in standard form, then $x - r$ is a factor of a related quadratic function.

Since each solution r corresponds to a zero of the related quadratic function, you can use the zeros to find the factors of the function.

For example, suppose a quadratic function $y = f(x)$ has zeros of 3 and -1. Then $x - 3$ and $x + 1$ are factors of $f(x)$. So the factored form of any quadratic function with zeros 3 and -1 can be written as shown below:

$$\mathbf{f(x) = a\,(x-3)(x+1)}$$

Practice

1. What are the linear factors of the quadratic expression $x^2 + 14x + 40$? What are the zeros of the related quadratic function? Explain the reasoning between the factors of the expression and the zeros of the function.

2. What are the solutions of $x^2 + 14x - 15 = 0$? Use the solutions to help you graph the related quadratic function.

Tell whether each of the following is a *quadratic expression*, a *quadratic equation*, or a *quadratic function*. State the linear factors any expressions, the solutions of any equations, and the zeros of any functions.

3. $f(x) = x^2 + 14x + 40$

4. $x^2 - 14x - 120$

5. $x^2 + 8x + 16 = 0$

6. $0 = x^2 - 5x + 4$

7. $x^2 - 7x - 60$

8. $y = x^2 + 9x - 22$

The solutions a quadratic equation are given. Write a related quadratic function in factored form.

9. 3, 7

10. -2, 5

11. What are the solutions of the equation $x^2 - 13x + 40 = 0$?

12. Sketch the graph of the function $f(x) = x^2 + 9x + 18$. Use the graph to find the zeros of the function.

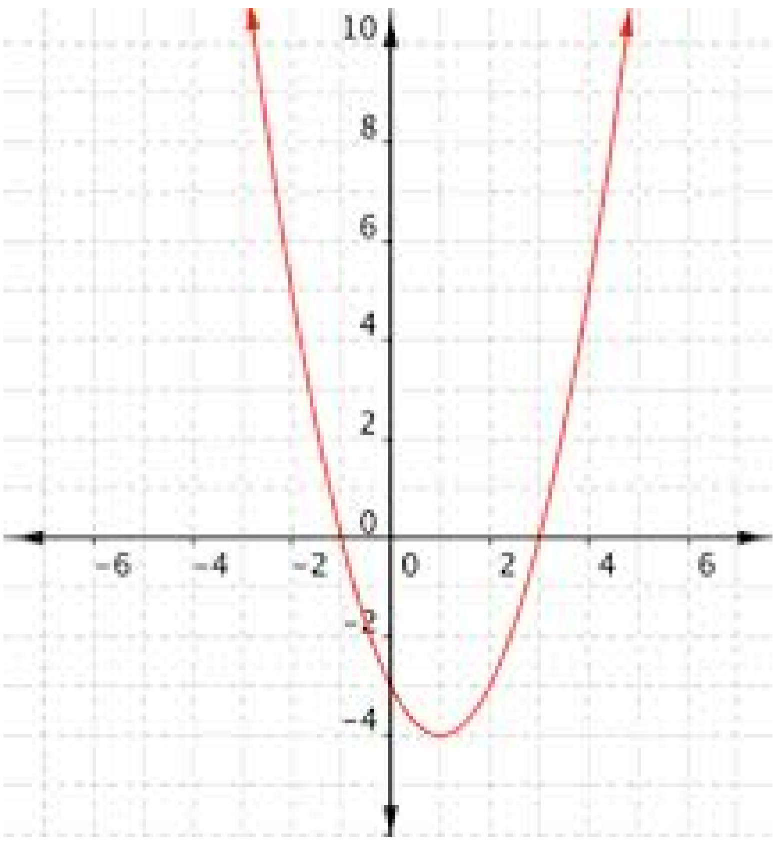


FIGURE 8.25

8.10 Completing the Square of Quadratic Functions

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Guidance

You saw in the last section that if you have a quadratic equation of the form $(x-2)^2 = 5$, you can easily solve it by taking the square root of each side:

$$x-2 = \sqrt{5} \qquad \text{and} \qquad x-2 = -\sqrt{5}$$

Simplify to get:

$$x = 2+\sqrt{5} \approx 4.24 \qquad \text{and} \qquad x = 2-\sqrt{5} \approx -0.24$$

So what do you do with an equation that isn't written in this nice form? In this section, you'll learn how to rewrite any quadratic equation in this form by **completing the square.**

Complete the Square of a Quadratic Expression

Completing the square lets you rewrite a quadratic expression so that it contains a perfect square trinomial that you can factor as the square of a binomial.

Remember that the square of a binomial takes one of the following forms:

$$(x+a)^2 = x^2+2ax+a^2$$
$$(x-a)^2 = x^2-2ax+a^2$$

So in order to have a perfect square trinomial, we need two terms that are perfect squares and one term that is twice the product of the square roots of the other terms.

Example A

Complete the square for the quadratic expression x^2+4x.

Solution

To complete the square we need a constant term that turns the expression into a perfect square trinomial. Since the middle term in a perfect square trinomial is always 2 times the product of the square roots of the other two terms, we re-write our expression as:

$$x^2+2(2)(x)$$

We see that the constant we are seeking must be 2^2 :

$$x^2 + 2(2)(x) + 2^2$$

Answer: By adding 4 to both sides, this can be factored as: $(x+2)^2$

Notice, though, that we just changed the value of the whole expression by adding 4 to it. If it had been an equation, we would have needed to add 4 to the other side as well to make up for this.

Also, this was a relatively easy example because a, the coefficient of the x^2 term, was 1. When that coefficient doesn't equal 1, we have to factor it out from the whole expression before completing the square.

Example B

Complete the square for the quadratic expression $4x^2 + 32x$.

Solution

Factor the coefficient of the x^2 term:

$4(x^2 + 8x)$

Re-write the expression:

$4(x^2 + 2(4)(x))$

We complete the square by adding the constant 4^2:

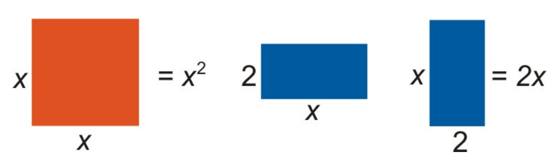
$4(x^2 + 2(4)(x) + 4^2)$

Factor the perfect square trinomial inside the parenthesis:

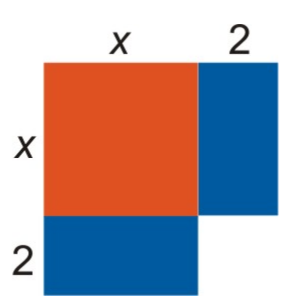
$4(x+4)^2$

The expression **"completing the square"** comes from a geometric interpretation of this situation. Let's revisit the quadratic expression in Example 1: $x^2 + 4x$.

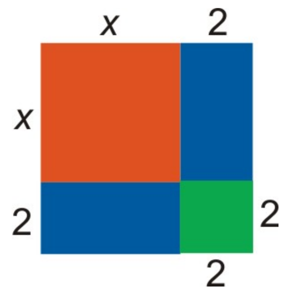
We can think of this expression as the sum of three areas. The first term represents the area of a square of side x. The second expression represents the areas of two rectangles with a length of 2 and a width of x:



We can combine these shapes as follows:



We obtain a square that is not quite complete. To complete the square, we need to add a smaller square of side length 2.



We end up with a square of side length $(x+2)$; its area is therefore $(x+2)^2$. Let's demonstrate the method of **completing the square** with an example.

Example C

Solve the following quadratic equation: $3x^2-10x=-1$

Solution

Divide all terms by the coefficient of the x^2 term:

$$x^2-\frac{10}{3}x=-\frac{1}{3}$$

Rewrite: $x^2-2\left(\frac{5}{3}\right)(x)=-\frac{1}{3}$

In order to have a perfect square trinomial on the right-hand-side we need to add the constant $\left(\frac{5}{3}\right)^2$. Add this constant to **both** sides of the equation:

$$x^2-2\left(\frac{5}{3}\right)(x)+\left(\frac{5}{3}\right)^2=-\frac{1}{3}+\left(\frac{5}{3}\right)^2$$

Factor the perfect square trinomial and simplify:

$$\left(x-\frac{5}{3}\right)^2=-\frac{1}{3}+\frac{25}{9}$$
$$\left(x-\frac{5}{3}\right)^2=\frac{22}{9}$$

Take the square root of both sides:

$$x-\frac{5}{3}=\sqrt{\frac{22}{9}} \quad \text{and} \quad x-\frac{5}{3}=-\sqrt{\frac{22}{9}}$$
$$x=\frac{5}{3}+\sqrt{\frac{22}{9}}\approx 3.23 \quad \text{and} \quad x=\frac{5}{3}-\sqrt{\frac{22}{9}}\approx 0.1$$

Answer: $x=3.23$ and $x=0.1$

Solving Quadratic Equations in Standard Form

If an equation is in standard form $(ax^2+bx+c=0)$, we can still solve it by the method of completing the square. All we have to do is start by moving the constant term to the right-hand-side of the equation.

Example D

Solve the following quadratic equation: $x^2+15x+12=0$

Solution

Move the constant to the other side of the equation:

$$x^2+15x=-12$$

Rewrite: $x^2+2\left(\frac{15}{2}\right)(x)=-12$

Add the constant $\left(\frac{15}{2}\right)^2$ to both sides of the equation:

$$x^2+2\left(\frac{15}{2}\right)(x)+\left(\frac{15}{2}\right)^2=-12+\left(\frac{15}{2}\right)^2$$

Factor the perfect square trinomial and simplify:

$$\left(x+\frac{15}{2}\right)^2=-12+\frac{225}{4}$$
$$\left(x+\frac{15}{2}\right)^2=\frac{177}{4}$$

Take the square root of both sides:

$$x+\frac{15}{2}=\sqrt{\frac{177}{4}} \quad \text{and} \quad x+\frac{15}{2}=-\sqrt{\frac{177}{4}}$$
$$x=-\frac{15}{2}+\sqrt{\frac{177}{4}}\approx -0.85 \quad \text{and} \quad x=-\frac{15}{2}-\sqrt{\frac{177}{4}}\approx -14.15$$

Answer: $x=-0.85$ and $x=-14.15$

Vocabulary

- A ***perfect square trinomial*** has the form $a^2+2(ab)+b^2$, which factors into $(a+b)^2$.

Guided Practice

Solve the following quadratic equation: $-x^2+22x=5$

Solution

Divide all terms by the coefficient of the x^2 term:

$$x^2-22x=-6$$

Rewrite: $x^2 - 2(11)(x) = -6$.

In order to have a perfect square trinomial on the right-hand-side we need to add the constant $(11)^2$. Add this constant to **both** sides of the equation:

$$x^2 - 2(11)(x) + (11)^2 = -6 + (11)^2$$

Factor the perfect square trinomial and simplify:

$$(x-11)^2 = -6 + (11)^2$$
$$\left(x - \frac{5}{3}\right)^2 = 16$$

Take the square root of both sides:

$$x - 11 = \sqrt{16} \qquad \text{and} \qquad x - 11 = -\sqrt{16}$$
$$x = 11 + \sqrt{16} = 15 \qquad \text{and} \qquad x = 11 - \sqrt{4} = 7$$

Answer: $x = 15$ and $x = 7$

Practice

Complete the square for each expression.

1. $x^2 + 5x$
2. $x^2 - 2x$
3. $x^2 + 3x$
4. $x^2 - 4x$
5. $3x^2 + 18x$
6. $2x^2 - 22x$
7. $8x^2 - 10x$
8. $5x^2 + 12x$

Solve each quadratic equation by completing the square.

9. $x^2 - 4x = 5$
10. $x^2 - 5x = 10$
11. $x^2 + 10x + 15 = 0$
12. $x^2 + 15x + 20 = 0$
13. $2x^2 - 18x = 3$
14. $4x^2 + 5x = -1$
15. $10x^2 - 30x - 8 = 0$
16. $5x^2 + 15x - 40 = 0$

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8.11 Quadratic Formula and the Discriminant: Using the Quadratic Formula

Guidance

Previous Concepts have presented three methods to solve a quadratic equation:

- By graphing to find the zeros;
- By solving using square roots; and
- By using completing the square to find the solutions

This Concept will present a fourth way to solve a quadratic equation: using the quadratic formula.

History of the Quadratic Formula

As early as 1200 BC, people were interested in solving quadratic equations. The Babylonians solved simultaneous equations involving quadratics. In 628 AD, Brahmagupta, an Indian mathematician, gave the first explicit formula to solve a quadratic equation. The quadratic formula was written as it is today by the Arabic mathematician Al-Khwarizmi. It is his name upon which the word "Algebra" is based.

The solution to any quadratic equation in standard form, $0 = ax^2 + bx + c$, is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example A

Solve $x^2 + 10x + 9 = 0$ *using the quadratic formula.*

Solution:

We know from the last Concept the answers are $x = -1$ *or* $x = -9$.

By applying the quadratic formula and $a = 1, b = 10$, and $c = 9$, we get:

$$\begin{aligned}
x &= \frac{-10 \pm \sqrt{(10)^2 - 4(1)(9)}}{2(1)} \\
x &= \frac{-10 \pm \sqrt{100 - 36}}{2} \\
x &= \frac{-10 \pm \sqrt{64}}{2} \\
x &= \frac{-10 \pm 8}{2} \\
x &= \frac{-10 + 8}{2} \text{ or } x = \frac{-10 - 8}{2} \\
x &= -1 \text{ or } x = -9
\end{aligned}$$

Example B

Solve $-4x^2+x+1=0$ *using the quadratic formula.*

Solution:

$$\begin{aligned}
\text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
\text{Plug in the values } a=-4, b=1, c=1: \quad & x=\frac{-1\pm\sqrt{(1)^2-4(-4)(1)}}{2(-4)}\\
\text{Simplify:} \quad & x=\frac{-1\pm\sqrt{1+16}}{-8}=\frac{-1\pm\sqrt{17}}{-8}\\
\text{Separate the two options:} \quad & x=\frac{-1+\sqrt{17}}{-8} \text{ and } x=\frac{-1-\sqrt{17}}{-8}\\
\text{Solve:} \quad & x\approx -.39 \text{ and } x\approx .64
\end{aligned}$$

Example C

Solve $8t^2+10t+3=0$ *using the quadratic formula.*

Solution:

$$\begin{aligned}
\text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
\text{Plug in the values } a=8, b=10, c=3: \quad & x=\frac{-10\pm\sqrt{(10)^2-4(8)(3)}}{2(8)}\\
\text{Simplify:} \quad & x=\frac{-10\pm\sqrt{100+96}}{16}=\frac{-10\pm\sqrt{196}}{16}\\
\text{Separate the two options:} \quad & x=\frac{-10+\sqrt{196}}{16} \text{ and } x=\frac{-10-\sqrt{196}}{16}\\
\text{Separate the two options:} \quad & x=\frac{-10+14}{16} \text{ and } x=\frac{-10-14}{16}\\
\text{Solve:} \quad & x=\frac{1}{4} \text{ and } x=-\frac{3}{2}
\end{aligned}$$

Guided Practice

Solve $3k^2+11k=4$ *using the quadratic formula.*

Solution:

First, we must make it so one side is equal to zero:

$3k^2+11k=4 \Rightarrow 3k^2+11k-4=0$

Now it is in the correct form for using the quadratic formula.

Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = 3, b = 11, c = -4$: $x = \frac{-11 \pm \sqrt{(11)^2 - 4(3)(-4)}}{2(3)}$

Simplify: $x = \frac{-11 \pm \sqrt{121 + 48}}{6} = \frac{-11 \pm \sqrt{169}}{6}$

Separate the two options: $x = \frac{-11 + \sqrt{169}}{6}$ and $x = \frac{-11 - \sqrt{169}}{6}$

Separate the two options: $x = \frac{-11 + 13}{6}$ and $x = \frac{-11 - 13}{6}$

Solve: $x = \frac{1}{3}$ and $x = -4$

Practice

1. What is the quadratic formula? When is the most appropriate situation to use this formula?
2. When was the first known solution of a quadratic equation recorded?

Solve the following quadratic equations using the quadratic formula.

3. $x^2 + 4x - 21 = 0$
4. $x^2 - 6x = 12$
5. $3x^2 - \frac{1}{2}x = \frac{3}{8}$
6. $2x^2 + x - 3 = 0$
7. $-x^2 - 7x + 12 = 0$
8. $-3x^2 + 5x = 0$
9. $4x^2 = 0$
10. $x^2 + 2x + 6 = 0$

8.12 The Quadratic Formula and the Discriminant: Using the Discriminant

Guidance

From the previous concept, the Quadratic Formula is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. The expression under the radical, b^2-4ac, is called the **discriminant.** You can use the discriminant to determine the number and type of solutions an equation has.

Investigation: Solving Equations with Different Types of Solutions

1. Solve $x^2-8x-20=0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{144}}{2} \\ &= \frac{8 \pm 12}{2} \rightarrow 10, -2 \end{aligned}$$

2. Solve $x^2-8x+6=0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{0}}{2} \\ &= \frac{8 \pm 0}{2} \rightarrow 4 \end{aligned}$$

3. Solve $x^2-8x+20=0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{-16}}{2} \\ &= \frac{8 \pm 4i}{2} \rightarrow 4 \pm 2i \end{aligned}$$

4. Look at the values of the discriminants from Steps 1-3. How do they differ? How does that affect the final answer?

From this investigation, we can conclude:

- If $b^2-4ac > 0$, then the equation has two real solutions.
- If $b^2-4ac = 0$, then the equation has one real solution; a double root.
- If $b^2-4ac < 0$, then the equation has two imaginary solutions.

Example A

Determine the type of solutions $4x^2-5x+17=0$ has.

Solution: Find the discriminant.

$$\begin{aligned} b^2-4ac &= (-5)^2-4(4)(17) \\ &= 25-272 \end{aligned}$$

At this point, we know the answer is going to be negative, so there is no need to continue (unless we were solving the problem). This equation has two imaginary solutions.

Example B

Solve the equation from Example A to prove that it does have two imaginary solutions.

Solution: Use the Quadratic Formula.

$$x = \frac{5 \pm \sqrt{25-272}}{8} = \frac{5 \pm \sqrt{-247}}{8} = \frac{5}{8} \pm \frac{\sqrt{247}}{8} i$$

Example C

Find the value of the determinant and state how many solutions the quadratic has.

$3x^2-5x-12=0$

Solution: Use the discriminant. a = 3, b = -5, and c = -12

$$\sqrt{(-5)^2-4(3)(-12)} = \sqrt{25+144} = \sqrt{169} = 13$$

This quadratic has two real solutions.

Intro Problem Revisit Set the expression $-5p^2+400p-8000$ equal to zero and then find the discriminant.

$-5p^2+400p-8000=0$

$$\begin{aligned} b^2-4ac &= (400)^2-4(-5)(-8000) \\ &= 160000-160000=0 \end{aligned}$$

At this point, we know the answer is zero, so the equation has one real solution. Therefore, there is one real break-even point.

Guided Practice

1. Use the discriminant to determine the type of solutions $-3x^2-8x+16=0$ has.
2. Use the discriminant to determine the type of solutions $25x^2-80x+64=0$ has.
3. Solve the equation from #1.

Answers

1.

$$\begin{aligned} b^2-4ac &= (-8)^2-4(-3)(16) \\ &= 64+192 \\ &= 256 \end{aligned}$$

This equation has two real solutions.

2.

$$\begin{aligned} b^2-4ac &= (-80)^2-4(25)(64) \\ &= 6400-6400 \\ &= 0 \end{aligned}$$

This equation has one real solution.

3. $x = \frac{8 \pm \sqrt{256}}{-6} = \frac{8 \pm 16}{-6} = -4, \frac{4}{3}$

Practice

Determine the number and type of solutions each equation has.

1. $x^2-12x+36=0$
2. $5x^2-9=0$
3. $2x^2+6x+15=0$
4. $-6x^2+8x+21=0$
5. $x^2+15x+26=0$
6. $4x^2+x+1=0$

Solve the following equations using the Quadratic Formula.

7. $x^2-17x-60=0$
8. $6x^2-20=0$
9. $2x^2+5x+11=0$

Challenge Determine the values for c that make the equation have a) two real solutions, b) one real solution, and c) two imaginary solutions.

10. $x^2+2x+c=0$
11. $x^2-6x+c=0$
12. $x^2+12x+c=0$
13. What is the discriminant of $x^2+2kx+4=0$? Write your answer in terms of k.
14. For what values of k will the equation have two real solutions?
15. For what values of k will the equation have one real solution?
16. For what values of k will the equation have two imaginary solutions?

Vocabulary:

Quadratic Graphs and Properties: Axis of Symmetry, Falling Object Model, Maximum, Minimum, Parabola, Quadratic Function, Quadratic Parent Function, Standard Form of Quadratic Function, Vertex

Quadratic Functions: Formula, y-Intercept, x-Intercept, Vertical Motion Model

Quadratic Function Transformations: Reflection, Compression, Sketch

Vertex Form of Quadratic Function: Maximum, Minimum, Translation, Vertex Form

Solving Quadratic Equations: Quadratic Equation, Root of an Equation, Standard Form of a Quadratic Equation, Zero of a Function

Factoring and Solving Quadratic Equations: Quadratic Equation, Standard Form of a Quadratic Equation, Zero-Product Property

Writing Quadratic Functions: Quadratic Function, Root of an Equation, Zero of a Function

*Completing the Square of Quadratic Function*s: Perfect Square Trinomial, Trinomial, Completing the Square

The Quadratic Formula and the Discriminant: Completing the Square, Discriminant, Quadratic Formula

CHAPTER 9 Exponential Equations and Functions

Chapter Outline

Exponential Functions: A.9a, A.9c, A.9d, A.12b

Exponential Growth and Decay: A.9b, A.9c, A.9d

Modeling Exponential Data: A.9a, A.9b, A.9c, A.9d, A.9e

9.1 Exponential Functions

Here you'll recognize, evaluate and graph exponential functions

Guidance

Let's think about exponential functions by looking at the following situation.

Two girls in a small town once shared a secret, just between the two of them. They couldn't stand it though, and each of them told three friends. Of course, their friends couldn't keep secrets, either, and each of them told three of their friends. Those friends told three friends, and those friends told three friends, and son on... and pretty soon the whole town knew the secret. There was nobody else to tell!

These girls experienced the startling effects of an exponential function.

If you start with the two girls who each told three friends, you can see that they told six people or $2 \cdot 3$.

Those six people each told three others, so that $6 \cdot 3$ or $2 \cdot 3 \cdot 3$—they told 18 people.

Those 18 people each told 3, so that now is $18 \cdot 3$ or $2 \cdot 3 \cdot 3 \cdot 3$ or 54 people.

You can see how this is growing and you could show the number of people told in each round of gossip with a function: $y = ab^x$ where y is the number of people told, a is the two girls who started the gossip, b is the number of friends that they each told, and x is the number of rounds of gossip that occurred.

This is called an *exponential function* —any function that can be written in the form $y = ab^x$, where a and b are constants, $a \neq 0, b > 0$, and $b \neq 1$.

As we did with linear and quadratic functions, we could make a table of values and calculate the number of people told after each round of gossip. Use the function $y = 2 \cdot 3^x$ where y is the number of people told and x is the number of rounds of gossip that occurred.

x rounds of gossip	0	1	2	3	4	5
y people told	2	6	18	54	162	486

How can you tell if a function is an exponential function?

If your function can be written in the form $y = ab^x$, where a and b are constants, $a \neq 0, b > 0$, and $b \neq 1$, then it must be exponential. In quadratic equations, your functions were always to the 2^{nd} power. In exponential functions, the exponent is a variable. Their graphs will have a characteristic curve either upward or downward.

Exponential Functions

1. $y = 2^x$
2. $c = 4 \cdot 10^d$
3. $y = 2 \cdot \left(\frac{2}{3}\right)^x$
4. $t = 4 \cdot 10^u$

Not Exponential Functions

	1. $y = 3 \cdot 1^x$	2. $n = 0 \cdot 3^p$	3. $y = (-4)^x$	4. $y = -6 \cdot 0^x$
because	$b = 1$	$a = 0$	$b < 0$	$b \leq 1$

Exponential functions can be graphed by using a table of values like we did for quadratic functions. Substitute values for x and calculate the corresponding values for y.

Take a look at this one.

Graph $y = 2^x$.

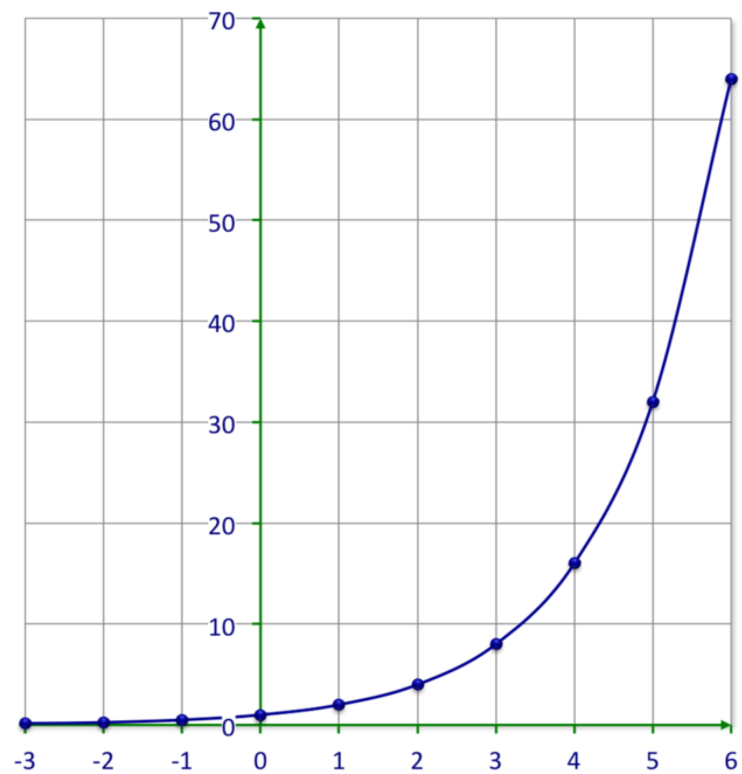
Here is the table.

TABLE 9.1:

x	$y = 2^x$	y
-3	$y = 2^{-3}$	$\frac{1}{8}$
-2	$y = 2^{-2}$	$\frac{1}{4}$
-1	$y = 2^{-1}$	$\frac{1}{2}$
0	$y = 2^0$	0

TABLE 9.1: (continued)

x	$y = 2^x$	y
1	$y = 2^1$	2
2	$y = 2^2$	4
3	$y = 2^3$	8
4	$y = 2^4$	16
5	$y = 2^5$	32
6	$y = 2^6$	64



Notice that the shapes of the graphs are not parabolic like the graphs of quadratic functions. Also, as the x value gets lower and lower, the y value approaches zero but never reaches it. As the x value gets even smaller, the y value may get infinitely close to zero but will never cross the x-axis.

Identify each function.

Example A

$y = 4^x$

Solution: Exponential function

Example B

$f(x) = 2x - 1$

Solution: Linear function

Example C

$y = ax^2 - bx + c$

Solution: Quadratic function

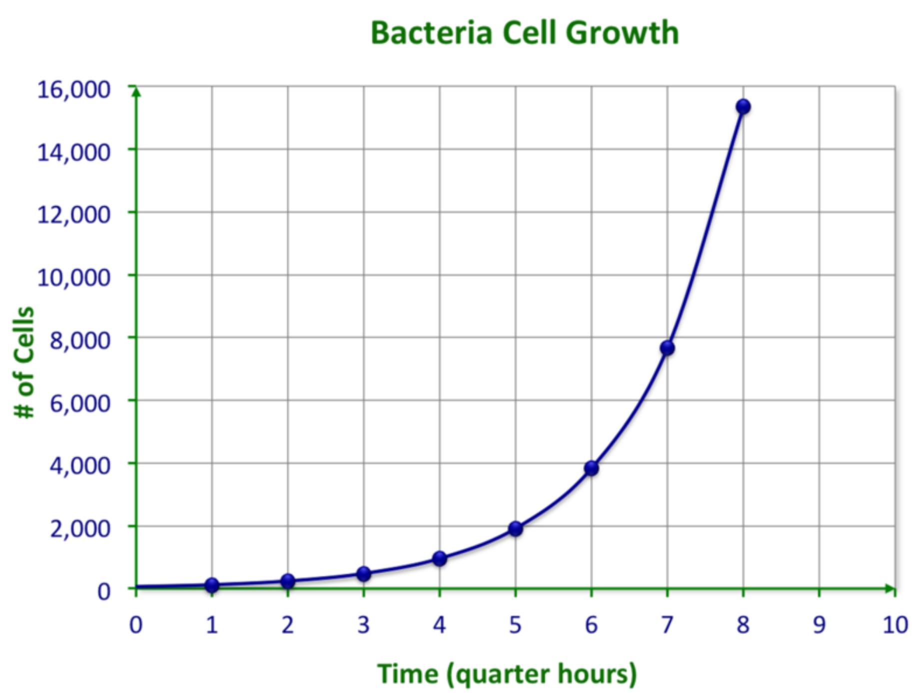
Now let's go back to the dilemma from the beginning of the Concept.

First, we can create a t-table to go with the equation of the function. Here are the values in that table.

TABLE 9.2:

q	b
0	60
1	120
2	240
3	480
4	960
5	1920
6	3840
7	7680
8	15360

Now here is our graph.



Guided Practice

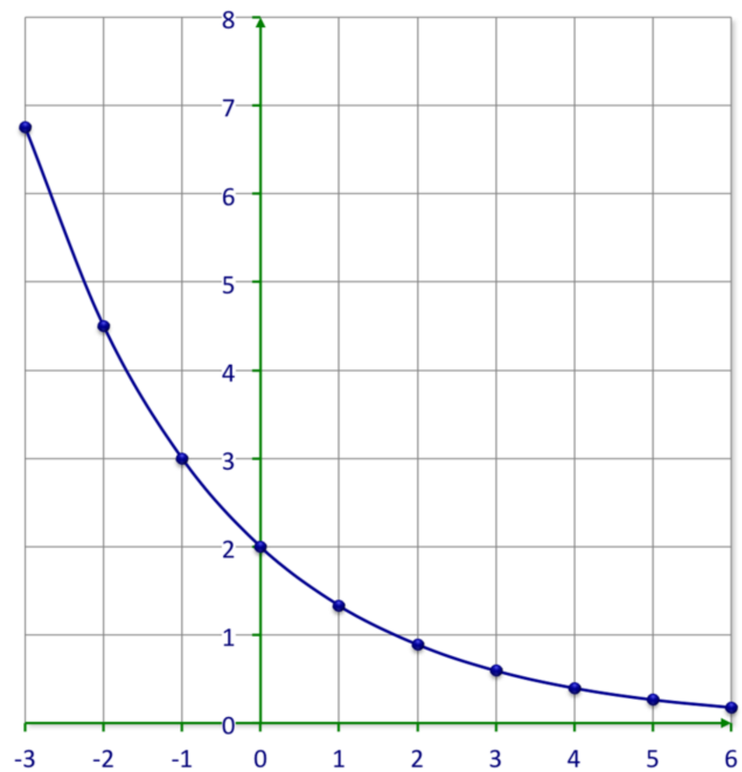
Here is one for you to try on your own.

Graph $y = 2 \cdot \left(\frac{2}{3}\right)^x$

TABLE 9.3:

x	y
-3	$\frac{27}{4}$
-2	$\frac{9}{2}$
-1	3
0	2
1	$\frac{4}{3}$
2	$\frac{8}{9}$
3	$\frac{16}{27}$
4	$\frac{32}{81}$
5	$\frac{64}{243}$
6	$\frac{128}{729}$

Solution



Practice

Directions: Classify the following functions as exponential or not exponential. If it is not exponential, state the reason why.

1. $y = 7^x$
2. $c = -2 \cdot 10^d$
3. $y = 1^x$
4. $y = 4^x$

5. $n = 0 \cdot \left(\frac{1}{2}\right)^x$
6. $y = 5 \cdot \left(\frac{4}{3}\right)^x$
7. $y = (-7)^x$
8. Use a table of values to graph the function $y = 3^x$.
9. Use a table of values to graph the function $y = \left(\frac{1}{3}\right)^x$.
10. What type of graph did you make in number 7?
11. What type of graph did you make in number 8?
12. Use a table of values to graph the function $y = -2^x$.
13. Use a table of values to graph the function $y = 5^x$.
14. Use a table of values to graph the function $y = -5^x$.
15. Use a table of values to graph the function $y = 6^x$.

9.2 Exponential Growth and Decay

Here you'll distinguish between exponential growth and exponential decay.

Guidance

Do you know how to tell if a function is an exponential function?

If your function can be written in the form $y = ab^x$, where a and b are constants, $a \neq 0, b > 0$, and $b \neq 1$, then it must be exponential. In quadratic equations, your functions were always to the 2^{nd} power. In exponential functions, the exponent is a variable. Their graphs will have a characteristic curve either upward or downward.

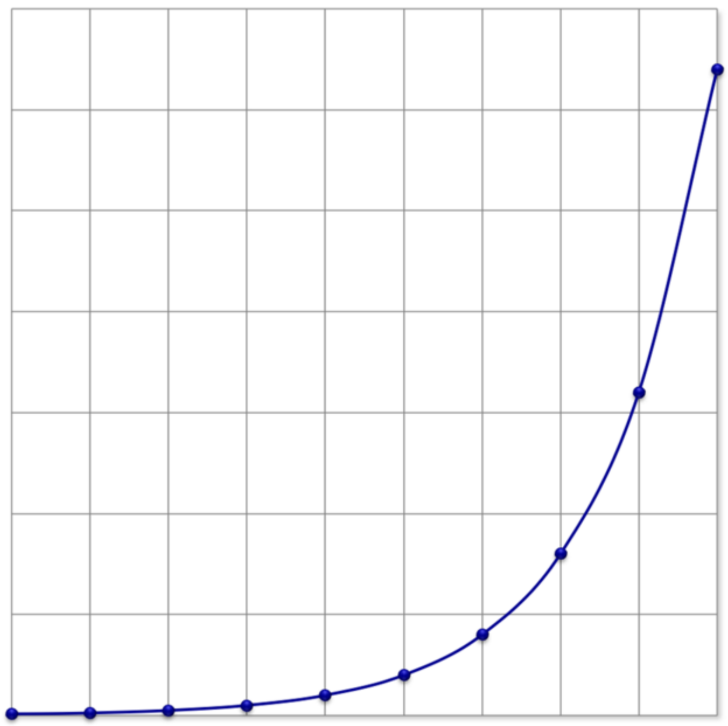
Exponential Functions

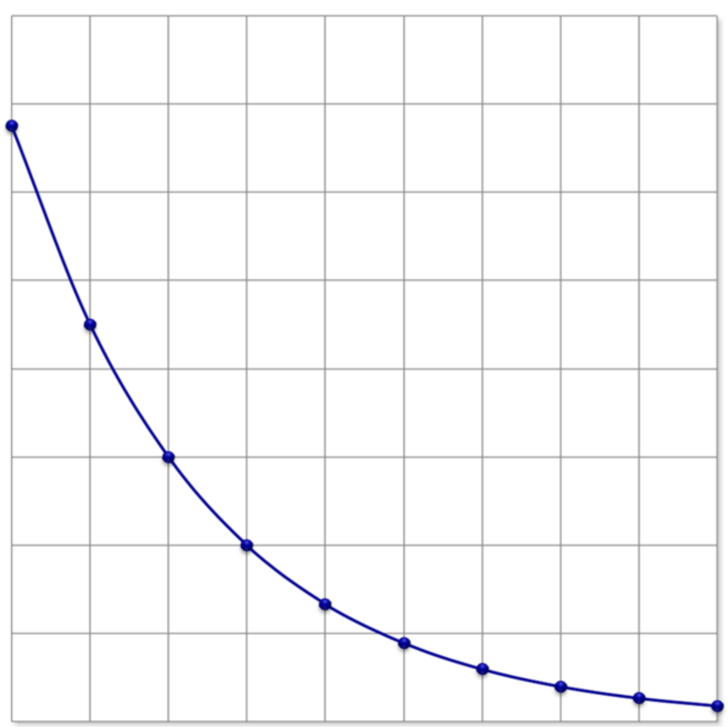
1. $y = 2^x$
2. $c = 4 \cdot 10^d$
3. $y = 2 \cdot \left(\frac{2}{3}\right)^x$
4. $t = 4 \cdot 10^u$

Not Exponential Functions

	1. $y = 3 \cdot 1^x$	2. $n = 0 \cdot 3^p$	3. $y = (-4)^x$	4. $y = -6 \cdot 0^x$
because	$b = 1$	$a = 0$	$b < 0$	$b \leq 1$

In some cases with exponential functions, as the x value increased, the y value increased, too. **This was a *direct* relationship known as *exponential growth* .** As the x value increases, the y value grows at a very fast rate! The other graph you saw showed the opposite—as the x value increased, the y value decreased. **This relationship is an *inverse* relationship known as *decay* .** The graphs of these functions are opposites, reflected on the y-axis.





We can also analyze growth and decay functions in real – life situations.

A famous story tells about a courtier who presented a Persian king with a beautiful handmade chessboard. The king asked him what he would like in return for his gift and the courtier surprised the king by asking him for one grain of rice on the first square of the chessboard, two grains of rice on the second, four grains on the third, etc. The king agreed and ordered for the rice to be brought. By the 21^{st} square, over a million grains of rice were required and, by the 41^{st} square, over a quadrillion grains of rice were needed. There was simply not enough rice in all the world for the final squares.

This story reminds us of the drastic increases that we can see in exponential functions. Although this story is a fable, there are many instances in the real-world where exponential growth can be seen.

Graph each function and tell whether it will represent exponential growth or decay.

Example A

$y = \frac{1}{2}^x$

Solution: Exponential decay

Example B

$y = 4^x$

Solution: Exponential growth

Example C

$y = 5^x$

Solution: Exponential growth

Now let's go back to the dilemma from the beginning of the Concept.

First, make a table of values.

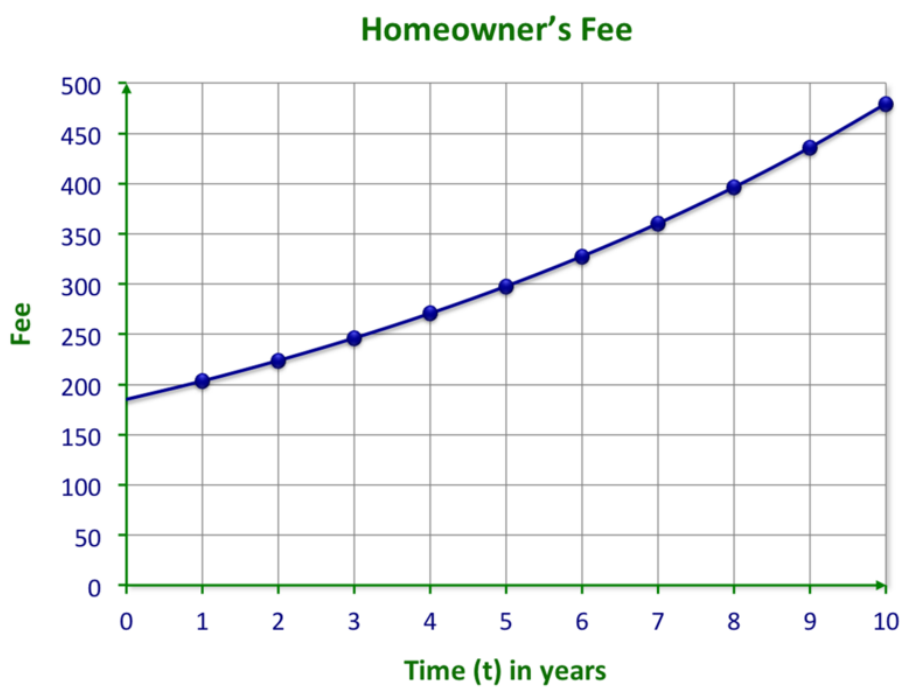
TABLE 9.4:

t	f
0	185

TABLE 9.4: (continued)

t	f
1	203.5
2	223.85
3	246.24
4	270.86
5	297.94
6	327.74
7	360.51
8	396.56
9	436.22
10	479.84

Now we graph the function.



Guided Practice

Here is one for you to try on your own.

Does the following function represent an exponential function?

$y = 3 \times 1^x$

Solution

No, it does not represent an exponential function because the b value is 1.

Practice

Directions: Graph each function. Then say whether it represents economic growth or decay. There will be two answers for each problem.

1. $y = 4^x$
2. $y = \frac{1}{2}^x$
3. $y = \frac{1}{3}^x$
4. $y = 7^x$
5. $y = 5^x$
6. $y = 2^x$
7. $y = \frac{1}{4}^x$
8. $y = \frac{3}{4}^x$
9. $y = 6^x$
10. $y = 11^x$
11. $y = 9^x$
12. $y = \frac{1}{8}^x$
13. $y = 12^x$
14. $y = \frac{2}{5}^x$
15. $y = 13^x$

9.3 Modeling Exponential Data

Here you will explore real-world applications of exponential functions.

Guidance

Exponential growth can be a bit surprising, as it can seem to be rather slow at first. At some point though, an exponential function will (sometimes rather suddenly) begin to increase very rapidly.

Population growth can often be modeled with an exponential function (assuming population grows as a percentage of the current population, i.e. 8% per year).

Example A

The population of a small town was 2,000 in the year 1950. The population increased over time, as shown by the values in the table below.

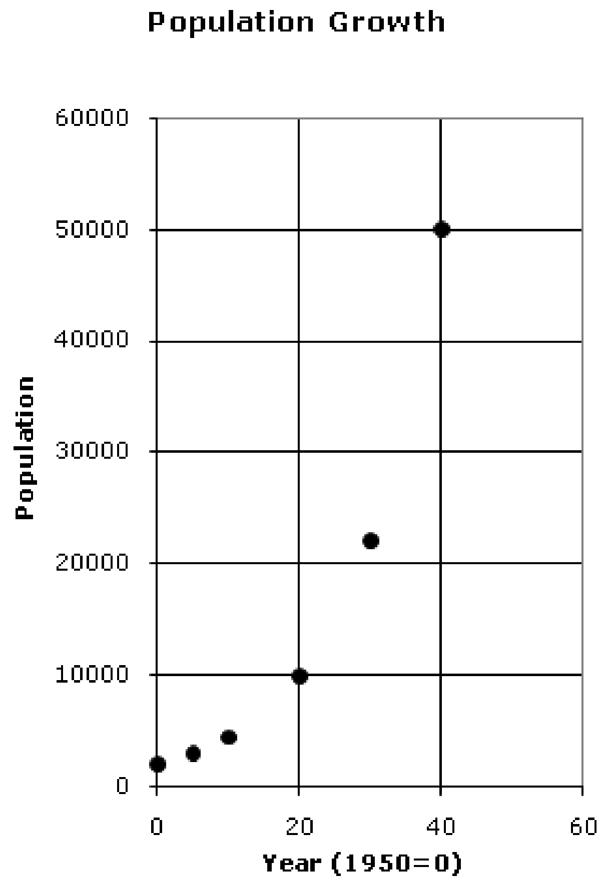
What is the number of people added to the population yearly? Why is this question more complex than it seems?

TABLE 9.5:

Year (1950 = 0)	**Population**
0	2000
5	2980
10	4450
20	9900
30	22,000
40	50,000

Solution:

If you plot these data points, you will see that the growth pattern is non-linear:



The population does not continue to increase by the same number of people each year, it rather increases by a percentage of the population at the end of each year, an exponential function.

Example B

Use a graphing calculator to find a function of the form $y = \mathrm{a}(\mathrm{b}^x)$ that fits the data in the table.

TABLE 9.6:

Year (1950 = 0)	Population
0	2000
5	2980
10	4450
20	9900
30	22,000
40	50,000

Solution:

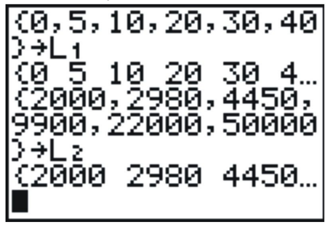
Using a TI-83/84 graphing calculator to find an exponential function that best fits a set of data

1. Entering the data

   a. Data must be entered into "lists". The calculator has six named lists, L1, L2, ... L6. We will enter the x values in L1 and the y values in L2. One way to do this is shown below:

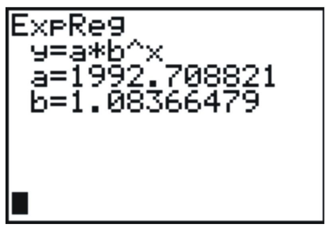
Press <TI font_2nd>[{] and then enter the numbers separated by commas, and close by pressing the following: <TI font_2nd>[{]<TI font_STO><TI font_2nd>[L1].

The top three lines of the figure below show the entry into list L1, followed by the entry of the y values into list L2.



Now press <TI font_STAT>, and move to the right to the CALC menu. Scroll down to option 10, **ExpReg.** Press <TI font_ENTER>, and you will return to the home screen. You should see **ExpReg** on the screen. As long as the numbers are in L1 and L2, the calculator will proceed to find an exponential function to fit the data you listed in List L1 and List L2. You should see on the home screen the values for a and b in the exponential function (See figure below).

Therefore the function* y *= 1992.7(1.0837)* ^{x} *is an approximate model for the data.



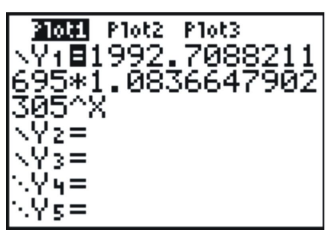
2. Plotting the data and the equation

To view plots of the data points and the equation on the same screen, do the following.

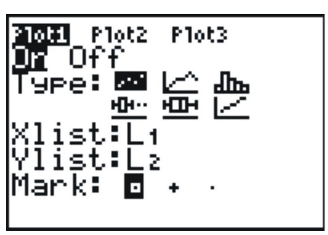
a. First, press <TI font_Y=>and clear any equations.

You can type in the equation above, or to get the equation from the calculator, do the following:

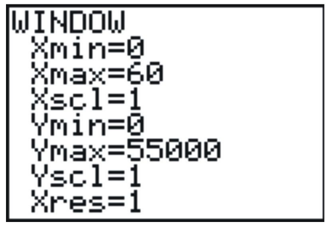
b. Enter the above rounded-off equation in Y1, or use the following procedure to get the full equation from the calculator: put the cursor in Y1, press <TI font_VARS>, 5, EQ, and 1. This should place the equation in Y1 (see figure below).



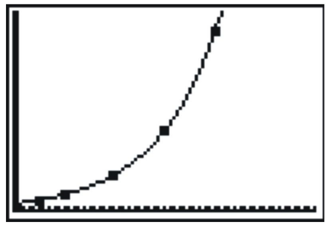
c. Now press <TI font_2nd>[STAT PLOT] and complete the items as shown in the figure below.



d. Now set your window. (Hint: use the range of the data to choose the window – the figure below shows our choices.)

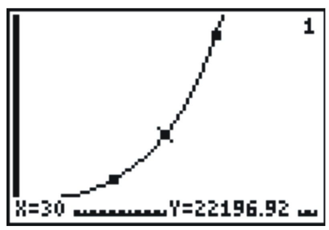


e. Press <TI font GRAPH>and you will to see the function and the data points as shown in the figure below.



3. Comparing the real data with the modeled results

It looks as if the data points lie on the function. However, using the TRACE function you can determine how close the modeled points are to the real data. Press <TI font_TRACE>to enter the TRACE mode. Then press the right arrow to move from one data point to another. Do this until you land on the point with value Y=22000. To see the corresponding modeled value, press the up or down arrow. See the figure below. The modeled value is approximately 22197, which is quite close to the actual data. You can verify any of the other data points using the same method.



Example C

(from the introduction)

You need to find a town that will have a minimum population of 100,000 by the third year from now. The town you are considering has a population of 89,000, with a yearly growth rate of 6%. Will it work?

Solution:

Final population is equal to initial population multiplied by the growth rate once each year.

That indicates that final population is: $[(P_i \cdot growth) \cdot growth] \cdot growth\ldots$ etc. Where P_i is **i**nitial population.

Using r for growth **r**ate, and x for years passed, this simplifies to the exponential function:

$P(f) = P_i \cdot r^x$

In our town, the population after x years would be: $P(x) = 89,000 \cdot (1.06)^x$

The beginning of the 3rd year will occur after 2 years have passed, substituting 2 in for x gives: $P(2) = 89,000 \cdot (1.06)^2$

$\therefore P(2) = 100,000.4$

The projected population is 100,000 (and 4/10... someone is pregnant!) in the 3rd year, just big enough. –>

Guided Practice

1) Tiny Town, CO, currently (year 2012) has a population of 26 people, but it is growing at a rate of 17% per year.

a) What is the *growth factor* for Tiny Town?
b) What will the population be in 2030?

2) Abbi invests $4000 in a savings account with an APR of 6.5% compounded yearly.

a) How much will she have after 2 years?
b) How much will she have after 15 years?
c) How many years will it take to reach $50,000?

3) Brandon bought a new car for $30,000. It wasn't until he drove away that his friend Kyle mentioned that the car was going to depreciate at a rate of 50% per year!

a) What is the decay factor of the car?
b) How much will the car be worth in 5 years?
c) Using your calculation from "b)", apx how long will it take before the car is only worth $100?

Answers

1) Recall from the lesson that the simplified function for population growth is $P_f = P_i \cdot r^t$ Where " P_f" is final population, "P_i" is initial (starting) population, "r" is growth rate, and "t" is time (in years).

a) The growth factor is 1.17, since the population each year is the entire population from the year before: $1 \cdot P$ added to the new population: $.17 \cdot P$.

b) The population in 2030 will be apx 375:
$P_f = 26 \cdot 1.17^17$
$P_f = 26 \cdot 14.426$
$P_f = 375$

2) Abbi's money can be calculated with the same type of formula as above: $A = P \cdot r^t$ Where " A" is final Amount, "'P*" is principal (starting money), "*r*" is growth rate (interest), and "*t*" is time (in years).*

a) After 2 years, Abbi will have apx $4500
$A = \$4000 \cdot 1.065^2$
$A = \$4000 \cdot 1.134$
$A = \$4536$

b) After 15 years, Abbi will have apx $10,300
$A = \$4000 \cdot 1.065^15$
$A = \$4000 \cdot 2.5718$
$A = \$10287.20$

c) To calculate how long it will take to reach $50,000, we use the formula with $A = \$50,000$ and x (in the exponent) is the number of years.
$\$50,000 = \$4000 \cdot 1.065^x$
$12.5 = 1.065^x$: Divide both sides by $4000
$log 12.5 = log 1.065^x$: Take the log of both sides

$log12.5 = xlog1.065$: Using $logx^y = ylogx$ from prior lesson
$\frac{log12.5}{log1.065} = x$: Divide both sides by $log1.065$
$\frac{1.096}{.0273} = x$: With a calculator
$40.14 = x$: With a calculator

It will take just over 40 years for Abbi's initial \$4000 to become \$50,000 at 6.5% interest compounded yearly.

3) The formula for calculating decay is again very similar: $V_f = V_i \cdot r^t$ Where " V_f" is final value, "V_i" is initial value, "r" is decay factor (depreciation rate), and "t" is time (in years).

a) The decay rate is simply $1 - .5 = .5$ since the car's value decays at a rate of 50% per year.

b) In 5 years, the car will be worth apx
$V_f = \$30,000 \cdot .5^5$
$V_f = \$30,000 \cdot .03125$
$V_f = \$937.50$ OUCH!

c) If the car loses 1/2 of its value each year, and it is worth apx \$1000 after 5 years:
Year 6 = $\$1000 \cdot .5 = \$500$
Year 7 = $\$500 \cdot .5 = \$250$
Year 8 = $\$250 \cdot .5 = \$125$

It will take only about 8 years before the car is only worth \$100. Brandon **may** have made a questionable purchase!

Practice

Calculate the following values using: $A = P \cdot r^t$

Assume all rates are x% per year, compounded yearly unless specified otherwise

1. What is the value of a \$5000 investment after 5 years at a rate of 5% ?
2. What is the value of a \$15000 investment after 3 years at a rate of 8% ?
3. What is the value of a \$3500 investment after 12 years at a rate of 2% ?
4. What is the value of a \$7550 investment after 7 years at a rate of 4.3% ?
5. What is the value of a \$42,340 investment after 13 years at a rate of 5.034% ?

For problems 6-10, calculate:

a) The growth factor

b) The final population

6. If a population starts at 5,000 people in 1995, and increases at a rate of 7% per year, what is the population in 2032?
7. If a population starts at 15,000 people in 2000, and increases at a rate of 3% per year, what is the population in 2027?
8. If a population starts at 25,500 people in 1900, and increases at a rate of 2% per year, what is the population in 2008?
9. If a population starts at 87,432 people in 1940, and increases at a rate of 4.3% per year, what is the population in 2040?
10. If a population starts at 126,352 people in 1776, and increases at a rate of 1.067% per year, what is the population in 2012?

For problems 11-15, calculate:

a) The decay factor (recall that decay factor = 1 - %decay as a decimal)

b) The final value, using $V_f = V_i \cdot r^t$ from the lesson.

11. A car is worth \$4000, and loses value at a rate of 12% per year, what will it be worth in 5 years?
12. A boat is purchased for \$14,000, and loses value at a rate of 16% per year, what will it be worth in 7 years?
13. A car is purchased for \$40,500, and loses value at a rate of 21% per year, what will it be worth in 4 years?
14. A motorcycle is worth \$9350, and loses value at a rate of 6.5% per year, what will it be worth in 3.5 years?
15. A plane is purchased for \$342,137, and loses value at a rate of 4.67% per year, what will it be worth in 13 years?

For problems 16-20, calculate the number of years required before the value reaches the specified total, using $A_f = A_i \cdot r^t$ and beginning with $A_f =$ final amount, and x (in the exponent) as the number of years.

16. How many years before a population of 5,000 reaches at least 8,000 at a growth rate of 6%?
17. How many years before a value of \$4,000 reaches at least \$7,000 at a growth rate of 4%?
18. How many years before a value of \$12,000 reaches at least \$25,000 at a growth rate of 12%?
19. How many years before a population of 15,500 reaches at least 46,000 at a growth rate of 8.5%?
20. How many years before the value of a car currently worth \$52,138 depreciates to at least \$8,000 at a depreciation rate of 14.7% ?

Vocabulary:

Exponential Functions: Domain, Range, Asymptote, Exponential Function

Exponential Growth and Decay: Compound Interest, Decay Factor, Exponential Decay, Exponential Growth, Growth Factor

Modeling Exponential Data: Exponential Decay, Exponential Function, Exponential Growth, Regression