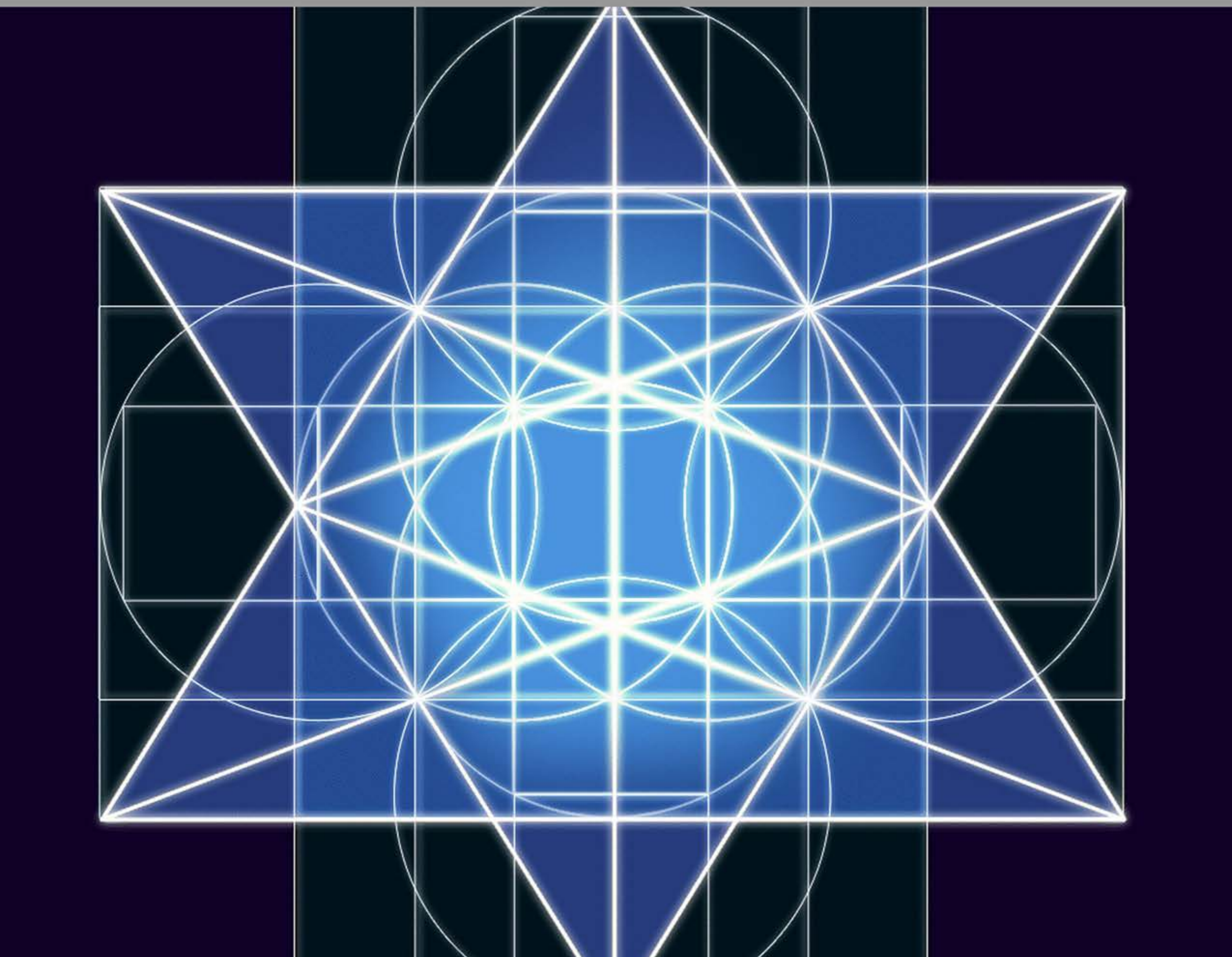
SFDRCISD Geometry



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Printed: October 20, 2016

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Contents

CHAPTER 1 Basics Geometry

Chapter Outline

In this chapter, students will learn about the building blocks of geometry. We will start with what the basic terms: point, line and plane. From here, students will learn about segments, midpoints, angles, bisectors, angle relationships, and how to classify polygons.

1.1 Points, Lines, and Planes

TEKS **G(1)A, G(1)C, G(1)E, G(4)A**

Learning Objectives

- Understand the terms *point, lines,* and *plane*
- Draw and label terms in a diagram

Vocabulary

- point
- line
- plane
- collinear
- noncollinear
- coplanar
- noncoplanar
- line segment
- ray
- postulate
- theorem

Warm-Up

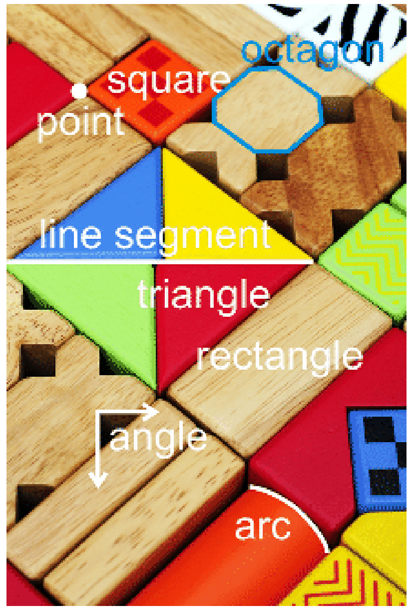
1. List and draw pictures of five geometric figures you are familiar with.
2. What shape is a yield sign?

3. Solve the algebraic equations.
   a. $4x-7=29$
   b. $-3x+5=17$

Know What? Geometry is everywhere. Remember these wooden blocks that you played with as a kid? If you played with these blocks, then you have been "studying" geometry since you were a child.

How many sides does the octagon have? What is something in-real life that is an octagon?

Geometry: The study of shapes and their spatial properties.



Building Blocks

In Geometry there three undefined terms, which means there is no formal definition but instead an "agreed" definition. These three words are *point, line,* and *plane.*

Point: An exact location in space.

A point describes a **location**, but has no size. Examples:

Points are labeled by a single capital letter.

TABLE 1.1:

Label It	***Say It***
A	point A

Line: Infinitely many points that extend forever in both directions.

A line has **direction** and **location** and is <u>always straight</u>.

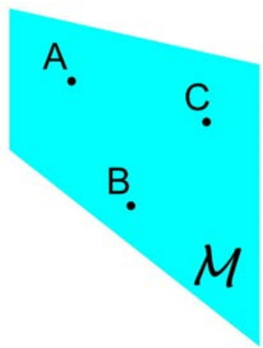
Lines are labeled by a single lower case letter or by any two points on the line.

TABLE 1.2:

Label It	*Say It*
line g	line g
$\overleftrightarrow{PQ}$	line PQ
$\overleftrightarrow{QP}$	line QP

Plane: Infinitely many intersecting lines that extend forever in all directions.

Think of a plane as a huge sheet of paper that goes on forever.



Planes are labeled by a single cursive capital letter or by any three points on the plane.

TABLE 1.3:

Label It	*Say It*
Plane $\mathcal{M}$	Plane M
Plane ABC	Plane ABC

Example 1: What best describes San Diego, California on a globe?

A. point

B. line

C. plane

Solution: A city is usually labeled with a dot, or point, on a globe.

Example 2: What best describes the surface of a movie screen?

A. point

B. line

C. plane

Solution: The surface of a movie screen is most like a plane.

Beyond the Basics Now we can use **point**, **line**, and **plane** to define new terms.

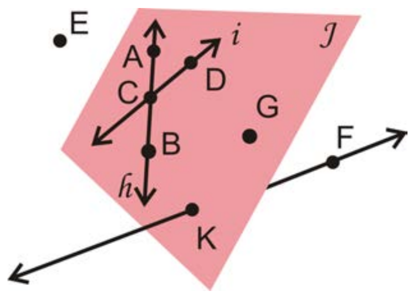
Space: The set of all points expanding in ***three*** dimensions.

Think back to the plane. It goes up and down, and side to side. If we add a third direction, we have space, something three-dimensional.

Collinear: Points that lie on the same line.

P, Q, R, S, and T are collinear because they are all on line w. If a point U was above or below line w, it would be **non-collinear**.

Coplanar: Points and/or lines within the same plane.



Lines h and i and points A, B, C, D, G, and K are **coplanar** in Plane $\mathcal{J}$.

Line $\overleftrightarrow{KF}$ and point E are **non-coplanar** with Plane $\mathcal{J}$.

Example 3: Use the picture above to answer these questions.

a) List another way to label Plane $\mathcal{J}$.

b) List another way to label line h.

c) Are K and F collinear? Explain.

d) Are E, B and F coplanar? Explain.

Solution:

a) Plane BDG. Any combination of three coplanar points that are not collinear would be correct.

b) $\overleftrightarrow{AB}$. Any combination of two of the letters A, C or B would also work.

c) Yes. It takes at least 2 points to create a line, therefor any 2 points will always form a single line.

d) Yes. It takes at least 3 points not on the same line to create a plane, therefore any 3 points not on the same line will always form a plane.

Endpoint: A point at the end of a line.

Line Segment: A line with two endpoints. Or, a line that stops at both ends.

Line segments are labeled by their endpoints. Order does not matter.

TABLE 1.4:

Label It	*Say It*
$\overline{AB}$	Segment AB
$\overline{BA}$	Segment BA

Ray: A line with one endpoint and extends forever in the other direction.

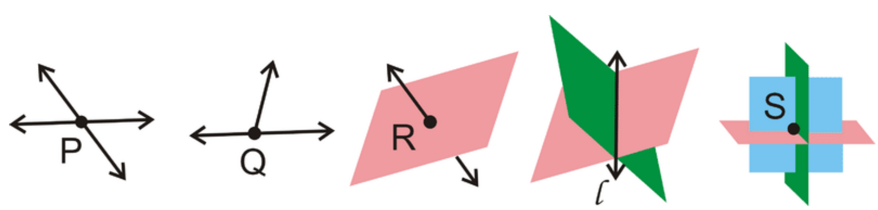
A ray is labeled by its endpoint and one other point on the line. For rays, order matters. When labeling, put endpoint under the side WITHOUT an arrow.

TABLE 1.5:

Label It	*Say It*
$\overrightarrow{CD}$	Ray CD
$\overleftarrow{DC}$	Ray CD

Use the following link to investigate lines, line segments, and rays. http://www.mathsisfun.com/definitions/line.html

Intersection: A point or line where lines, planes, segments or rays cross.



Example 4: What best describes a straight road connecting two cities?

A. ray

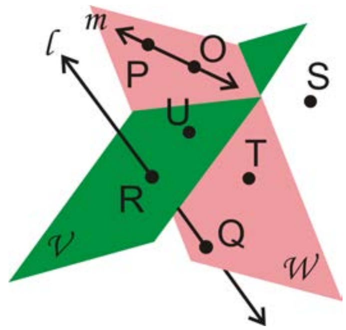
B. line

C. segment

D. plane

Solution: The straight road connects two cities, which are like endpoints. The best term is segment, or C.

Example 5: Answer the following questions about the picture to the right.



a) Is line l coplanar with Plane $\mathcal{V}$ or $\mathcal{W}$? Explain.

b) Are R and Q collinear? Explain.

c) What point is non-coplanar with either plane?

d) List three coplanar points in Plane $\mathcal{W}$.

Solution:

a) No. Line l and Plane $\mathcal{V}$ or $\mathcal{W}$ do not all lie in the same plane.

b) Yes. Through any two points there is a line.

c) S

d) Any combination of P, O, T and Q would work.

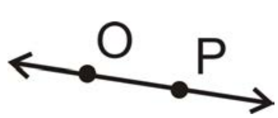
Further Beyond This section introduces a few basic postulates.

Postulates: Basic rules of geometry. ***We can assume that all postulates are true***.

Theorem: A statement that is ***proven true*** using postulates, definitions, and previously proven theorems.

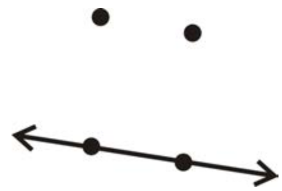
Postulate 1-1

There is exactly one (straight) line through any two points



Through points O and P is exactly one line.

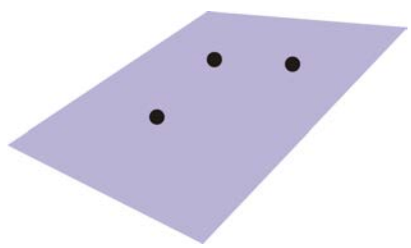
Investigation 1-1: Line Investigation



1. Draw two points anywhere on a piece of paper.
2. Use a ruler to connect these two points.
3. How many lines can you draw to go through these two points?

Postulate 1-2

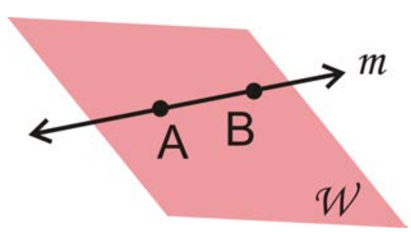
One plane contains any three non-collinear points



Any three points will always make a plane.

Postulate 1-3

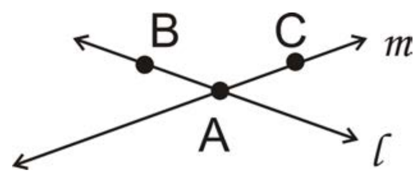
A line with points in a plane is also contained in that plane.



Line *m* is also in *plane W.*

Postulate 1-4

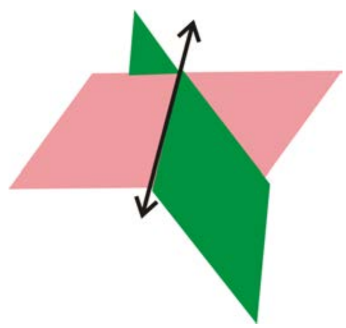
The intersection of two lines will be one point



Lines l and m intersect at point A.

Postulate 1-5

The intersection of two planes is a line

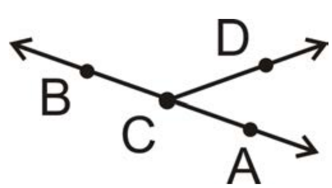


The pink and green plane intersect along a single line.

When making geometric drawings, you need to be clear and label all points and lines.

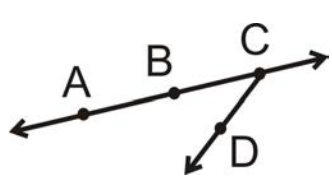
Example 6a: Draw and label the intersection of line $\overleftrightarrow{AB}$ and ray $\overrightarrow{CD}$ at point C.

Solution: It does not matter where you put A or B on the line, nor the direction that $\overrightarrow{CD}$ points.



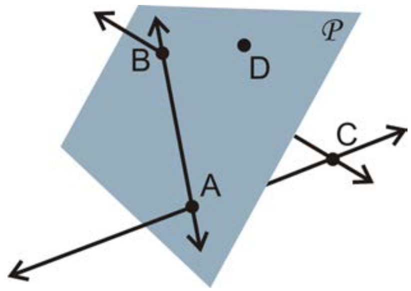
Example 6b: Redraw Example 6a, so that it looks different but is still true.

Solution:



Example 7: Describe the picture below using the geometric terms you have learned.

Solution: $\overleftrightarrow{AB}$ and D are coplanar in Plane $\mathcal{P}$, while $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ intersect at point C which is non-coplanar.



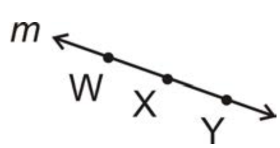
Know What? Revisited The octagon has 8 sides. In Latin, "octo" or "octa" means 8, so octagon, literally means "8-sided figure." An octagon in real-life would be a stop sign.

Practice Problems

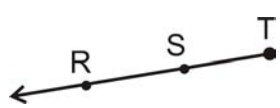
- Questions 1-5 are similar to Examples 6a and 6b.
- Questions 6-8 are similar to Examples 3 and 5.
- Questions 9-12 are similar to Examples 1, 2, and 4.
- Questions 13-16 are similar to Example 7.
- Questions 17-25 use the definitions and postulates learned in this lesson.

For questions 1-5, draw and label an image to fit the descriptions.

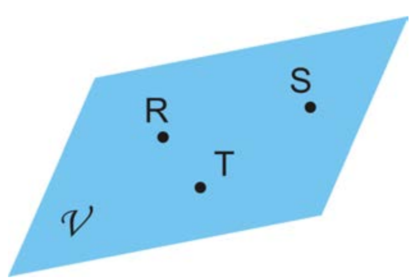
1. $\overrightarrow{CD}$ intersecting $\overline{AB}$ and Plane P containing $\overline{AB}$ but not $\overrightarrow{CD}$.
2. Three collinear points A, B, and C and B is also collinear with points D and E.
3. $\overrightarrow{XY}, \overrightarrow{XZ}$, and $\overrightarrow{XW}$ such that $\overrightarrow{XY}$ and $\overrightarrow{XZ}$ are coplanar, but $\overrightarrow{XW}$ is not.
4. Two intersecting planes, $\mathcal{P}$ and $\mathcal{Q}$, with $\overline{GH}$ where G is in plane $\mathcal{P}$ and H is in plane $\mathcal{Q}$.
5. Four non-collinear points, I, J, K, and L, with line segments connecting all points to each other.
6. Name this line in five ways.



7. Name the geometric figure in three different ways.



8. Name the geometric figure below in two different ways.

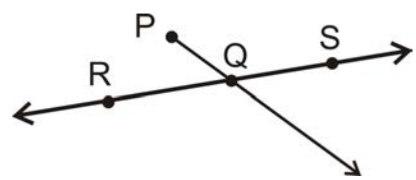


9. What is the best possible geometric model for a soccer field? Explain your answer.

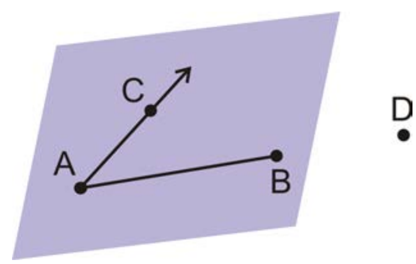
10. List two examples of where you see rays in real life.
11. What type of geometric object is the intersection of a line and a plane? Draw your answer.
12. What is the difference between a postulate and a theorem?

For 13-16, use geometric notation to explain each picture in as much detail as possible.

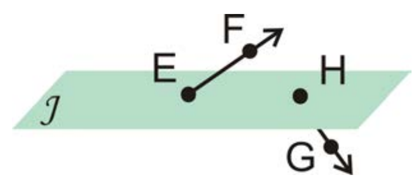
13.



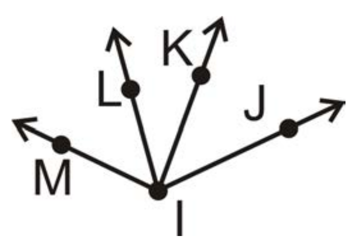
14.



15.



16.



For 17-25, determine if the following statements are true or false.

17. Any two points are collinear.
18. Any three points determine a plane.
19. A line is to two rays with a common endpoint.
20. A line segment is infinitely many points between two endpoints.
21. A point takes up space.
22. A line is one-dimensional.
23. Any four points are coplanar.
24. $\overrightarrow{AB}$ could be read "ray *AB*" or "ray *BA*."
25. $\overleftrightarrow{AB}$ could be read "line *AB*" or "line *BA*."

Use the diagram below for problems 26 – 28.

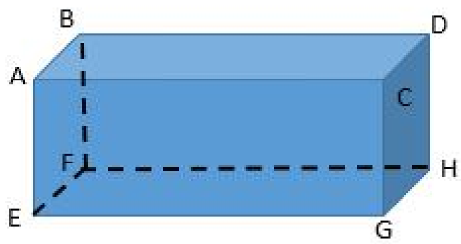


FIGURE 1.1

26. Name the intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{BF}$.

27. Name the intersection of plane ACD and BDH.

28. Name three planes that intersect at point E.

26.

29. **CHALLENGE** In the town depicted below, each street intersects every existing street one time. Each intersection consists of only two streets.

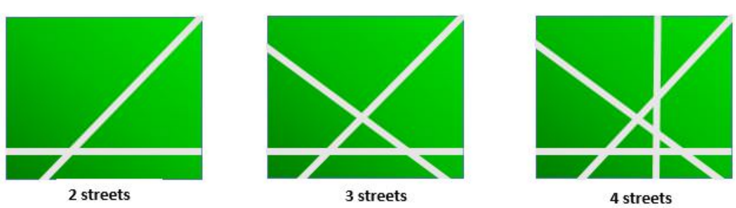


FIGURE 1.2

a. Each intersection requires one traffic light. If there are 5 streets in the town, how many traffic lights are needed? 6 streets?
b. What pattern could be used to determine the number of traffic lights needed for every additional street added to the town?

30. Describe how a set of points on a given line can be coplanar or non-coplanar.

31. Select 5 different real objects and describe how the objects can represent the 3 undefined terms in geometry.

Warm-Up Answers

1. Examples could be triangles, squares, rectangles, lines, circles, points, pentagons, stop signs (octagons), boxes (prisms), or dice (cubes).
2. A yield sign is a triangle with equal sides.
3.

   a. $4x - 7 = 29$
   $4x = 36$
   $x = 9$

b. $$\begin{aligned} -3x+5 &= 17 \\ -3x &= 12 \\ x &= -4 \end{aligned}$$

1.2 Segments and Distance

TEKS G(1)B, G(1)D, G(2)B

Learning Objectives

- Use the ruler postulate
- Use the segment addition postulate
- Plot line segments on the x-y plane
- find distances on a coordinate plane

Vocabulary

- distance

Warm-Up

1. Draw a line segment with endpoints C and D.
2. How would you label the following figure? List 2 different ways.

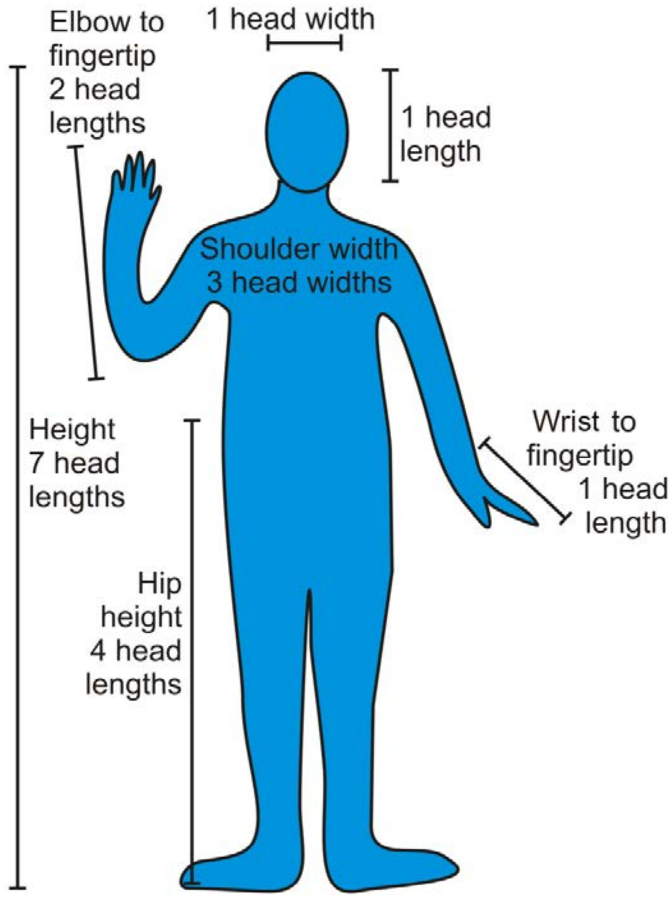
3. Draw three collinear points and a fourth that is coplanar.
4. Plot the following points on the $x-y$plane.

   a. (3, -3)
   b. (-4, 2)
   c. (0, -7)
   d. (6, 0)

Know What? The average adult human body can be measured in "heads." For example, the average human is 7-8 heads tall. When doing this, each person uses their own head to measure their own body. Other measurements are in the picture to the right.

See if you can find the following measurements:

- The length from the wrist to the elbow
- The length from the top of the neck to the hip
- The width of each shoulder



Measuring Distances

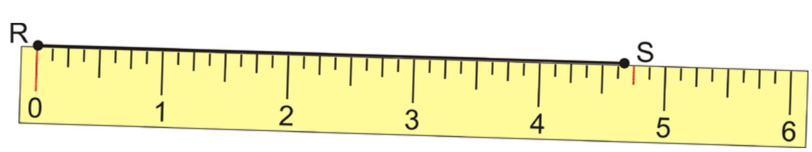
Distance: The length between two points.

Measure: To determine how far apart two geometric objects are.

The most common way to measure distance is with a ruler. In this text we will use inches and centimeters.

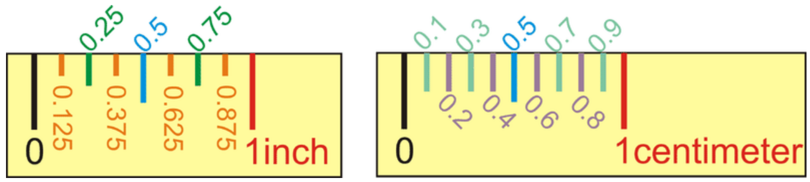
Example 1: Determine how long the line segment is, in inches. Round to the nearest quarter-inch.

Solution: To measure this line segment, it is very important to line up the "0" with the one of the endpoints. DO NOT USE THE EDGE OF THE RULER.



From this ruler, it looks like the segment is 4.75 inches (in) long.

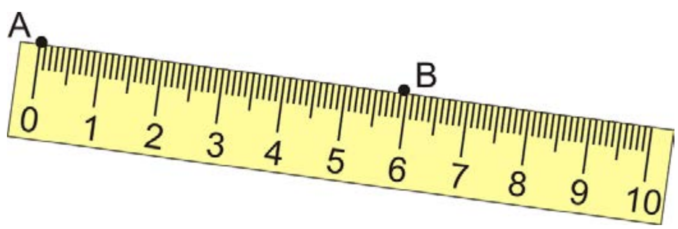
Inch-rulers are usually divided up by eight-inch (or 0.125 in) segments. Centimeter rulers are divided up by tenth-centimeter (or 0.1 cm) segments.



The two rulers above are **NOT DRAWN TO SCALE**, which means that the measured length is not the distance apart that it is labeled.

Example 2: Determine the measurement between the two points to the nearest tenth of a centimeter.

Solution: Even though there is no line segment between the two points, we can still measure the distance using a ruler.



It looks like the two points are 6 centimeters (cm) apart.

NOTE: We label a line segment, $\overline{AB}$ and the ***distance*** between A and B is shown below. m means measure. The two can be used interchangeably.

TABLE 1.6:

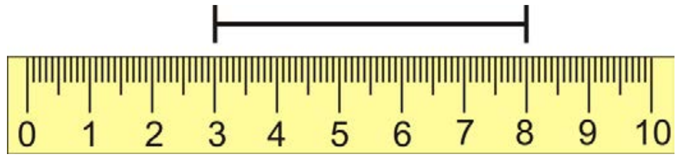
Label It	***Say It***
AB	The distance between A and B The measure of $\overline{AB}$

Ruler Postulate

The distance between two points is the absolute value of the difference between the numbers shown on the ruler

The ruler postulate implies that you do not need to start measuring at "0", as long as you subtract the first number from the second. "Absolute value" is used because ***distance is always positive***.

Example 3: What is the distance marked on the ruler below? The ruler is in centimeters.

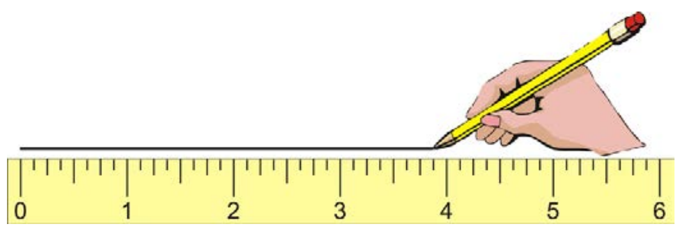


Solution: Subtract one endpoint from the other. The line segment spans from 3 cm to 8 cm. $|8-3| = |5| = 5$

The line segment is 5 cm long. Notice that you also could have done $|3-8| = |-5| = 5$.

Example 4: Draw $\overline{CD}$, such that $CD = 3.825$ *in*.

Solution: To draw a line segment, start at "0" and draw a segment to 3.825 in.



Put points at each end and label.

Segment Addition Postulate

If A, B, and C are collinear and B is between A and C, then $AB + BC = AC$.

First, in the picture below, is between and . As long as is , it can be considered to be the endpoints.

For example, if $AB = 5$ *cm* and $BC = 12$ *cm*, then AC must equal $5+12$ or 17 cm (in the picture above).

Example 5: Make a sketch of $\overline{OP}$, where Q is between O and P.

Solution: Draw $\overline{OP}$ first, then place Q on the segment.

Example 6: In the picture from Example 5, if $OP = 17$ and $QP = 6$, what is OQ?

Solution: Use the Segment Additional Postulate.

$$\begin{aligned} OQ + QP &= OP \\ OQ + 6 &= 17 \\ OQ &= 17 - 6 \\ OQ &= 11 \end{aligned}$$

Example 7: Make a sketch of: S is between T and V. R is between S and T. $TR = 6, RV = 23$, and $TR = SV$.

Solution: Interpret the first sentence first: S is between T and V.

Then add in what we know about R: It is between S and T. Put markings for $TR = SV$.

Example 8: Find SV, TS, RS and TV from Example 7.

Solution:

<u>*For SV*</u>: It is equal to TR, so $SV = 6\ cm$.

$$\begin{aligned} \underline{For\ RS}:\ RV &= RS + SV \\ 23 &= RS + 6 \\ RS &= 17\ cm \end{aligned} \qquad \begin{aligned} \underline{For\ TS}:\ TS &= TR + RS \\ TS &= 6 + 17 \\ TS &= 23\ cm \end{aligned} \qquad \begin{aligned} \underline{For\ TV}:\ TV &= TR + RS + SV \\ TV &= 6 + 17 + 6 \\ TV &= 29\ cm \end{aligned}$$

Example 9: ***Algebra Connection*** For $\overline{HK}$, suppose that J is between H and K. If $HJ = 2x + 4$, $JK = 3x + 3$, and $KH = 22$, find x.

Solution: Use the Segment Addition Postulate.

$$\begin{aligned} HJ \quad + \quad JK \quad &= KH \\ (2x + 4) + (3x + 3) &= 22 \\ 5x + 7 &= 22 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

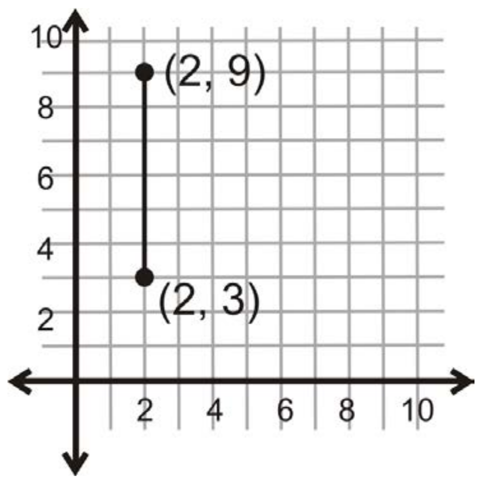
Distances on a Grid

You can now find the distances between points in the $x-y$ plane if the lines are horizontal or vertical.

If the line is vertical, find the change in the $y-$coordinates.

If the line is horizontal, find the change in the $x-$ coordinates.

Example 10: What is the distance between the two points shown below?

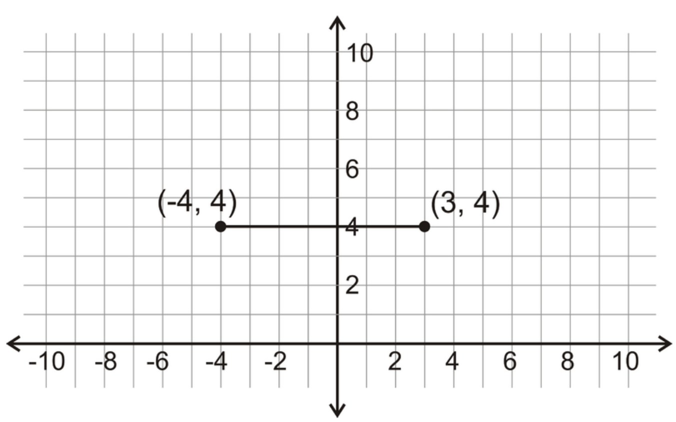


Solution: Because this line is vertical, look at the change in the $y-$coordinates.

$$|9-3| = |6| = 6$$

The distance between the two points is 6 units.

Example 11: What is the distance between the two points shown below?

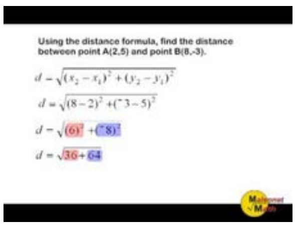


Solution: Because this line is horizontal, look at the change in the $x-$coordinates.

$$|(-4)-3| = |-7| = 7$$

The distance between the two points is 7 units.

If the line is NOT vertical* or *horizontal, you can compute the distance using Pythagorean Theorem or the Distance Formula. The following video will show you how.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/145622

Example 12: What is the distance between the two points shown below?

Solution: By using Pythagorean Theorem, you can determine the lengths of the legs are 9 and 6, and the length of the hypotenuse will be the length of the segment indicated. Therefore using Pythagorean Theorem says

$leg^2 + leg^2 = hypotenuse^2$

$9^2 + 6^2 = hyp^2$

$81 + 36 = hyp^2$

$117 = hyp^2$

$\sqrt{117} = \sqrt{hyp^2}$

$hyp = \sqrt{117} units$

$hyp = \sqrt{117} units$

The distance between the 2 points is $\sqrt{117}$ units long which is approximately 10.817 units.

You can find the same distance between 2 points by using the distance formula.

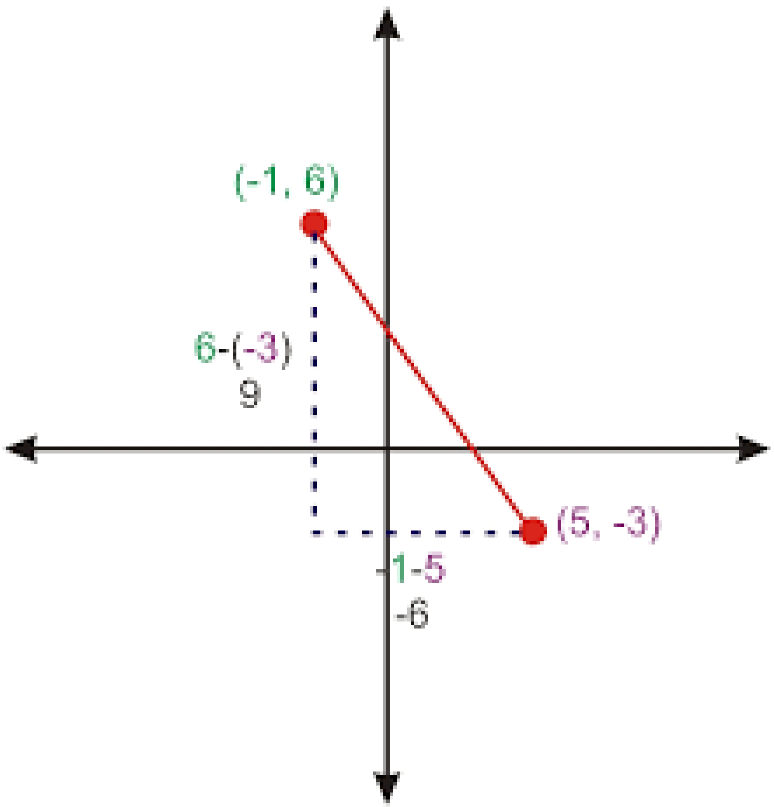


FIGURE 1.3

The Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) can be defined as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This formula will be derived in Chapter 9.

Example 13: Find the distance between (4, -2) and (-10, 3).

Solution: Plug in (4, -2) for (x_1, y_1) and (-10, 3) for (x_2, y_2) and simplify.

$$\begin{aligned} d &= \sqrt{(-10-4)^2 + (3+2)^2} \\ &= \sqrt{(-14)^2 + (5)^2} \\ &= \sqrt{196+25} \\ &= \sqrt{221} \approx 14.87 \ units \end{aligned}$$

Example 14: Find the distance between (-2, -3) and (3, 9).

Solution: Use the distance formula, plug in the points, and simplify.

$$\begin{aligned} d &= \sqrt{(-2-3)^2+(-3-9)^2} \\ &= \sqrt{(-5)^2+(-12)^2} \\ &= \sqrt{25+144} \\ &= \sqrt{169} = 13 \ units \end{aligned}$$

Know What? Revisited The length from the wrist to the elbow is one head, the length from the top of the neck to the hip is two heads, and the width of each shoulder one head width.

Practice Problems

- Questions 1-8 are similar to Examples 1 and 2.
- Questions 9-12 are similar to Example 3.
- Questions 13-17 are similar to Examples 5 and 6.
- Questions 18 and 19 are similar to Example 7 and 8.
- Questions 20 and 21 are similar to Example 9.
- Questions 22-24 are similar to Examples 10 and 11.
- Questions 25-28 are similar to Example 12, 13, and 14

For 1-4, find the length of each line segment in inches. Round to the nearest $\frac{1}{8}$ of an inch.

1.

2.

3.

4.

For 5-8, find the distance between each pair of points in centimeters. Round to the nearest tenth.

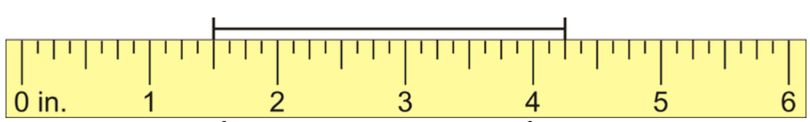
5.

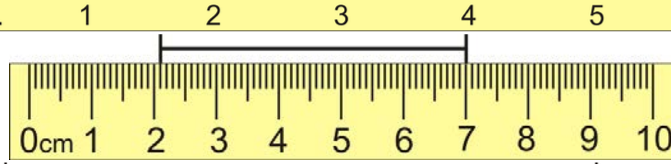
6.

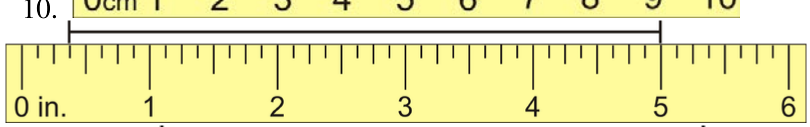
7.

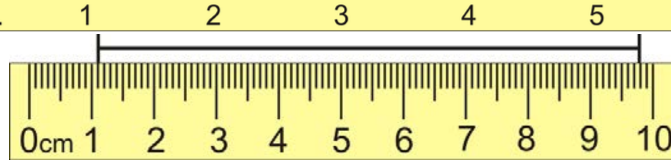
8.

For 9-12, use the ruler in each picture to determine the length of the line segment.

9.


10.


11.


12.


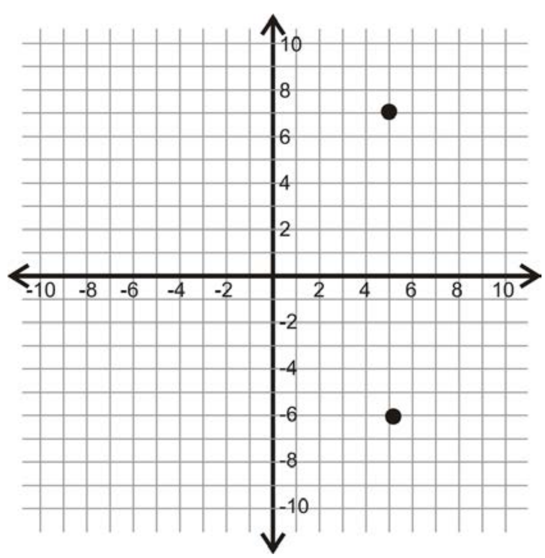
13. Make a sketch of $\overline{BT}$, with A between B and T.
14. If O is in the middle of $\overline{LT}$, where exactly is it located? If $LT = 16$ *cm*, what is LO and OT?
15. For three collinear points, A between T and Q.
    a. Draw a sketch.
    b. Write the Segment Addition Postulate.
    c. If $AT = 10$ *in* and $AQ = 5$ *in*, what is TQ?
16. For three collinear points, M between H and A.
    a. Draw a sketch.
    b. Write the Segment Addition Postulate.
    c. If $HM = 18$ *cm* and $HA = 29$ *cm*, what is AM?
17. For three collinear points, I between M and T.
    a. Draw a sketch.
    b. Write the Segment Addition Postulate.
    c. If $IT = 6$ *cm* and $MT = 25$ *cm*, what is IM?
18. Make a sketch that matches the description: B is between A and D. C is between B and D. $AB = 7$ *cm*, $AC = 15$ *cm*, and $AD = 32$ *cm*. Find BC, BD, and CD.
19. Make a sketch that matches the description: E is between F and G. H is between F and E. $FH = 4$ *in*, $EG = 9$ *in*, and $FH = HE$. Find FE, HG, and FG.

For 20 and 21, Suppose J is between H and K. Use the Segment Addition Postulate to solve for x. Then find the length of each segment.

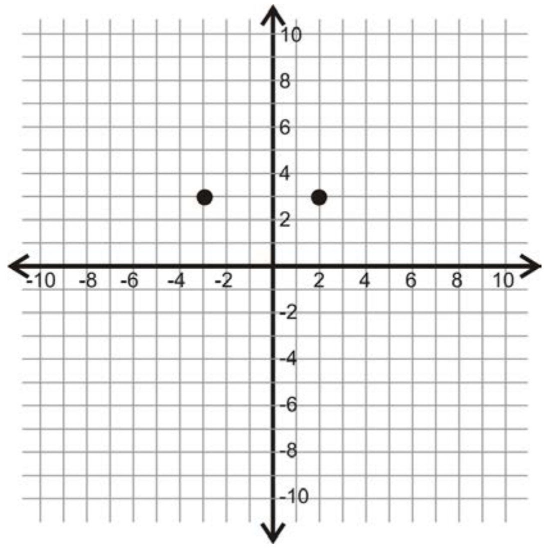
20. $HJ = 4x+9,\ JK = 3x+3,\ KH = 33$
21. $HJ = 5x-3,\ JK = 8x-9,\ KH = 131$

For 22-24, determine the vertical or horizontal distance between the two points.

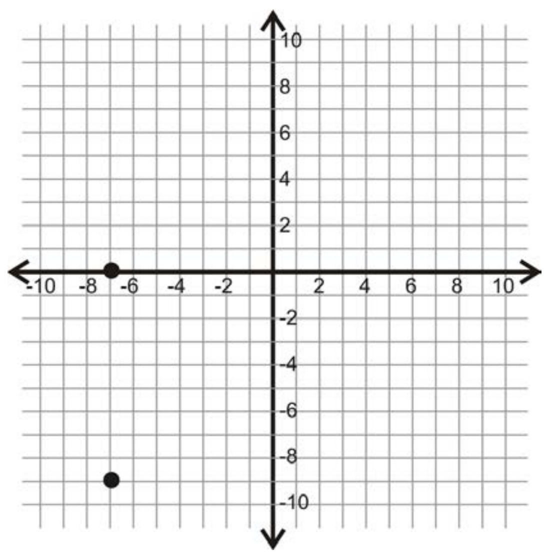
22.



23.



24.



For 25-28, determine the distance between the two points. Use either Pythagorean Theorem or the Distance Formula.

25. (2 , -5) and (-4 , 3)
26. (9 , 1) and (4 , 4)
27. (-6 , 8) and the origin
28. (3 , 7) and (-1 , 2)

Review and Reflect

34. Analyze the two different ways to find the distance between 2 points, and explain which method you prefer and why?

ALGEBRA Point *B* is between *A* and *C* on $\overline{AC}$. Use the given information to write an equation in terms of *x*. Solve the equation. Then find *AB* and *BC*.

29. $AB = 7x + 1$
 $BC = 4x - 4$
 $AC = 30$

30. $AB = 8x - 4$
 $BC = 6x - 6$
 $AC = 60$

31. $AB = 5x + 1$
 $BC = 3x - 6$
 $AC = 19$

32. **CHALLENGE** In the diagram, $\overline{AB} \cong \overline{BC}$, $\overline{AC} \cong \overline{CD}$, and $AD = 20$. Find the lengths of all the segments in the diagram. Suppose you choose one of the segments at random. What is the probability that the measure of the segment is greater than 5? *Explain.*

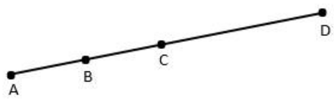


FIGURE 1.4

33. **PROBLEM SOLVING**

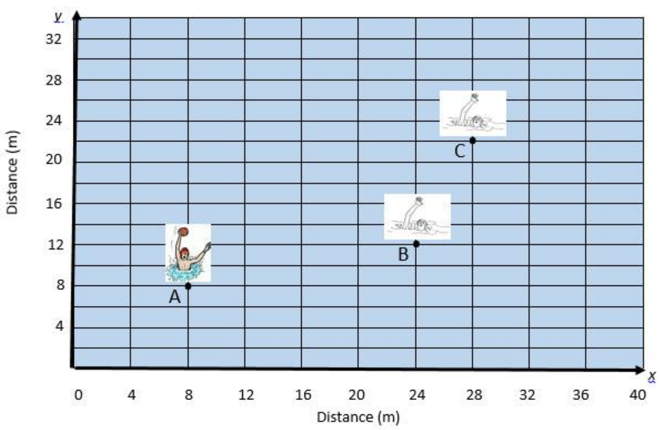


FIGURE 1.5

The diagram above shows the positions of three players during part of a water polo match. Player *A* throws the ball to Player *B*, who then throws it to Player *C*. How far did Player *A* throw the ball? How far did Player *B* throw the ball? How far would Player *A* have thrown the ball if he had thrown it directly to Player *C*? Round all answers to the nearest tenth of a meter.

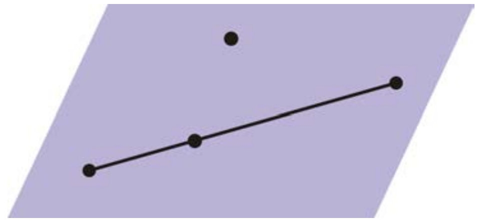
FIGURE 1.6

35. The word 'between' was used multiple times throughout this topic, based on your knowledge give your best definition for the word, between.

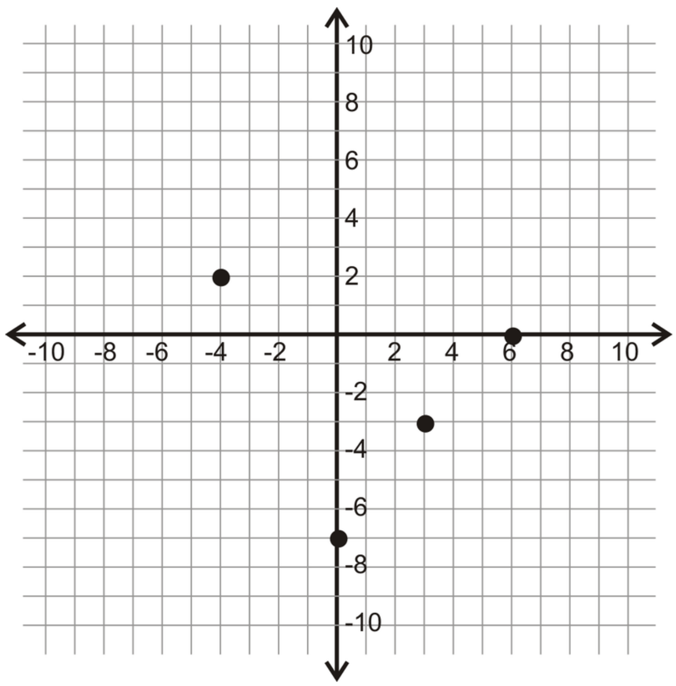
Warm-Up Answers

1.

2. line l, $\overline{MN}$
3.



4.



1.3 Angles and Measurement

TEKS G(1)C, G(1)F, G(5)A

Learning Objectives

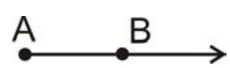
- Classify angles
- Apply the Protractor Postulate and the Angle Addition Postulate

Vocabulary

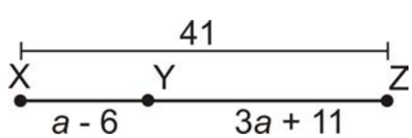
- angle
- vertex
- sides
- straight angle
- right angle
- acute angle
- obtuse angle
- perpendicular lines

Warm-Up

1. Label the following geometric figure. What is it called?



2. Find a, XY and YZ.



3. B is between A and C on $\overline{AC}$. If $AB = 4$ and $BC = 9$, what is AC?

Know What? Back to the building blocks. Every block has its own dimensions, angles and measurements. Using a protractor, find the measure of the three outlined angles in the "castle" below.

Two Rays = One Angle

In #1 above, the figure was a ray. It is labeled $\overrightarrow{AB}$, with the arrow over the point that is NOT the endpoint. When two rays have the same endpoint, an angle is created.

Angle: When two rays have the same endpoint.

Vertex: The common endpoint of the two rays that form an angle.

Sides: The two rays that form an angle.

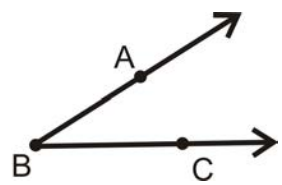


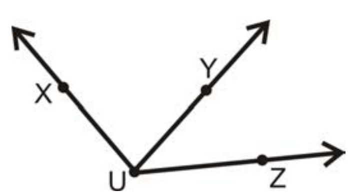
TABLE 1.7:

Label It	***Say It***
$\angle ABC$	Angle ABC
$\angle CBA$	Angle CBA

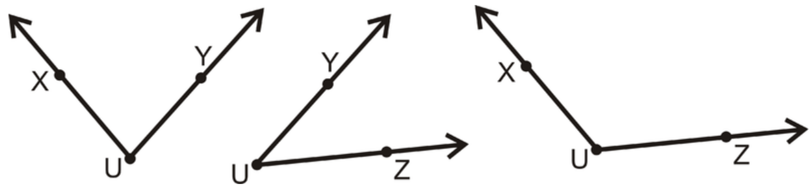
The vertex is B and the sides are $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Always use three letters to name an angle, $\angle$ SIDE-VERTEX-SIDE.

Example 1: How many angles are in the picture below? Label each one.



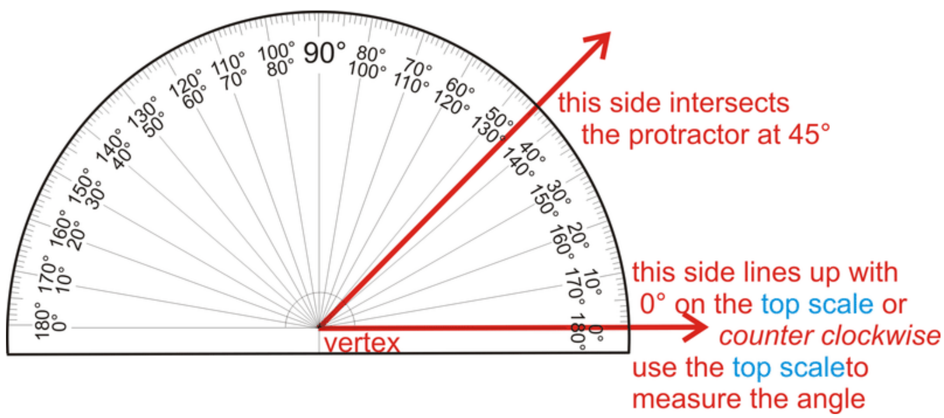
Solution: There are three angles with vertex U. It might be easier to see them all if we separate them.



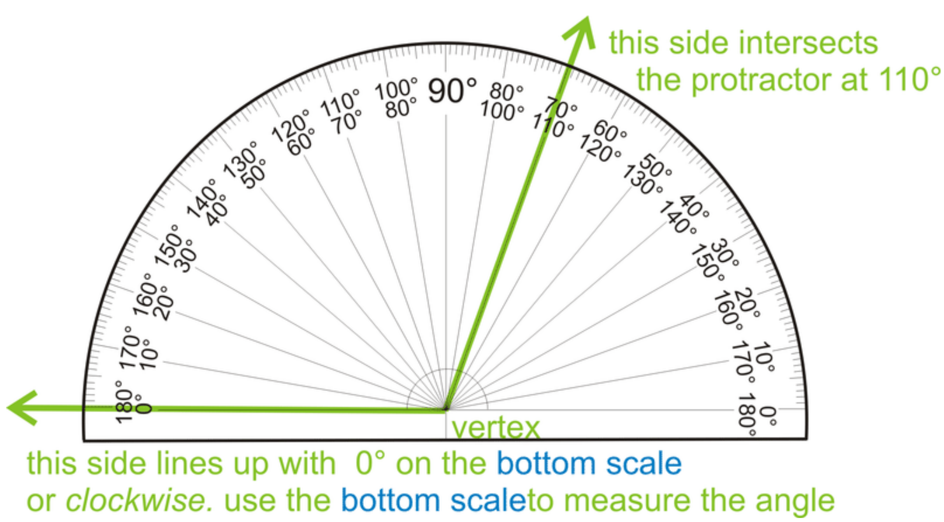
So, the three angles can be labeled, $\angle XUY$ (or $\angle YUX$), $\angle YUZ$ (or $\angle ZUY$), and $\angle XUZ$ (or $\angle ZUX$).

Protractor Postulate

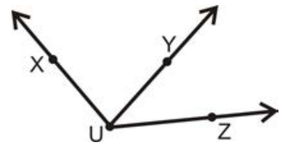
We measure a line segment's *length* with a ruler. Angles are measured with something called a ***protractor***. A protractor is a measuring device that measures how "open" an angle is. Angles are measured in degrees, and labeled with a $^\circ$ symbol.



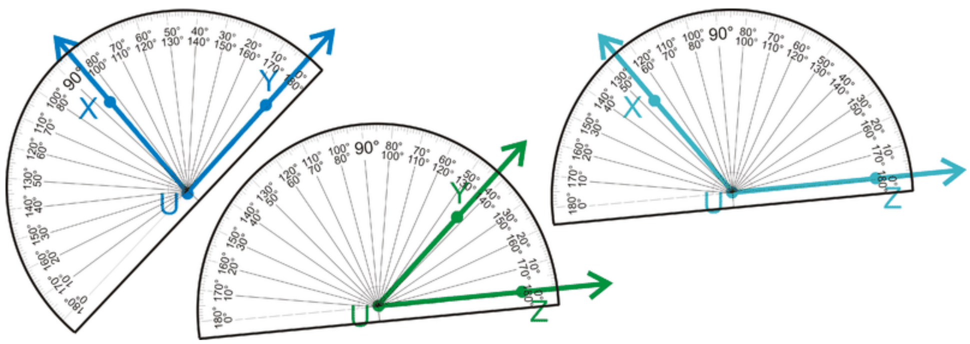
There are two sets of measurements, one starting on the left and the other on the right side of the protractor. Both go around from 0° to 180°. When measuring angles, always line up one side with 0°, and see where the other side hits the protractor. The vertex lines up in the middle of the bottom line.



Example 2: Measure the three angles from Example 1, using a protractor.



Solution: Just like in Example 1, it might be easier to measure these three angles if we separate them.



With measurement, we put an m in front of the $\angle$ sign to indicate measure. So, $m\angle XUY = 84^\circ$, $m\angle YUZ = 42^\circ$ and $m\angle XUZ = 126^\circ$.

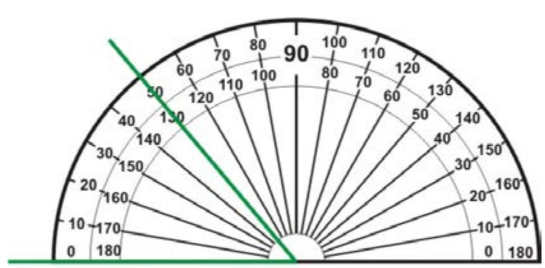
Just like the Ruler Postulate for line segments, there is a Protractor Postulate for angles.

Protractor Postulate

For every angle there is a number between 0° and 180° that is the measure of the angle. The angle's measure is the difference of the degrees where the sides of the angle intersect the protractor. ***For now, angles are always positive***.

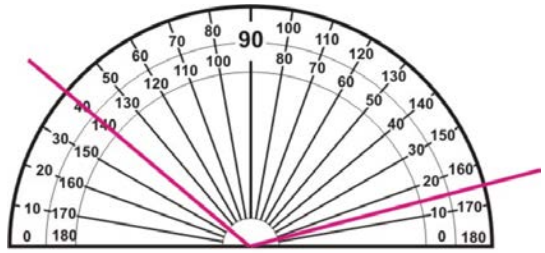
In other words, you do not have to start measuring an angle at 0°, as long as you subtract one measurement from the other.

Example 3: What is the measure of the angle shown below?



Solution: This angle is lined up with 0°, so where the second side intersects the protractor is the angle measure, which is 50°.

Example 4: What is the measure of the angle shown below?

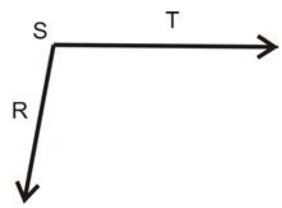


Solution: This angle is not lined up with 0°, so use subtraction to find its measure. It does not matter which scale you use.

Inner scale: $140^\circ - 15^\circ = 125^\circ$

Outer scale: $165^\circ - 40^\circ = 125^\circ$

Example 5: Use a protractor to measure $\angle RST$ below.

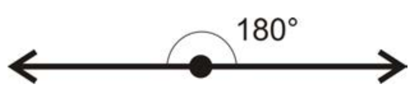


Solution: Lining up one side with 0° on the protractor, the other side hits 100°.

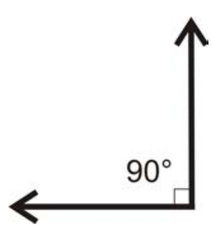
Classifying Angles

Angles can be grouped into four different categories.

Straight Angle: An angle that measures exactly 180°.

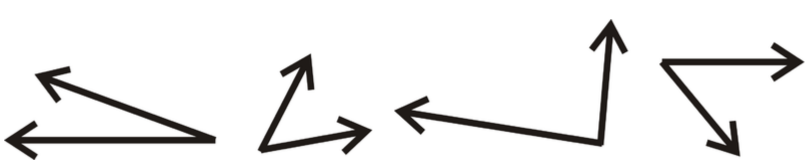


Right Angle: An angle that measures exactly 90°.

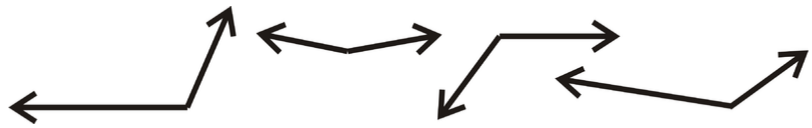


This "box" marks right, or 90°, angles.

Acute Angles: Angles that measure between 0° and less than 90°.

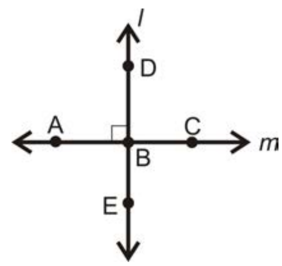


Obtuse Angles: Angles that measure more than 90° and less than 180°.



Use the following link to explore different angles.http://www.mathsisfun.com/angles.html

Perpendicular: When two lines intersect to form four right angles.



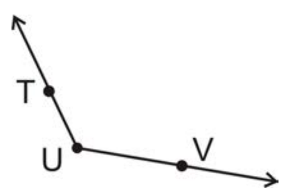
Even though all four angles are 90°, only one needs to be marked with the half-square.

The symbol for perpendicular is $\perp$.

TABLE 1.8:

Label It	***Say It***
$l \perp m$	Line l is perpendicular to line m.
$\overleftrightarrow{AC} \perp \overleftrightarrow{DE}$	Line AC is perpendicular to line DE.

Example 6: Name the angle and determine what type of angle it is.



Solution: The vertex is U. So, the angle can be $\angle TUV$ or $\angle VUT$. To determine what type of angle it is, compare it to a right angle.

Because it opens wider than a right angle, and less than a straight angle it is **obtuse**.

Example 7: What type of angle is 84°? What about 165°?

Solution: 84° is less than 90°, so it is **acute**. 165° is greater than 90°, but less than 180°, so it is **obtuse**.

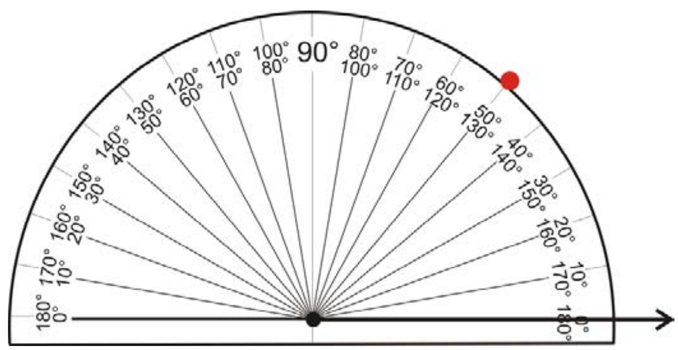
Drawing an Angle

Investigation 1-2: Drawing a 50° Angle with a Protractor

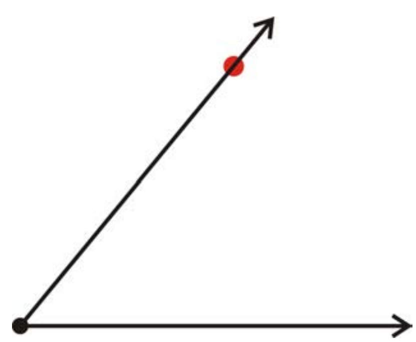
1. Start by drawing a horizontal line across the page, 2 in long.

2. Place an endpoint at the left side of your line.

3. Place the protractor on this point, such that the bottom line of the protractor is on the line and the endpoint is at the center. Mark 50° on the appropriate scale.



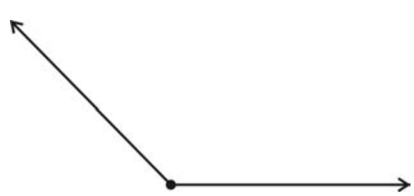
4. Remove the protractor and connect the vertex and the 50° mark.



This process can be used to draw any angle between 0° and 180°. See http://www.mathsisfun.com/geometry/protractor-using.html for an **animation** of this investigation.

Example 8: Draw a 135° angle.

Solution: Following the steps from above, your angle should look like this:



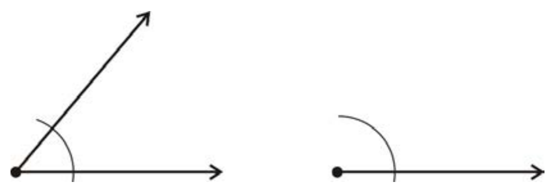
Now that we know how to draw an angle, we can also copy that angle with a compass and a ruler. Anytime we use a compass and ruler to draw geometric figures, it is called a **construction**.

Compass: A tool used to draw circles and arcs.

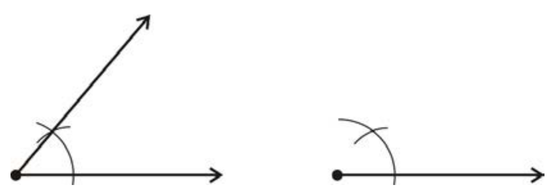
Investigation 1-3: Copying an Angle with a Compass and Ruler

1. We are going to copy the 50° angle from Investigation 1-2. First, draw a straight line, 2 inches long, and place an endpoint at one end.

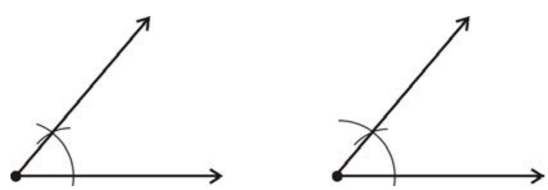
2. With the point (non-pencil side) of the compass on the vertex, draw an arc that passes through both sides of the angle. Repeat this arc with the line we drew in #1.



3. Move the point of the compass to the horizontal side of the angle we are copying. Place the point where the arc intersects this side. Open (or close) the "mouth" of the compass so that you can draw an arc that intersects the other side and the arc drawn in #2. Repeat this on the line we drew in #1.



4. Draw a line from the new vertex to the arc intersections.

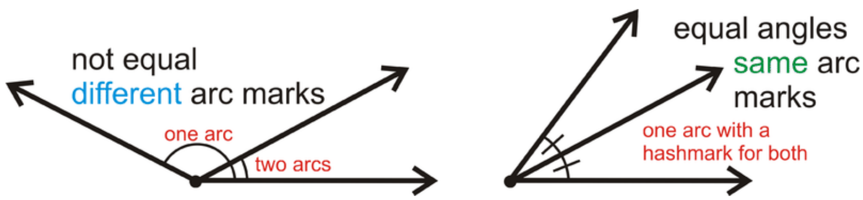


To watch an **animation** of this construction, see http://www.mathsisfun.com/geometry/construct-anglesame.html

Marking Angles and Segments in a Diagram

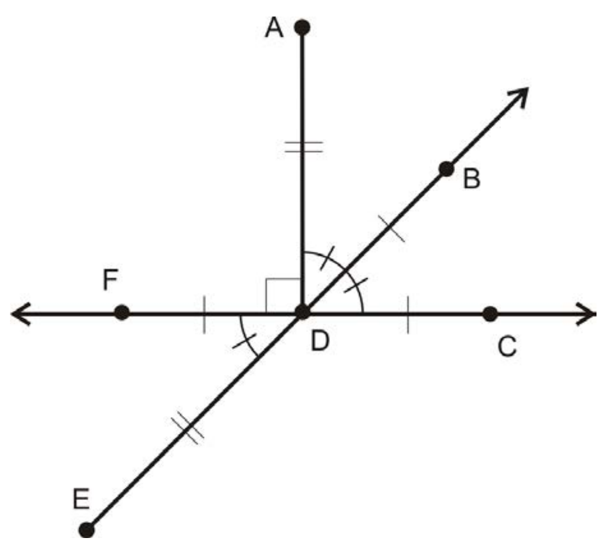
With all these segments and angles, we need to have different ways to label equal angles and segments.

Angle Markings



Segment Markings

Example 9: Write all equal angle and segment statements.



Solution: $\overline{AD} \perp \overleftrightarrow{FC}$

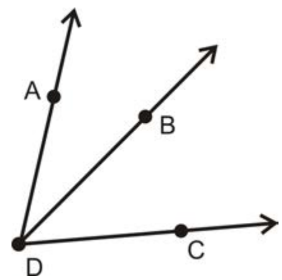
$$\begin{aligned} m\angle ADB = m\angle BDC = m\angle FDE = 45^\circ \\ AD = DE \\ FD = DB = DC \\ m\angle ADF = m\angle ADC = 90^\circ \end{aligned}$$

Like the Segment Addition Postulate, there is an Angle Addition Postulate.

Angle Addition Postulate

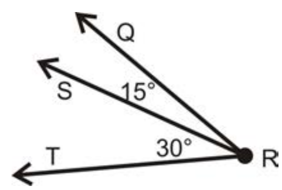
If B is on the interior of $\angle ADC$, then

$$m\angle ADC = m\angle ADB + m\angle BDC$$



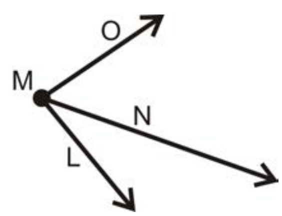
If B is in the interior of $\angle ADC$, then $m\angle ADB + m\angle BDC = m\angle ADC$

Example 10: What is $m\angle QRT$ in the diagram below?



Solution: Using the Angle Addition Postulate, $m\angle QRT = 15^\circ + 30^\circ = 45^\circ$.

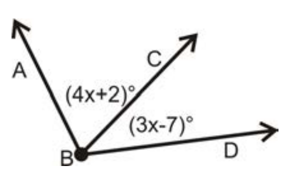
Example 11: What is $m\angle LMN$ if $m\angle LMO = 85^\circ$ and $m\angle NMO = 53^\circ$?



Solution: $m\angle LMO = m\angle NMO + m\angle LMN$, so $85^\circ = 53^\circ + m\angle LMN$.

$$m\angle LMN = 32^\circ.$$

Example 12: ***Algebra Connection*** If $m\angle ABD = 100^\circ$, find x.



Solution: $m\angle ABD = m\angle ABC + m\angle CBD$. Write an equation.

$$\begin{aligned} 100^\circ &= (4x+2)^\circ + (3x-7)^\circ \\ 100^\circ &= 7x^\circ - 5^\circ \\ 105^\circ &= 7x^\circ \\ 15^\circ &= x \end{aligned}$$

Know What? Revisited Using a protractor, the measurement marked in the red triangle is 90°, the measurement in the green triangle is 45° and the measurement in the blue square is 90°.

Practice Problems

- Questions 1-10 use the definitions, postulates and theorems from this section.
- Questions 11-16 are similar to Investigation 1-2 and Examples 7 and 8.
- Questions 17 and 18 are similar to Investigation 1-3.
- Questions 19-22 are similar to Examples 2-5.
- Question 23 is similar to Example 9.
- Questions 24-28 are similar to Examples 10 and 11.
- Questions 29 and 30 are similar to Example 12.

For questions 1-10, determine if the statement is true or false. If false, explain why or draw a diagram to show your reasoning.

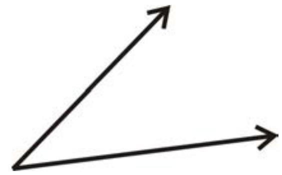
1. Two angles always add up to be greater than 90°.
2. 180° is an obtuse angle.
3. 180° is a straight angle.
4. Two perpendicular lines intersect to form four right angles.
5. A construction uses a protractor and a ruler.
6. For an angle $\angle ABC, C$ is the vertex.
7. For an angle $\angle ABC, \overline{AB}$ and $\overline{BC}$ are the sides.
8. The m in front of $m\angle ABC$ means measure.
9. Angles are always measured in degrees.
10. The Angle Addition Postulate says that an angle is equal to the sum of the smaller angles around it.

For 11-16, draw the angle with the given degree, using a protractor and a ruler. Also, state what type of angle it is.

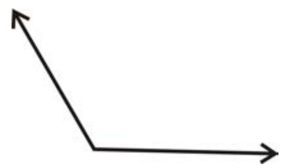
11. 55°
12. 92°
13. 178°
14. 5°
15. 120°
16. 73°
17. ***Construction*** Copy the angle you made from #12, using a compass and a ruler.
18. ***Construction*** Copy the angle you made from #16, using a compass and a ruler.

For 19-22, use a protractor to determine the measure of each angle. Also, state what type of angle it is.

19.

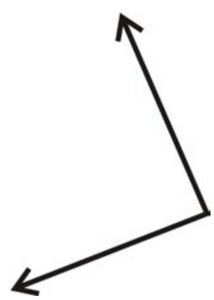


20.

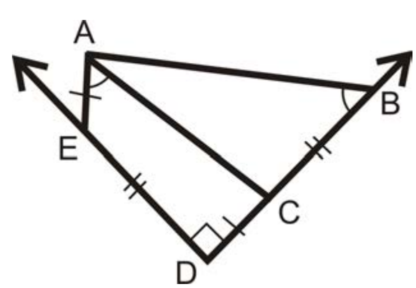


21.

22.



23. Interpret the picture to the right. Write down all equal angles, segments and if any lines are perpendicular.



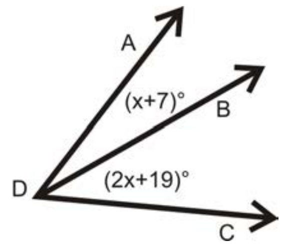
In Exercises 24-29, use the following information: Q is in the interior of $\angle ROS$. S is in the interior of $\angle QOP$. P is in the interior of $\angle SOT$. S is in the interior of $\angle ROT$ and $m\angle ROT = 160^\circ$, $m\angle SOT = 100^\circ$, and $m\angle ROQ = m\angle QOS = m\angle POT$.

24. Make a sketch.
25. Find $m\angle QOP$
26. Find $m\angle QOT$
27. Find $m\angle ROQ$

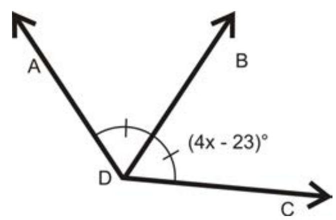
28. Find $m\angle SOP$

Algebra Connection Solve for x.

29. $m\angle ADC = 56^\circ$



30. $m\angle ADC = 130^\circ$



32. When might it be appropriate to use the Angle Addition Postulate in real life? Explain.

33. Describe the different ways to classify angles, and give at least one real world example of each.

Warm-Up Answers

1. $\overrightarrow{AB}$, a ray

2. $XY = 3, YZ = 38$

$$\begin{aligned} a-6+3a+11 &= 41 \\ 4a+5 &= 41 \\ 4a &= 36 \\ a &= 9 \end{aligned}$$

3. Use the Segment Addition Postulate, $AC = 13$.

1.4 Midpoints and Bisectors

TEKS G(1)D, G(1)F, G(2)A, G(2)B, G(5)A, G(5)B

Learning Objectives

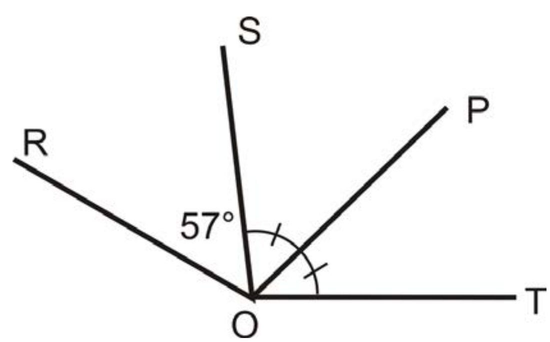
- Identify the midpoint of line segments
- Identify the bisector of a line segment
- Understand and use the Angle Bisector Postulate

Vocabulary

- congruent
- midpoint
- bisector
- perpendicular bisector
- angle bisector

Warm-Up

1. $m\angle SOP = 38^\circ$, find $m\angle POT$ and $m\angle ROT$.



2. Find the slope between the two numbers.
   a. (-4, 1) and (-1, 7)
   b. (5, -6) and (-3, -4)
3. Find the average of these numbers: 23, 30, 18, 27, and 32.

Know What? The building to the right is the Transamerica Building in San Francisco. This building was completed in 1972 and, at that time was one of the tallest buildings in the world. In order to make this building as tall as it is and still abide by the building codes, the designer used this pyramid shape.

It is very important in designing buildings that the angles and parts of the building are equal. What components of this building look equal? Analyze angles, windows, and the sides of the building.

Congruence

You could argue that another word for ***equal*** is ***congruent***. But, the two are a little different.

Congruent: When two geometric figures have the same shape and size.

TABLE 1.9:

Label It	***Say It***
$\overline{AB} \cong \overline{BA}$	line segment AB is congruent to line segment BA

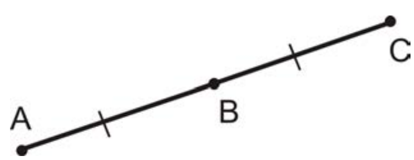
TABLE 1.10:

Equal	***Congruent***
$=$	$\cong$
used with ***measurement***	used with ***figures***
$AB = 5cm$	$\overline{AB} \cong \overline{BA}$
$m\angle ABC = 60^\circ$	$\angle ABC \cong \angle CBA$

If two segments or angles are congruent, then they are also equal.

Midpoints

Midpoint: A point on a line segment that divides it into two congruent segments.

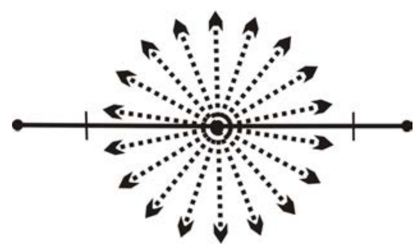


Because $AB = BC$, B is the midpoint of $\overline{AC}$.

Midpoint Postulate

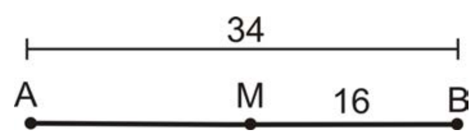
Any line segment will have exactly one midpoint

This postulate is referring to the *midpoint*, not the lines that pass through the midpoint.



There are infinitely many lines that pass through the midpoint.

Example 1: Is M a midpoint of $\overline{AB}$?

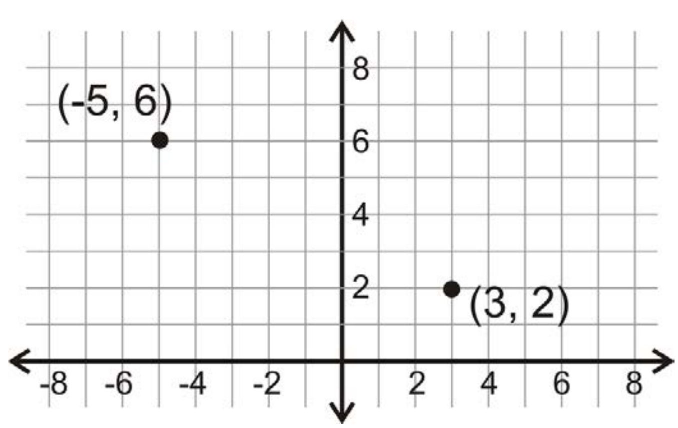


Solution: No, it is not $MB = 16$ and $AM = 34 - 16 = 18$. AM must equal MB in order for M to be the midpoint of $\overline{AB}$.

Midpoint Formula

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/147997

When points are plotted in the coordinate plane, we can use a formula to find the midpoint between them. Here are two points, (-5, 6) and (3, 4).



It follows that the midpoint should be halfway between the points on the line. Just by looking, it seems like the midpoint is (-1, 4).

Midpoint Formula: For two points, (x_1, y_1) and x_2, y_2, the midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Let's use the formula to make sure (-1, 4) is the midpoint between (-5, 6) and (3, 2).

$$\left(\frac{-5+3}{2}, \frac{6+2}{2}\right) = \left(\frac{-2}{2}, \frac{8}{2}\right) = (-1, 4)$$

Always use this formula to determine the midpoint.

Example 2: Find the midpoint between (9, -2) and (-5, -1).

Solution: Plug the points into the formula.

$$\left(\frac{9+(-5)}{2}, \frac{-2+(-1)}{2}\right) = \left(\frac{4}{2}, \frac{-3}{2}\right) = (2, -\frac{3}{2})$$

Example 3: If $M(3, -1)$ is the midpoint of $\overline{AB}$ and $B(7, -6)$, find A.

Solution: Plug in what you know into the midpoint formula.

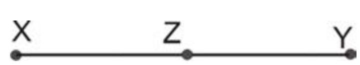
$$\left(\frac{7+x_A}{2}, \frac{-6+y_A}{2}\right) = (3, -1)$$
$$\frac{7+x_A}{2} = 3 \text{ and } \frac{-6+y_A}{2} = -1$$
$$7+x_A = 6 \text{ and } -6+y_A = -2$$
$$x_A = -1 \text{ and } y_A = 4$$
$$\text{So, } A \text{ is } (-1, 4).$$

Segment Bisectors

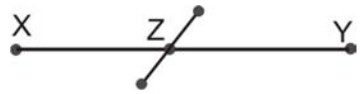
Segment Bisector: A bisector cuts a line segment into two congruent parts and passes through the midpoint.

Example 4: Use a ruler to draw a bisector of the segment.

Solution: First, find the midpoint. Measure the line segment. It is 4 cm long. To find the midpoint, divide 4 cm by 2 because we want 2 equal pieces. Measure 2 cm from one endpoint and draw the midpoint.



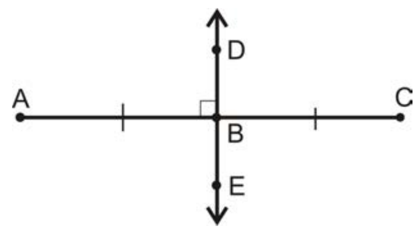
To finish, draw a line that passes through Z.



A specific type of segment bisector is called a perpendicular bisector.

Perpendicular Bisector: A line, ray or segment that passes through the midpoint of another segment and intersects the segment at a right angle.

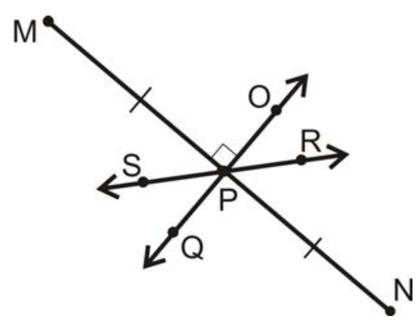
$$\overline{AB} \cong \overline{BC}$$
$$\overline{AC} \perp \overleftrightarrow{DE}$$



Perpendicular Bisector Postulate

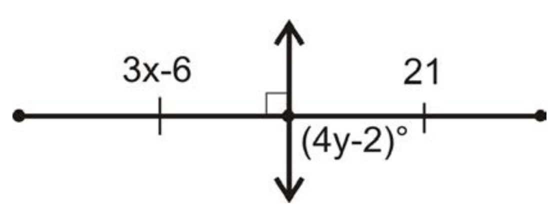
For every line segment, there is one perpendicular bisector

Example 5: Which line is the perpendicular bisector of $\overline{MN}$?



Solution: The perpendicular bisector must bisect $\overline{MN}$ and be perpendicular to it. Only $\overleftrightarrow{OQ}$ fits this description. $\overleftrightarrow{SR}$ is a bisector, but is not perpendicular.

Example 6: ***Algebra Connection*** Find x and y.



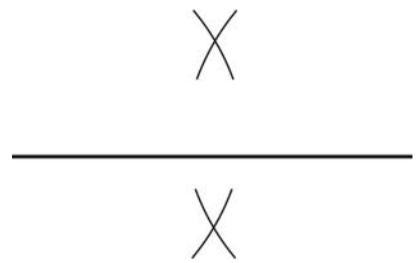
Solution: The line shown is the perpendicular bisector.

$$\begin{aligned} \text{So, } 3x-6 &= 21 \\ 3x &= 27 \\ x &= 9 \end{aligned} \qquad \begin{aligned} \text{And, } (4y-2)^\circ &= 90^\circ \\ 4y &= 92^\circ \\ y &= 23^\circ \end{aligned}$$

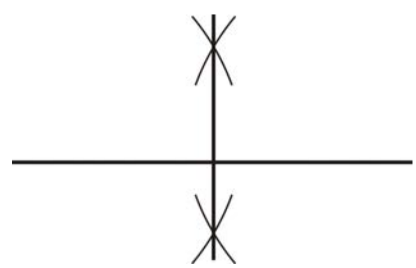
Investigation 1-4: Constructing a Perpendicular Bisector

1. Draw a line that is 6 cm long, halfway down your page.

2. Place the pointer of the compass at an endpoint. Open the compass to be greater than half of the segment. Make arc marks above and below the segment. Repeat on the other endpoint. Make sure the arc marks intersect.



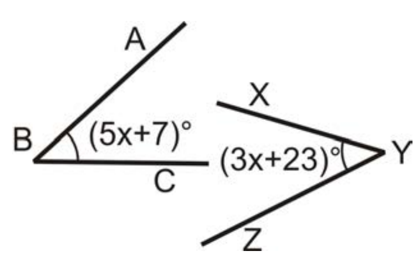
3. Use your straightedge to draw a line connecting the arc intersections.



This constructed line bisects the line you drew in #1 **and** intersects it at 90°. To see an animation of this investigation, go to http://www.mathsisfun.com/geometry/construct-linebisect.html .

Congruent Angles

Example 7: ***Algebra Connection*** What is the measure of each angle?



Solution: From the picture, we see that the angles are equal.

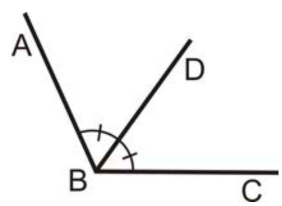
Set the angles equal to each other and solve.

$$\begin{aligned}(5x+7)^\circ &= (3x+23)^\circ\\ 2x^\circ &= 16^\circ\\ x &= 8^\circ\end{aligned}$$

To find the measure of $\angle ABC$, plug in $x = 8^\circ$ to $(5x+7)^\circ \rightarrow (5(8)+7)^\circ = (40+7)^\circ = 47^\circ$. Because $m\angle ABC = m\angle XYZ$, $m\angle XYZ = 47^\circ$ too.

Angle Bisectors

Angle Bisector: A ray that divides an angle into two congruent angles, each having a measure exactly half of the original angle.



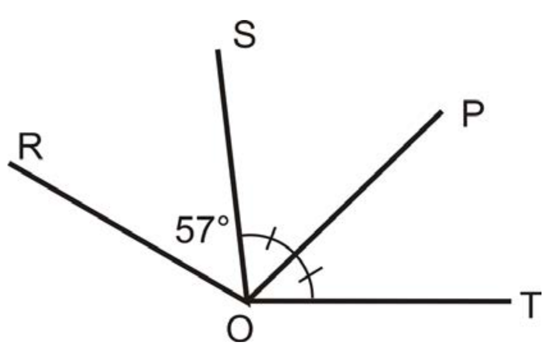
$\overline{BD}$ is the angle bisector of $\angle ABC$

$$\begin{aligned}\angle ABD &\cong \angle DBC\\ m\angle ABD &= \frac{1}{2}m\angle ABC\end{aligned}$$

Angle Bisector Postulate

Every angle has exactly one angle bisector

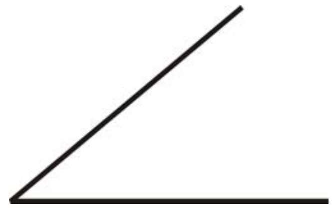
Example 8: Let's take a look at Review Queue #1 again. Is $\overline{OP}$ the angle bisector of $\angle SOT$?



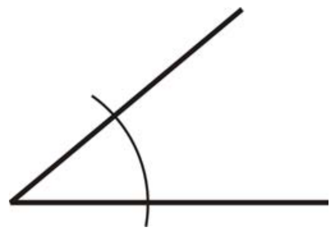
Solution: Yes, $\overline{OP}$ is the angle bisector of $\angle SOT$ from the markings in the picture.

Investigation 1-5: Constructing an Angle Bisector

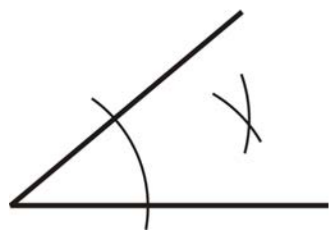
1. Draw an angle on your paper. Make sure one side is horizontal.



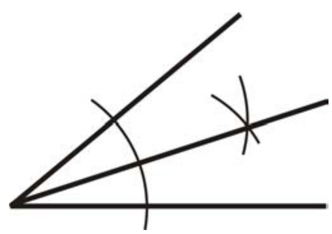
2. Place the pointer on the vertex. Draw an arc that intersects both sides.



3. Move the pointer to the arc intersection with the horizontal side. Make a second arc mark on the interior of the angle. Repeat on the other side. Make sure they intersect.



4. Connect the arc intersections from #3 with the vertex of the angle.

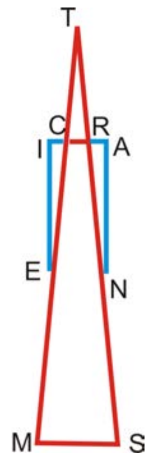


To see an animation of this construction, view http://www.mathsisfun.com/geometry/construct-anglebisect.html .

Know What? Revisited The image to the right is an outline of the Transamerica Building from earlier in the lesson. From this outline, we can see the following parts are congruent:

$$\begin{array}{lcl} \overline{TR} \cong \overline{TC} & & \angle TCR \cong \angle TRC \\ \overline{RS} \cong \overline{CM} & & \angle CIE \cong \angle RAN \\ \overline{CI} \cong \overline{RA} & \text{and} & \angle TMS \cong \angle TSM \\ \overline{AN} \cong \overline{IE} & & \angle IEC \cong \angle ANR \\ \overline{TS} \cong \overline{TM} & & \angle TCI \cong \angle TRA \end{array}$$

All the four triangular sides of the building are congruent to each other as well.

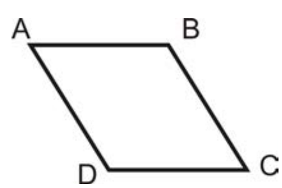


Practice Problems

- Questions 1-18 are similar to Examples 1, 4, 5 and 8.
- Questions 19-22 are similar to Examples 6 and 7.
- Question 23 is similar to Investigation 1-5.
- Question 24 is similar to Investigation 1-4.
- Questions 25-28 are similar to Example 2.
- Question 29 and 30 are similar to Example 3.

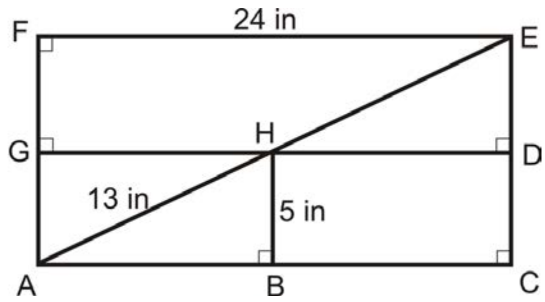
1. Copy the figure below and label it with the following information:

$$\angle A \cong \angle C$$
$$\angle B \cong \angle D$$
$$\overline{AB} \cong \overline{CD}$$
$$\overline{AD} \cong \overline{BC}$$



For 2-10, us the following picture to answer the questions.

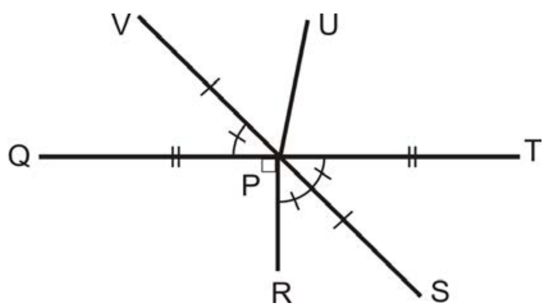
H is the midpoint of $\overline{AE}$ and $\overline{DG}$, B is the midpoint of $\overline{AC}$, $\overline{GD}$ is the perpendicular bisector of $\overline{FA}$ and $\overline{EC}$, $\overline{AC} \cong \overline{FE}$ and $\overline{FA} \cong \overline{EC}$



Find:

2. AB
3. GA
4. ED
5. HE
6. $m\angle HDC$
7. FA
8. GD
9. $m\angle FED$
10. How many copies of triangle AHB can fit inside rectangle $FECA$ without overlapping?

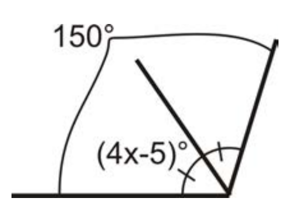
For 11-18, use the following picture to answer the questions.



11. Name the angle bisector of $\angle TPR$?
12. P is the midpoint of what two segments?
13. What is $m\angle QPR$?
14. What is $m\angle TPS$?
15. How does $\overline{VS}$ relate to $\overline{QT}$?
16. How does $\overline{QT}$ relate to $\overline{VS}$?
17. What is $m\angle QPV$?
18. Is $\overline{PU}$ a bisector? If so, of what?

Algebra Connection For 19-24, use algebra to determine the value of variable in each problem.

21.



19. Find AM.

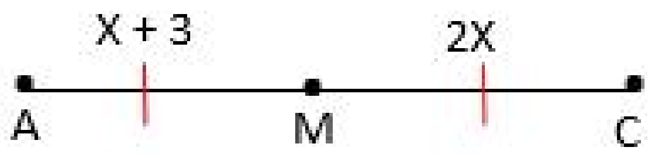


FIGURE 1.7

20. Find XZ.

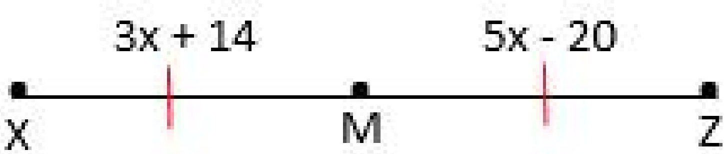
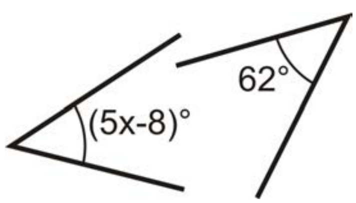


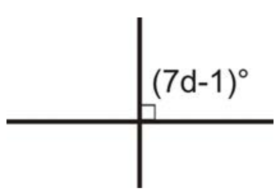
FIGURE 1.8

22.



23.

24.



25. Construction Using your protractor, draw an angle that is 110°. Then, use your compass to construct the angle bisector. What is the measure of each angle?

26. Construction Using your ruler, draw a line segment that is 7 cm long. Then use your compass to construct the perpendicular bisector. What is the measure of each segment?

For questions 27-30, find the midpoint between each pair of points.

27. (-2, -3) and (8, -7)

28. (9, -1) and (-6, -11)

29. (-4, 1) and (14, 0)

30. (0, -5) and (-9, 9)

Given the midpoint (M) and either endpoint of $\overline{AB}$, find the other endpoint.

31. $A(-1,2)$and $M(3,6)$

32. $B(1,-7)$ and $M(\frac{3}{2},-6)$

33. Points A, B, and C lie on a number line. Their coordinates are 0, 1, and x, respectively. Given AC = CB, what is the value of x?

34. CHALLENGE... M is the midpoint of $\overline{AB}$, AM = $\frac{x}{8}$, and AB = $\frac{3x}{4}$- 6. Find MB.

35. Compare and contrast midpoints and bisectors.

36. How many perpendicular bisectors can be drawn for a given line segment? Explain.

Warm-Up Answers

1. $m\angle POT = 38^\circ$, $m\angle ROT = 57^\circ + 38^\circ + 38^\circ = 133^\circ$
   a. $\frac{7-1}{-1+4} = \frac{6}{3} = 2$
   b. $\frac{-4+6}{-3-5} = \frac{2}{-8} = -\frac{1}{4}$
2. $\frac{23+30+18+27+32}{5} = \frac{130}{5} = 26$ [U+EFFE]

1.5 Angle Pairs

TEKS G(1)B, G(1)D, G(5)A, G(5)B, G(6)A

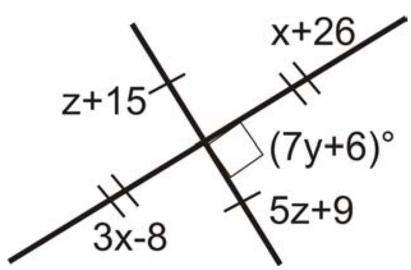
Learning Objectives

- Recognize complementary angles, supplementary angles, linear pairs,and vertical angles
- Apply the Linear Pair Postulate and the Vertical Angles Theorem

Vocabulary

- complementary angles
- supplementary angles
- adjacent angles
- linear pair
- vertical angles

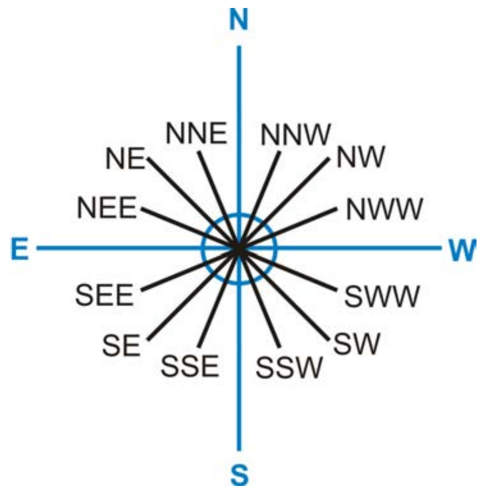
Warm-Up



1. Find x.
2. Find y.
3. Find z.

Know What? A compass (as seen to the right) is used to determine the direction a person is traveling. The angles between each direction are very important because they enable someone to be more specific with their direction. A direction of 45° NW, would be straight out along that northwest line.

What headings have the same angle measure? What is the angle measure between each compass line?



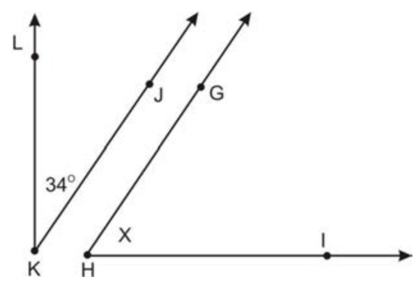
Complementary Angles

Complementary: Two angles that add up to 90°.

Complementary angles **do not** have to be:

- congruent
- adjacent (next to each other)

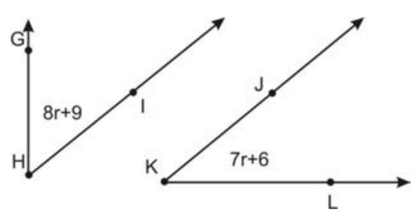
Example 1: The two angles below are complementary. $m\angle GHI = x$. What is x?



Solution: Because the two angles are complementary, they add up to 90°. Make an equation.

$$\begin{aligned} x + 34^\circ &= 90^\circ \\ x &= 56^\circ \end{aligned}$$

Example 2: The two angles below are complementary. Find the measure of each angle.



Solution: The two angles add up to 90°. Make an equation.

$$
\begin{aligned}
8r+9^\circ+7r+6^\circ &= 90^\circ \\
15r+15^\circ &= 90^\circ \\
15r &= 75^\circ \\
r &= 5^\circ
\end{aligned}
$$

However, you need to find each angle. Plug r back into each expression.

$$
\begin{aligned}
m\angle GHI &= 8(5^\circ)+9^\circ = 49^\circ \\
m\angle JKL &= 7(5^\circ)+6^\circ = 41^\circ
\end{aligned}
$$

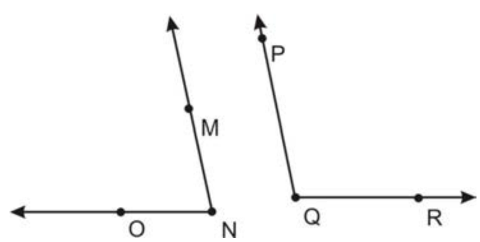
Supplementary Angles

Supplementary: Two angles that add up to 180°.

Supplementary angles **do not** have to be:

- congruent
- adjacent (next to each other)

Example 3: The two angles below are supplementary. If $m\angle MNO = 78^\circ$ what is $m\angle PQR$?



Solution: Set up an equation. However, instead of equaling 90°, now it is 180°.

$$
\begin{aligned}
78^\circ + m\angle PQR &= 180^\circ \\
m\angle PQR &= 102^\circ
\end{aligned}
$$

Example 4: What is the measure of two congruent, supplementary angles?

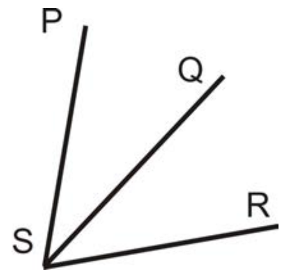
Solution: Supplementary angles add up to 180°. Congruent angles have the same measure. So, $180^\circ \div 2 = 90^\circ$, which means two congruent, supplementary angles are right angles, or 90°.

Linear Pairs

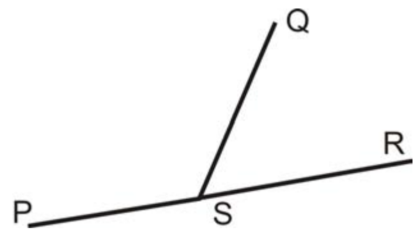
Adjacent Angles: Two angles that have the same vertex, share a side, and do not overlap.

$\angle PSQ$ and $\angle QSR$ are adjacent.

$\angle PQR$ and $\angle PQS$ are NOT adjacent because they overlap.



Linear Pair: Two angles that are adjacent and the non-common sides form a straight line.

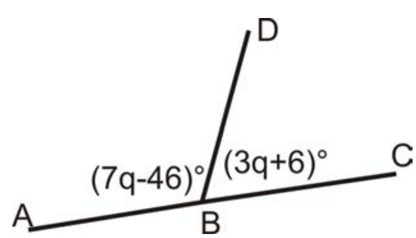


$\angle PSQ$ and $\angle QSR$ are a linear pair.

Linear Pair Postulate

If two angles are a linear pair, then the angles are supplementary

Example 5: ***Algebra Connection*** What is the measure of each angle?



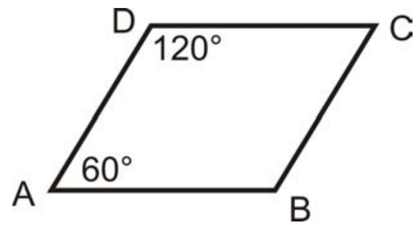
Solution: These two angles are a linear pair, so they add up to 180°.

$$\begin{aligned}(7q-46)^\circ+(3q+6)^\circ &= 180^\circ\\ 10q-40^\circ &= 180^\circ\\ 10q &= 220^\circ\\ q &= 22^\circ\end{aligned}$$

Plug in q to get the measure of each angle. $m\angle ABD = 7(22^\circ)-46^\circ = 108^\circ$ $m\angle DBC = 180^\circ - 108^\circ = 72^\circ$

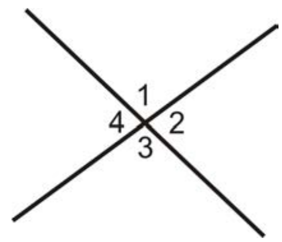
Example 6: Are $\angle CDA$ and $\angle DAB$ a linear pair? Are they supplementary?

Solution: The two angles are not a linear pair because they do not have the same vertex. They are supplementary, $120^\circ + 60^\circ = 180^\circ$.



Vertical Angles

Vertical Angles: Two non-adjacent angles formed by intersecting lines.



$\angle 1$ and $\angle 3$ are vertical angles

$\angle 2$ and $\angle 4$ are vertical angles

These angles are labeled with numbers. You can tell that these are labels because they do not have a degree symbol.

Investigation 1-6: Vertical Angle Relationships

1. Draw two intersecting lines on your paper. Label the four angles created $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$, just like the picture above.
2. Use your protractor to find $m\angle 1$.
3. What is the angle relationship between $\angle 1$ and $\angle 2$ called? Find $m\angle 2$.
4. What is the angle relationship between $\angle 1$ and $\angle 4$ called? Find $m\angle 4$.
5. What is the angle relationship between $\angle 2$ and $\angle 3$ called? Find $m\angle 3$.
6. Are any angles congruent? If so, write them down.

From this investigation, you should find that $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.

Vertical Angles Theorem

If two angles are vertical angles, then the two angles are congruent

We can prove the Vertical Angles Theorem using the same process we used in the investigation. We will not use any specific values for the angles.

From the picture above:

$$\begin{array}{lll} \angle 1 \text{ and } \angle 2 \text{ are a linear pair} & \rightarrow m\angle 1 + m\angle 2 = 180^\circ & \text{Equation 1} \\ \angle 2 \text{ and } \angle 3 \text{ are a linear pair} & \rightarrow m\angle 2 + m\angle 3 = 180^\circ & \text{Equation 2} \\ \angle 3 \text{ and } \angle 4 \text{ are a linear pair} & \rightarrow m\angle 3 + m\angle 4 = 180^\circ & \text{Equation 3} \end{array}$$

All of the equations $= 180^\circ$, so Equation 1 = Equation 2 and Equation 2 = Equation 3.

$$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3 \qquad \text{and} \qquad m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$$

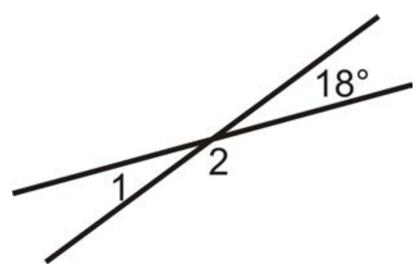
Cancel out the like terms

$$m\angle 1 = m\angle 3 \qquad \text{and} \qquad m\angle 2 = m\angle 4$$

Recall that anytime the measures of two angles are equal, the angles are also congruent. So, $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$ too.

To see an animation of this investigation, go to http://www.mathsisfun.com/geometry/vertical-angles.html

Example 7: Find $m\angle 1$ and $m\angle 2$.

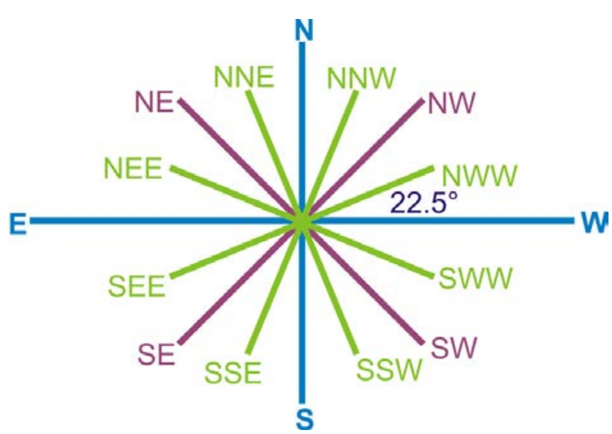


Solution: $\angle 1$ is vertical angles with 18°, so $m\angle 1 = 18^\circ$.

$\angle 2$ is a linear pair with $\angle 1$ or 18°, so $18^\circ + m\angle 2 = 180^\circ$.

$m\angle 2 = 180^\circ - 18^\circ = 162^\circ$.

Know What? Revisited The compass has several vertical angles and all of the smaller angles are $22.5^\circ, 180^\circ \div 8$. Directions that are opposite each other have the same angle measure, but of course, a different direction. All of the green directions have the same angle measure, 22.5°, and the purple have the same angle measure, 45°. N, S, E and W all have different measures, even though they are all 90° apart.



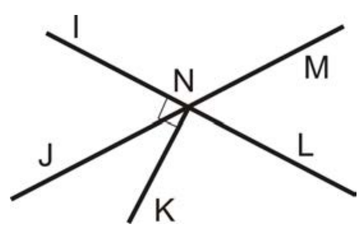
Practice Problems

- Questions 1 and 2 are similar to Examples 1, 2, and 3.

- Questions 3-8 are similar to Examples 3, 4, 6 and 7.
- Questions 9-16 use the definitions, postulates and theorems from this section.
- Questions 17-25 are similar to Example 5.

1. Find the measure of an angle that is complementary to $\angle ABC$ if $m\angle ABC$is
   a. 45°
   b. 82°
   c. 19°
   d. z°
2. Find the measure of an angle that is supplementary to $\angle ABC$ if $m\angle ABC$is
   a. 45°
   b. 118°
   c. 32°
   d. x°

Use the diagram below for exercises 3-8. Note that $\overline{NK} \perp \overleftrightarrow{IL}$.



3. Name one pair of vertical angles.
4. Name one linear pair of angles.
5. Name two complementary angles.
6. Name two supplementary angles.

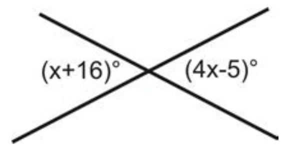
7. What is:
   a. $m\angle INL$
   b. $m\angle LNK$
8. If $m\angle INJ = 63^\circ$, find:
   a. $m\angle JNL$
   b. $m\angle KNJ$
   c. $m\angle MNL$
   d. $m\angle MNI$

For 9-16, determine if the statement is true or false. If false, explain why or draw a diagram.

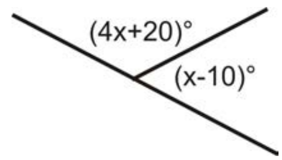
9. Vertical angles are congruent.
10. Linear pairs are congruent.
11. Complementary angles add up to 180°.
12. Supplementary angles add up to 180°
13. Adjacent angles share a vertex.
14. Adjacent angles overlap.
15. Complementary angles are always 45°.
16. Vertical angles have the same vertex.

For 17-23, find the value of x or y.

17.



18.



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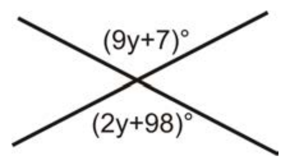
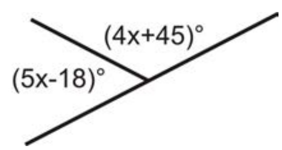
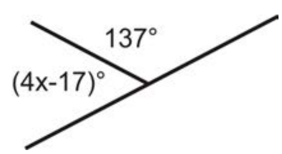


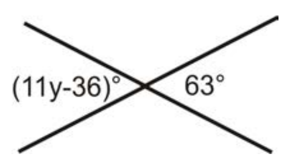
20.



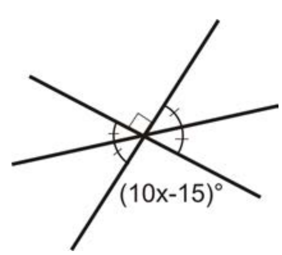
21.



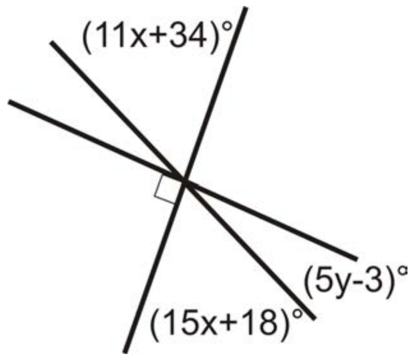
22.



23.



For 24-25, use the figure below:



24. Find x.
25. Find y.
26. CHALLENGE...$\overrightarrow{BD}$bisects $\angle ABC$, $\overrightarrow{BE}$biscets$\angle ABD$, and $\overrightarrow{BF}$bisects $\angle ABE$.The measure of $\angle FBE$is 17°. Find $m\angle CBD$.

Review and Reflect

27. Describe how supplementary angles migh also be called a linear pair.

28. Given two interesting lines, if you know the measure of one angle how can you determine the measure of the three remaining angles?

Warm-Up Answers

1. $$\begin{aligned} x+26 &= 3x-8 \\ 34 &= 2x \\ 17 &= x \end{aligned}$$
2. $$\begin{aligned} (7y+6)^\circ &= 90^\circ \\ 7y &= 84^\circ \\ y &= 12^\circ \end{aligned}$$
3. $$\begin{aligned} z+15 &= 5z+9 \\ 6 &= 4z \\ 1.5 &= z \end{aligned}$$

1.6 Classifying Polygons

TEKS G(1)A, G(1)G, G(5)A

Learning Objectives

- Define triangle and polygon
- Classify triangles by their sides and angles
- Understand the difference between convex and concave polygons
- Classify polygons by number of sides

Vocabulary

- triangle
- right triangle
- obtuse triangle
- acute triangle
- equiangular triangle
- scalene triangle
- isosceles triangle
- equilateral triangle
- polygon
- convex
- concave
- diagonals
- regular polygon

Warm-Up

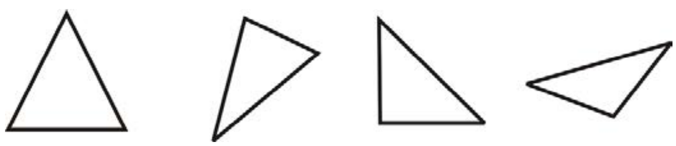
1. Draw a triangle.
2. Where have you seen 4, 5, 6 or 8 - sided polygons in real life? List 3 examples.
3. Fill in the blank.
   a. Vertical angles are always _____________.
   b. Linear pairs are _____________.
   c. The parts of an angle are called _____________ and a _____________.

Know What? The pentagon in Washington DC is a pentagon with congruent sides and angles. There is a smaller pentagon inside of the building that houses an outdoor courtyard. Looking at the picture, the building is divided up into 10 smaller sections. What are the shapes of these sections? Are any of these division lines diagonals? How do you know?

Triangles

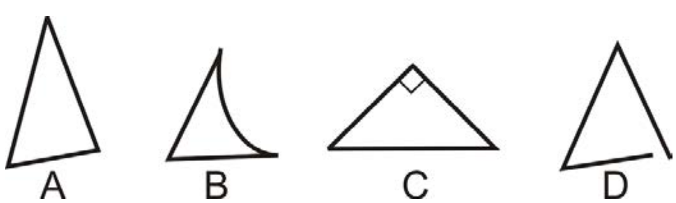
Triangle: Any closed figure made by three line segments intersecting at their endpoints.

Every triangle has three **vertices** (the points where the segments meet), three **sides** (the segments), and three **interior angles** (formed at each vertex). All of the following shapes are triangles.



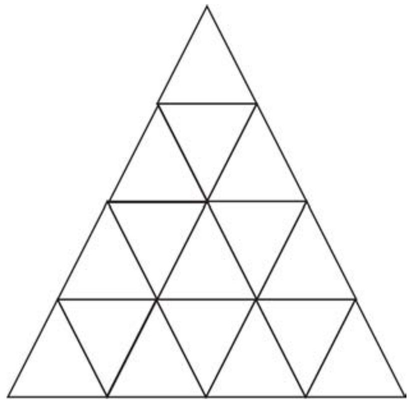
You might have also learned that the sum of the interior angles in a triangle is 180°. Later we will prove this, but for now you can use this fact to find missing angles.

Example 1: Which of the figures below are not triangles?



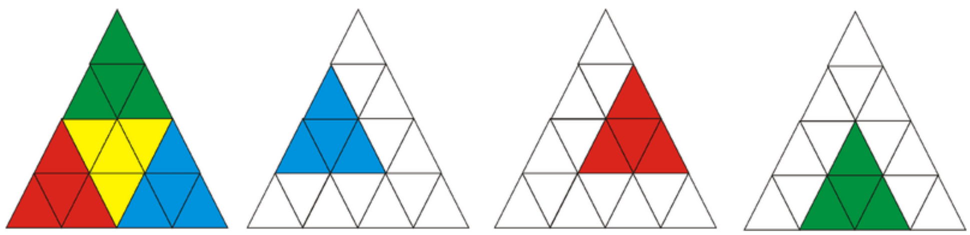
Solution: B is not a triangle because it has one curved side. D is not closed, so it is not a triangle either.

Example 2: How many triangles are in the diagram below?

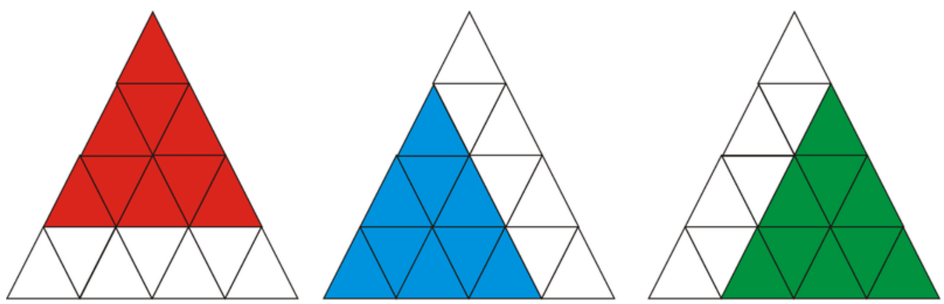


Solution: Start by counting the smallest triangles, 16.

Now count the triangles that are formed by 4 of the smaller triangles, 7.



Next, count the triangles that are formed by 9 of the smaller triangles, 3.

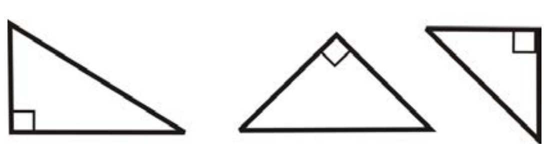


Finally, there is the one triangle formed by all 16 smaller triangles. Adding these numbers together, we get $16+7+3+1=27$.

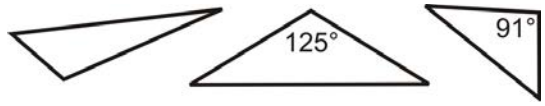
Classifying by Angles

Angles can be grouped by their angles; acute, obtuse or right. In any triangle, two of the angles will always be acute. The third angle can be acute, obtuse, or right. ***We classify each triangle by this angle***.

Right Triangle: A triangle with one right angle.

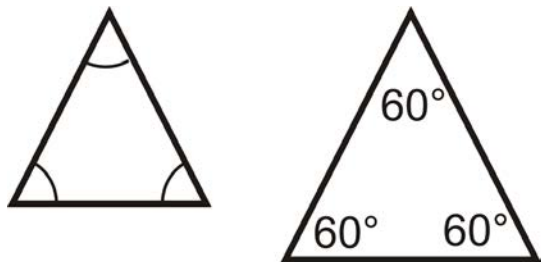


Obtuse Triangle: A triangle with one obtuse angle.

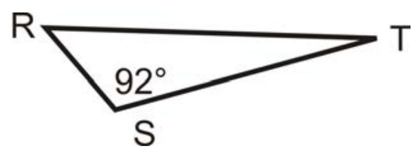


Acute Triangle: A triangle where all three angles are acute.

Equiangular Triangle: When all the angles in a triangle are congruent.



Example 3: Which term best describes $\triangle RST$ below?

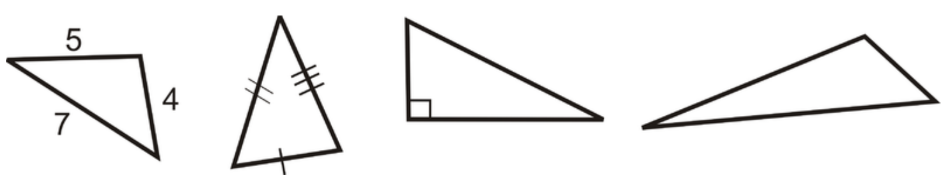


Solution: This triangle has one labeled obtuse angle of 92°. Triangles can only have one obtuse angle, so it is an obtuse triangle.

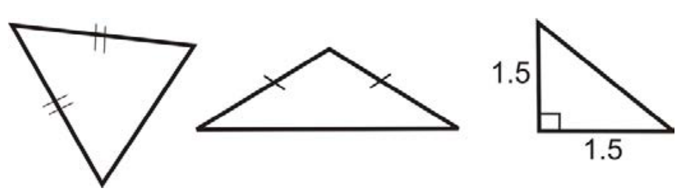
Classifying by Sides

You can also group triangles by their sides.

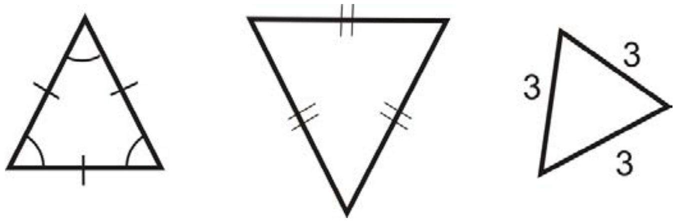
Scalene Triangle: A triangles where all three sides are different lengths.



Isosceles Triangle: A triangle with *at least* two congruent sides.

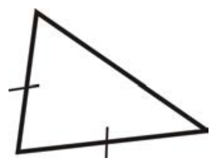


Equilateral Triangle: A triangle with three congruent sides.



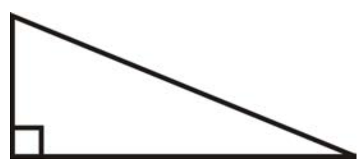
From the definitions, an equilateral triangle is also an isosceles triangle.

Example 4: Classify the triangle by its sides and angles.



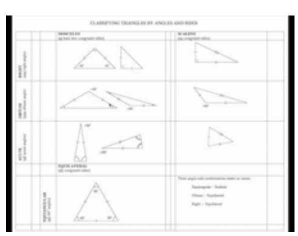
Solution: We see that there are two congruent sides, so it is isosceles. By the angles, they all look acute. We say this is an acute isosceles triangle.

Example 5: Classify the triangle by its sides and angles.



Solution: This triangle has a right angle and no sides are marked congruent. So, it is a right scalene triangle.

The following is a musical video of these classifications.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/148002

Polygons

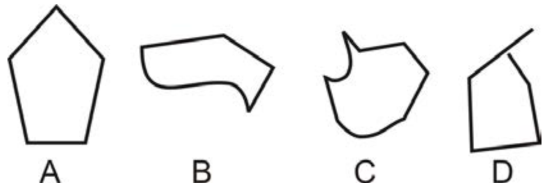
Polygon: Any closed, 2-dimensional figure that is made entirely of line segments that intersect at their endpoints.

Polygons can have any number of sides and angles, but the sides can never be curved.

The segments are called the **sides** of the polygons, and the points where the segments intersect are called **vertices**.

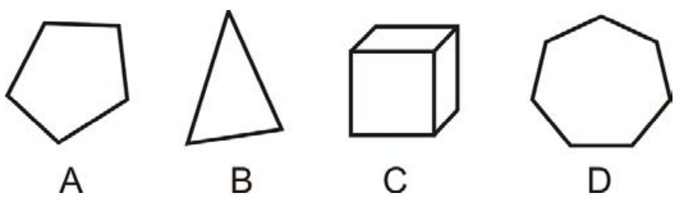
Regular Polygon: A convex polygon that is both equilateral and equiangular.

Example 6: Which of the figures below is a polygon?



Solution: The easiest way to identify the polygon is to identify which shapes are **not** polygons. B and C each have at least one curved side, so they are not be polygons. D has all straight sides, but one of the vertices is not at the endpoint, so it is not a polygon. A is the only polygon.

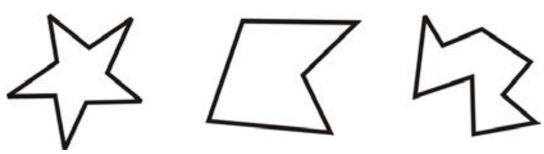
Example 7: Which of the figures below is *not* a polygon?



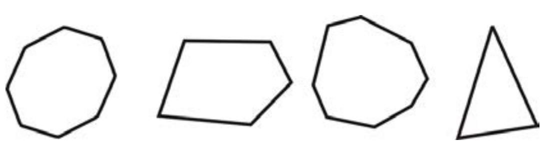
Solution: C is a three-dimensional shape, so it does not lie within one plane, so it is not a polygon.

Convex and Concave Polygons

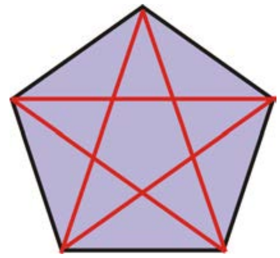
Polygons can be either **convex** or **concave**. The term concave refers to a cave, or the polygon is "caving in". All stars are concave polygons.



A convex polygon does not do this. Convex polygons look like:



Diagonals: Line segments that connect the vertices of a convex polygon that are not sides.



The red lines are all diagonals.

This pentagon has 5 diagonals.

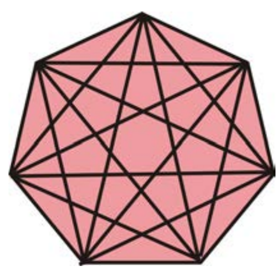
The Number of Diagonals of an n-sided Polygon is:

$$\frac{n(n-3)}{2}$$

Example 8: Determine if the shapes below are convex or concave.

Solution: To see if a polygon is concave, look at the polygons and see if any angle "caves in" to the interior of the polygon. The first polygon does not do this, so it is convex. The other two do, so they are concave.

Example 9: How many diagonals does a 7-sided polygon have?



Solution: Draw a 7-sided polygon, also called a heptagon.

Drawing in all the diagonals and counting them, we see there are 14.

Classifying Polygons

Whether a polygon is convex or concave, it is always named by the number of sides.

TABLE 1.11:

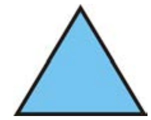
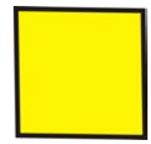
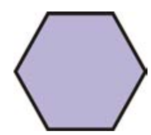
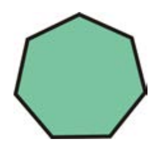
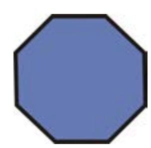
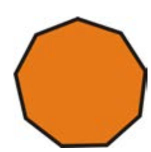
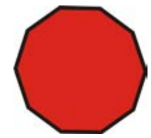
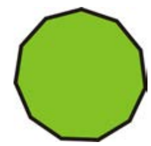
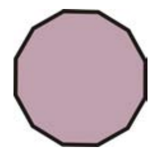
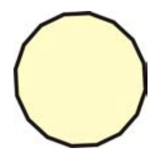
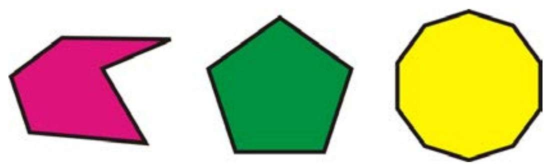
Polygon Name	Number of Sides	Number of Diagonals	Convex Example
Triangle	3	0	
Quadrilateral	4	2	
Pentagon	5	5	
Hexagon	6	9	
Heptagon	7	14	
Octagon	8	?	
Nonagon	9	?	

Table 1.11: (continued)

Polygon Name	Number of Sides	Number of Diagonals	Convex Example
Decagon	**10**	**?**	
Undecagon or hendecagon	**11**	**?**	
Dodecagon	**12**	**?**	
n-gon	n **(where** $n > 12$**)**	**?**	

Example 10: Name the three polygons below by their number of sides and if it is convex or concave.



Solution: The pink polygon is a concave hexagon (6 sides).

The green polygon convex pentagon (5 sides).

The yellow polygon is a convex decagon (10 sides).

Know What? Revisited The pentagon is divided up into 10 sections, all quadrilaterals. None of these dividing lines are diagonals because they are not drawn from vertices.

Practice Problems

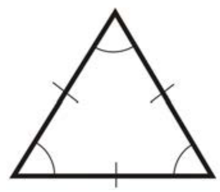
- Questions 1-8 are similar to Examples 3, 4 and 5.
- Questions 9-14 are similar to Examples 8 and 10
- Question 15 is similar to Example 6.
- Questions 16-19 are similar to Example 9 and the table.

- Questions 20-25 use the definitions, postulates and theorems in this section.

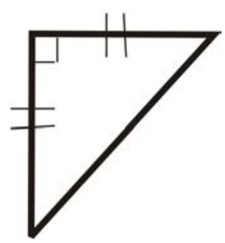
For questions 1-6, classify each triangle by its sides and by its angles.

1.

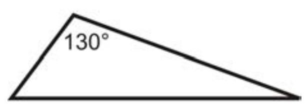
2.



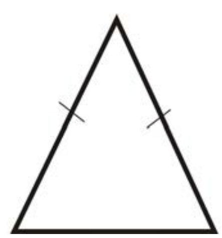
3.



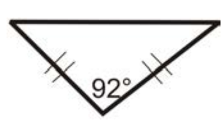
4.



5.



6.

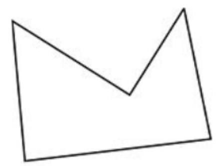


7. Can you draw a triangle with a right angle and an obtuse angle? Why or why not?

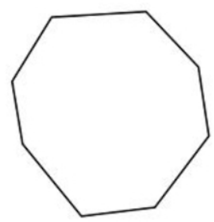
8. In an isosceles triangle, can the angles opposite the congruent sides be obtuse?

In problems 9-14, classify each polygon and determine if it's convex or concave.

9.

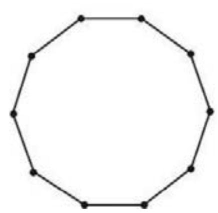


10.

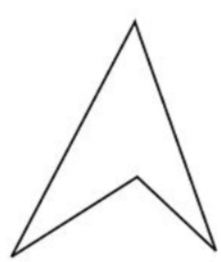


11.

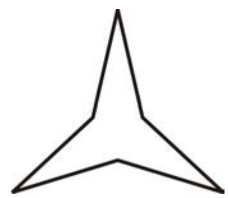
12.



13.



14.



15. Explain why the following figures are NOT polygons:

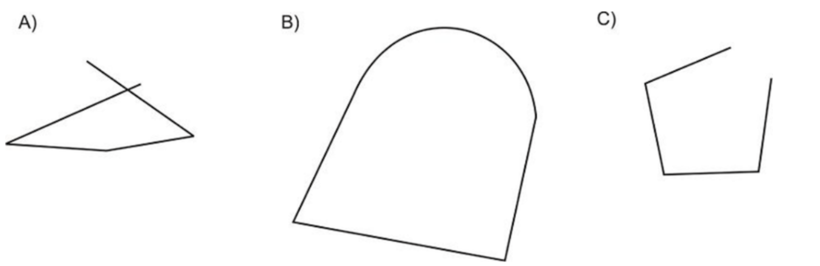


FIGURE 1.9

16. How many diagonals can you draw from **one vertex** of a pentagon? Draw a sketch of your answer.
17. How many diagonals can you draw from **one vertex** of an octagon? Draw a sketch of your answer.
18. How many diagonals can you draw from **one vertex** of a dodecagon?
19. Determine the number of total diagonals for an octagon, nonagon, decagon, undecagon, and dodecagon.

For 20-25, determine if the statement is true or false.

20. Obtuse triangles can be isosceles.
21. A polygon must be enclosed.
22. A star is a convex polygon.
23. A right triangle is acute.
24. An equilateral triangle is equiangular.
25. A quadrilateral is always a square.
26. A 5-point star is a decagon
    GEBRA CONNECTION
27. The lengths (in inches) of two sides of a regular pentagon are represented by the expressions 5x - 33 and 2x - 6. Find the length of a side of the pentagon.
28. The expressions $(3x + 7)^\circ$ and $(10x - 14)^\circ$ represent the measures of two angles of a regular nonagon. Find the measure of an angle of the nonagon.
    figure is a regular polygon. Expressions are given for two side lengths. Find the value of *x*.

Review and Reflect

31. Describe the difference between and convex and concave polygon. Give at least 2 examples of each.
32. Is an equilateral triangle also isosceles? Explain.

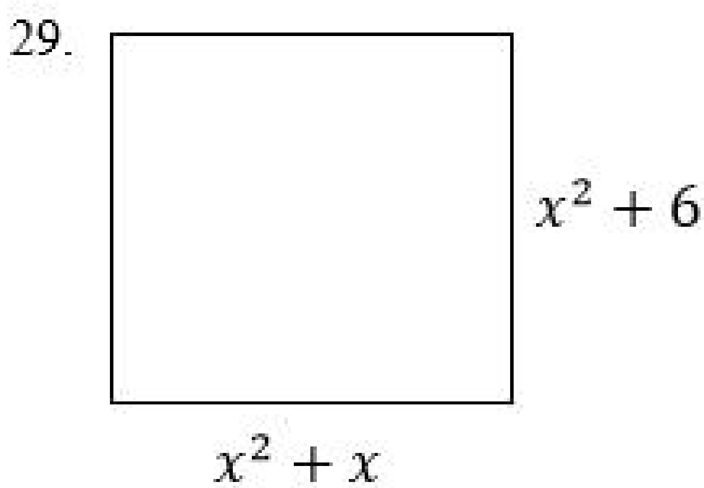


FIGURE 1.10

29.

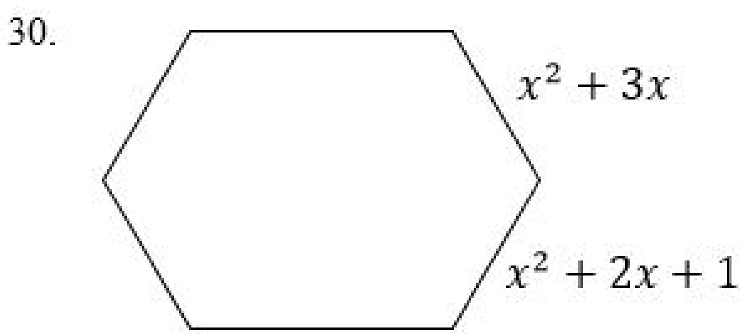
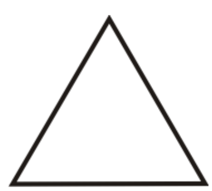


FIGURE 1.11

Warm-Up Answers

1. 
2. Examples include: stop sign (8), table top (4), the Pentagon (5), snow crystals (6), bee hive combs (6), soccer ball pieces (5 and 6)
   a. congruent or equal
   b. supplementary
   c. sides, vertex

1.7 Chapter 1 Review

Symbol Toolbox

$\overleftrightarrow{AB}, \overrightarrow{AB}, \overline{AB}$ - Line, ray, line segment

$\angle ABC$ - Angle with vertex B

$m\overline{AB}$ or AB - Distance between A and B

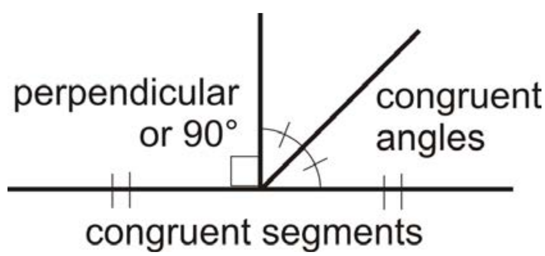
$m\angle ABC$ - Measure of $\angle ABC$

$\perp$ - Perpendicular

$=$ - Equal

$\cong$ - Congruent

Markings



Keywords, Postulates, and Theorems

Points, Lines, and Planes

- Geometry
- Point
- Line
- Plane
- Space
- Collinear
- Coplanar
- Endpoint
- Line Segment
- Ray
- Intersection
- Postulates
- Theorem
- Postulate 1-1
- Postulate 1-2
- Postulate 1-3
- Postulate 1-4
- Postulate 1-5

Segments and Distance

- Distance
- Measure
- Ruler Postulate
- Segment Addition Postulate

Angles and Measurement

- Angle
- Vertex
- Sides
- Protractor Postulate
- Straight Angle
- Right Angle
- Acute Angles
- Obtuse Angles
- Convex
- Concave
- Polygon
- Perpendicular
- Construction
- Compass
- Angle Addition Postulate

Midpoints and Bisectors

- Congruent
- Midpoint.
- Midpoint Postulate
- Bisector
- Segment Bisector
- Perpendicular Bisector
- Perpendicular Bisector Postulate
- Angle Bisector
- Angle Bisector Postulate

Angle Pairs

- Complementary
- Supplementary
- Adjacent Angles
- Linear Pair
- Linear Pair Postulate
- Vertical Angles
- Vertical Angles Theorem

Classifying Polygons

- Triangle
- Right Triangle

- Obtuse Triangle
- Acute Triangle
- Equiangular Triangle
- Scalene Triangle
- Isosceles Triangle
- Equilateral Triangle
- Vertices
- Diagonals

Review

Match the definition or description with the correct word.

1. When three points lie on the same line. — A. Measure
2. All vertical angles are ________. — B. Congruent
3. Linear pairs add up to _______. — C. Angle Bisector
4. The m in from of $m\angle ABC$. — D. Ray
5. What you use to measure an angle. — E. Collinear
6. When two sides of a triangle are congruent. — F. Perpendicular
7. $\perp$ — G. Line
8. A line that passes through the midpoint of another line. — H. Protractor
9. An angle that is greater than 90°. — I. Segment Addition Postulate
10. The intersection of two planes is a ___________. — J. Obtuse
11. $AB + BC = AC$ — K. Point
12. An exact location in space. — L. 180°
13. A sunbeam, for example. — M. Isosceles
14. Every angle has exactly one. — N. Pentagon
15. A closed figure with 5 sides. — O. Hexagon

P. Bisector

To review in a Jeopardy style game, go to **https://www.superteachertools.net/jeopardyx/jeopardy-review-game.php?gamefile=1408206073#.VKObUtLF_eQ**

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9686 .

1.8 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1^{st} **Section: Points, Lines, and Planes**

Geometry

Point

Line

Plane

Space

Collinear

Coplanar

Endpoint

Line Segment

Ray

Intersection

Postulates

Theorem

Postulate 1-1

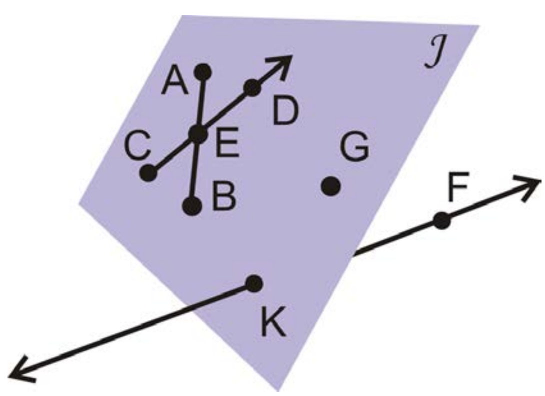
Postulate 1-2

Postulate 1-3

Postulate 1-4

Postulate 1-5

Use this picture to identify the geometric terms in this section.



Homework:

2^{nd} **Section: Segments and Distance**

Distance

Measure

Ruler Postulate

Segment Addition Postulate

Homework:

3^{rd} **Section: Angles and Measurement**

Angle

Vertex

Sides

Protractor Postulate

Straight Angle

Right Angle

Acute Angles

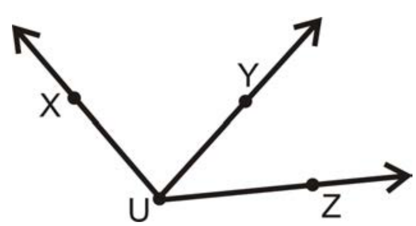
Obtuse Angles

Perpendicular

Construction

Compass

Angle Addition Postulate



Homework:

4^{th} **Section: Midpoints and Bisectors**

Congruent

Midpoint.

Midpoint Postulate

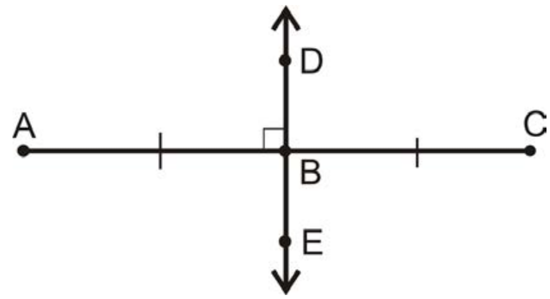
Segment Bisector

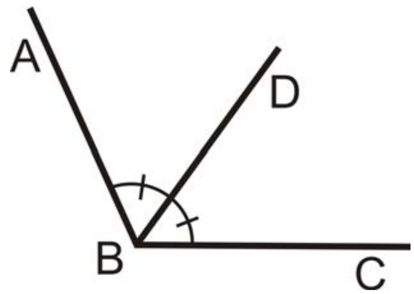
Perpendicular Bisector

Perpendicular Bisector Postulate

Angle Bisector

Angle Bisector Postulate





Homework:

5^{th} **Section: Angle Pairs**

Complementary

Supplementary

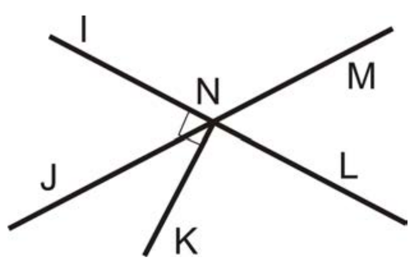
Adjacent Angles

Linear Pair

Linear Pair Postulate

Vertical Angles

Vertical Angles Theorem



Homework:

6^{th} **Section: Classifying Polygons**

Draw your own pictures for this section

Triangle

Right Triangle

Obtuse Triangle

Acute Triangle

Equiangular Triangle

Scalene Triangle

Isosceles Triangle

Equilateral Triangle

Vertices

Sides

Polygon

Convex Polygon

Concave Polygon

Quadrilateral, Pentagon, Hexagon, Heptagon, Octagon, Nonagon, Decagon...

Diagonals

Homework:

CHAPTER 2 Reasoning and Proof

Chapter Outline

This chapter explains how to use reasoning to prove theorems about angle pairs and segments. This chapter also introduces the properties of congruence, which will be used in 2-column proofs.

2.1 Inductive Reasoning

TEKS G(1)D, G(4)C, G(1)G

Learning Objectives

- Recognize visual and number patterns
- Write a counterexample

Vocabulary

- inductive reasoning
- conjecture
- counterexample

Warm-Up

1. Look at the patterns of numbers below. Determine the next three numbers in the list.
   a. 1, 2, 3, 4, 5, 6, ______, ______, ______
   b. 3, 6, 9, 12, 15, ______, ______, ______
   c. 5, 1, -3, -7, -11, ______, ______, ______
2. Are the statements below true or false? If they are false, state why.
   a. Perpendicular lines form four right angles.
   b. Linear pairs are always congruent.
3. For the line, $y = 3x + 1$:
   a. Find the slope.
   b. Find the $y-$intercept.
   c. Make an $x - y$ table for $x = 1, 2, 3, 4$, and 5.

Know What? This is the "famous" locker problem:

A new high school has just been completed. There are 100 lockers that are numbered 1 to 100. During recess, the students decide to try an experiment. The first student opens all of the locker doors. The second student closes all of the lockers with even numbers. The 3^{rd} student changes every 3^{rd} locker (*change means closing lockers that are open, and opening lockers that are closed*). The 4^{th} student changes every 4^{th} locker and so on.

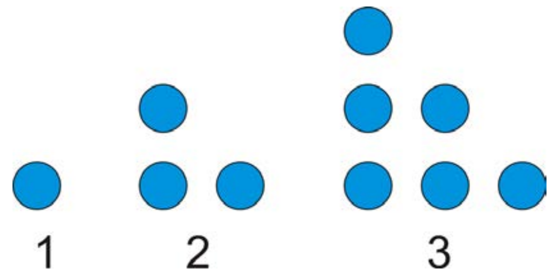
Imagine that this continues until the 100 students have followed the pattern with the 100 lockers. At the end, which lockers will be open and which will be closed?

Visual Patterns

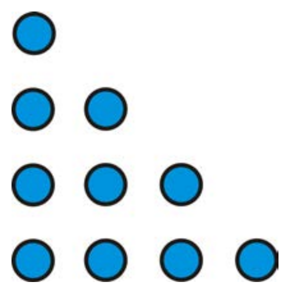
Inductive Reasoning: Making conclusions based upon examples and patterns.

Let's look at some patterns to get a feel for what inductive reasoning is.

Example 1: A dot pattern is shown below. How many dots would there be in the 4^{th} figure? How many dots would be in the 6^{th} figure?

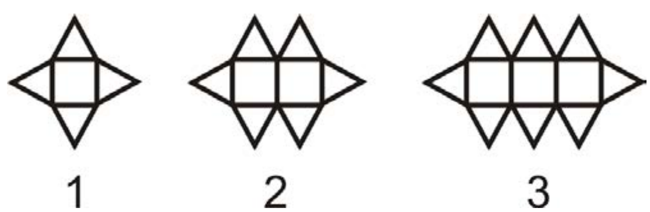


Solution: Draw a picture. Counting the dots, there are $4+3+2+1=10$ *dots*.



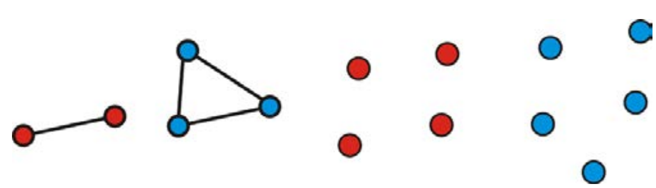
For the 6^{th} figure, we can use the same pattern, $6+5+4+3+2+1$. There are 21 dots in the 6^{th} figure.

Example 2: How many *triangles* would be in the 10^{th} figure?

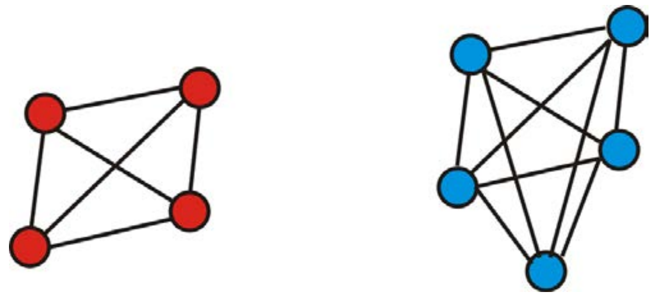


Solution: There would be 10 squares in the 10^{th} figure, with a triangle above and below each one. There is also a triangle on each end of the figure. That makes $10+10+2=22$ triangles in all.

Example 3: For two points, there is one line segment between them. For three non-collinear points, there are three segments. For four points, how many line segments are between them? If you add a fifth point, how many line segments are between the five points?



Solution: Draw a picture of each and count the segments.



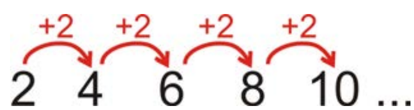
For 4 points there are 6 line segments and for 5 points there are 10 line segments.

Number Patterns

Let's look at a few examples.

Example 4: Look at the pattern 2, 4, 6, 8, 10, ... What is the 19^{th} term in the pattern?

Solution: For part a, each term is 2 more than the previous term.



You could count out the pattern until the 19^{th} term, but that could take a while. Notice that the 1^{st} term is $2 \cdot 1$, the 2^{nd} term is $2 \cdot 2$, the 3^{rd} term is $2 \cdot 3$, and so on. So, the 19^{th} term would be $2 \cdot 19$ or 38.

Example 5: Look at the pattern 1, 3, 5, 7, 9, 11, ... What is the 34^{th} term in the pattern?

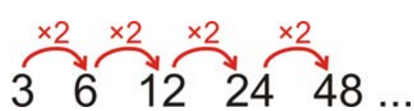
Solution: The next term would be 13 and continue go up by 2. Comparing this pattern to Example 4, each term is one less. So, we can reason that the 34^{th} term would be $34 \cdot 2$ minus 1, which is 67.

Example 6: Look at the pattern: 3, 6, 12, 24, 48, ...

a) What is the next term in the pattern?

b) The 10^{th} term?

Solution: This pattern is different than the previous two examples. Here, each term is *multiplied* by 2 to get the next term.



Therefore, the next term will be $48 \cdot 2$ or 96. To find the 10^{th} term, continue to multiply by 2, or $3 \cdot \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{2^9} = 1536$.

Example 7: Find the 8^{th} term in the list of numbers: $2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \ldots$

Solution: First, change 2 into a fraction, or $\frac{2}{1}$. So, the pattern is now $\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \ldots$ The top is 2, 3, 4, 5, 6. It increases by 1 each time, so the 8^{th} term's numerator is 9. The denominators are the square numbers, so the 8^{th} term's denominator is 8^2 or 64. The 8^{th} term is $\frac{9}{64}$.

Conjectures and Counterexamples

Conjecture: An "educated guess" that is based on examples in a pattern.

Example 8: Here's an algebraic equation and a table of values for n and the result, t.

$$t = (n-1)(n-2)(n-3)$$

TABLE 2.1:

n	$(n-1)(n-2)(n-3)$	t
1	$(0)(-1)(-2)$	0
2	$(1)(0)(-1)$	0
3	$(2)(1)(0)$	0

After looking at the table, Pablo makes this conjecture:

The value of $(n-1)(n-2)(n-3)$ is 0 for any number n.

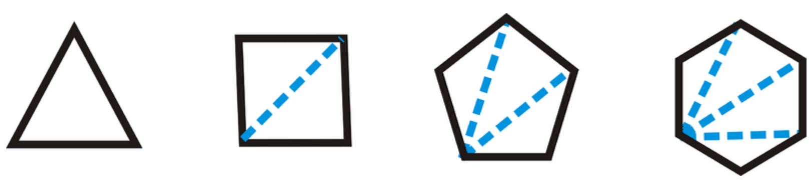
Is this a true conjecture?

Solution: This is not a valid conjecture. If Pablo were to continue the table to $n = 4$, he would have see that $(n-1)(n-2)(n-3) = (4-1)(4-2)(4-3) = (3)(2)(1) = 6$

In this example $n = 4$ is called a counterexample.

Counterexample: An example that disproves a conjecture.

Example 9: Arthur is making figures for an art project. He drew polygons and some of their diagonals.



From these examples, Arthur made this conjecture:

If a convex polygon has n sides, then there are $n-3$ triangles drawn from any vertex of the polygon.

Is Arthur's conjecture correct? Or, can you find a counterexample?

Solution: The conjecture appears to be correct. If Arthur draws other polygons, in every case he will be able to draw $n-3$ triangles if the polygon has n sides.

Notice that we have ***not proved*** Arthur's conjecture, but only found several examples that hold true. So, at this point, we say that the conjecture is true.

Watch this!

MEDIA
Click image to the left or use the URL below.
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Know What? Revisited The table below is the start of the 100 lockers and students. Students are vertical and the lockers are horizontal. X means the locker is closed, O means the locker is open.

TABLE 2.2:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
1	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
2		X		X		X		X		X		X		X		X
3			X			O			X			O			X	
4				O				O				X				O
5					X					O					O	
6						X						O				
7							X							O		
8								X								X
9									O							
10										X						
11											X					
12												X				
13													X			
14														X		
15															X	
16																O

If you continue on in this way, the numbers that follow the pattern: 1, 4, 9, 16, ... are going to be the only open lockers. These numbers are called square numbers and they are: 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100.

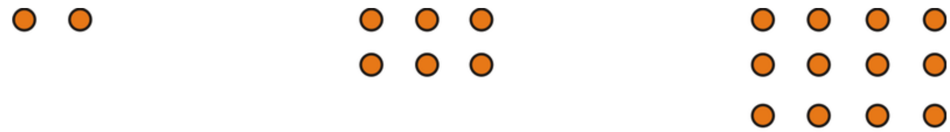
Practice Problems

- Questions 1-5 are similar to Examples 1, 2a, and 3.
- Questions 6-17 are similar to Examples 4-7.
- Questions 18-25 are similar to Examples 8 and 9.

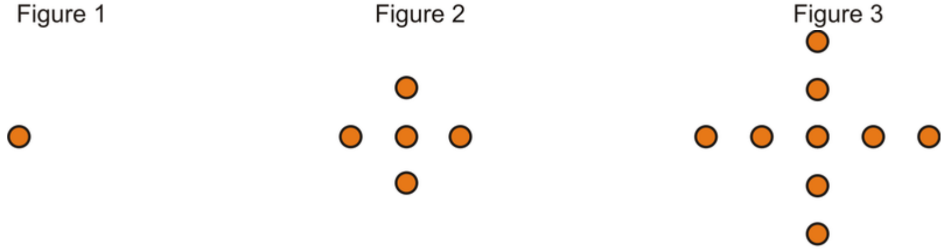
For questions 1-3, determine how many dots there would be in the 4^{th} and the 10^{th} pattern of each figure below.

1. 

Figure 1 Figure 2 Figure 3



2. Figure 1 Figure 2 Figure 3

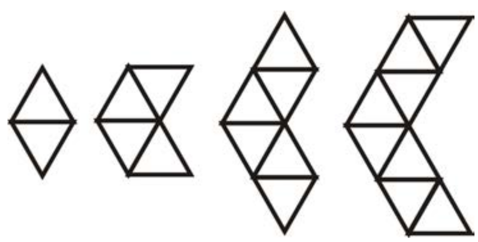


3. Figure 1 Figure 2 Figure 3

4. Use the pattern below to answer the questions.

   a. Draw the next figure in the pattern.
   b. How does the number of points in each star relate to the figure number?

5. Use the pattern below to answer the questions. All the triangles are equilateral triangles.



   a. Draw the next figure in the pattern. How many triangles does it have?
   b. Determine how many triangles are in the 24^{th} figure.

For questions 6-13, determine: the next three terms in the pattern.

6. 5, 8, 11, 14, 17, ...
7. 6, 1, -4, -9, -14, ...
8. 2, 4, 8, 16, 32, ...
9. 67, 56, 45, 34, 23, ...
10. 9, -4, 6, -8, 3, ...
11. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots$
12. $\frac{2}{3}, \frac{4}{7}, \frac{6}{11}, \frac{8}{15}, \frac{10}{19}, \ldots$
13. -1, 5, -9, 13, -17, ...

For questions 14-17, determine the next two terms **and** describe the pattern.

14. 3, 6, 11, 18, 27, ...
15. 3, 8, 15, 24, 35, ...
16. 1, 8, 27, 64, 125, ...
17. 1, 1, 2, 3, 5, ...

For questions 18-23, give a counterexample for each of the following statements.

18. If n is a whole number, then $n^2 > n$.
19. Every prime number is an odd number.
20. All numbers that end in 1 are prime numbers.
21. All positive fractions are between 0 and 1.
22. Any three points that are coplanar are also collinear.
23. Congruent supplementary angles are also linear pairs.

Use the following story for questions 24 and 25.

A car salesman sold 5 used cars to five different couples. He noticed that each couple was under 30 years old. The following day, he sold a new, luxury car to a couple in their 60's. The salesman determined that only younger couples by used cars.

24. Is the salesman's conjecture logical? Why or why not?
25. Can you think of a counterexample?

PROBLEM SOLVING

26. **BASEBALL** You are watching a pitcher who throws two types of pitches, a fastball (F) and a knuckle ball (K). You notice that the order of pitches was F, K, F, F, K, K, F, F, F. Assuming that this pattern continues, predict the next five pitches.

FIGURE 2.1

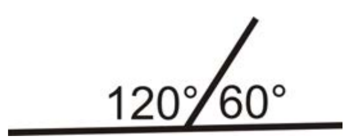
Review and Reflect

27. When is it necessary to use a counterexample?

28. Describe the process of inductive reasoning.

Warm-Up Answers

1.
   a. 7, 8, 9
   b. 18, 21, 24
   c. 36, 49, 64
2.
   a. true
   b. false,



3.

a. $m = 3$
b. $b = 1$
c.

TABLE 2.3:

x	y
1	4
2	7
3	10
4	13
5	16

2.2 Writing Conditional Statements

TEKS G(1)F, G(4)B, G(4)C

Learning Objectives

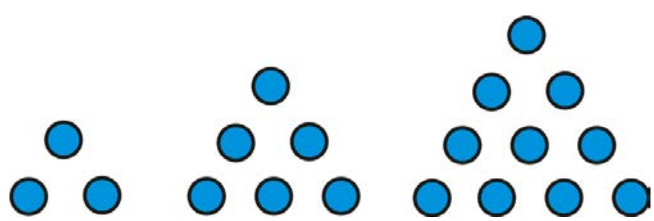
- Identify the hypothesis and conclusion of an if-then statment
- Write the converse, inverse, and contrapositive of an if-then statement

Vocabulary

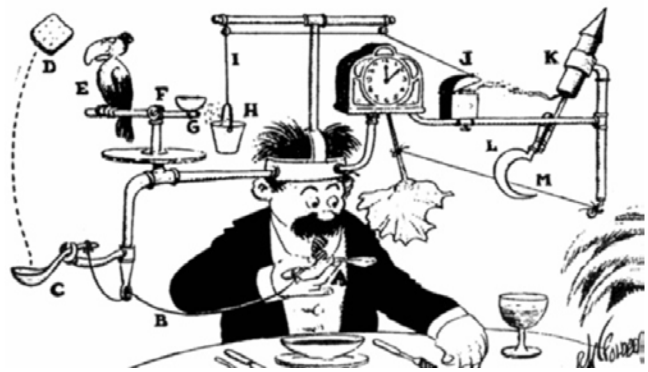
- conditional
- hypothesis
- conclusion
- converse
- inverse
- contraposative
- biconditional

Warm-Up

Find the next figure or term in the pattern.

1. 5, 8, 12, 17, 23, ...
2. $\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{9}, \frac{6}{10}, \ldots$
3. 
4. Find a counterexample for the following conjectures.
   a. If it is April, then it is Spring Break.
   b. If it is June, then I am graduating.

Know What? Rube Goldberg was a cartoonist in the 1940s who drew crazy inventions to do very simple things. The invention to the right has a series of smaller tasks that leads to the machine wiping the man's face with a napkin.



Describe each step, from A to M.

If-Then Statements

Conditional Statement (also called an **If-Then Statement**): A statement with a hypothesis followed by a conclusion.

Hypothesis: The first, or "if," part of a conditional statement.

Conclusion: The second, or "then," part of a conditional statement. The conclusion is the result of a hypothesis.

If-then statements might not always be written in the "if-then" form.

Statement 1: If you work overtime, then you'll be paid time-and-a-half.

Statement 2: I'll wash the car if the weather is nice.

Statement 3: If 2 divides evenly into x, then x is an even number.

Statement 4: I'll be a millionaire when I win monopoly.

Statement 5: All equiangular triangles are equilateral.

Statements 1 and 3 are written in the "if-then" form. The hypothesis of Statement 1 is "you work overtime." The conclusion is "you'll be paid time-and-a-half."

So, if Sarah works overtime, then what will happen? From **Statement 1**, we can conclude that she will be paid time-and-a-half.

If 2 goes evenly into 16, what can you conclude? From **Statement 3**, we know that 16 must be an even number.

Statement 2 has the hypothesis after the conclusion. If the word "if" is in the middle of the statement, then the hypothesis is after it. The statement can be rewritten:

If the weather is nice, then I will wash the car.

Statement 4 uses the word "when" instead of "if" and is like Statement 2. It can be written:

If I win monopoly, then I will be a millionaire.

Statement 5 "if" and "then" are not there. It can be rewritten:

If a triangle is equiangular, then it is equilateral.

Example 1: Use the statement: *I will graduate when I pass Calculus.*

a) Rewrite in if-then form.

b) Determine the hypothesis and conclusion.

Solution: This statement is like Statement 4 above. It should be:

If I pass Calculus, then I will graduate.

The hypothesis is "I pass Calculus," and the conclusion is "I will graduate."

Converse, Inverse, and Contrapositive

Look at **Statement 2** again: *If the weather is nice, then I'll wash the car.*

This can be rewritten using letters, known as **symbolic notation**, to represent the hypothesis and conclusion.

$$p = \text{the weather is nice} \qquad\qquad q = \text{I'll wash the car}$$

Now the statement is: If p, then q.

An arrow can also be used in place of the "if-then": $p \rightarrow q$

We can also make the negations, or "nots" of p and q. The symbolic version of not p, is $\sim p$.

$$\sim p = \text{the weather is not nice} \qquad\qquad \sim q = \text{I won't wash the car}$$

Using these "nots" and switching the order of p and q, we can create three new statements.

Converse	$q \rightarrow p$	$\underbrace{\text{If I wash the car}}_{q}, \underbrace{\text{then the weather is nice}}_{p}.$
Inverse	$\sim p \rightarrow \sim q$	$\underbrace{\text{If the weather is not nice}}_{\sim p}, \underbrace{\text{then I won't wash the car}}_{\sim q}.$
Contrapositive	$\sim q \rightarrow \sim p$	$\underbrace{\text{If I don't wash the car}}_{\sim q}, \underbrace{\text{then the weather is not nice}}_{\sim p}.$

If the "if-then" statement is true, then the contrapositive is also true. The contrapositive is ***logically equivalent*** to the original statement. The converse and inverse may or may not be true.

Example 2: If $n > 2$, then $n^2 > 4$.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

$\underline{\text{Converse}}$:	If $n^2 > 4$, then $n > 2$.	*False*. If $n^2 = 9, n = -3$ or 3. $(-3)^2 = 9$
$\underline{\text{Inverse}}$:	If $n < 2$, then $n^2 < 4$.	*False*. If $n = -3$, then $n^2 = 9$.
$\underline{\text{Contrapositive}}$:	If $n^2 < 4$, then $n < 2$.	*True*. the only $n^2 < 4$ is 1. $\sqrt{1} = \pm 1$ which are both less then 2.

Before going on to the next example,watch the following video example below.

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/148831

Example 3: If I am at Disneyland, then I am in California.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

Converse :	If I am in California, then I am at Disneyland.
	False. I could be in San Francisco.
Inverse :	If I am not at Disneyland, then I am not in California.
	False. Again, I could be in San Francisco.
Contrapositive :	If I am not in California, then I am not at Disneyland.
	True. If I am not in the state, I couldn't be at Disneyland.

Notice for the converse and inverse ***we can use the same counterexample.***

Example 4: Any two points are collinear.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: First, change the statement into an "if-then" statement:

If two points are on the same line, then they are collinear.

Converse :	If two points are collinear, then they are on the same line. *True*.
Inverse :	If two points are not on the same line, then they are not collinear. *True*.
Contrapositive :	If two points are not collinear, then they do not lie on the same line. *True*.

Biconditional Statements

Example 4 is an example of a biconditional statement.

Biconditional Statement: When the original statement and converse are both true.

$p \rightarrow q$ is true

$q \rightarrow p$ is true

then, $p \leftrightarrow q$, said "p if and only if q"

Example 5: Rewrite Example 4 as a biconditional statement.

Solution: *If two points are on the same line, then they are collinear* can be rewritten as:

Two points are on the same line if and only if they are collinear.

Replace the "if-then" with "if and only if" in the middle of the statement.

Example 6: The following is a true statement:

$m\angle ABC > 90^\circ$ if and only if $\angle ABC$ is an obtuse angle.

Determine the two true statements within this biconditional.

Solution: *Statement 1:* If $m\angle ABC > 90^\circ$, then $\angle ABC$ is an obtuse angle.

Statement 2: If $\angle ABC$ is an obtuse angle, then $m\angle ABC > 90^\circ$.

This is the definition of an obtuse angle. ***All geometric definitions are biconditional statements.***

To go further in depth with biconditionals and truth tables, visit the following site. http://www.mathgoodies.com/lessons/vol9/biconditional.html

Example 7: $p : x < 10 \quad q : 2x < 50$

a) Is $p \rightarrow q$ true? If not, find a counterexample.

b) Is $q \rightarrow p$ true? If not, find a counterexample.

c) Is $\sim p \rightarrow \sim q$ true? If not, find a counterexample.

d) Is $\sim q \rightarrow \sim p$ true? If not, find a counterexample.

Solution:

a) If $x < 10$, then $2x < 50$. ***True***.

b) If $2x < 50$, then $x < 10$. ***False***, $x = 15$

c) If $x > 10$, then $2x > 50$. ***False***, $x = 15$

d) If $2x > 50$, then $x > 10$. ***True***, $x \geq 25$

Know What? Revisited The series of events is as follows:

If the man raises his spoon, then it pulls a string, which tugs the spoon back, then it throws a cracker into the air, the bird will eat it and turns the pedestal. Then the water tips over, which goes into the bucket, pulls down the string, the string opens the box, where a fire lights the rocket and goes off. This allows the hook to pull the string and then the man's face is wiped with the napkin.

Practice Problems

- Questions 1-6 are similar to Statements 1-5 and Example 1.
- Questions 7-16 are similar to Examples 2, 3, and 4.
- Questions 17-22 are similar to Examples 5 and 6.
- Questions 23-25 are similar to Example 7.

For questions 1-6, determine the hypothesis and the conclusion.

1. If 5 divides evenly into x, then x ends in 0 or 5.
2. If a triangle has three congruent sides, it is an equilateral triangle.
3. Three points are coplanar if they all lie in the same plane.
4. If $x = 3$, then $x^2 = 9$.
5. If you take yoga, then you are relaxed.
6. All baseball players wear hats.
7. Write the converse, inverse, and contrapositive of #1. Determine if they are true or false. If they are false, find a counterexample.
8. Write the converse, inverse, and contrapositive of #5. Determine if they are true or false. If they are false, find a counterexample.
9. Write the converse, inverse, and contrapositive of #6. Determine if they are true or false. If they are false, find a counterexample.
10. Find the converse of #2. If it is true, write the biconditional of the statement.
11. Find the converse of #3. If it is true, write the biconditional of the statement.
12. Find the converse of #4. If it is true, write the biconditional of the statement.

For questions 13-16, use the statement:

If $AB = 5$ and $BC = 5$, then B is the midpoint of $\overline{AC}$.

13. Is this a true statement? If not, what is a counterexample?
14. Find the converse of this statement. Is it true?
15. Find the inverse of this statement. Is it true?
16. Find the contrapositive of #14. Which statement is it the same as?

Find the converse of each true if-then statement. If the converse is true, write the biconditional statement.

17. An acute angle is less than 90°.
18. If you are at the beach, then you are sun burnt.
19. If $x > 4$, then $x + 3 > 7$.

For questions 20-22, determine the two true conditional statements from the given biconditional statements.

20. A U.S. citizen can vote if and only if he or she is 18 or more years old.
21. A whole number is prime if and only if its factors are 1 and itself.
22. $2x = 18$ if and only if $x = 9$.

For questions 23-25, determine if:

(a) $p \rightarrow q$ is true.

(b) $q \rightarrow p$ is true.

(c) $\sim p \rightarrow \sim q$ is true.

(d) $\sim q \rightarrow \sim p$ is true.

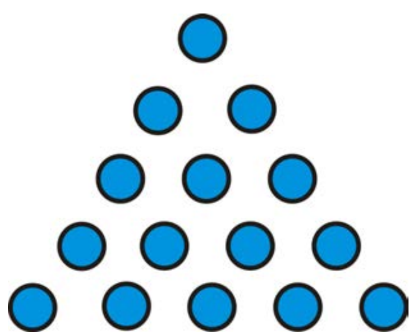
If any are false, find a counterexample.

23. p : Joe is 16. - q : He has a driver's license.
24. p : A number ends in 5. - q : It is divisible by 5.
25. $p : x = 4$ - $q : x^2 = 16$

Review and Reflect

26. What is true about all biconditionals?
27. Give an example of a conditional, its converse, and then write the biconditional.

Warm-Up Answers

1. 30
2. $\frac{7}{11}$
3. 
4.
   a. It could be another day that isn't during Spring Break. Spring Break doesn't last the entire month.
   b. You could be a freshman, sophomore or junior. There are several counterexamples.

2.3 Deductive Reasoning

TEKS G(1)A, G(1)G, G(4)B, G(4)C

Learning Objectives

- Apply basic rules of logic
- Compare inductive reasoning and deductive reasoning

Vocabulary

- logic
- deductive reasoning
- Law of Detachment
- Law of Contrapositive
- Law of Syllogism

Warm-Up

1. Write the converse, inverse, and contrapositive of the following statement: Football players wear shoulder pads.
2. Are the converse, inverse or contrapositive of #1 true? If not, find a counterexample.
3. An if-then statement is $p \rightarrow q$.
   a. What is the inverse of $p \rightarrow q$?
   b. What is the converse of the inverse of $p \rightarrow q$?

Know What? In a fictitious far-away land, a peasant is awaiting his fate from the king. He is standing in a stadium, with two doors in front of him. Both doors have signs on them, which are below:

TABLE 2.4:

Door A	Door B
IN THIS ROOM THERE IS A LADY, AND IN THE OTHER ROOM THERE IS A TIGER.	IN ONE OF THESE ROOMS THERE IS A LADY, AND IN ONE OF THE OTHER ROOMS THERE IS A TIGER.

The king states, "Only one of these statements is true. If you pick correctly, you will find the lady. If not, the tiger will be waiting for you." Which door should the peasant pick?

Deductive Reasoning

Logic: The study of reasoning.

In the first section, you learned about inductive reasoning, making conclusions based upon patterns. Now, we will learn about deductive reasoning.

Deductive Reasoning: Drawing conclusion from facts. Conclusions are usually drawn from general statements about something more specific.

Example 1: Suppose Bea makes the following statements, which are known to be true.

If Central High School wins today, they will go to the regional tournament. Central High School won today.

What is the logical conclusion?

Solution: This is an example of deductive reasoning. These are true statements that we can take as facts. The conclusion is: *Central High School will go to the regional tournament*.

Example 2: Here are two true statements.

Every odd number is the sum of an even and an odd number.

5 is an odd number.

What can you conclude?

Solution: Based on only these two true statements, there is one conclusion: *5 is the sum of an even and an odd number.* (This is true, $5 = 3+2$ or $4+1$).

Law of Detachment

Let's look at Example 2 and change it into symbolic form.

p : A number is odd $\qquad$ q : It is the sum of an even and odd number

The first statement is $p \rightarrow q$.

- The second statement in Example 2, "5 is an odd number," is a specific example of p. "A number" is 5.
- The conclusion is q, "5 is the sum of an even and an odd number."

The symbolic form of Example 2 is:

$$\begin{aligned} &p \rightarrow q \\ &p \\ &\therefore q \qquad\qquad \therefore \text{ symbol for therefore} \end{aligned}$$

All deductive arguments that follow this pattern have a special name, the Law of Detachment.

Law of Detachment: If $p \rightarrow q$ is true, and p is true, then q is true.

Example 3: Here are two true statements.

If $\angle ABC$ *and* $\angle CBD$ *are a linear pair, then* $m\angle ABC + m\angle CBD = 180^\circ$.

$\angle ABC$ *and* $\angle CBD$ *are a linear pair*.

What conclusion can you draw from this?

Solution: This is an example of the Law of Detachment, therefore:

$$m\angle ABC + m\angle CBD = 180^\circ$$

Example 4: Here are two true statements. *Be careful!*

If $\angle 1$ *and* $\angle 2$ *are a linear pair, then* $m\angle 1 + m\angle 2 = 180^\circ$.

$m\angle 1 = 90^\circ$ *and* $m\angle 2 = 90^\circ$.

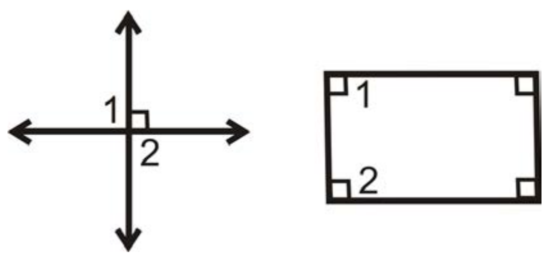
What conclusion can you draw from these two statements?

Solution: Here there is NO conclusion. These statements are in the form:

$$\begin{aligned} &p \rightarrow q \\ &q \end{aligned}$$

We ***cannot*** conclude that $\angle 1$ and $\angle 2$ are a linear pair.

Here are two counterexamples:



Law of Contrapositive

Example 5: The following two statements are true.

If a student is in Geometry, then he or she has passed Algebra I.

Daniel has not passed Algebra I.

What can you conclude from these two statements?

Solution: These statements are in the form:

$$\begin{aligned} &p \rightarrow q \\ &\sim q \end{aligned}$$

Not q is the beginning of the contrapositive $(\sim q \rightarrow \sim p)$, therefore the logical conclusion is *not* p: *Daniel is not in Geometry*.

This example is called the Law of Contrapositive.

Law of Contrapositive: If $p \rightarrow q$ is true and $\sim q$ is true. Then, you can conclude $\sim p$.

The Law of Contrapositive is a logical argument.

Example 6: Determine the conclusion from the true statements below.

Babies wear diapers.

My little brother does not wear diapers.

Solution: The second statement is the equivalent of $\sim q$. Therefore, the conclusion is $\sim p$, or: *My little brother is not a baby*.

Example 7: Determine the conclusion from the true statements below.

If you are not in Chicago, then you can't be on the L.

Bill is in Chicago.

Solution: If we were to rewrite this symbolically, it would look like:

$$\begin{array}{l} \sim p \rightarrow \sim q \\ p \end{array}$$

This is not in the form of the Law of Contrapositive or the Law of Detachment, so there is no logical conclusion.

Example 8: Determine the conclusion from the true statements below.

If you are not in Chicago, then you can't be on the L.

Sally is on the L.

Solution: If we were to rewrite this symbolically, it would look like:

$$\begin{array}{l} \sim p \rightarrow \sim q \\ q \end{array}$$

Even though it looks a little different, this is an example of the Law of Contrapositive. Therefore, the logical conclusion is: *Sally is in Chicago*.

Law of Syllogism

Example 9: Determine the conclusion from the following true statements.

If Pete is late, Mark will be late.

If Mark is late, Karl will be late.

So, if Pete is late, what will happen?

Solution: If Pete is late, this starts a domino effect of lateness. Mark will be late and Karl will be late too. So, if Pete is late, then *Karl will be late*, is the logical conclusion.

Each "then" becomes the next "if" in a chain of statements. This is called the Law of Syllogism.

Law of Syllogism: If $p \rightarrow q$ and $q \rightarrow r$ are true, then $p \rightarrow r$ is true.

Inductive vs. Deductive Reasoning

Inductive Reasoning: Using Patterns

Deductive Reasoning: Using Facts

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Example 10: Solving an equation for x is an example of inductive or deductive reasoning?

Solution: Deductive reasoning. Solving an equation uses Properties of Equality (facts) to solve a problem for x.

Example 11: 1, 10, 100, 1000, . . . is an example of inductive or deductive reasoning?

Solution: Inductive reasoning. This is a pattern.

Example 12: Doing an experiment and writing a hypothesis is an example of inductive or deductive reasoning?

Solution: Inductive reasoning. Making a hypothesis comes from the patterns found in the experiment. These are not facts.

Example 13: Proving the experiment from Example 12 is true is an example of inductive or deductive reasoning?

Solution: Deductive reasoning. Here you would have to use facts to prove what happened in the experiment is supposed to happen.

Know What? Revisited Analyze the two statements on the doors.

Door A: IN THIS ROOM THERE IS A LADY, AND IN THE OTHER ROOM THERE IS A TIGER.

Door B: IN ONE OF THESE ROOMS THERE IS A LADY, AND IN ONE OF THE OTHER ROOMS THERE IS A TIGER.

We know that one door is true, so the other one must be false. Read Door B carefully, it says "in one of these rooms," which means the lady could be behind either door, which has to be true. So, because Door B is the true statement, Door A is false and the tiger is behind it. The peasant should pick Door B.

Practice Problems

Determine the logical conclusion and state which law you used (Law of Detachment, Law of Contrapositive, or Law of Syllogism). If no conclusion can be drawn, write "no conclusion."

1. People who vote for Jane Wannabe are smart people. I voted for Jane Wannabe.
2. If Rae is the driver today then Maria is the driver tomorrow. Ann is the driver today.
3. All equiangular triangles are equilateral. $\triangle ABC$ is equiangular.
4. If North wins, then West wins. If West wins, then East loses.
5. If $z > 5$, then $x > 3$. If $x > 3$, then $y > 7$.
6. If I am cold, then I wear a jacket. I am not wearing a jacket.
7. If it is raining outside, then I need an umbrella. It is not raining outside.
8. If a shape is a circle, then it never ends. If it never ends, then it never starts. If it never starts, then it doesn't exist. If it doesn't exist, then we don't need to study it.
9. If you text while driving, then you are unsafe. You are a safe driver.
10. If you wear sunglasses, then it is sunny outside. You are wearing sunglasses.
11. If you wear sunglasses, then it is sunny outside. It is cloudy.
12. I will clean my room if my mom asks me to. I am not cleaning my room.
13. Write the symbolic representation of #8. Include your conclusion. Does this argument make sense?
14. Write the symbolic representation of #10. Include your conclusion.
15. Write the symbolic representation of #11. Include your conclusion.

Determine if the problems below represent inductive or deductive reasoning. Briefly explain your answer.

16. John is watching the weather. As the day goes on it gets more and more cloudy and cold. He concludes that it is going to rain.
17. Beth's 2-year-old sister only eats hot dogs, blueberries and yogurt. Beth decides to give her sister some yogurt because she is hungry.
18. Nolan Ryan has the most strikeouts of any pitcher in Major League Baseball. Jeff debates that he is the best pitcher of all-time for this reason.
19. Ocean currents and waves are dictated by the weather and the phase of the moon. Surfers use this information to determine when it is a good time to hit the water.
20. As Rich is driving along the 405, he notices that as he gets closer to LAX the traffic slows down. As he passes it, it speeds back up. He concludes that anytime he drives past an airport, the traffic will slow down.

Determine if the following statements are true or false.

21. The Law of Detachment uses an if-then statement and its hypothesis to draw a conclusion.
22. There is a Law of Inverse.
23. Sometimes arguments can be valid, but not make sense.
24. The Law of Syllogism takes the conclusion from a statement and makes it the hypothesis of the next.
25. Number patterns are an example of deductive reasoning.

Review and Reflect

26. Describe in your own words how to use the Law of Detachment.

27. Describe in your own words how to use the Law of Syllogism.

Warm-Up Answers

1. Converse: If you wear shoulder pads, then you are a football player.

Inverse: If you are not a football player, then you do not wear shoulder pads.

Contrapositive: If you do not wear shoulder pads, then you are not a football player.

2. The converse and inverse are both false. A counterexample for both could be a woman from the 80's. They definitely wore shoulder pads!

3. (a) $\sim p \rightarrow \sim q$

(b) The converse of the $\sim p \rightarrow \sim q$ is $\sim q \rightarrow \sim p$, or the contrapositive.

2.4 Algebraic and Congruence Properties

TEKS G(1)B, G(6)A

Learning Objectives

- Understand basic properties of equality and congruence
- Solve equations and justify each step
- Fill in the blanks of a 2-column proof

Vocabulary

Warm-Up

Solve the following problems.

1. Solve $2x-3=9$.

2. *If two angles are a linear pair, they are supplementary.*

If two angles are supplementary, their sum is 180°.

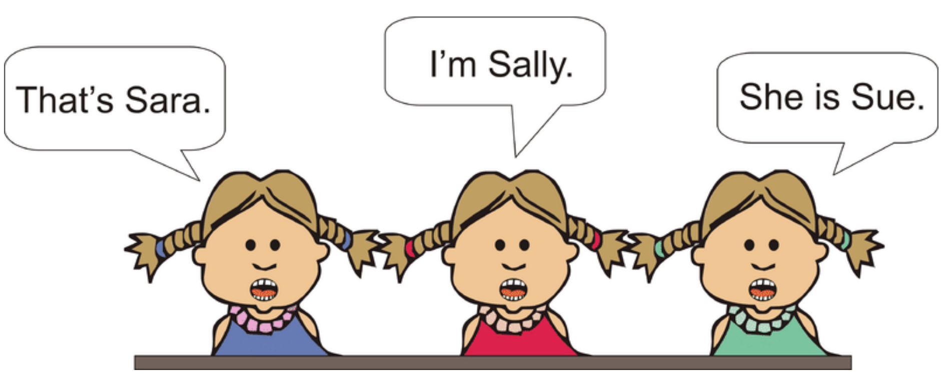
What can you conclude? By which law?

3. Draw a picture with the following:

$\angle LMN$ is bisected by $\overline{MO}$	$\overline{LM} \cong \overline{MP}$
$\angle OMP$ is bisected by $\overline{MN}$	N is the midpoint of $\overline{MQ}$

Know What? Three identical triplets are sitting next to each other. The oldest is Sara and she **always tells the truth**. The next oldest is Sue and she **always lies**. Sally is the youngest of the three. She **sometimes lies and sometimes tells the truth**.

Scott came over one day and didn't know who was who, so he asked each sister who was sitting in the middle. Who is who?



Properties of Equality

Recall from Chapter 1 that the = sign and the word "equality" are used with numbers. The basic properties of equality were introduced to you in Algebra I. Here they are again:

Reflexive Property of Equality: $AB = AB$

Symmetric Property of Equality: $m\angle A = m\angle B$ and $m\angle B = m\angle A$

Transitive Property of Equality: $AB = CD$ and $CD = EF$, then $AB = EF$

Substitution Property of Equality: If $a = 9$ and $a - c = 5$, then $9 - c = 5$

Addition Property of Equality: If $2x - 5 = 7$, then $2x = 12$

Subtraction Property of Equality: If $m\angle x + 15^\circ = 65^\circ$, then $m\angle x = 50^\circ$

Multiplication Property of Equality: If $\frac{y}{2} = 10$, then y = 20.

Division Property of Equality: If $3b = 18$, then $b = 6$.

Distributive Property: If $5(2x - 7) = 55$, then $10x - 35 = 55$.

Properties of Congruence

Recall that $\overline{AB} \cong \overline{CD}$ if and only if $AB = CD$ and $\angle ABC \cong \angle DEF$ if and only if $m\angle ABC = m\angle DEF$. The Properties of Equality work for $AB, CD, m\angle ABC$ and $m\angle DEF$.

Just like the properties of equality, there are properties of congruence. These properties hold for figures and shapes.

TABLE 2.5:

	For Line Segments	*For Angles*
Reflexive Property of Congruence	$\overline{AB} \cong \overline{AB}$	$\angle B \cong \angle B$
Symmetric Property of Congruence	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$	If $\angle ABC \cong \angle DEF$, then $\angle DEF \cong \angle ABC$
Transitive Property of Congruence	If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$	If $\angle ABC \cong \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle GHI$

Using Properties of Equality with Equations

When you solve equations in algebra you use properties of equality. You might not write out the property for each step, but you should know that there is an equality property that justifies that step.

Example 1: Solve $2(3x - 4) + 11 = x - 27$ and write the property for each step (also called "to justify each step").

Solution:

$$\begin{aligned}
2(3x-4)+11 &= x-27 \\
6x-8+11 &= x-27 && \text{Distributive Property} \\
6x+3 &= x-27 && \text{Combine like terms} \\
6x+3-3 &= x-27-3 && \text{Subtraction Property of Equality} \\
6x &= x-30 && \text{Simplify} \\
6x-x &= x-x-30 && \text{Subtraction Property of Equality} \\
5x &= -30 && \text{simplify} \\
\frac{5x}{5} &= \frac{-30}{5} && \text{Division Property of Equality} \\
x &= -6 && \text{Simplify}
\end{aligned}$$

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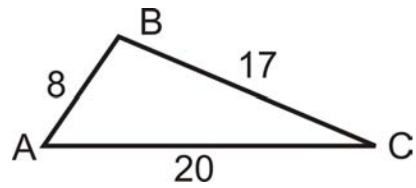
Example 2: $AB = 8, BC = 17$, and $AC = 20$. Are points A, B, and C collinear?

Solution: Set up an equation using the Segment Addition Postulate.

$$\begin{aligned}
AB+BC &= AC && \text{Segment Addition Postulate} \\
8+17 &= 20 && \text{Substitution Property of Equality} \\
25 &\neq 20 && \text{Combine like terms}
\end{aligned}$$

Because the two sides are not equal, A, B and C are not collinear.



Example 3: If $m\angle A + m\angle B = 100^\circ$ and $m\angle B = 40^\circ$, prove that $\angle A$ is an acute angle.

Solution: We will use a 2-column format, with statements in one column and their reasons next to it, just like Example 1.

$$m\angle A + m\angle B = 100^\circ \text{Given Information}$$
$$m\angle B = 40^\circ \text{Given Information}$$
$$m\angle A + 40^\circ = 100^\circ \text{Substitution Property of Equality}$$
$$m\angle A = 60^\circ \text{Subtraction Property of Equality}$$
$$\angle A \text{ is an acute angle} \text{Definition of an acute angle}, m\angle A < 90^\circ$$

Two-Column Proof

Examples 1 and 3 are examples of two-column proofs. They both have the left side, the statements, and on the right side are the reason for these statements. Here we will continue with more proofs and some helpful tips for completing one.

Example 4: Write a two-column proof for the following:

If A, B, C, and D are points on a line, in the given order, and $AB = CD$, then $AC = BD$.

Solution: When the statement is given in this way, the "if" part is the given and the "then" part is what we are trying to prove.

Always start with drawing a picture of what you are given.

Plot the points in the order A, B, C, D on a line.

Add the given, $AB = CD$.

Draw the 2-column proof and start with the given information.

TABLE 2.6:

Statement	*Reason*
1. A, B, C, and D are collinear, in that order.	Given
2. $AB = CD$	Given
3. $BC = BC$	Reflexive Property of Equality

TABLE 2.6: (continued)

Statement	***Reason***
4. $AB+BC=BC+CD$	Addition Property of Equality
5. $AB+BC=AC$ $BC+CD=BD$	Segment Addition Postulate
6. $AC=BD$	Substitution or Transitive Property of Equality

Once we reach what we wanted to prove, we are done.

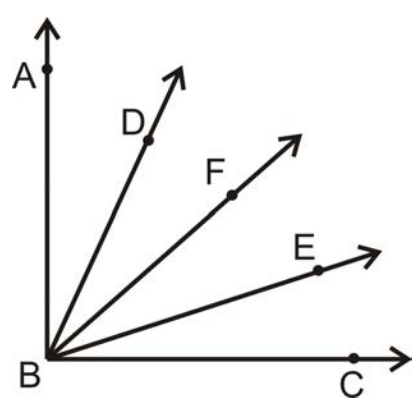
When completing a proof, these keep things in mind:

- **Number each step.**
- **Start with the given information.**
- **Statements with the same reason can (or cannot) be combined into one step**. It is up to you. For example, steps 1 and 2 above could have been one step. And, in step 5, the two statements could have been written separately.
- **Draw a picture and mark it with the given information.**
- **You must have a reason for EVERY statement.**
- **The order of the statements in the proof is not fixed.** For example, steps 3, 4, and 5 could have been interchanged and it would still make sense.
- **Reasons will be definitions, postulates, properties and previously proven theorems.** "Given" is only used as a reason if the information in the statement column was told in the problem.

Example 5: Write a two-column proof.

Given: $\overrightarrow{BF}$ bisects $\angle ABC$; $\angle ABD \cong \angle CBE$

Prove: $\angle DBF \cong \angle EBF$



Solution: First, put the appropriate markings on the picture. Recall, that bisect means "to cut in half." Therefore, $m\angle ABF = m\angle FBC$.

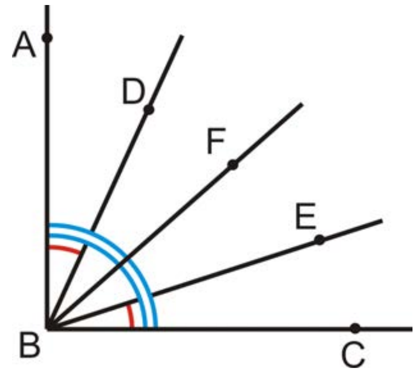
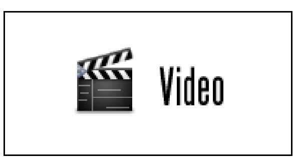


TABLE 2.7:

Statement	***Reason***
1. $\overrightarrow{BF}$ bisects $\angle ABC, \angle ABD \cong \angle CBE$	Given
2. $m\angle ABF = m\angle FBC$	Definition of an Angle Bisector
3. $m\angle ABD = m\angle CBE$	If angles are $\cong$, then their measures are equal.
4. $m\angle ABF = m\angle ABD + m\angle DBF$	Angle Addition Postulate
$m\angle FBC = m\angle EBF + m\angle CBE$	
5. $m\angle ABD + m\angle DBF = m\angle EBF + m\angle CBE$	Substitution Property of Equality
6. $m\angle ABD + m\angle DBF = m\angle EBF + m\angle ABD$	Substitution Property of Equality
7. $m\angle DBF = m\angle EBF$	Subtraction Property of Equality
8. $\angle DBF \cong \angle EBF$	If measures are equal, the angles are $\cong$.

Use symbols and abbreviations for words within proofs. For example, $\cong$ was used in place of the word *congruent* above. You could also use $\angle$ for the word angle.

Watch this!



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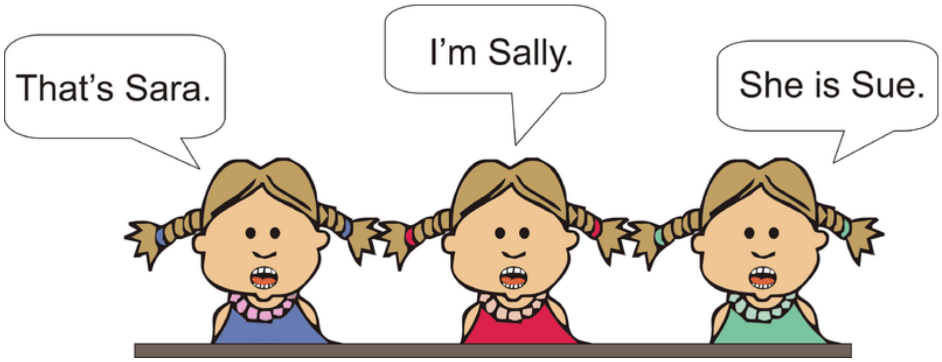
Know What? Revisited Analyzing the picture and what we know the sister on the left cannot be Sara because she lied (if we take what the sister in the middle said as truth). So, let's assume that the sister in the middle is telling the truth, she is Sally. However, we know this is impossible, because that would have to mean that the sister on the right is lying and Sarah does not lie. From this, that means that the sister on the right is Sara and she is telling the truth, the sister in the middle is Sue. So, the first sister is Sally. The order is: Sally, Sue, Sara.



Practice Problems

- Questions 1-8 are similar to Examples 1 and 3.
- Questions 9-14 use the Properties of Equality.
- Questions 15-17 are similar to Example 2.
- Questions 18 and 19 are similar to Examples 8 and 9.
- Questions 20-34 are review.

For questions 1-8, solve each equation and justify each step.

1. $3x+11=-16$
2. $7x-3=3x-35$
3. $\frac{2}{3}g+1=19$
4. $\frac{1}{2}MN=5$
5. $5m\angle ABC=540^\circ$
6. $10b-2(b+3)=5b$
7. $\frac{1}{4}y+\frac{5}{6}=\frac{1}{3}$
8. $\frac{1}{4}AB+\frac{1}{3}AB=12+\frac{1}{2}AB$

For questions 9-14, use the given property or properties of equality to fill in the blank. x, y, and z are real numbers.

9. Symmetric: If $x=3$, then ______________.
10. Distributive: If $4(3x-8)$, then ______________.
11. Transitive: If $y=12$ and $x=y$, then ______________.
12. Symmetric: If $x+y=y+z$, then ______________.
13. Transitive: If $AB=5$ and $AB=CD$, then ______________.
14. Substitution: If $x=y-7$ and $x=z+4$, then ______________.
15. Given points E, F, and G and $EF=16$, $FG=7$ and $EG=23$. Determine if E, F and G are collinear.
16. Given points H, I and J and $HI=9$, $IJ=9$ and $HJ=16$. Are the three points collinear? Is I the midpoint?
17. If $m\angle KLM=56^\circ$ and $m\angle KLM+m\angle NOP=180^\circ$, explain how $\angle NOP$ must be an obtuse angle.

Fill in the blanks in the proofs below.

18. Given: $\angle ABC\cong\angle DEF$ $\angle GHI\cong\angle JKL$ Prove: $m\angle ABC+m\angle GHI=m\angle DEF+m\angle JKL$

TABLE 2.8:

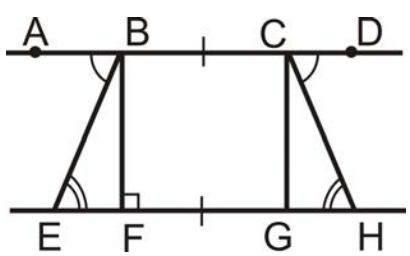
Statement	Reason
1.	Given
2. $m\angle ABC = m\angle DEF$	
$m\angle GHI = m\angle JKL$	
3.	Addition Property of Equality
4. $m\angle ABC + m\angle GHI = m\angle DEF + m\angle JKL$	

19. Given: M is the midpoint of $\overline{AN}$. N is the midpoint $\overline{MB}$Prove: $AM = NB$

TABLE 2.9:

Statement	Reason
1.	Given
2.	Definition of a midpoint
3. $AM = NB$	

Use the diagram to answer questions 20-25.



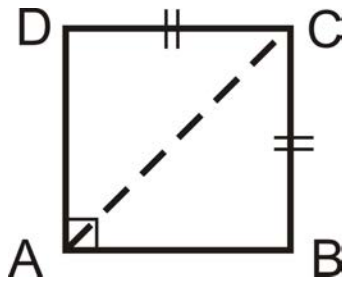
20. Name a right angle.
21. Name two perpendicular lines.
22. Given that $EF = GH$, is $EG = FH$ true? Explain your answer.
23. Is $\angle CGH$ a right angle? Why or why not?
24. Fill in the blanks:

$$m\angle ABF = m\angle ABE + m\angle\_\_\_\_$$
$$m\angle DCG = m\angle DCH + m\angle\_\_\_\_$$

25. Fill in the blanks:

$$AB + \_\_ = AC$$
$$\_\_ + CD = BD$$

Use the diagram to answer questions 26-31.

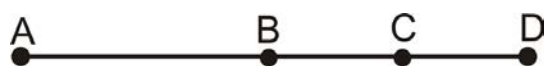


Which of the following must be true from the diagram?

Take each question separately, they do not build upon each other.

26. $\overline{AD} \cong \overline{BC}$
27. $\overline{AB} \cong \overline{CD}$
28. $\overline{CD} \cong \overline{BC}$
29. $\overline{AB} \perp \overline{AD}$
30. $ABCD$ is a square
31. $\overline{AC}$ bisects $\angle DAB$

Use the diagram to answer questions 32-34.



Given: B bisects $\overline{AD}$, C is the midpoint of $\overline{BD}$ and $AD = 12$.

What is the value of each of the following?

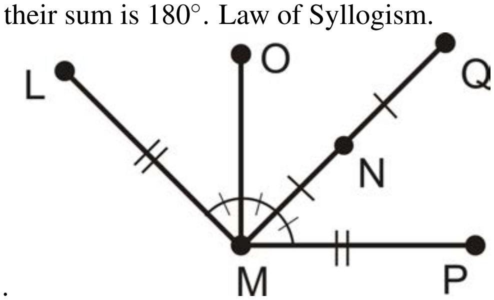
32. AB
33. BC
34. AC

Review and Reflect

35. Proofs can be written in different formats such as a two-column proof, paragraph proof, and flow chart. Write a flow chart proofs for example 5.

36. Solve the following equation and justify each step in a paragraph proof, $\frac{1}{2}x - 14 = -5$

Warm-Up Answers

1. $x = 3$
2. If 2 angles are a linear pair, then their sum is 180°. Law of Syllogism.
3. 

2.5 Proofs about Angle Pairs and Segments

TEKS G(1)G, G(6)A

Learning Objectives

- Use theorems about pairs of angles, right angles, and midpoints

Warm-Up

Fill in the 2-column proof.

1. Given: $\overline{VX}$ is the angle bisector of $\angle WVY$.

$\overline{VY}$ is the angle bisector of $\angle XVZ$.

Prove: $\angle WVX \cong \angle YVZ$

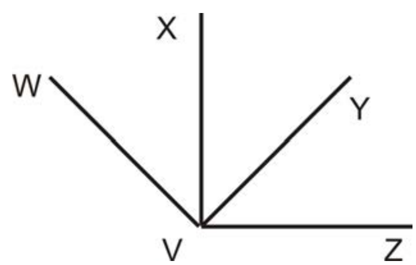
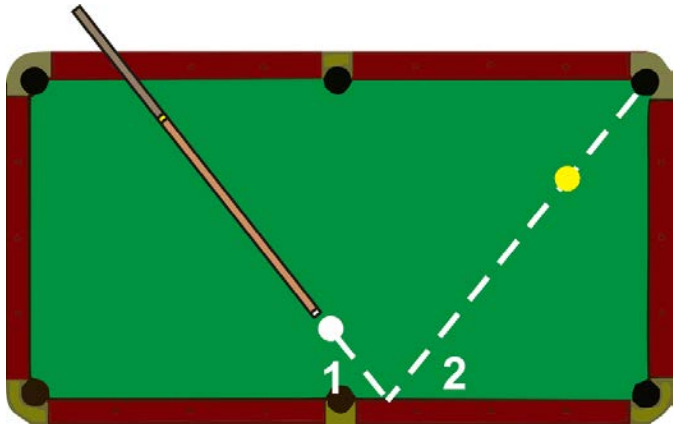


TABLE 2.10:

Statement	***Reason***
1.	Given
2.	Definition of an Angle Bisector
3. $m\angle WVX = m\angle YVZ$	
4. $\angle WVX \cong \angle YVZ$	

Know What? The game of pool relies heavily on angles. The angle at which you hit the cue ball with your cue determines if you hit the yellow ball and if you can hit it into the pocket.

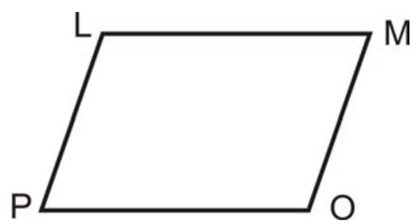


The best path to get the yellow ball into the corner pocket is to use the path in the picture to the right. You measure and need to hit the cue ball so that it hits the side of the table at a 50° angle (this would be $m\angle 1$). Find $m\angle 2$ and how it relates to $\angle 1$.

If you would like to play with the angles of pool, click the link for an interactive game. http://www.coolmath-games.com/0-poolgeometry/index.html

Naming Angles

As we learned in Chapter 1, angles can be addressed by numbers and three letters, where the letter in the middle is the vertex. **We can shorten this label to only the middle letter if there is only one angle with that vertex.**



All of the angles in this parallelogram can be labeled by one letter, the vertex, instead of all three.

$\angle MLP$ is $\angle L$	$\angle LMO$ is $\angle M$
$\angle MOP$ is $\angle O$	$\angle OPL$ is $\angle P$

Right Angle Theorem

If two angles are right angles, then the angles are congruent

Proof of the Right Angle Theorem

Given: $\angle A$ and $\angle B$ are right angles

Prove: $\angle A \cong \angle B$

TABLE 2.11:

Statement	*Reason*
1. $\angle A$ and $\angle B$ are right angles	Given
2. $m\angle A = 90^\circ$ and $m\angle B = 90^\circ$	Definition of right angles
3. $m\angle A = m\angle B$	Transitive Property of Equality
4. $\angle A \cong \angle B$	$\cong$ angles have = measures

Anytime right angles are mentioned in a proof, you will need to use this theorem to say the angles are congruent.

Watch this!

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Same Angle Supplements Theorem

If two angles are supplementary to the same angle then the angles are congruent

$$m\angle A + m\angle B = 180^\circ \text{ and}$$
$$m\angle C + m\angle B = 180^\circ$$
$$\text{then } m\angle A = m\angle C$$

Proof of the Same Angles Supplements Theorem

Given: $\angle A$ and $\angle B$ are supplementary angles. $\angle B$ and $\angle C$ are supplementary angles.

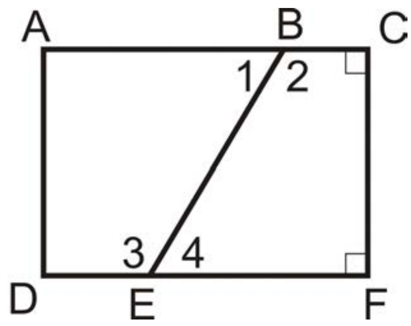
Prove: $\angle A \cong \angle C$

TABLE 2.12:

Statement	*Reason*
1. $\angle A$ and $\angle B$ are supplementary	Given
$\angle B$ and $\angle C$ are supplementary	
2. $m\angle A + m\angle B = 180^\circ$	Definition of supplementary angles
$m\angle B + m\angle C = 180^\circ$	
3. $m\angle A + m\angle B = m\angle B + m\angle C$	Substitution Property of Equality
4. $m\angle A = m\angle C$	Subtraction Property of Equality
5. $\angle A \cong \angle C$	$\cong$ angles have = measures

Example 1: $\angle 1 \cong \angle 4$ and $\angle C$ and $\angle F$ are right angles.

Which angles are congruent and why?



Solution: By the Right Angle Theorem, $\angle C \cong \angle F$. Also, $\angle 2 \cong \angle 3$ by the Same Angles Supplements Theorem because $\angle 1 \cong \angle 4$ and they are linear pairs with these congruent angles.

Same Angle Complements Theorem

If two angles are complementary to the same angle then the angles are congruent

$$\begin{aligned} m\angle A + m\angle B &= 90^\circ \text{ and} \\ m\angle C + m\angle B &= 90^\circ \\ \text{then } m\angle A &= m\angle C. \end{aligned}$$

The proof of the Same Angles **Complements** Theorem is in the Warm-Up Questions. Use the proof of the Same Angles **Supplements** Theorem to help you.

Recall the Vertical Angles Theorem from Chapter 1. We will do a proof here.

Given: Lines k and m intersect.

Prove: $\angle 1 \cong \angle 3$

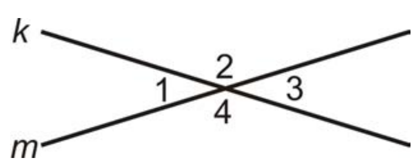


TABLE 2.13:

Statement	***Reason***
1. Lines k and m intersect	Given
2. $\angle 1$ and $\angle 2$ are a linear pair $\angle 2$ and $\angle 3$ are a linear pair	Definition of a Linear Pair
3. $\angle 1$ and $\angle 2$ are supplementary $\angle 2$ and $\angle 3$ are supplementary	Linear Pair Postulate
4. $m\angle 1 + m\angle 2 = 180^\circ$ $m\angle 2 + m\angle 3 = 180^\circ$	Definition of Supplementary Angles
5. $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$	Substitution Property of Equality
6. $m\angle 1 = m\angle 3$	Subtraction Property of Equality
7. $\angle 1 \cong \angle 3$	$\cong$ angles have = measures

You can also do a proof for $\angle 2 \cong \angle 4$, which would be exactly the same.

Example 2: In the picture $\angle 2 \cong \angle 3$ and $k \perp p$.

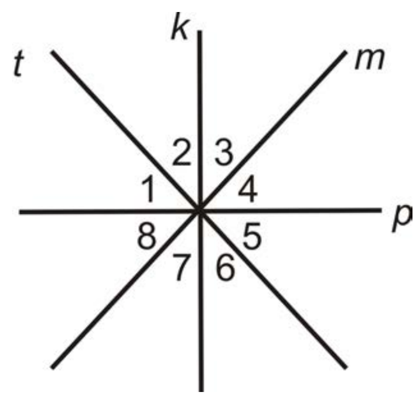
Each pair below is congruent. State why.

a) $\angle 1$ and $\angle 5$

b) $\angle 1$ and $\angle 4$

c) $\angle 2$ and $\angle 6$

d) $\angle 6$ and $\angle 7$



Solution:

a) and

b) Same Angles Complements Theorem

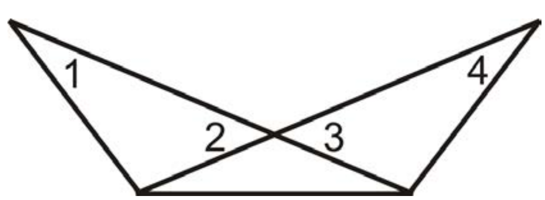
c) Vertical Angles Theorem

d) Vertical Angles Theorem followed by the Transitive Property

Example 3: Write a two-column proof.

Given: $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$

Prove: $\angle 1 \cong \angle 4$



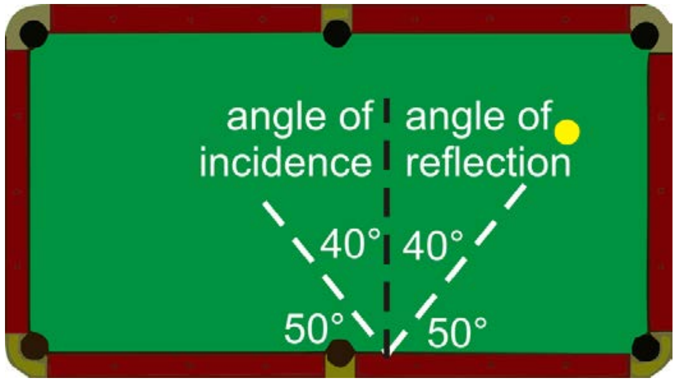
Solution:

TABLE 2.14:

Statement	***Reason***
1. $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$	Given
2. $\angle 2 \cong \angle 3$	Vertical Angles Theorem
3. $\angle 1 \cong \angle 4$	Transitive Property of Congruence

Know What? Revisited If $m\angle 1 = 50^\circ$, then $m\angle 2 = 50^\circ$.

Draw a perpendicular line at the point of reflection. The laws of reflection state that the angle of incidence is equal to the angle of reflection (see picture). This is an example of the Same Angles Complements Theorem.



Practice Problems

Fill in the blanks in the proofs below.

1. Given: $\overline{AC} \perp \overline{BD}$ and $\angle 1 \cong \angle 4$Prove: $\angle 2 \cong \angle 3$

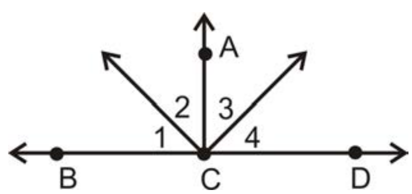


TABLE 2.15:

Statement	*Reason*
1. $\overline{AC} \perp \overline{BD}, \angle 1 \cong \angle 4$	
2. $m\angle 1 = m\angle 4$	
3.	$\perp$ lines create right angles
4. $m\angle ACB = 90^\circ$ $m\angle ACD = 90^\circ$	
5. $m\angle 1 + m\angle 2 = m\angle ACB$ $m\angle 3 + m\angle 4 = m\angle ACD$	
6.	Substitution
7. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	
8.	Substitution
9.	Subtraction Property of Equality
10. $\angle 2 \cong \angle 3$	

2. Given: $\angle MLN \cong \angle OLP$Prove: $\angle MLO \cong \angle NLP$

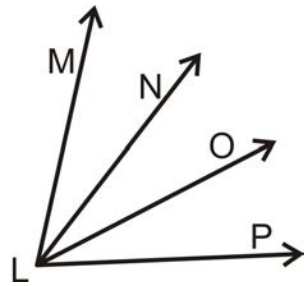


TABLE 2.16:

Statement	Reason
1.	
2.	$\cong$ angles have = measures
3.	Angle Addition Postulate
4.	Substitution
5. $m\angle MLO = m\angle NLP$	
6.	$\cong$ angles have = measures

3. Given: $\overline{AE} \perp \overline{EC}$ and $\overline{BE} \perp \overline{ED}$Prove: $\angle 1 \cong \angle 3$

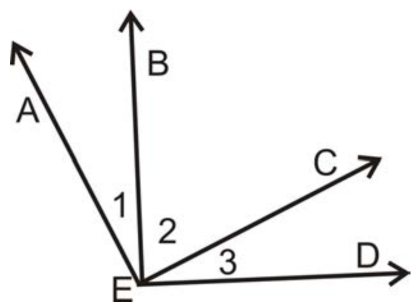


TABLE 2.17:

Statement	Reason
1.	
2.	$\perp$ lines create right angles
3. $m\angle BED = 90^\circ$	
$m\angle AEC = 90^\circ$	
4.	Angle Addition Postulate
5.	Substitution
6. $m\angle 2 + m\angle 3 = m\angle 1 + m\angle 3$	
7.	Subtraction Property of Equality
8.	$\cong$ angles have = measures

4. Given: $\angle L$ is supplementary to $\angle M$$\angle P$ is supplementary to $\angle O$$\angle L \cong \angle O$Prove: $\angle P \cong \angle M$

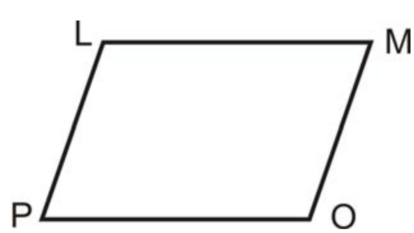


TABLE 2.18:

Statement	Reason
1.	
2. $m\angle L = m\angle O$	
3.	Definition of supplementary angles
4.	Substitution
5.	Substitution
6.	Subtraction Property of Equality
7. $\angle M \cong \angle P$	

5. Given: $\angle 1 \cong \angle 4$Prove: $\angle 2 \cong \angle 3$

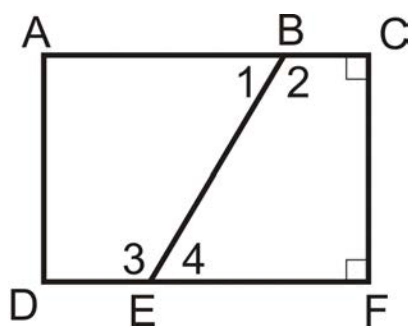


TABLE 2.19:

Statement	Reason
1.	
2. $m\angle 1 = m\angle 4$	
3.	Definition of a Linear Pair
4. $\angle 1$ and $\angle 2$ are supplementary	
$\angle 3$ and $\angle 4$ are supplementary	
5.	Definition of supplementary angles
6. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	
7. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 1$	
8. $m\angle 2 = m\angle 3$	
9. $\angle 2 \cong \angle 3$	

6. Given: $\angle C$ and $\angle F$ are right angles Prove: $m\angle C + m\angle F = 180^\circ$

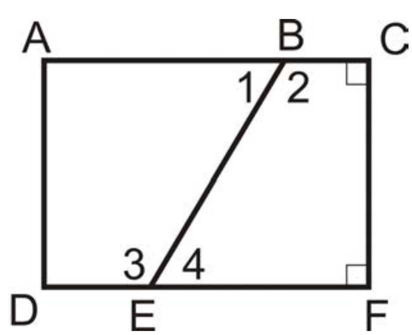


TABLE 2.20:

Statement	Reason
1.	
2. $m\angle C = 90^\circ, m\angle F = 90^\circ$	
3. $90^\circ + 90^\circ = 180^\circ$	
4. $m\angle C + m\angle F = 180^\circ$	

7. Given: $l \perp m$ Prove: $\angle 1 \cong \angle 2$

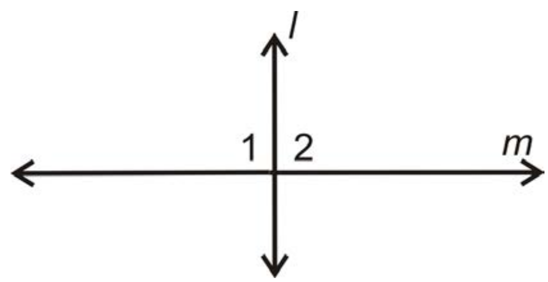


TABLE 2.21:

Statement	Reason
1. $l \perp m$	
2. $\angle 1$ and $\angle 2$ are right angles	
3.	

8. Given: $m\angle 1 = 90^\circ$ Prove: $m\angle 2 = 90^\circ$

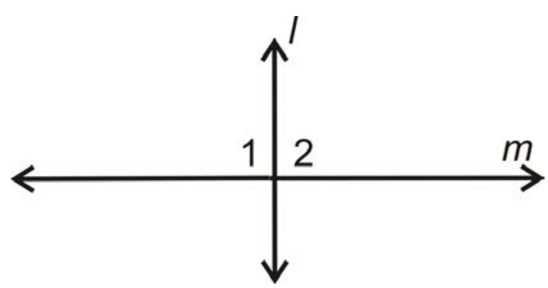


TABLE 2.22:

Statement	Reason
1.	
2. $\angle 1$ and $\angle 2$ are a linear pair	
3.	Linear Pair Postulate
4.	Definition of supplementary angles
5.	Substitution
6. $m\angle 2 = 90^\circ$	

9. Given: $l \perp m$ Prove: $\angle 1$ and $\angle 2$ are complements

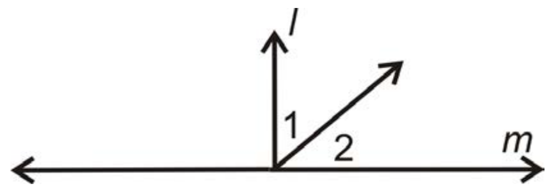


TABLE 2.23:

Statement	*Reason*
1.	
2.	$\perp$ lines create right angles
3. $m\angle 1 + m\angle 2 = 90^\circ$	
4. $\angle 1$ and $\angle 2$ are complementary	

10. Given: $l \perp m$ and $\angle 2 \cong \angle 6$Prove: $\angle 6 \cong \angle 5$

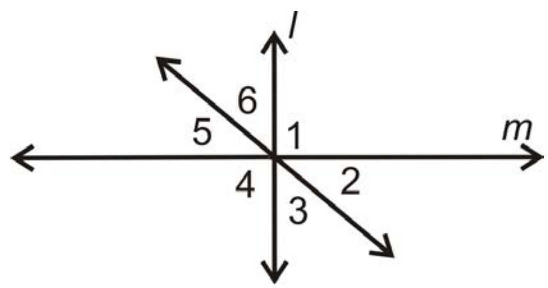
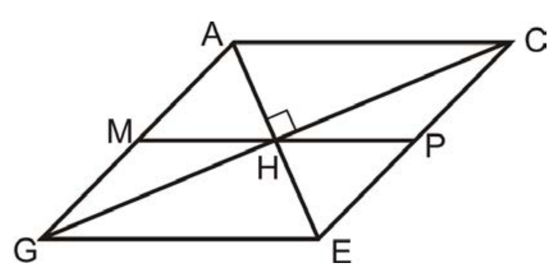


TABLE 2.24:

Statement	*Reason*
1.	
2. $m\angle 2 = m\angle 6$	
3. $\angle 5 \cong \angle 2$	
4. $m\angle 5 = m\angle 2$	
5. $m\angle 5 = m\angle 6$	

Use the picture for questions 11-20.



Given: H is the midpoint of $\overline{AE}, \overline{MP}$ and $\overline{GC}$

M is the midpoint of $\overline{GA}$

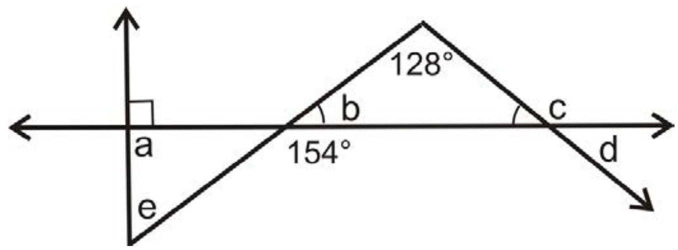
P is the midpoint of $\overline{CE}$

$\overline{AE} \cong \overline{GC}$

11. List two pairs of vertical angles.

12. List all the pairs of congruent segments.
13. List two linear pairs that do not have H as the vertex.
14. List a right angle.
15. List a pair of adjacent angles that are NOT a linear pair (do not add up to 180°).
16. What line segment is the perpendicular bisector of $\overline{AE}$?
17. Name a bisector of $\overline{MP}$.
18. List a pair of complementary angles.
19. If $\overline{GC}$ is an angle bisector of $\angle AGE$, what two angles are congruent?
20. Find $m\angle GHE$.

For questions 21-25, find the measure of the lettered angles in the picture below.



21. a
22. b
23. c
24. d
25. e (hint: e is complementary with b)

ALGEBRA Find the measure of each angle in the diagram. 26.

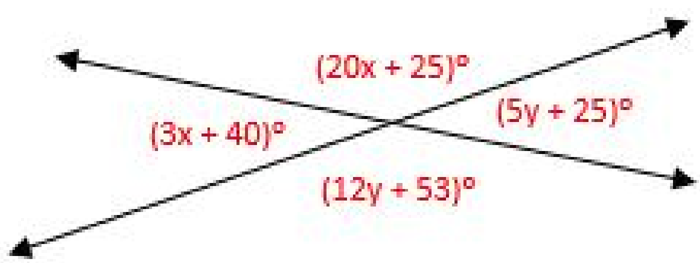


FIGURE 2.2

27.

Review and Reflect

28. Write a paragraph proof for example 3.

29.Justify your answer to number 25.

Warm-Up Answers

1.

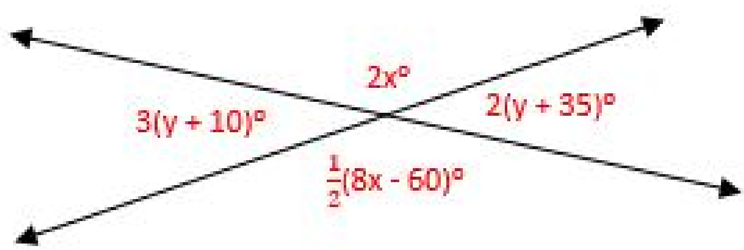


FIGURE 2.3

TABLE 2.25:

Statement	Reason
1. $\overline{VX}$ is an $\angle$ bisector of $\angle WVY$	Given
$\overline{VY}$ is an $\angle$ bisector of $\angle XVZ$	
2. $\angle WVX \cong \angle XVY$	Definition of an angle bisector
$\angle XVY \cong \angle YVZ$	
3. $\angle WVX \cong \angle YVZ$	Transitive Property

2.6 Chapter 2 Review

Symbol Toolbox

$\rightarrow$	if-then	$\sim$ not
$\therefore$	therefore	

Keywords and Vocabulary

Inductive Reasoning

- Inductive Reasoning
- Conjecture
- Counterexample

Conditional Statements

- Conditional Statement (If-Then Statement)
- Hypothesis
- Conclusion
- Converse
- Inverse
- Contrapositive
- Biconditional Statement

Deductive Reasoning

- Logic
- Deductive Reasoning
- Law of Detachment
- Law of Contrapositive
- Law of Syllogism

Algebraic Congruence Properties

- Reflexive Property of Equality
- Symmetric Property of Equality
- Transitive Property of Equality
- Substitution Property of Equality
- Addition Property of Equality
- Subtraction Property of Equality

- Multiplication Property of Equality
- Division Property of Equality
- Distributive Property
- Reflexive Property of Congruence
- Symmetric Property of Congruence
- Transitive Property of Congruence

Proofs about Angle Pairs Segments

- Right Angle Theorem
- Same Angle Supplements Theorem
- Same Angle Complements Theorem

Review

Match the definition or description with the correct word.

1. $5 = x$ and $y+4 = x$, then $5 = y+4$ —A. Law of Contrapositive
2. An educated guess —B. Inductive Reasoning
3. $6(2a+1) = 12a+12$ —C. Inverse
4. $2, 4, 8, 16, 32, \ldots$ —D. Transitive Property of Equality
5. $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{AB}$ —E. Counterexample
6. $\sim p \rightarrow \sim q$ —F. Conjecture
7. Conclusions drawn from facts. —G. Deductive Reasoning
8. If I study, I will get an "A" on the test. I did not get an A. Therefore, I didn't study. —H. Distributive Property
9. $\angle A$ and $\angle B$ are right angles, therefore $\angle A \cong \angle B$. —I. Symmetric Property of Congruence
10. 2 disproves the statement: "All prime numbers are odd." —J. Right Angle Theorem

K. Definition of Right Angles

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9687 .

2.7 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1^{st} **Section: Inductive Reasoning**

Inductive Reasoning

Conjecture

Counterexample

Homework:

2^{nd} **Section: Conditional Statements**

Conditional Statement (If-Then Statement)

Hypothesis

Conclusion

Converse

Inverse

Contrapositive

Biconditional Statement

Homework:

3^{rd} **Section: Deductive Reasoning**

Logic

Deductive Reasoning

Law of Detachment

Law of Contrapositive

Law of Syllogism

Homework:

4^{th} **Section: Algebraic Congruence Properties**

Reflexive Property of Equality

Symmetric Property of Equality

Transitive Property of Equality

Substitution Property of Equality

Addition Property of Equality

Subtraction Property of Equality

Multiplication Property of Equality

Division Property of Equality

Distributive Property

Reflexive Property of Congruence

Symmetric Property of Congruence

Transitive Property of Congruence

Homework:

5^{th} **Section: Proofs about Angle Pairs Segments**

Right Angle Theorem

Same Angle Supplements Theorem

Same Angle Complements Theorem

Homework:

CHAPTER 3 Parallel and Perpendicular Lines

Chapter Outline

In this chapter, you will explore the different relationships formed by parallel and perpendicular lines and planes. Different angle relationships will also be explored and what happens when lines are parallel. You will start to prove lines parallel or perpendicular using a fill-in-the-blank 2-column proof. There is also an algebra review of the equations of lines, slopes, and how that relates to parallel and perpendicular lines in geometry.

3.1 Lines and Angles

TEKS **G(4)D, G(5)A, G(5)B**

Learning Objectives

- Define parallel lines, skew lines,and perpendicular planes
- Understand the Parallel Line Postulate and the Perpendicular Line Postulate
- Identify angles made by two lines and a transversal

Vocabulary

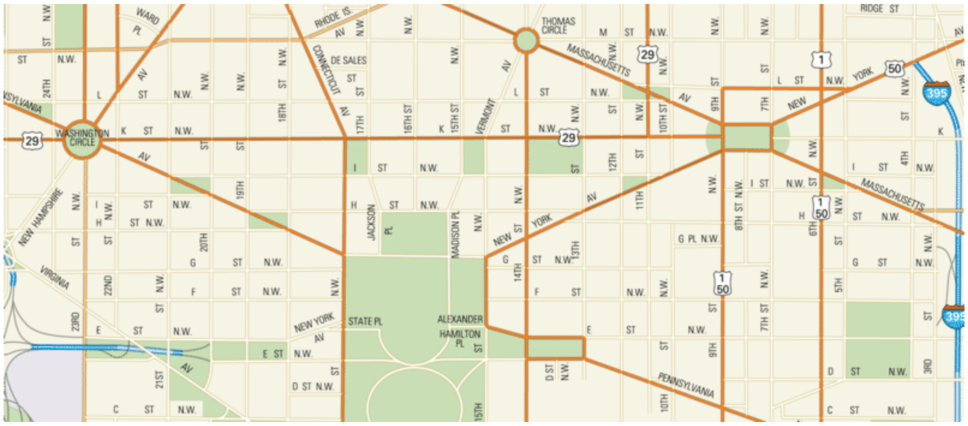
- parallel lines
- skew lines
- transversal
- corresponding angles
- alternate interior angles
- alternate exterior angles
- same side interior angles

Warm-Up

1. What is the equation of a line with slope of -2 and y-intercept of 3?
2. What is the slope of the line that passes through (3, 2) and (5, -6)?
3. Find the y-intercept of the line from #2 and write the equation in slope intercept form.
4. Define parallel in your own words.

Know What? To the right is a partial map of Washington DC. The streets are designed on a grid system, where lettered streets, A through Z run east to west and numbered streets, 1^{st} to 30^{th} run north to south. Every state also has its own street that runs diagonally through the city.

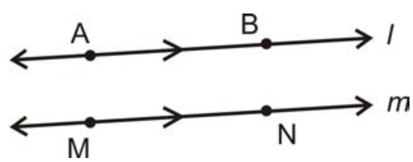
Which streets are parallel? Which streets are perpendicular? How do you know?



If you are having trouble viewing this map, look at the interactive map: http://www.travelguide.tv/washington/map .html

Defining Parallel and Skew

Parallel: Two or more lines that lie in the same plane and never intersect.



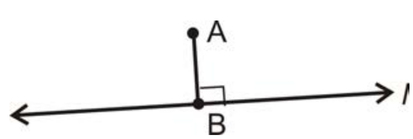
To show that lines are parallel, arrows are used.

Table 3.1:

Label It	***Say It***
$\overleftrightarrow{AB} \| \overleftrightarrow{MN}$	Line *AB* is parallel to line *MN*
$l \| m$	Line *l* is parallel to line *m*.

Lines must be marked parallel with the arrows in order to say they are parallel. Just because two lines LOOK parallel, does not mean that they are.

Recall the definition of perpendicular from Chapter 1. Two lines are perpendicular when they intersect to form a 90° angle. Below $l \perp \overline{AB}$.



In the definitions of parallel and perpendicular, the word "line," is used. Line segments, rays and planes can also be parallel or perpendicular.

The image to the below shows two parallel planes, with a third blue plane that is perpendicular to both of them.

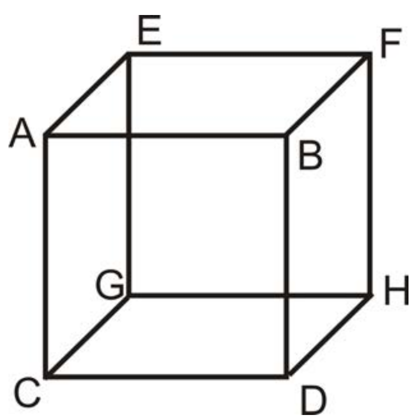
An example of parallel planes could be the top of a table and the floor. The legs would be in perpendicular planes to the table top and the floor.

Skew lines: Lines that are in different planes and never intersect.

In the cube:

$\overline{AB}$ and $\overline{FH}$ are skew

$\overline{AC}$ and $\overline{EF}$ are skew



Example 1: Using the cube above to find:

(a) A pair of parallel planes

(b) A pair of perpendicular planes

(c) A pair of skew lines.

Solution: Remember, you only need to use three points to label a plane. Below are answers, but there are other possibilities too.

(a) Planes ABC and EFG

(b) Planes ABC and CDH

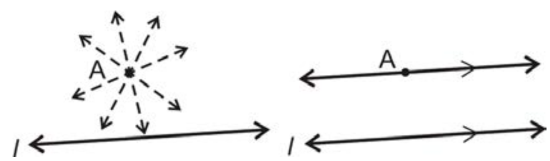
(c) $\overline{BD}$ and $\overline{CG}$

Parallel Line Postulate

Parallel Postulate

For any line and a point ***not*** on the line, there is one line parallel to this line through the point

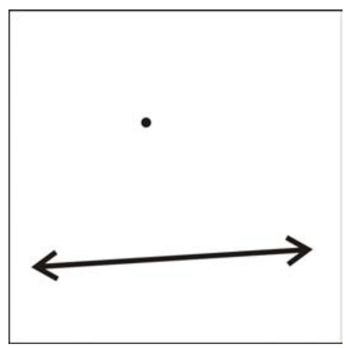
There are infinitely many lines that go through A, but only ***one*** that is parallel to l.



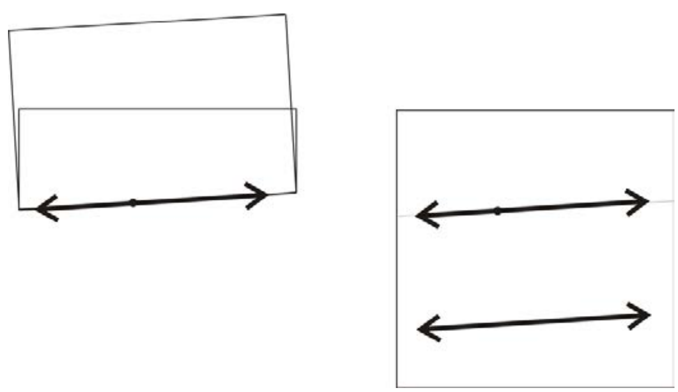
Investigation 3-1: Patty Paper and Parallel Lines

Tools Needed: Patty paper, pencil, ruler

1. Get a piece of patty paper (a translucent square piece of paper). Draw a line and a point above the line.



2. Fold up the paper so that the line is over the point. Crease the paper and unfold.



3. Are the lines parallel?

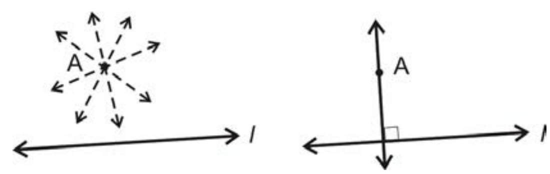
Yes. This investigation duplicates the line we drew in #1 over the point. This means that there is only one parallel line through this point.

Perpendicular Line Postulate

Perpendicular Line Postulate

For any line and a point ***not*** on the line, there is one line perpendicular to this line passing through the point

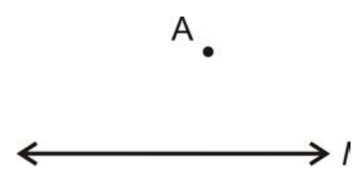
There are infinitely many lines that pass through A, but only ***one*** that is perpendicular to l.



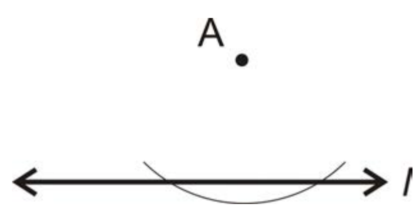
Investigation 3-2: Perpendicular Line Construction; through a Point NOT on the Line

Tools Needed: Pencil, paper, ruler, compass

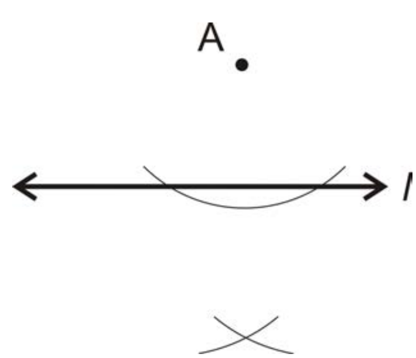
1. Draw a horizontal line and a point above that line. Label the line l and the point A.



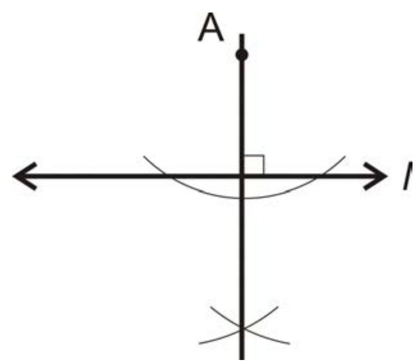
2. Take the compass and put the pointer on A. Open the compass so that it reaches past line l. Draw an arc that intersects the line twice.



3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc below the line. Repeat this on the other side so that the two arc marks intersect.



4. Take your straightedge and draw a line from point A to the arc intersections below the line. This line is perpendicular to l and passes through A.



To see a demonstration of this construction, go to:

http://www.mathsisfun.com/geometry/construct-perpnotline.html

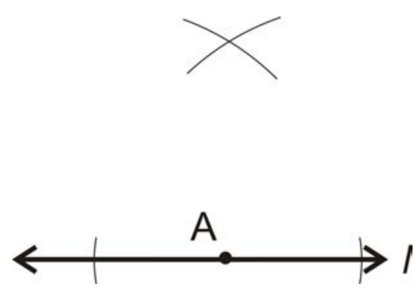
Investigation 3-3: Perpendicular Line Construction; through a Point on the Line

Tools Needed: Pencil, paper, ruler, compass

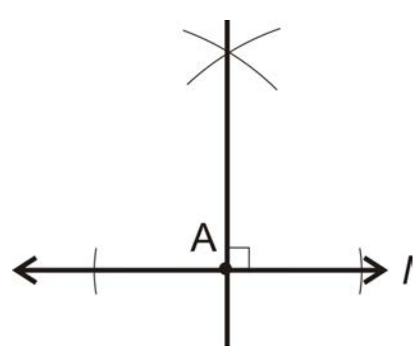
1. Draw a horizontal line and a point on that line. Label the line l and the point A.

2. Take the compass and put the pointer on A. Open the compass so that it reaches out horizontally along the line. Draw two arcs that intersect the line on either side of the point.

3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc above or below the line. Repeat this on the other side so that the two arc marks intersect.



4. Take your straightedge and draw a line from point A to the arc intersections above the line. This line is perpendicular to l and passes through A.

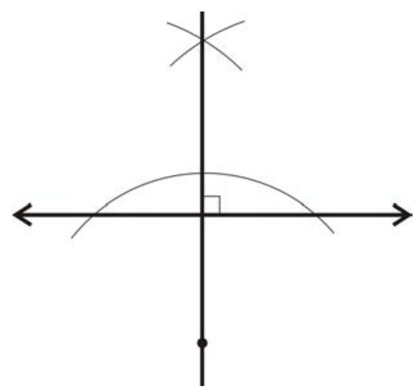


To see a demonstration of this construction, go to:

http://www.mathsisfun.com/geometry/construct-perponline.html

Example 2: Construct a perpendicular line through the point below.

Solution: Even though the point is below the line, the construction is the same as Investigation 3-2. However, draw the arc marks in step 3 *above* the line.

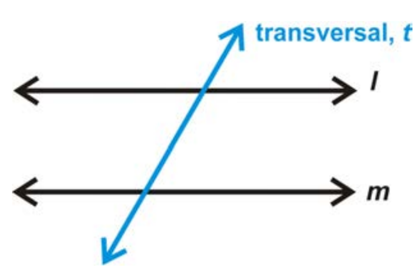


Angles and Transversals

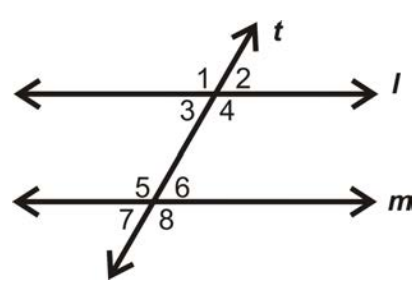
Transversal: A line that intersects two other lines. To learn more about transversals, go to https://www.mathsisfun.com/geometry/transversal.html

The area *between* l and m is said to be in the *interior*.

The area *outside* l and m is said to be in the *exterior*.



Looking at t, l, and m, there are 8 angles formed. They are labeled below.



In this figure there are 8 linear pairs and 4 pairs of vertical angles.

An example of a linear pair would be $\angle 1$ and $\angle 2$.

An example of vertical angles would be $\angle 5$ and $\angle 8$.

Example 3: List all the other linear pairs and vertical angle pairs in the picture above.

Solution:

Linear Pairs: $\angle 2$ and $\angle 4$, $\angle 4$ and $\angle 3$, $\angle 3$ and $\angle 1$, $\angle 5$ and $\angle 6$, $\angle 6$ and $\angle 8$, $\angle 8$ and$\angle 7$, $\angle 7$ and $\angle 5$.$\angle$

Vertical Angles: $\angle 1$ and $\angle 4$, $\angle 2$ and $\angle 3$, $\angle 6$ and $\angle 7$

When two or more lines are cut by a transversal such as the picture below, there are also 4 new angle relationships.

Corresponding Angles: Two angles that are on the same side of the transversal and in similar positions. Imagine sliding the four angles formed with line *l* down to line *m*. The angles which match up are corresponding.

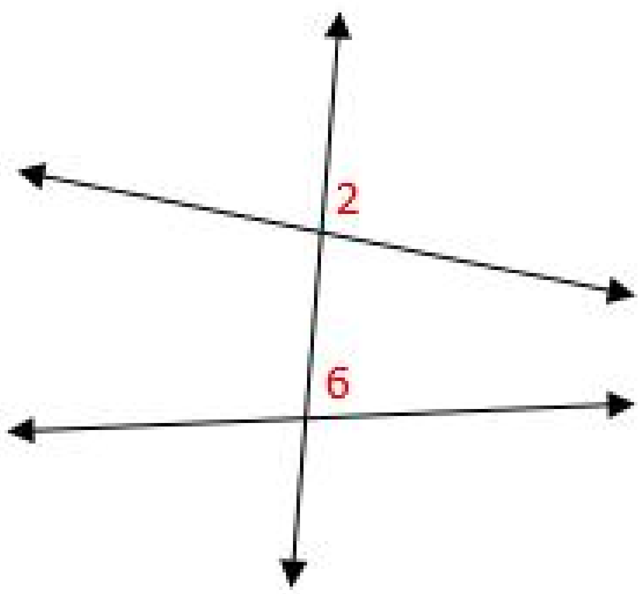


FIGURE 3.1

The figure above displays one pair of corresponding angles.

Alternate Interior Angles: Two angles that are on the interior, but on opposite sides of the transversal.

The figure above displays one pair of alternate interior angles.

Alternate Exterior Angles: Two angles that are on the exterior, but also on opposite sides of the transversal.

The figure above displays one pair of alternate exterior angles.

Consecutive Interior Angles (Same Side Interior Angles): Two angles that are on the same side of the transversal and on the interior. Same side interior angles may also be referred to as consecutive angles.

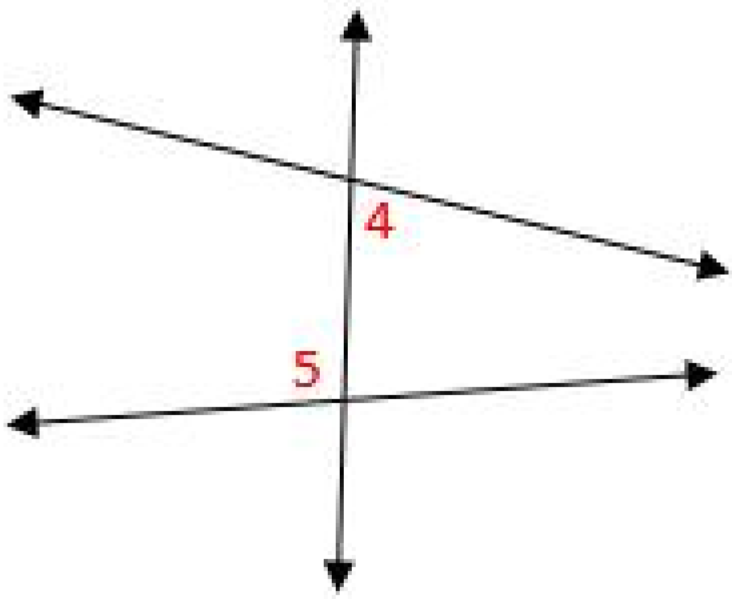


FIGURE 3.2

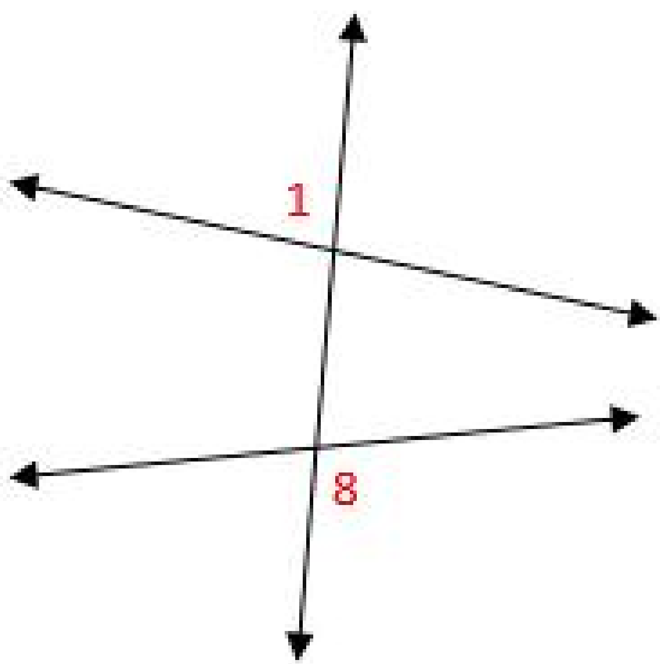


FIGURE 3.3

The figure above displays one pair of consecutive interior angles.

Example 4: Using the picture below, list all the other pairs of each of the newly defined angle relationships.

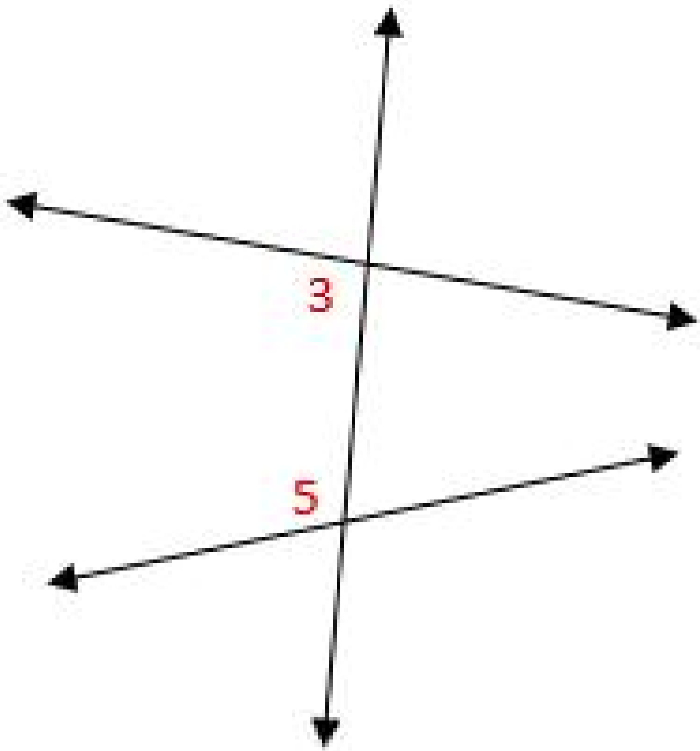


FIGURE 3.4

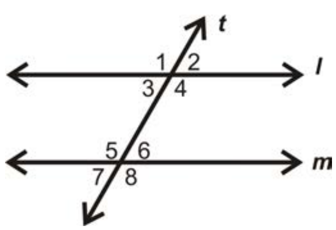


FIGURE 3.5

Solution:

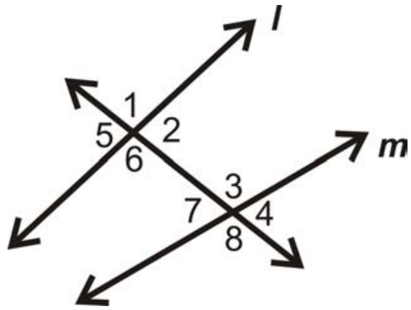
Corresponding Angles: $\angle 1$ and $\angle 5$, $\angle 7$ and $\angle 3$, $\angle 2$ and $\angle 6$, $\angle 4$ and $\angle 8$. $\angle$

Alternate Interior Angles: $\angle 4$ and $\angle 5$

Alternate Exterior Angles: $\angle 2$ and $\angle 7$

Same Side Interior Angles: $\angle 4$ and $\angle 6$

Example 5: For the picture below, determine:



(a) A corresponding angle to $\angle 3$?

(b) An alternate interior angle to $\angle 7$?

(c) An alternate exterior angle to $\angle 4$?

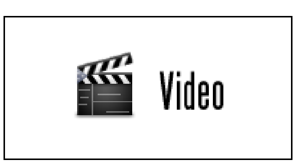
Solution:

(a) $\angle 1$

(b) $\angle 2$

(c) $\angle 5$

There are many studies of Geometry, in high school we study Euclidean Geometry. Euclidean Geometry comes from the geometry book written by Euclid of Alexandria titled Elements. Non Euclidean Geometry consists of geometry studied on spherical and hyperbolic surfaces. The following videos shows the relationship between parallel lines in Spherical Geometry.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/161395

The following link can be used to study more Spherical Geometry. http://en.wikipedia.org/wiki/Spherical_geometry

Know What? Revisited For Washington DC, all of the lettered and numbered streets are parallel. The lettered streets are perpendicular to the numbered streets. We do not have enough information about the state-named streets to say if they are parallel or perpendicular.

Practice Problems

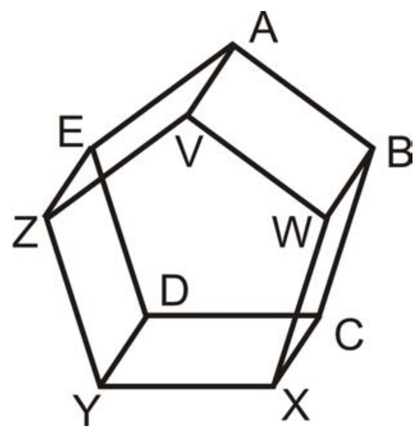
- Questions 1-3 use the definitions of parallel, perpendicular, and skew lines.
- Question 4 asks about the Parallel Line Postulate and the Perpendicular Line Postulate.
- Questions 5-9 use the definitions learned in this section and are similar to Example 1.
- Questions 10-20 are similar to Examples 4 and 5.
- Question 21 is similar to Example 2 and Investigation 3-2.
- Questions 22-30 are Algebra I review.

1. Which of the following is the best example of parallel lines?
   a. Railroad Tracks
   b. Lamp Post and a Sidewalk
   c. Longitude on a Globe
   d. Stonehenge (the stone structure in Scotland)
2. Which of the following is the best example of perpendicular lines?
   a. Latitude on a Globe
   b. Opposite Sides of a Picture Frame
   c. Fence Posts
   d. Adjacent Sides of a Picture Frame
3. Which of the following is the best example of skew lines?
   a. Roof of a Home

b. Northbound Freeway and an Eastbound Overpass
c. Longitude on a Globe
d. The Golden Gate Bridge

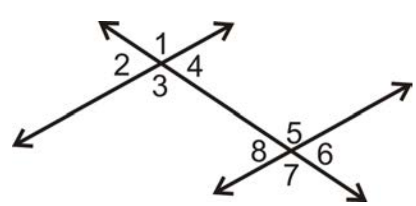
4. ***Writing*** What is the difference between the Parallel Line Postulate and the Perpendicular Line Postulate? How are they similar?

Use the figure below to answer questions 5-9. The two pentagons are parallel and all of the rectangular sides are perpendicular to both of them.



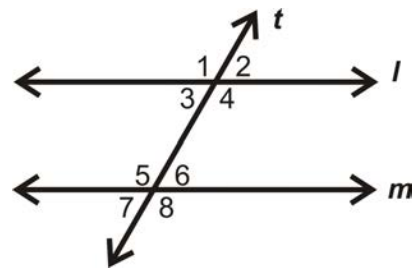
5. Find two pairs of skew lines.
6. List a pair of parallel lines.
7. List a pair of perpendicular lines.
8. For $\overline{AB}$, how many perpendicular lines would pass through point V? Name this line.
9. For $\overline{XY}$, how many parallel lines would pass through point D? Name this line.

For questions 10-16, use the picture below.



10. What is the corresponding angle to $\angle 4$?
11. What is the alternate interior angle with $\angle 5$?
12. What is the corresponding angle to $\angle 8$?
13. What is the alternate exterior angle with $\angle 7$?
14. What is the alternate interior angle with $\angle 4$?
15. What is the consecutive interior angle with $\angle 3$?
16. What is the corresponding angle to $\angle 1$?

Use the picture below for questions 17-20.



17. If $m\angle 2 = 55^\circ$, what other angles do you know and explain how you determined their measures?
18. If $m\angle 5 = 123^\circ$, what other angles do you know and explain how you determined their measures?
19. If $t \perp l$, is $t \perp m$? Why or why not?
20. Is $l||m$? Why or why not?

21. ***Construction*** Draw a line and a point not on the line. Construct a perpendicular line to the one your drew.

ANGLE RELATIONSHIPS Copy and complete the statement. List all possible correct answers.

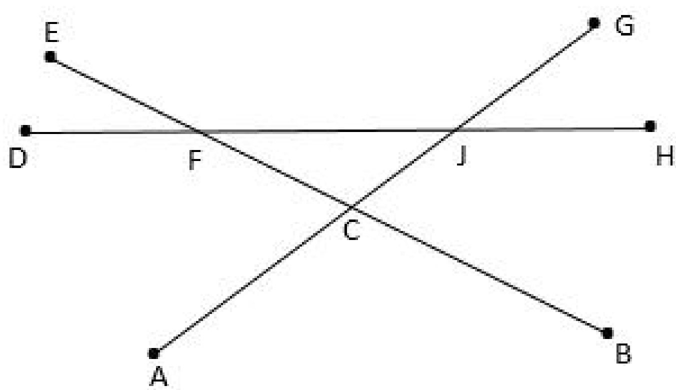


FIGURE 3.6

22. $\angle CFJ$and _____ are corresponding angles.

23. $\angle CJH$and _____ consecutive interior angles.

24. $\angle DFC$and _____ are alternate interior angles.

25. $\angle GJH$and _____are alternate exterior angles.

Algebra Review Find the slope of the line between the two points, $\frac{y_2-y_1}{x_2-x_1}$.

26. (-3, 2) and (-2, 1)
27. (5, -9) and (0, 1)
28. (2, -7) and (5, 2)
29. (8, 2) and (-1, 5)
30. Find the equation of the line in slope intercept form from #26.
31. Find the equation of the line in slope intercept form from #27.
32. Find the equation of the line in slope intercept form from #28.
33. Is the line $y = -x+3$ parallel to the line in #30? How do you know?
34. Is the line $y = -x+3$ perpendicular to the line in #30? How do you know?

Review and Reflect

35. Give your own definition of a transversal.

36. Choose an example of architecture and explain how parallel, perpendicular, or skew lines were used.
37. How can you use the construction of perpendicular lines to construct parallel lines?

Warm-Up Answers

1. $y = -2x + 3$
2. $m = \frac{-6-2}{5-3} = \frac{-8}{2} = -4$
3.

$$\begin{aligned} 2 &= -4(3) + b \\ 2 &= -12 + b \\ -14 &= b \\ y &= -4x + 14 \end{aligned}$$

4. Something like: Two lines that never touch or intersect and in the same plane. If we do not say "in the same plane," this definition could include skew lines.

3.2 Properties of Parallel Lines

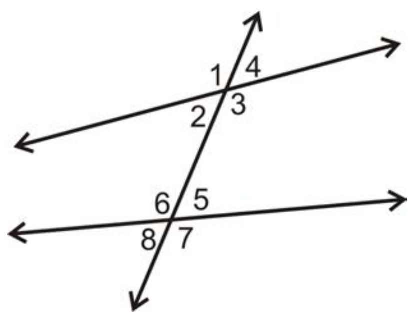
TEKS G(1)B, G(1)F, G(5)A, G(6)A

Learning Objectives

- Determine what happens to corresponding angles, alternate interior angles, alternate exterior angles, and consecutive interior angles when the two lines cut by a transversal are parallel

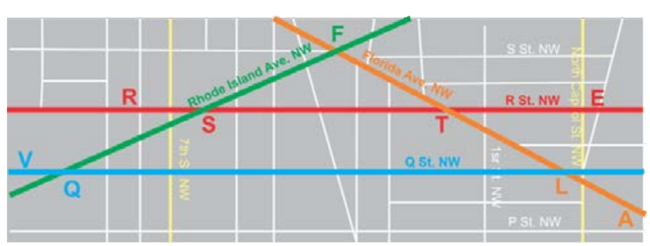
Warm-Up

Use the picture below to determine:



1. A pair of corresponding angles.
2. A pair of alternate interior angles.
3. A pair of alternate exterior angles.
4. A pair of consecutive interior angles.

Know What? The streets below are in Washington DC. The red street and the blue street are parallel. The transversals are the green and orange streets.

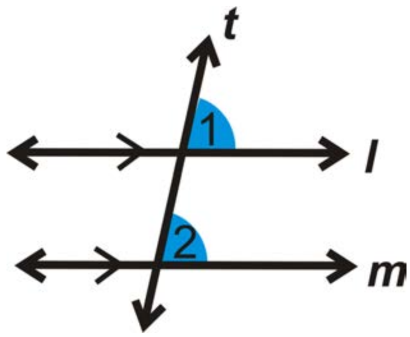


1. If $m\angle FTS = 35^\circ$, determine the other angles that are 35°.
2. If $m\angle SQV = 160^\circ$, determine the other angles that are 160°.

Corresponding Angles Postulate

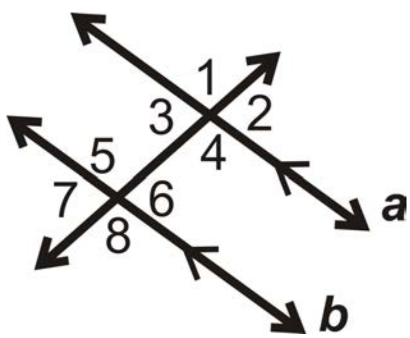
Corresponding Angles Postulate

If two parallel lines are cut by a transversal, then the corresponding angles are congruent



If $l||m$, then $\angle 1 \cong \angle 2$.

Example 1: If $a||b$, which pairs of angles are congruent by the Corresponding Angles Postulate?



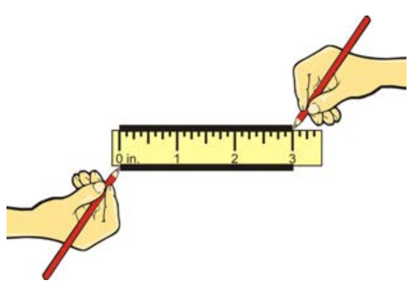
Solution: There are 4 pairs of congruent corresponding angles:

$\angle 1 \cong \angle 5,\ \angle 2 \cong \angle 6,\ \angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$.

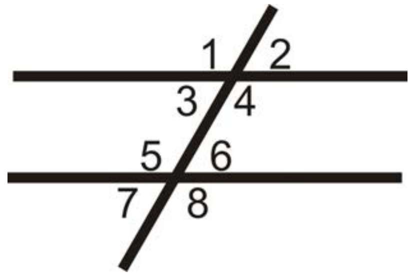
Investigation 3-4: Corresponding Angles Exploration

1. Place your ruler on the paper. On either side of the ruler, draw 2 lines, 3 inches long.

This is the easiest way to ensure that the lines are parallel.



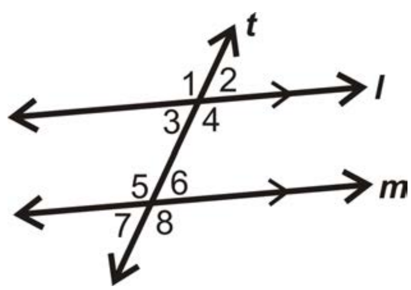
2. Remove the ruler and draw a transversal. Label the eight angles as shown.



3. Using your protractor, measure all of the angles. What do you notice?

You should notice that all the corresponding angles have equal measures.

Example 2: If $m\angle 2 = 76^\circ$, what is $m\angle 6$?



Solution: $\angle 2$ and $\angle 6$ are corresponding angles and $l||m$ from the arrows on them. $\angle 2 \cong \angle 6$ by the Corresponding Angles Postulate, which means that $m\angle 6 = 76^\circ$.

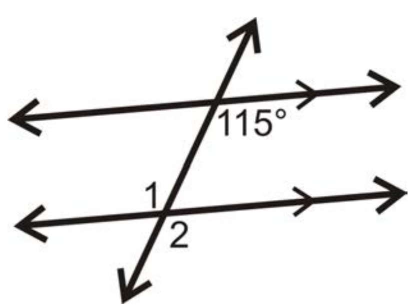
Example 3: Using the measures of $\angle 2$ and $\angle 6$ from Example 2, find all the other angle measures.

Solution: If $m\angle 2 = 76^\circ$, then $m\angle 1 = 180^\circ - 76^\circ = 104^\circ$ (linear pair). $\angle 3 \cong \angle 2$ (vertical angles), so $m\angle 3 = 76^\circ$. $m\angle 4 = 104^\circ$ (vertical angle with $\angle 1$).

By the Corresponding Angles Postulate, we know $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$, so $m\angle 5 = 104^\circ$, $m\angle 6 = 76^\circ$, $m\angle 7 = 76^\circ$, and $m\angle 104^\circ$.

Alternate Interior Angles Theorem

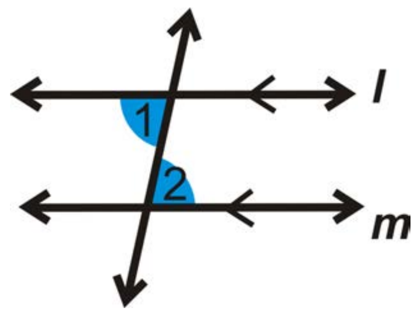
Example 4: Find $m\angle 1$.



Solution: $m\angle 2 = 115^\circ$ because they are corresponding angles and the lines are parallel. $\angle 1$ and $\angle 2$ are vertical angles, so $m\angle 1 = 115^\circ$.

Alternate Interior Angels Theorem

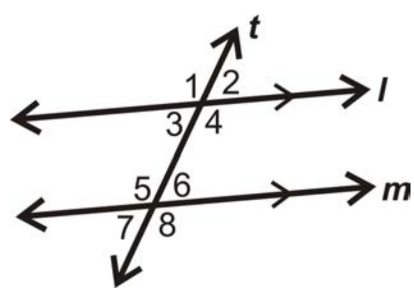
If two parallel lines are cut by a transversal, then the alternate interior angles are congruent



If $l||m$, then $\angle 1 \cong \angle 2$

Proof of Alternate Interior Angles Theorem

Given: $l||m$



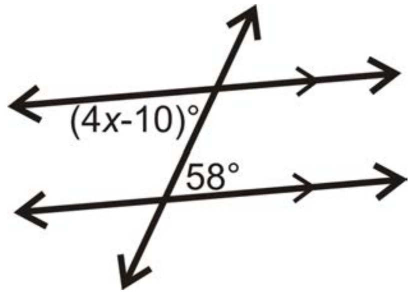
Prove: $\angle 3 \cong \angle 6$

TABLE 3.2:

Statement	***Reason***
1. $l\|\|m$	Given
2. $\angle 3 \cong \angle 7$	Corresponding Angles Postulate
3. $\angle 7 \cong \angle 6$	Vertical Angles Theorem
4. $\angle 3 \cong \angle 6$	Transitive Property of Congruence

We could have also proved that $\angle 4 \cong \angle 5$.

Example 5: ***Algebra Connection*** Find the measure of x.

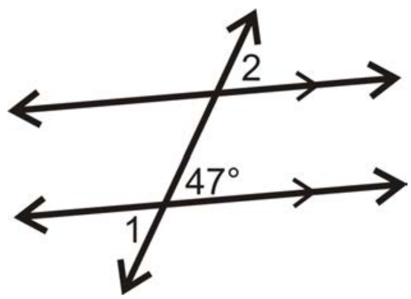


Solution: The two given angles are alternate interior angles and equal.

$$\begin{aligned}(4x-10)^\circ &= 58^\circ \\ 4x &= 68^\circ \\ x &= 17^\circ\end{aligned}$$

Alternate Exterior Angles Theorem

Example 6: Find $m\angle 1$ and $m\angle 2$.

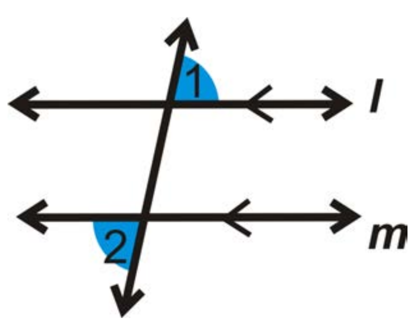


Solution: $m\angle 1 = 47^\circ$ by the vertical angles theorem. The lines are parallel, so $m\angle 2 = 47^\circ$ by the Corresponding Angles Postulate.

Here, $\angle 1$ and $\angle 2$ are alternate exterior angles.

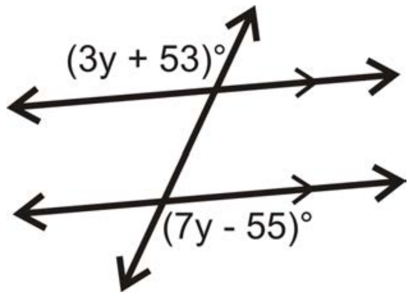
Alternate Exterior Angles Theorem

If two parallel lines are cut by a transversal, then the alternate exterior angles are congruent



If $l||m$, then $\angle 1 \cong \angle 2$.

Example 7: ***Algebra Connection*** Find the measure of each angle and the value of y.



Solution: The angles are alternate exterior angles. Because the lines are parallel, the angles are equal.

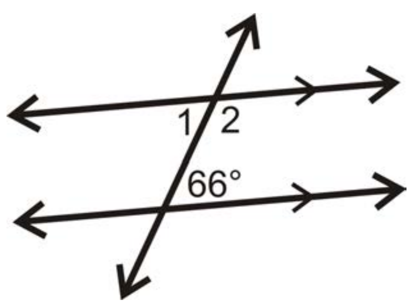
$$\begin{aligned}(3y+53)^\circ &= (7y-55)^\circ \\ 108^\circ &= 4y \\ 27^\circ &= y\end{aligned}$$

If $y = 27^\circ$, then each angle is $3(27^\circ) + 53^\circ = 134^\circ$.

Consecutive Interior Angles Theorem

Consecutive interior angles (or same side angles) are on the interior of the parallel lines and on the same side of the transversal. They have a different relationship that the other angle pairs.

Example 8: Find $m\angle 2$.



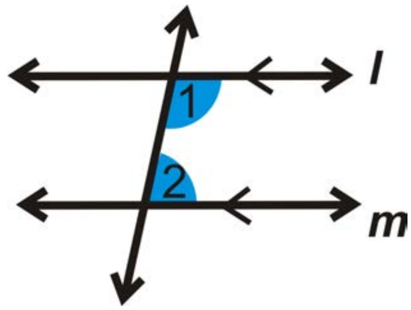
Solution: $\angle 1$ and 66° are alternate interior angles, so $m\angle 1 = 66^\circ$. $\angle 1$ and $\angle 2$ are a linear pair, so they add up to 180°.

$$\begin{aligned}m\angle 1 + m\angle 2 &= 180^\circ \\ 66^\circ + m\angle 2 &= 180^\circ \\ m\angle 2 &= 114^\circ\end{aligned}$$

This example shows that if two parallel lines are cut by a transversal, the consecutive interior angles add up to 180°.

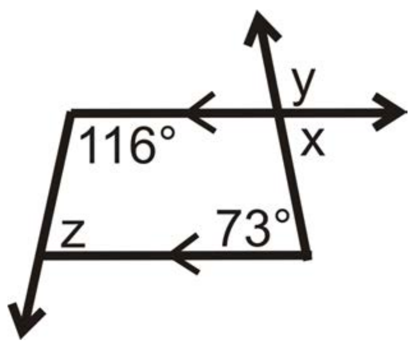
Consecutive Interior Angles Theorem

If two parallel lines are cut by a transversal, then the consecutive interior angles are supplementary



If $l || m$, then $m\angle 1 + m\angle 2 = 180^\circ$.

Example 9: Find x, y, and z.



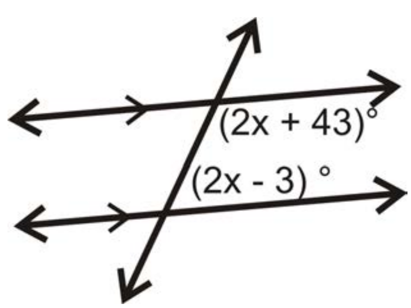
Solution: $x = 73^\circ$ by Alternate Interior Angles

$y = 107^\circ$ because it is a linear pair with x.

$z = 64^\circ$ by Consecutive Interior Angles.

For an interactive look at parallel lines and transversals, go to http://www.mathwarehouse.com/geometry/angle/parallel-lines-cut-transversal.php

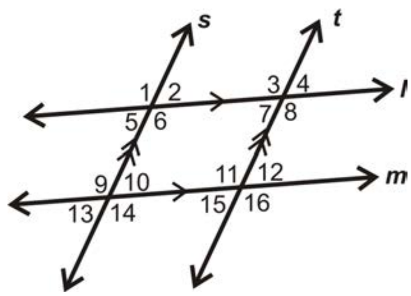
Example 10: ***Algebra Connection*** Find the measure of x.



Solution: The given angles are consecutive interior angles. Because the lines are parallel, the angles add up to 180°.

$$\begin{aligned}(2x+43)^\circ+(2x-3)^\circ &= 180^\circ\\ (4x+40)^\circ &= 180^\circ\\ 4x &= 140^\circ\\ x &= 35^\circ\end{aligned}$$

Example 11: $l||m$ and $s||t$. Explain how $\angle 1 \cong \angle 16$.



Solution: Because $\angle 1$ and $\angle 16$ are not on the same transversal, we cannot assume they are congruent. One path to showing congruence is listed below, there is more than one way to show $\angle 1 \cong \angle 16$.

$\angle 1 \cong \angle 3$ by Corresponding Angles

$\angle 3 \cong \angle 16$ by Alternate Exterior Angles

$\angle 1 \cong \angle 16$ by the Transitive Property

We can summarize this lesson by watching the following video.

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/149922

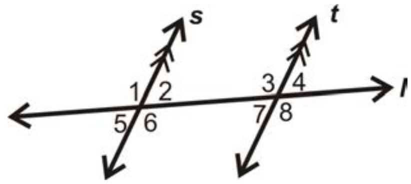
Know What? Revisited Using what we have learned in this lesson, the other angles that are 35° are $\angle TLQ$, $\angle ETL$, and the vertical angle with $\angle TLQ$. The other angles that are 160° are $\angle FSR$, $\angle TSQ$, and the vertical angle with $\angle SQV$.

Practice Problems

- Questions 1-7 use the theorems learned in this section.
- Questions 8-16 are similar to Example 11.
- Questions 17-20 are similar to Example 6, 8 and 9.
- Questions 21-25 are similar to Examples 5, 7, and 10.

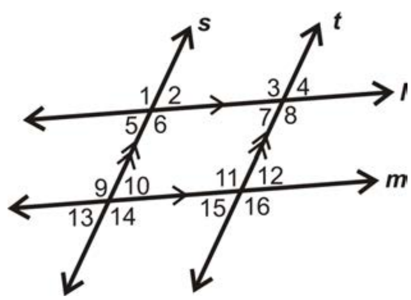
- Questions 26-29 are similar to the proof of the Alternate Interior Angles Theorem.
- Question 30 uses the theorems learned in this section.

For questions 1-7, determine if each angle pair below is congruent, supplementary or neither. If they are congruent of supplementary give the reason why.



1. $\angle 1$ and $\angle 7$
2. $\angle 4$ and $\angle 2$
3. $\angle 6$ and $\angle 3$
4. $\angle 5$ and $\angle 8$
5. $\angle 1$ and $\angle 6$
6. $\angle 4$ and $\angle 6$
7. $\angle 2$ and $\angle 3$

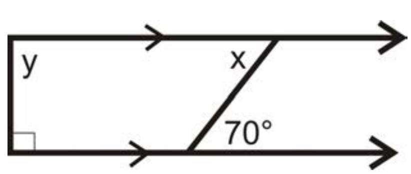
For questions 8-16, determine if the angle pairs below are: Corresponding Angles, Alternate Interior Angles, Alternate Exterior Angles, Consecutive Interior Angles, Vertical Angles, Linear Pair or None. Also state whether each pair is congruent or supplementary.



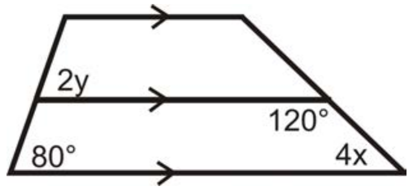
8. $\angle 2$ and $\angle 13$
9. $\angle 7$ and $\angle 12$
10. $\angle 1$ and $\angle 11$
11. $\angle 6$ and $\angle 10$
12. $\angle 14$ and $\angle 9$
13. $\angle 3$ and $\angle 11$
14. $\angle 4$ and $\angle 15$
15. $\angle 5$ and $\angle 16$
16. List all angles congruent to $\angle 8$.

For 17-20, find the values of x and y.

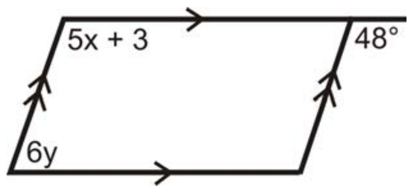
17.



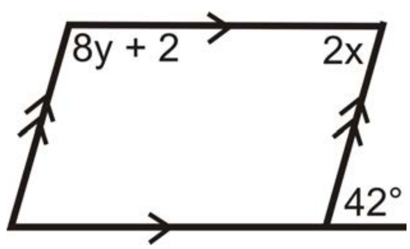
18.



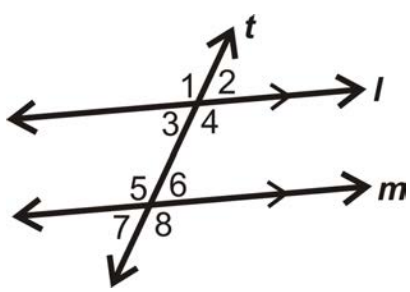
19.



20.



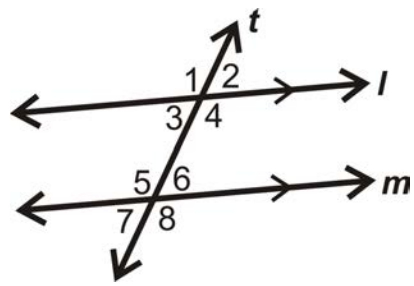
Algebra Connection For questions 21-25, use the picture to the right. Find the value of x and/or y.



21. $m\angle 1 = (4x+35)^\circ$, $m\angle 8 = (7x-40)^\circ$
22. $m\angle 2 = (3x+14)^\circ$, $m\angle 6 = (8x-76)^\circ$
23. $m\angle 3 = (3x+12)^\circ$, $m\angle 5 = (5x+8)^\circ$
24. $m\angle 4 = (5x-33)^\circ$, $m\angle 5 = (2x+60)^\circ$
25. $m\angle 1 = (11y-15)^\circ$, $m\angle 7 = (5y+3)^\circ$

Fill in the blanks in the proofs below.

26. Given: $l || m$

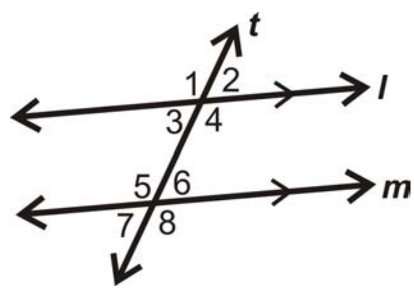


Prove: $\angle 3$ and $\angle 5$ are supplementary (Consecutive Interior Angles Theorem)

TABLE 3.3:

Statement	***Reason***
1.	Given
2. $\angle 1 \cong \angle 5$	
3.	$\cong$ angles have = measures
4.	Linear Pair Postulate
5.	Definition of Supplementary Angles
6. $m\angle 3 + m\angle 5 = 180^\circ$	
7. $\angle 3$ and $\angle 5$ are supplementary	

27. Given: $l || m$

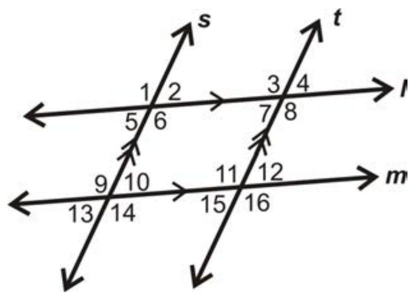


Prove: $\angle 1 \cong \angle 8$ (Alternate Exterior Angles Theorem)

TABLE 3.4:

Statement	***Reason***
1.	
2. $\angle 1 \cong \angle 5$	
3.	Vertical Angles Theorem
4. $\angle 1 \cong \angle 8$	

For 28 and 29, use the picture to the below.



28. Given: $l||m,\ s||t$ Prove: $\angle 2 \cong \angle 15$

TABLE 3.5:

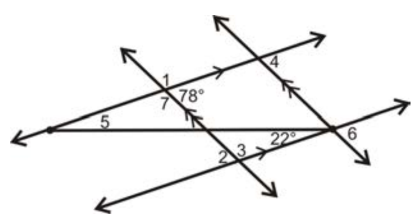
Statement	**Reason**
1. $l||m,\ s||t$	
2. $\angle 2 \cong \angle 13$	
3.	Corresponding Angles Postulate
4. $\angle 2 \cong \angle 15$	

29. Given: $l||m,\ s||t$ Prove: $\angle 4$ and $\angle 9$ are supplementary

TABLE 3.6:

Statement	**Reason**
1.	
2. $\angle 6 \cong \angle 9$	
3. $\angle 4 \cong \angle 7$	
4.	Consecutive Interior Angles
5. $\angle 9$ an $\angle 4$ are supplementary	

30. Find the measures of all the numbered angles in the figure below.



CHALLENGE Find the values of X and Y.

31.

32.

33. Given two parallel lines are cut by a transversal and the measure of one angle is given, describe how to find the remaining seven angles.

34. Describe at least 2 different ways to determine if a pair of lines cut by a transversal are parallel.

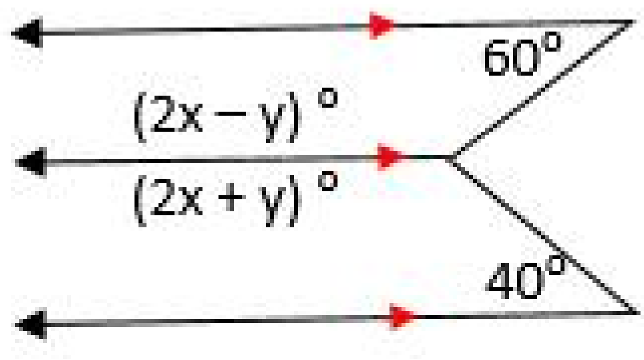


FIGURE 3.7

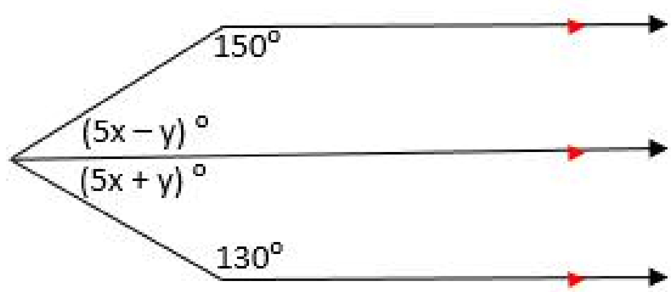


FIGURE 3.8

Warm-Up Answers

1. $\angle 1$ and $\angle 6$, $\angle 2$ and $\angle 8$, $\angle 3$ and $\angle 7$, or $\angle 4$ and $\angle 5$
2. $\angle 2$ and $\angle 5$ or $\angle 3$ and $\angle 6$
3. $\angle 1$ and $\angle 7$ or $\angle 4$ and $\angle 8$
4. $\angle 3$ and $\angle 5$ or $\angle 2$ and $\angle 6$

3.3 Proving Lines are Parallel

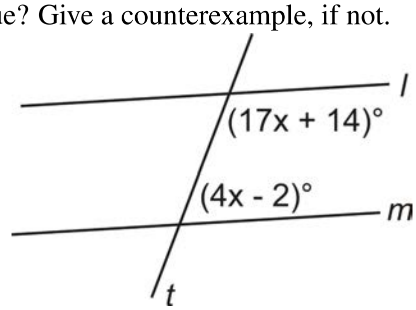
TEKS G(1)D, G(4)B, G(5)A, G(5)B

Learning Objectives

- Use the *converses* of the Corresponding Angles Postulate, Alternate Interior Angles Theorem, Alternate Exterior Angles Theorem, and the consecutive Interior Angles Theorem to show that line are parallel
- Construct parallel lines using the above converses
- Use the Parallel Lines Property

Warm-Up

Answer the following questions.

1. Write the converse of the following statements:
   a. *If it is summer, then I am out of school.*
   b. *I will go to the mall when I am done with my homework.*
2. Are any of the converses from #1 true? Give a counterexample, if not.
3. 

   Determine the value of x if $l||m$.
4. What is the measure of each angle in #3?

Know What? Here is a picture of the support beams for the Coronado Bridge in San Diego. To aid the strength of the curved bridge deck, the support beams should not be parallel.

This bridge was designed so that $\angle 1 = 92^\circ$ and $\angle 2 = 88^\circ$. Are the support beams parallel?

Corresponding Angles Converse

Recall that the converse of

If a, then b is

If b, then a

For the Corresponding Angles Postulate:

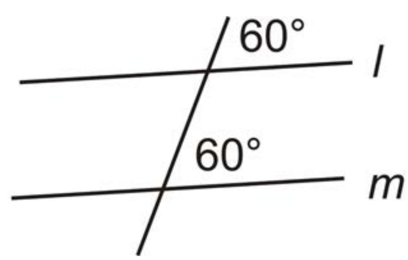
If *two lines are parallel*, then *the corresponding angles are congruent*.

BECOMES

If *corresponding angles are congruent*, then *the two lines are parallel*.

Is this true? If corresponding angles are both 60°, would the lines be parallel?

If

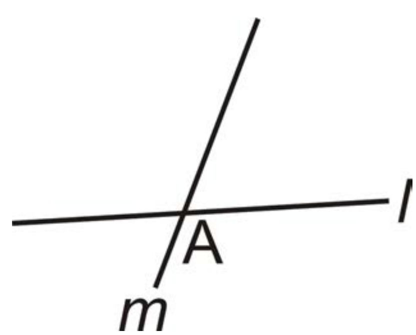


then, is $l||m$?

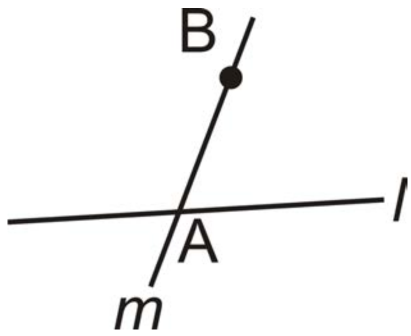
YES. Congruent corresponding angles make the slopes of l and m the same which makes the lines parallel.

Investigation 3-5: Creating Parallel Lines using Corresponding Angles

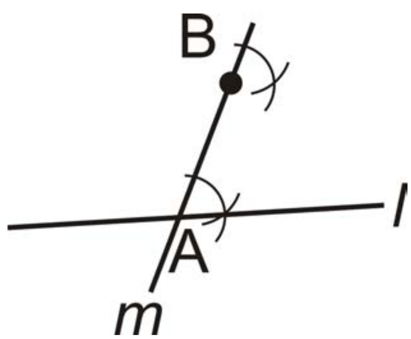
1. Draw two intersecting lines. Make sure they are not perpendicular. Label them l and m, and the point of intersection, A, as shown.



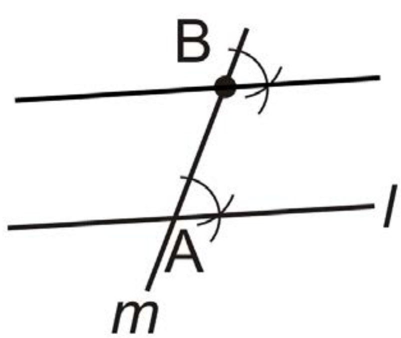
2. Create a point, B, on line m, above A.



3. Copy the acute angle at A (the angle to the right of m) at point B. See Investigation 2-2 in Chapter 2 for the directions on how to copy an angle.



4. Draw the line from the arc intersections to point B.

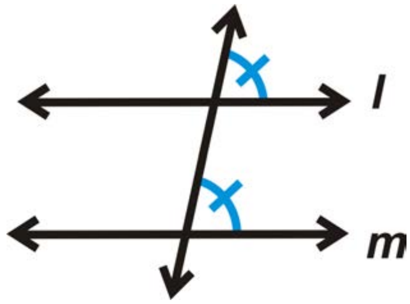


The copied angle allows the line through point B to have the same slope as line l, making the two lines parallel.

Converse of Corresponding Angles Postulate

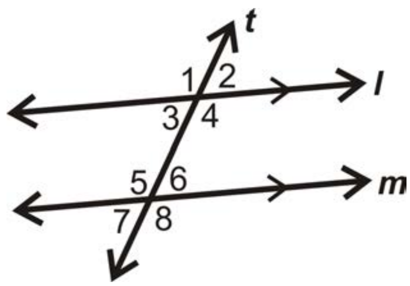
If corresponding angles are congruent when two lines are cut by a transversal, then the lines are parallel

If



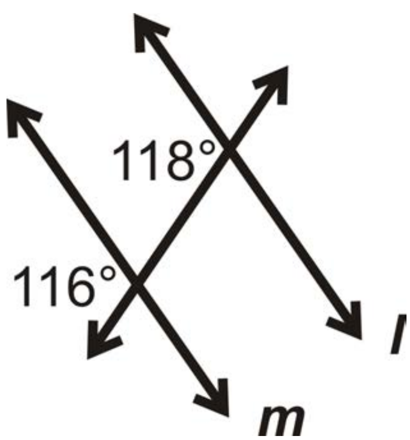
then $l||m$.

Example 1: If $m\angle 8 = 110^\circ$ and $m\angle 4 = 110^\circ$, then what do we know about lines l and m?



Solution: $\angle 8$ and $\angle 4$ are corresponding angles. Since $m\angle 8 = m\angle 4$, we can conclude that $l||m$.

Example 2: Is $l||m$?



Solution: The two angles are corresponding and must be equal to say that $l||m$. $116^\circ \neq 118^\circ$, so l is not parallel to m.

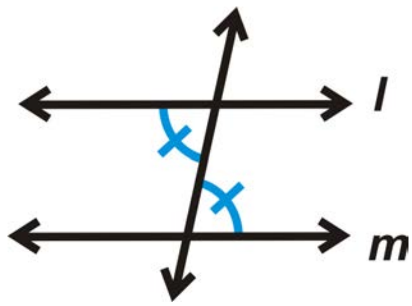
Alternate Interior Angles Converse

The converse of the Alternate Interior Angles Theorem is:

Converse of Alternate Interior Angles Theorem

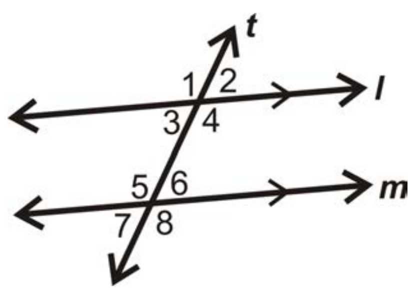
If two lines are cut by a transversal and the alternate interior angles are congruent, then the lines are parallel

If



then $l||m$.

Example 3: Prove the Converse of the Alternate Interior Angles Theorem.



Given: l and m and transversal t

$\angle 3 \cong \angle 6$

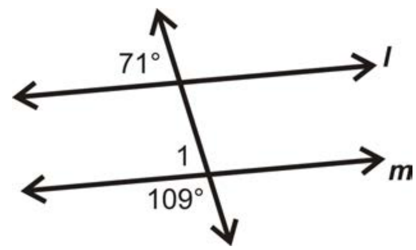
Prove: $l||m$

Solution:

TABLE 3.7:

Statement	***Reason***
1. l and m and transversal t and $\angle 3 \cong \angle 6$	Given
2. $\angle 3 \cong \angle 2$	Vertical Angles Theorem
3. $\angle 2 \cong \angle 6$	Transitive Property of Congruence
4. $l||m$	Converse of the Corresponding Angles Postulate

Example 4: Is $l||m$?

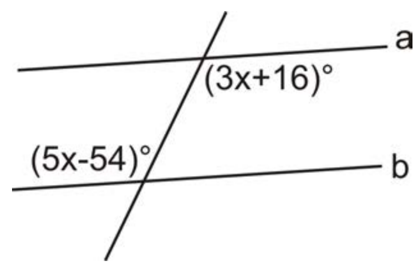


Solution: Find $m\angle 1$. We know its linear pair is 109°, so they add up to 180°.

$$\begin{aligned} m\angle 1 + 109^\circ &= 180^\circ \\ m\angle 1 &= 71^\circ. \end{aligned}$$

This means $l||m$.

Example 5: ***Algebra Connection*** What does x have to be to make $a||b$?



Solution: The angles are alternate interior angles, and must be equal for $a||b$. Set the expressions equal to each other and solve

$$\begin{aligned} 3x + 16^\circ &= 5x - 54^\circ \\ 70^\circ &= 2x \\ 35^\circ &= x \end{aligned}$$

To make $a||b$, $x = 35^\circ$.

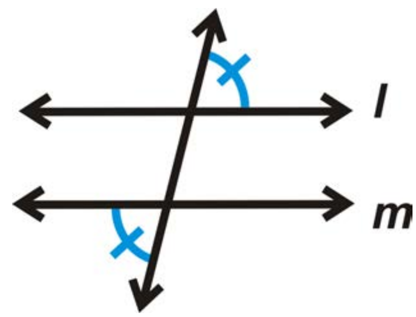
Converse of Alternate Exterior Angles Consecutive Interior Angles

You have probably guessed that the converse of the Alternate Exterior Angles Theorem and the Consecutive Interior Angles Theorem are true.

Converse of the Alternate Exterior Angles Theorem

If two lines are cut by a transversal and the alternate exterior angles are congruent, then the lines are parallel

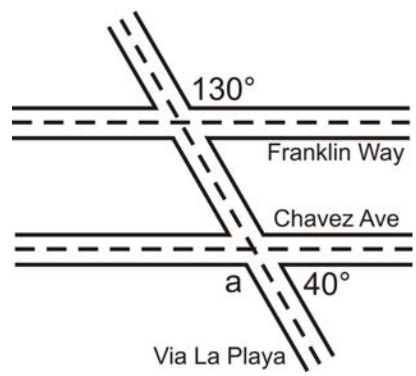
If



then $l||m$.

Click on the following link to explore relationships between angle pairs and slopes of line, https://tube.geogebra.org/student/m125444

Example 6: ***Real-World Situation*** The map below shows three roads in Julio's town.



Julio used a surveying tool to measure two angles at the intersections in this picture he drew (NOT to scale). ***Julio wants to know if Franklin Way is parallel to Chavez Avenue***.

Solution: The 130° angle and $\angle a$ are alternate exterior angles. If $m\angle a = 130^\circ$, then the lines are parallel.

$$\begin{aligned} \angle a + 40^\circ &= 180^\circ && \text{by the Linear Pair Postulate} \\ \angle a &= 140^\circ \end{aligned}$$

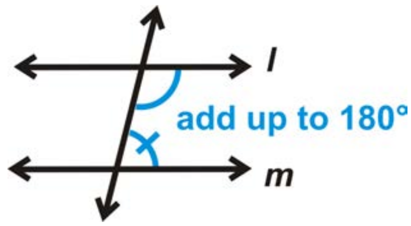
$140^\circ \neq 130^\circ$, so Franklin Way and Chavez Avenue are not parallel streets.

The final converse theorem is the Consecutive Interior Angles Theorem. Remember that these angles aren't congruent when lines are parallel, they are supplementary and **add up to** 180°.

Converse of the Consecutive Interior Angles Theorem (Same Side Interior Angles Theorem)

If two lines are cut by a transversal and the consecutive interior angles are supplementary, then the lines are parallel

If

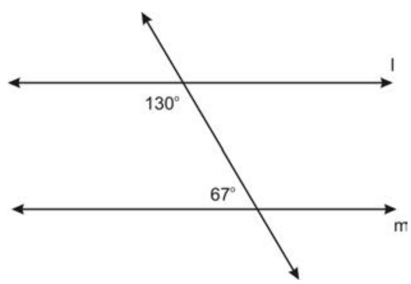


then $l||m$.

Example 7: Is $l||m$? How do you know?

Solution: These angles are Consecutive Interior Angles. So, if they add up to 180°, then $l||m$.

$130° + 67° \neq 180°$, therefore lines l and m are not parallel.



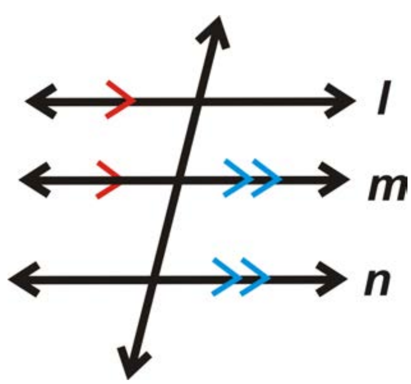
Use the following link and its interactive questions to test your knowledge of proving lines parallel, go to http://www.mathsisfun.com/geometry/parallel-lines.html

Parallel Lines Property

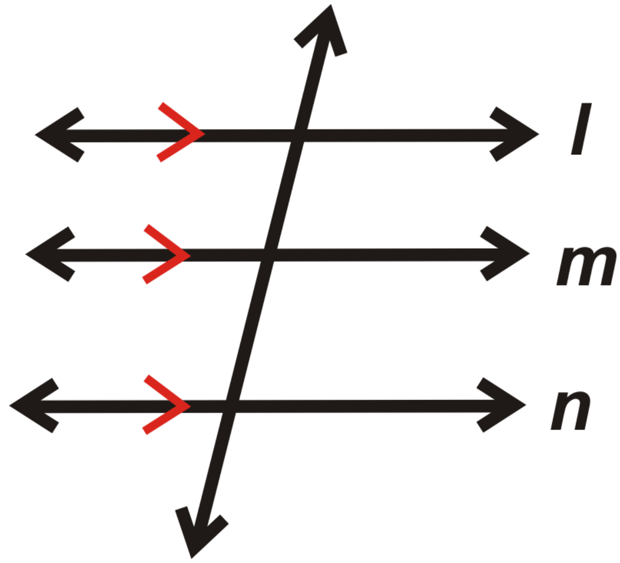
The Parallel Lines Property is a transitive property for parallel lines. The Transitive Property of Equality is: If $a = b$ and $b = c$, then $a = c$. The Parallel Lines Property changes = to $||$.

Parallel Lines Property: If lines $l||m$ and $m||n$, then $l||n$.

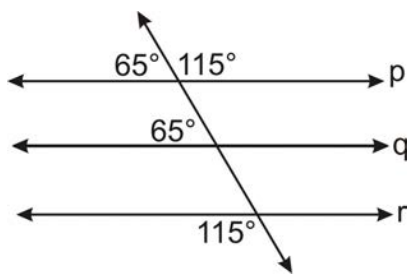
If



, then



Example 8: Are lines q and r parallel?



Solution: First find if $p||q$, then $p||r$. If so, $q||r$.

$p||q$ because the corresponding angles are equal.

$p||r$ because the alternate exterior angles are equal.

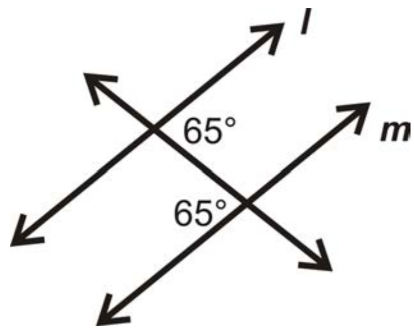
$q||r$ by the Parallel Lines Property.

Know What? Revisited: $\angle 1$ and $\angle 2$ are corresponding angles and must be equal for the beams to be parallel. $\angle 1 = 92^\circ$ and $\angle 2 = 88^\circ$, so they are not equal and the beams are not parallel, therefore the bridge is study and safe.

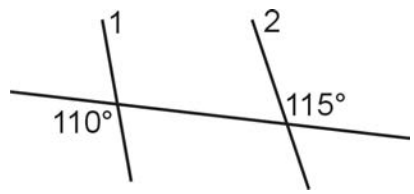
Practice Problems

- Questions 1-13 are similar to Examples 1, 2, 4, and 7.
- Question 14 is similar to Investigation 3-1.
- Question 15 uses the Corresponding Angles Postulate and its converse.
- Questions 16-22 are similar to Example 3.
- Questions 23-36 are similar to Examples 1, 2, 4, and 7.
- Questions 37-40 are similar to Example 5.

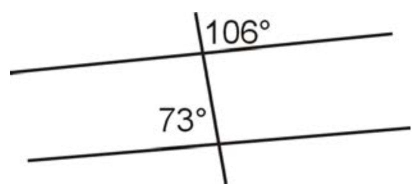
1. Are lines l and m parallel? If yes, how do you know?



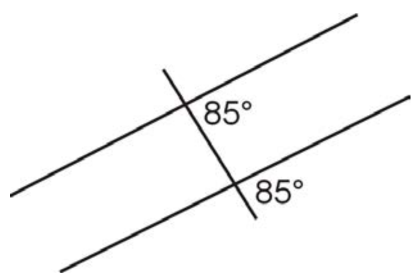
2. Are lines 1 and 2 parallel? Why or why not?



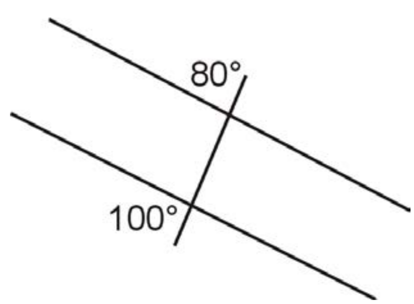
3. Are the lines below parallel? Why or why not?



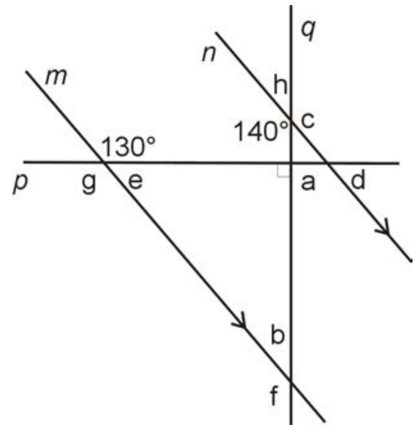
4. Are the lines below parallel? Justify your answer.



5. Are the lines below parallel? Justify your answer.



Use the following diagram. $m||n$ and $p \perp q$. Find each angle and give a reason for each answer.



6. $a =$ _____
7. $b =$ _____
8. $c =$ _____
9. $d =$ _____
10. $e =$ _____
11. $f =$ _____
12. $g =$ _____
13. $h =$ _____
14. ***Construction*** Using Investigation 3-1 to help you, show that two lines are parallel by constructing congruent alternate interior angles. HINT: Steps 1 and 2 will be the same, but at step 3, you will copy the angle in a different spot.
15. ***Writing*** Explain when you would use the Corresponding Angles Postulate and the Converse of the Corresponding Angles Postulate in a proof.

For Questions 16-22, fill in the blanks in the proofs below.

16. Given: $l||m,\ p||q$Prove: $\angle 1 \cong \angle 2$

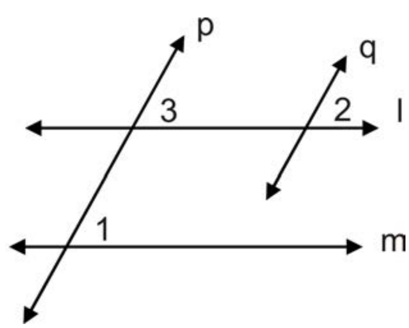


TABLE 3.8:

Statement	***Reason***
1. $l\|\|m$	1.
2.	2. Corresponding Angles Postulate
3. $p\|\|q$	3.
4.	4.
5. $\angle 1 \cong \angle 2$	5.

17. Given: $p||q,\ \angle 1 \cong \angle 2$Prove: $l||m$

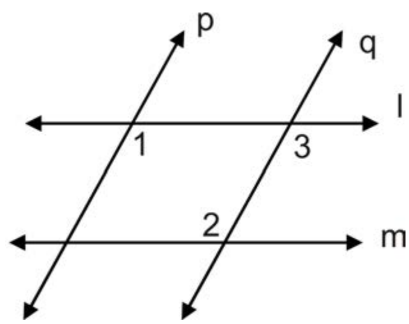


TABLE 3.9:

Statement	*Reason*
1. $p \| q$	1.
2.	2. Corresponding Angles Postulate
3. $\angle 1 \cong \angle 2$	3.
4.	4. Transitive Property of Congruence
5.	5. Converse of Alternate Interior Angles Theorem

18. Given: $\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4$Prove: $l || m$

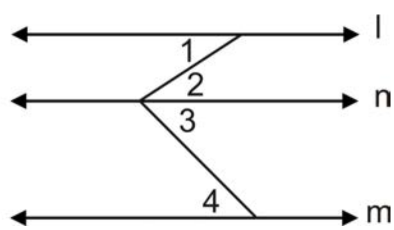


TABLE 3.10:

Statement	*Reason*
1. $\angle 1 \cong \angle 2$	1.
2. $l \| n$	2.
3. $\angle 3 \cong \angle 4$	3.
4.	4. Converse of Alternate Interior Angles Theorem
5. $l \| m$	5.

19. Given: $m \perp l,\ n \perp l$Prove: $m || n$

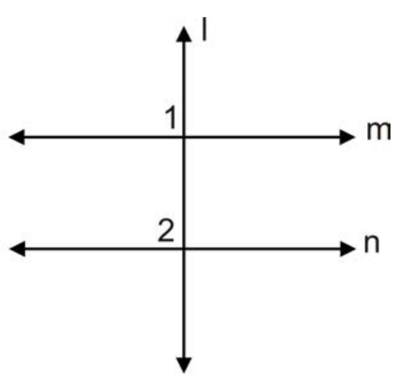


TABLE 3.11:

Statement	Reason
1. $m \perp l,\ n \perp l$	1.
2. $m\angle 1 = 90^\circ,\ m\angle 2 = 90^\circ$	2.
3.	3.Transitive Property
4. $m \vert\vert n$	4.

20. Given: $\angle 1 \cong \angle 3$Prove: $\angle 1$ and $\angle 4$ are supplementary

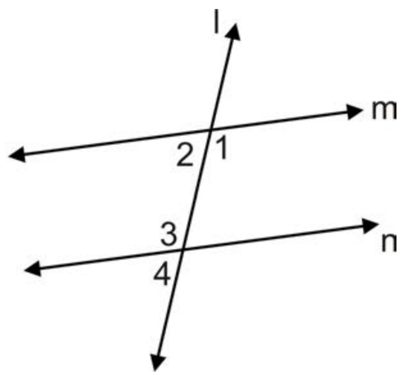


TABLE 3.12:

Statement	Reason
1.	1.
2. $m \vert\vert n$	2.
3.	3.Linear Pair Postulate
4.	4.Substitution
5. $\angle 1$ and $\angle 4$ are supplementary	5.

21. Given: $\angle 2 \cong \angle 4$Prove: $\angle 1 \cong \angle 3$

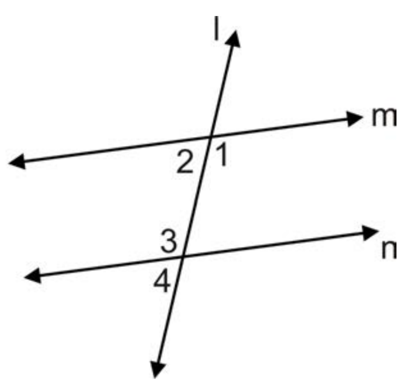


TABLE 3.13:

Statement	Reason
1.	1.
2. $m \vert\vert n$	2.
3. $\angle 1 \cong \angle 3$	3.

22. Given: $\angle 2 \cong \angle 3$Prove: $\angle 1 \cong \angle 4$

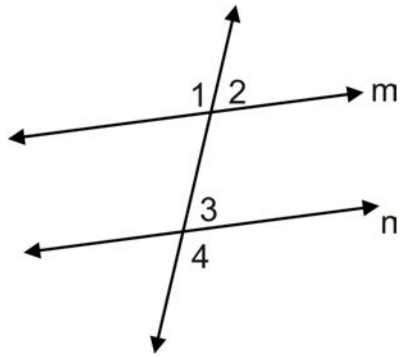
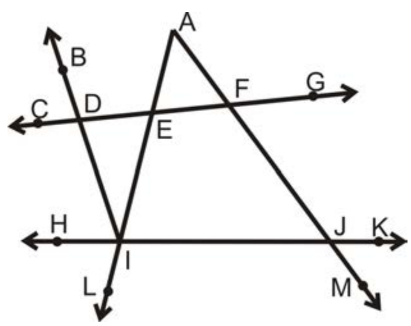


TABLE 3.14:

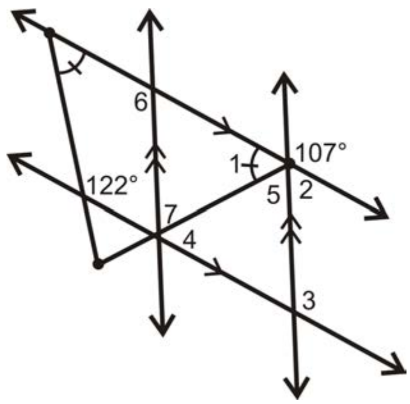
Statement	*Reason*
1.	1.
2.	2.Converse of Corresponding Angles Theorem
3. $\angle 1 \cong \angle 4$	3.

In 23-29, use the given information to determine which lines are parallel. If there are none, write none. Consider each question individually.



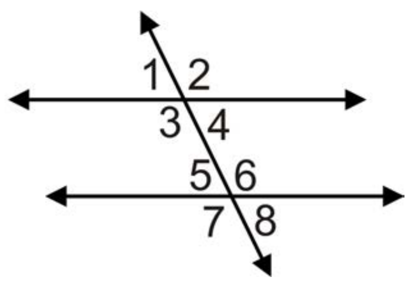
23. $\angle BDC \cong \angle JIL$
24. $\angle AFD$ *and* $\angle BDF$ are supplementary
25. $\angle EAF \cong \angle FJI$
26. $\angle EFJ \cong \angle FJK$
27. $\angle DIE \cong \angle EAF$
28. $\angle EDB \cong \angle KJM$
29. $\angle DIJ$ *and* $\angle FJI$ are supplementary

In 30-36, find the measure of the lettered angles below.



30. $m\angle 1$
31. $m\angle 2$
32. $m\angle 3$
33. $m\angle 4$
34. $m\angle 5$
35. $m\angle 6$
36. $m\angle 7$

Algebra Connection For 37-40, what does x have to measure to make the lines parallel?



37. $m\angle 3 = (3x + 25)^\circ$ and $m\angle 5 = (4x - 55)^\circ$
38. $m\angle 2 = (8x)^\circ$ and $m\angle 7 = (11x - 36)^\circ$
39. $m\angle 1 = (6x - 5)^\circ$ and $m\angle 5 = (5x + 7)^\circ$
40. $m\angle 4 = (3x - 7)^\circ$ and $m\angle 7 = (5x - 21)^\circ$

REASONING Can you prove that lines a and b are parallel? If so, *explain* how.

41.

42.

43.

Review and Reflect

44. When trying to prove lines are parallel, what are some of the angle relationships that must be true?

45. Why do we use the converse of each special angle pair relationship to prove the lines are parallel?

Warm-Up Answers

1.

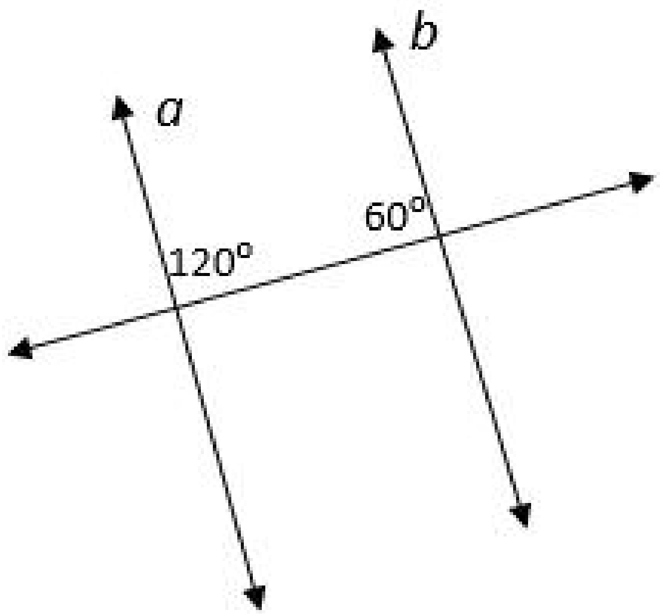


FIGURE 3.9

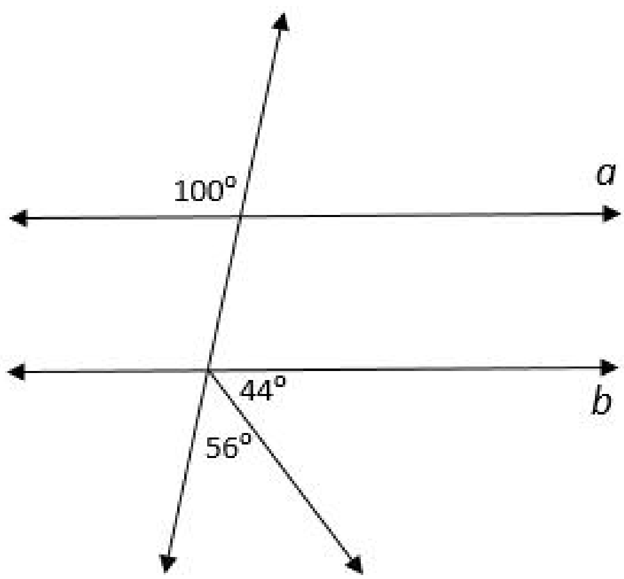


FIGURE 3.10

a. *If I am out of school, then it is summer.*
b. *If I go to the mall, then I am done with my homework.*

2.

a. Not true, I could be out of school on any school holiday or weekend during the school year.
b. Not true, I don't have to be done with my homework to go to the mall.

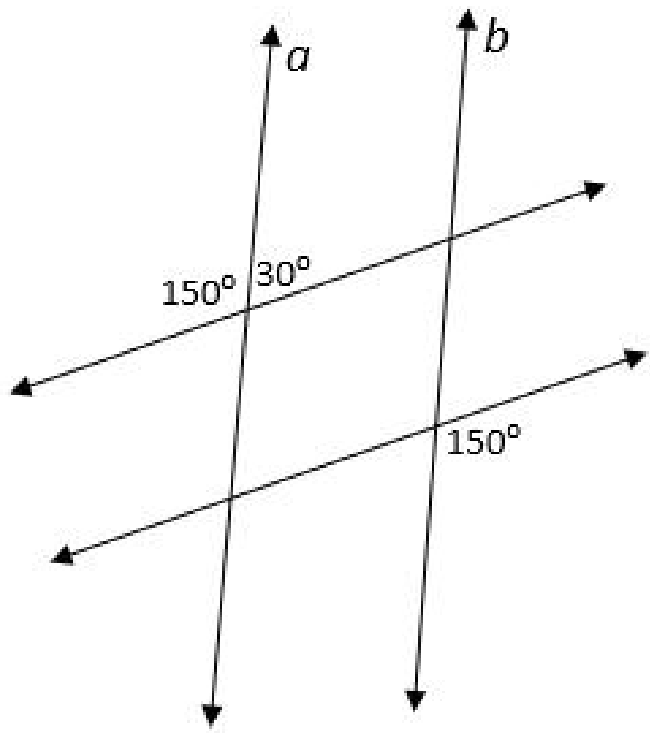


FIGURE 3.11

3. The two angles are supplementary.

$$\begin{aligned}(17x+14)^\circ+(4x-2)^\circ &= 180^\circ\\ 21x+12^\circ &= 180^\circ\\ 21x &= 168^\circ\\ x &= 8^\circ\end{aligned}$$

4. The angles are $17(8^\circ)+14^\circ=150^\circ$ and $180^\circ-150^\circ=30^\circ$

3.4 Properties of Perpendicular Lines

TEKS G(1)D, G(5)A, G(5)B

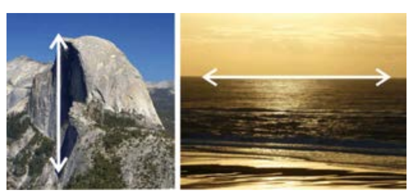
Learning Objectives

- Understand the properties of perpendicular lines

Warm-Up

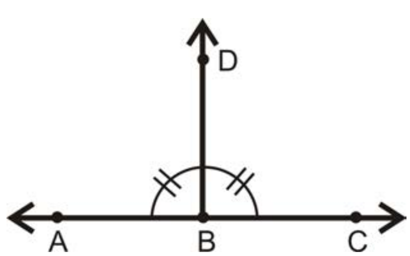
1. Draw a picture of two parallel lines, l and m, and a transversal that is perpendicular to l. Is the transversal perpendicular to m too?
2. $m\angle A = 35^\circ$ and is complementary to $\angle B$. What is $m\angle B$?
3. $m\angle C = 63^\circ$ and is complementary to $\angle D$. What is $m\angle D$?
4. Draw a picture of a linear pair where the angles are congruent. What is the measure of each angle?

Know What? There are several examples of slope in nature. To the right are pictures of Half Dome, in Yosemite National Park and the horizon over the Pacific Ocean. These are examples of horizontal and vertical lines in real life. What is the slope of these lines?



Congruent Linear Pairs

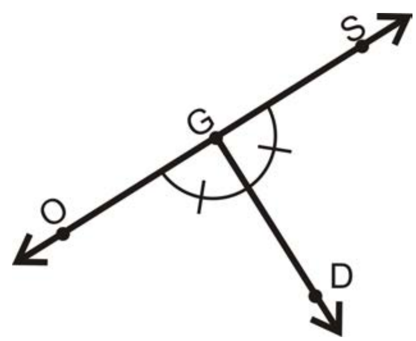
A linear pair is a pair of adjacent angles whose outer sides form a straight line. The Linear Pair Postulate says that the angles in a linear pair add up to 180°. What happens when the angles in a linear pair are congruent?



$$m\angle ABD + m\angle DBC = 180^\circ \text{ Linear Pair Postulate}$$
$$m\angle ABD = m\angle DBC \text{ The angles are congruent}$$
$$m\angle ABD + m\angle ABD = 180^\circ \text{ Substitution Property of Equality}$$
$$2m\angle ABD = 180^\circ \text{ Combine like terms}$$
$$m\angle ABD = 90^\circ \text{ Division Property of Equality}$$

***If a linear pair is congruent, the angles are both* 90°.**

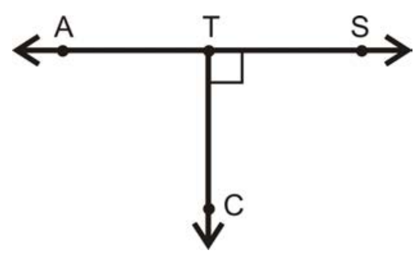
Example 1: Determine the measure of $\angle SGD$ and $\angle OGD$.



Solution: The angles are congruent and form a linear pair.

Both angles are 90°.

Example 2: Find $m\angle CTA$.

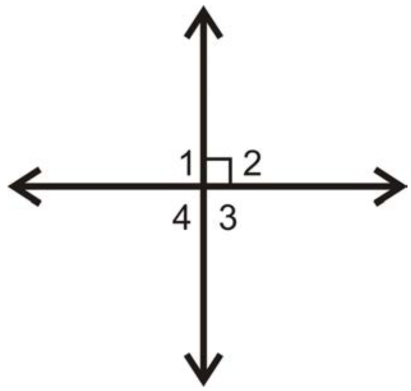


Solution: These two angles form a linear pair and $\angle STC$ is a right angle.

$$m\angle STC = 90^\circ$$
$$m\angle CTA \text{ is } 180^\circ - 90^\circ = 90^\circ$$

Perpendicular Transversals

When two lines intersect, four angles are created. If the two lines are perpendicular, then all four angles are right angles, *even though only one needs to be marked*. All four angles are 90°.

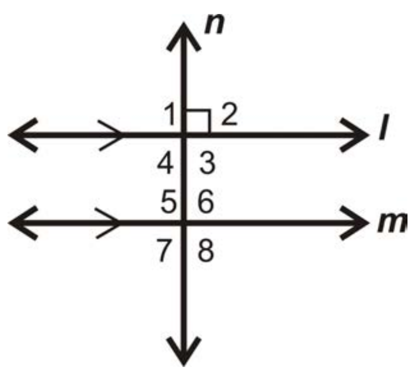


If $l || m$ and $n \perp l$, is $n \perp m$?

Example 3: Write a 2-column proof.

Given: $l || m, \ l \perp n$

Prove: $n \perp m$



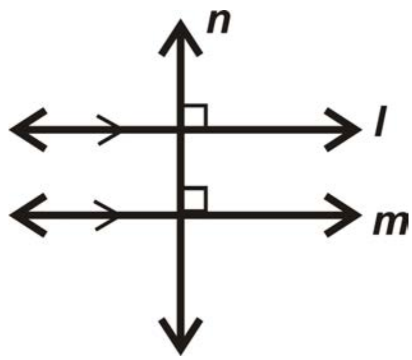
Solution:

TABLE 3.15:

Statement	***Reason***
1. $l \| m, \ l \perp n$	Given
2. $\angle 1, \ \angle 2, \ \angle 3$, and $\angle 4$ are right angles	Definition of perpendicular lines
3. $m\angle 1 = 90^\circ$	Definition of a right angle
4. $m\angle 1 = m\angle 5$	Corresponding Angles Postulate
5. $m\angle 5 = 90^\circ$	Transitive Property of Equality
6. $m\angle 6 = m\angle 7 = 90^\circ$	Congruent Linear Pairs
7. $m\angle 8 = 90^\circ$	Vertical Angles Theorem
8. $\angle 5, \ \angle 6, \ \angle 7$, and $\angle 8$ are right angles	Definition of right angle
9. $n \perp m$	Definition of perpendicular lines

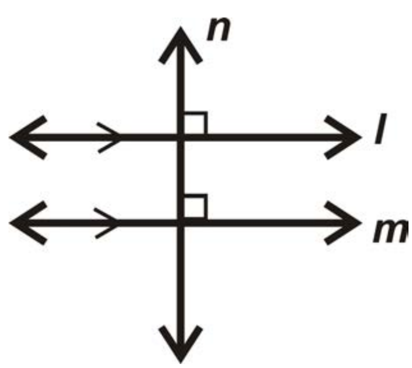
Perpendicular Transversal Theorem

If a transversal is perpendicular to one of two parallel lines, then it is perpendicular to the other
If $l||m$ and $l \perp n$, then $n \perp m$



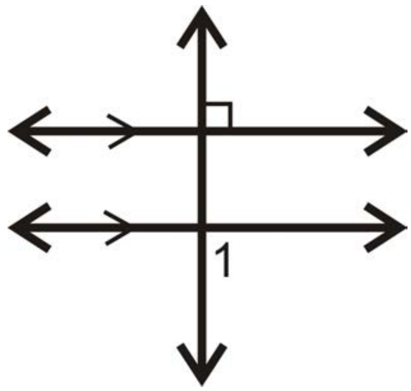
Lines Perpendicular to a Transversal Theorem

In a plane, if two lines are perpendicular to the same line, then they are parallel to each other If $l \perp n$ and $n \perp m$, then $l||m$



Every angle in the above two theorems will always be 90°.

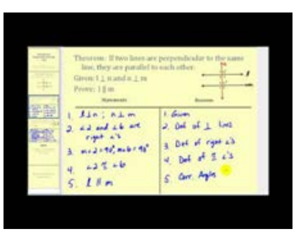
Example 4: Determine the measure of $\angle 1$.



Solution: From the Perpendicular Transversal Theorem, we know that both parallel lines are perpendicular to the transversal.

$$m\angle 1 = 90^\circ.$$

The following video proves lines are parallel given a perpendicular transversal.



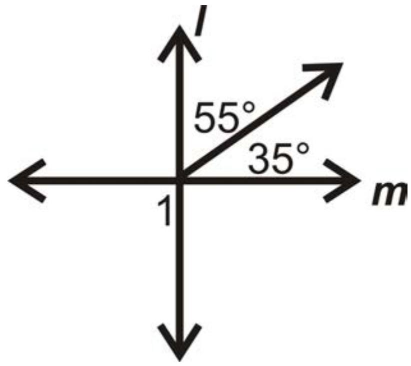
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/149924

Adjacent Complementary Angles If complementary angles are adjacent, their nonadjacent sides are perpendicular rays. What you have learned about perpendicular lines can be applied here.

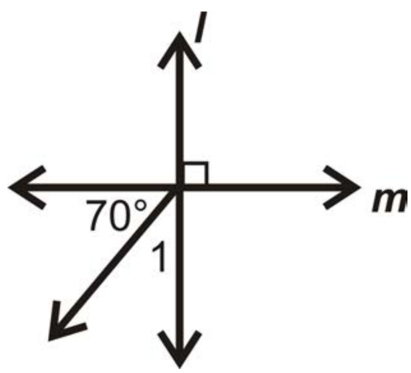
Example 5: Find $m\angle 1$.



Solution: The two adjacent angles add up to 90°, so $l \perp m$.

$$m\angle 1 = 90^\circ.$$

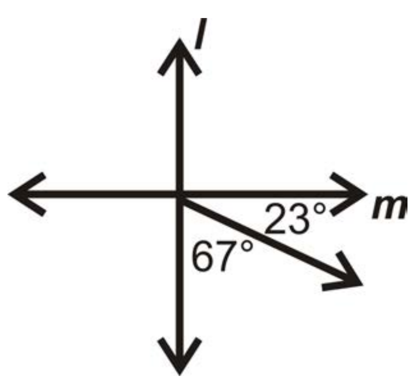
Example 6: What is the measure of $\angle 1$?



Solution: $l \perp m$

$$\begin{aligned} \text{So, } m\angle 1 + 70^\circ &= 90^\circ \\ m\angle 1 &= 20^\circ \end{aligned}$$

Example 7: Is $l \perp m$? Explain why or why not.



Solution: If the two adjacent angles add up to 90°, then l and m are perpendicular.

$$23^\circ + 67^\circ = 90^\circ$$

Therefore, $l \perp m$.

Know What? Revisited

Half Dome is vertical, so the slope is undefined.

http://www.nps.gov/yose/index.htm

Any horizon over an ocean is horizontal, which has a slope of zero, or no slope.

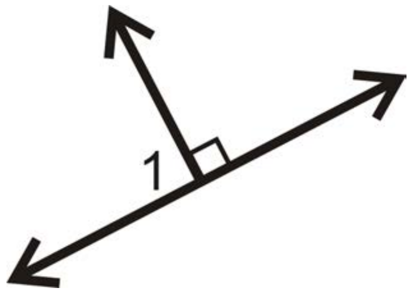
If Half Dome was placed on top of the ocean, the two would be perpendicular.

Practice Problems

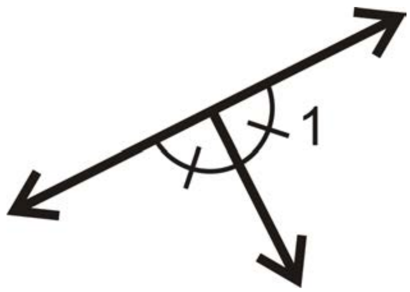
- Questions 1-25 are similar to Examples 1, 2, 4, 5, 6 and 7.
- Question 26 is similar to Example 3.
- Questions 27-30 are similar to Examples 5 and 6 and use Algebra.

Find the measure of $\angle 1$ for each problem below.

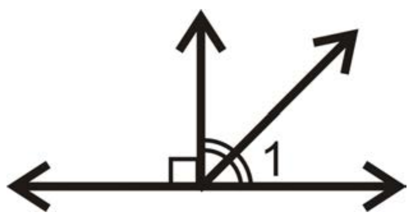
1.



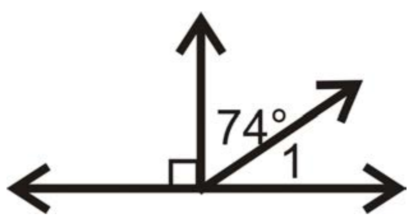
2.



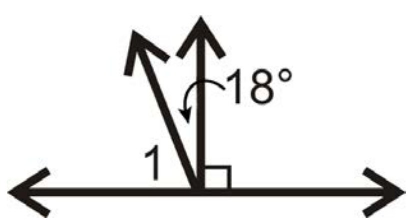
3.



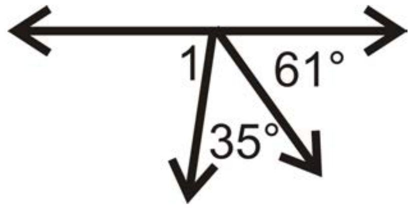
4.



5.



6.



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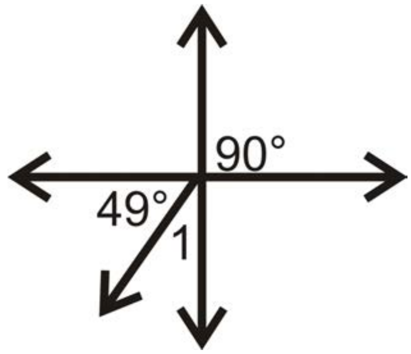

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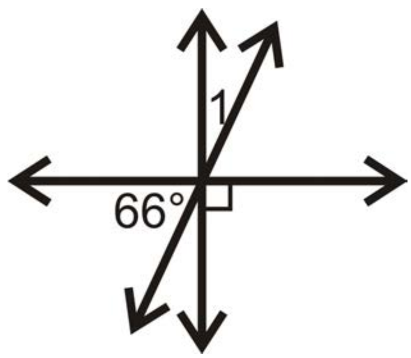
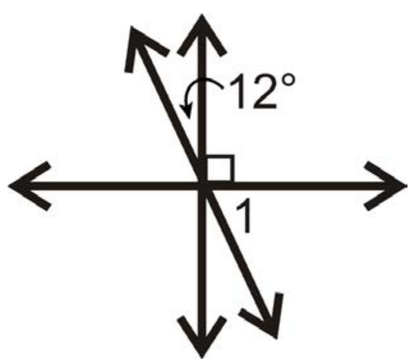
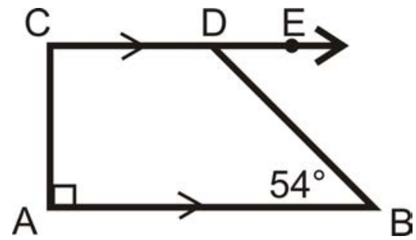


9.



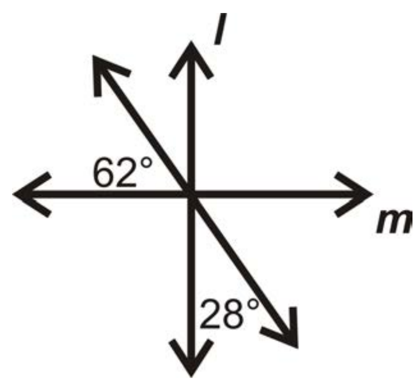
For questions 10-13, use the picture below.



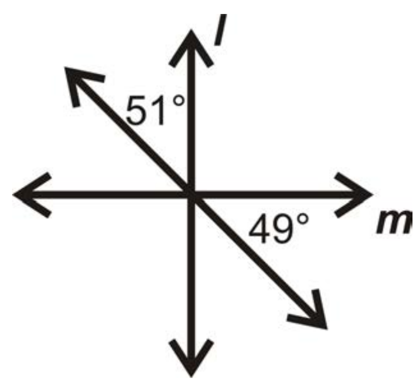
10. Find $m\angle ACD$.
11. Find $m\angle CDB$.
12. Find $m\angle EDB$.
13. Find $m\angle CDE$.

In questions 14-17, determine if $l \perp m$ and explain your reasoning.

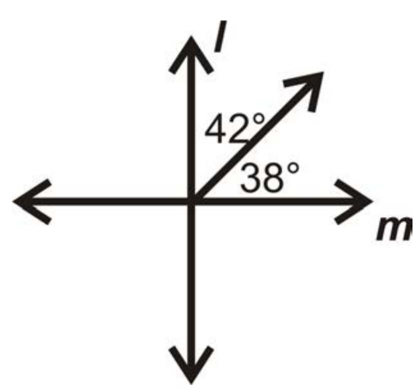
14.



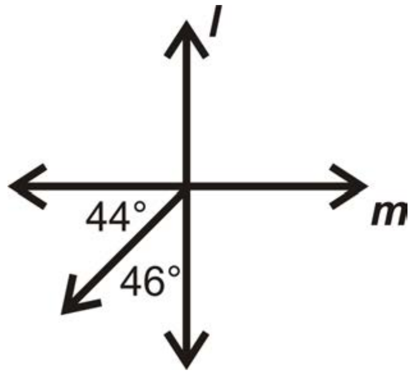
15.



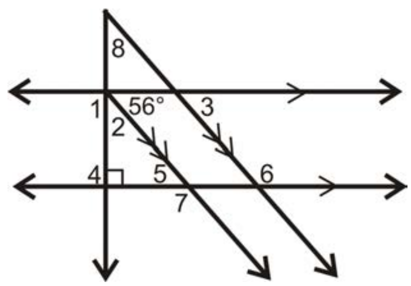
16.



17.



For questions 18-25, use the picture below.



18. Find $m\angle 1$.
19. Find $m\angle 2$.
20. Find $m\angle 3$.
21. Find $m\angle 4$.
22. Find $m\angle 5$.
23. Find $m\angle 6$.
24. Find $m\angle 7$.
25. Find $m\angle 8$.

Fill in the blanks in the proof below.

26. Given: $l \perp m$, $l \perp n$Prove: $m || n$

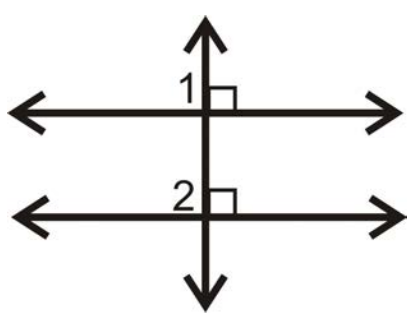
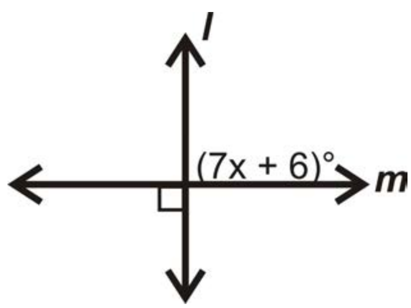


TABLE 3.16:

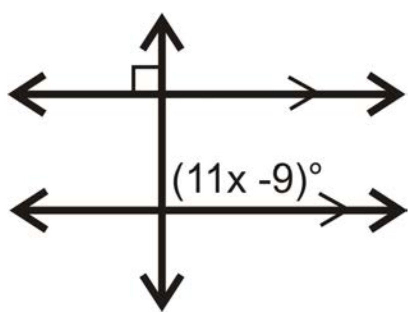
Statement	***Reason***
1.	
2. $\angle 1$ and $\angle 2$ are right angles	
3.	Definition of right angles
4.	Transitive Property of Equality
5. $m \| n$	

Algebra Connection Find the value of x.

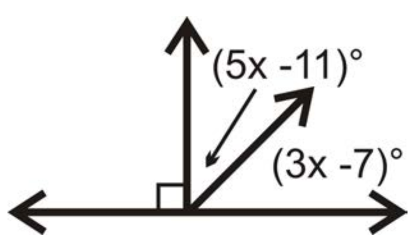
27.



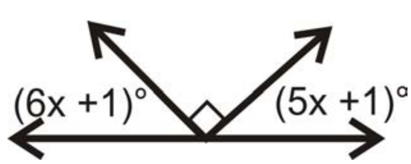
28.



29.



30.



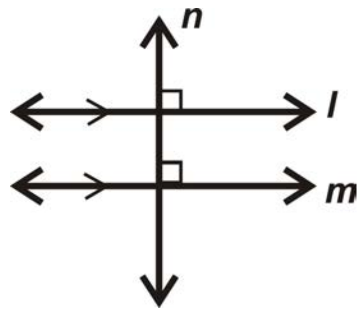
Review and Reflect

31. If two complementary angles are adjacent, why are their nonadjacent sides perpendicular?

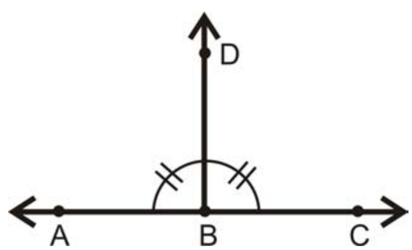
32. Given 2 parallel lines one of which is perpendicular to its transversal, prove the second line is also perpendicular to the transversal.

Warm-Up Answers

1. Yes, the transversal will be perpendicular to m.



2. $m\angle B = 55^\circ$
3. $m\angle D = 27^\circ$
4. Each angle is 90°.



3.5 Parallel and Perpendicular Lines on the Coordinate Plane

TEKS G(1)D, G(2)B, G(2)C

Learning Objectives

- Compute Slope
- Determine the equation of parallel and perpendicular lines

Vocabulary

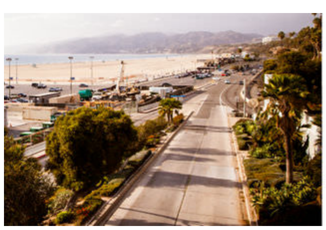
- slope
- parallel lines
- perpendicular lines
- point slope form

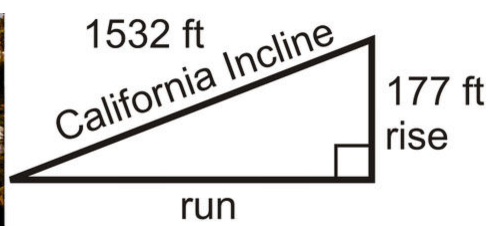
Warm-Up

Find the slope between the following points.

1. (-3, 5) and (2, -5)
2. (7, -1) and (-2, 2)
3. Is $x = 3$ horizontal or vertical? How do you know? What is the slope of the line?
4. Is $y = -1$ horizontal or vertical? How do you know? What is the slope of the line?
5. Graph $y = \frac{1}{4}x - 2$ on an $x - y$ plane.

Know What? The picture to the right is the California Incline, a short road that connects Highway 1 with Santa Monica. The length of the road is 1532 feet and has an elevation of 177 feet. *You may assume that the base of this incline is zero feet.* **Can you find the slope of the California Incline?**





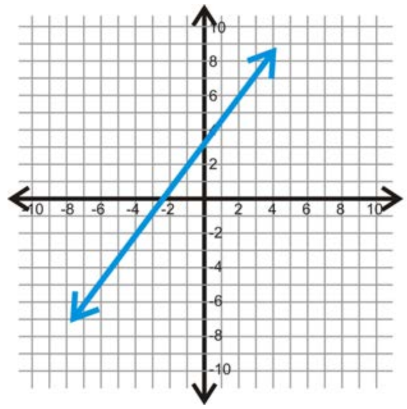
HINT: You will need to use the Pythagorean Theorem, which you may have seen in a previous math class.

Slope in the Coordinate Plane

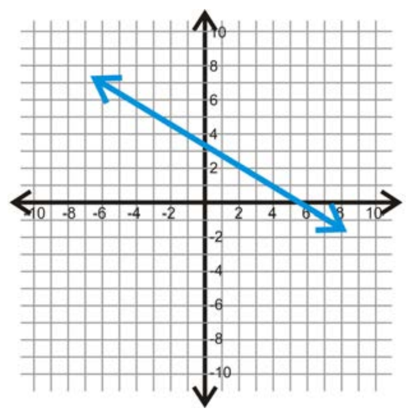
Recall from Algebra I, two points (x_1, y_1) and (x_2, y_2) have a slope of $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

Different Types of Slope:

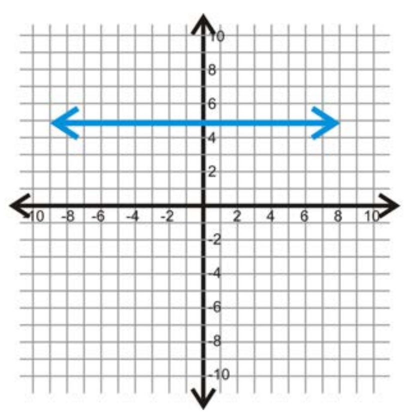
Positive



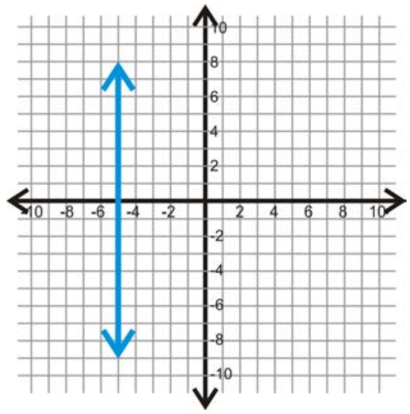
Negative



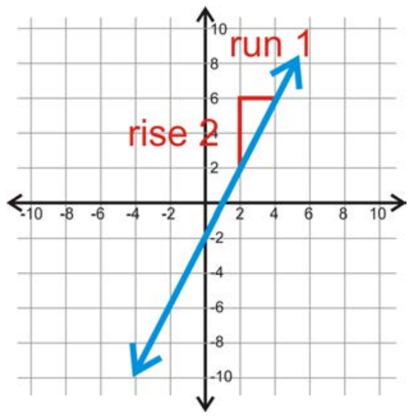
Zero



Undefined



Example 1: What is the slope of the line containing the points (2, 2) and (4, 6)?

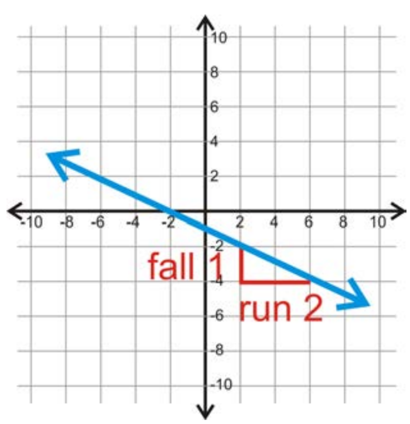


Solution: Use (2, 2) as (x_1, y_1) and (4, 6) as (x_2, y_2).

$$m = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

This slope is positive. Slope can also be written "rise over run." In this case we "rise" 2, and "run" in the positive direction 1.

Example 2: Find the slope between (-8, 3) and (2, -2).

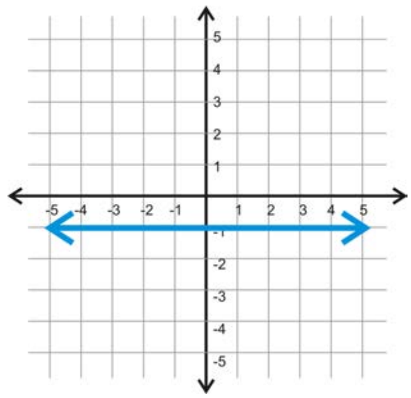


Solution: Use (-8, 3) as (x_1, y_1) and (2, -2) as (x_2, y_2).

$$m = \frac{-2-3}{2-(-8)} = \frac{-5}{10} = -\frac{1}{2}$$

This slope is negative. Here, we don't "rise," but "fall" one, and "run" in the positive direction 2.

Example 3: Find the slope between (-5, -1) and (3, -1).



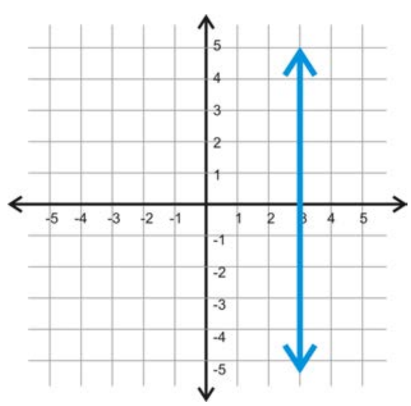
Solution: Use (-5, -1) as (x_1, y_1) and (3, -1) as (x_2, y_2).

$$m = \frac{-1-(-1)}{3-(-5)} = \frac{0}{8} = 0$$

The slope of this line is 0, or a horizontal line. Horizontal lines always pass through the $y-$axis. The $y-$coordinate for both points is -1.

So, the equation of this line is $y = -1$.

Example 4: What is the slope of the line through (3, 2) and (3, 6)?



Solution: Use (3, 2) as (x_1, y_1) and (3, 6) as (x_2, y_2).

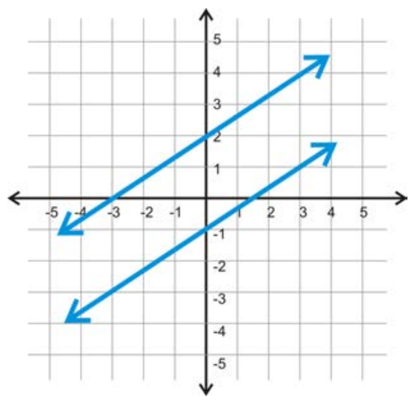
$$m = \frac{6-2}{3-3} = \frac{4}{0} = undefined$$

The slope of this line is undefined, which means that it is a vertical line. Vertical lines always pass through the $x-$axis. The $x-$coordinate for both points is 3.

So, the equation of this line is $x = 3$.

Slopes of Parallel Lines

Earlier in the chapter we defined parallel lines as two lines that never intersect. In the coordinate plane, that would look like this:



If we take a closer look at these two lines, the slopes are both $\frac{2}{3}$.

This can be generalized to any pair of parallel lines.

Parallel lines always have the same slope and different y−intercepts.

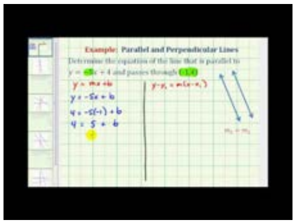
Example 5: Find the equation of the line that is parallel to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

Solution: Recall that the equation of a line is $y = mx + b$, where m is the slope and b is the $y-$intercept. We know that parallel lines have the same slope, so the line will have a slope of $-\frac{1}{3}$. Now that we have a slope and a point on the line we need to use point slope form to write out an equation. Plug in 9 for x_1, -5 for y_1, and $-\frac{1}{3}$ for m.

$$= m(x - x_1)y - (-5)$$

The equation of line is $y = -\frac{1}{3}x - 2$.

When given the slope of a line and a point on the line, point slope form is a good way to find the equation of the line. However, it is not the only way do determine the equation. The following video shows a side by side comparison in determining the equation of a line in 2 different ways.



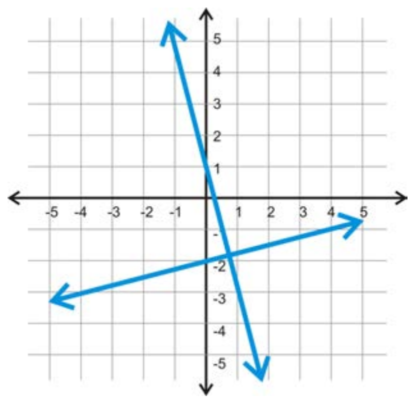
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/149927

Slopes of Perpendicular Lines

Perpendicular lines are two lines that intersect at a 90°, or right angle. In the coordinate plane, that would look like this:



If we take a closer look at these two lines, the slope of one is -4 and the other is $\frac{1}{4}$.

This can be generalized to any pair of perpendicular lines in the coordinate plane.

The slopes of perpendicular lines are opposite signs and reciprocals of each other.

Example 6: Find the slope of the line perpendicular to the given lines below.

(a) $y = 2x + 3$

(b) $y = -\frac{2}{3}x - 5$

(C) $y = x + 2$

Solution: Look at the slope of each of these.

(a) $m = 2$, so $m_\perp$ is the reciprocal and negative, $m_\perp = -\frac{1}{2}$.

(b) $m = -\frac{2}{3}$, take the reciprocal and make the slope positive, $m_\perp = \frac{3}{2}$.

(c) Because there is no number in front of x, the slope is 1. The reciprocal of 1 is 1, so the only thing to do is make it negative, $m_\perp = -1$.

Example 7: Find the equation of the line that is perpendicular to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

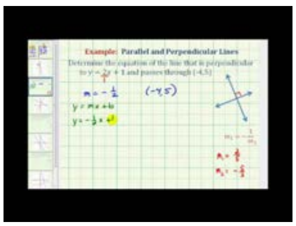
Solution: First, the slope needed is the reciprocal and opposite sign of $-\frac{1}{3}$. Now, use point slope form and plug in 9 for x_1, -5 for y_1, and 3 for m into point slope form.

$$= m(x - x_1)y - (-5)$$

$m_{\perp} = \frac{3}{2}$

Therefore, the equation of line is $y = 3x - 32$.

Here is another side by side video showing 2 different ways to find the equation of a line perpendicular to a given lines. The first method uses only slope intercept form, and the second method uses the same information but using point slope form.



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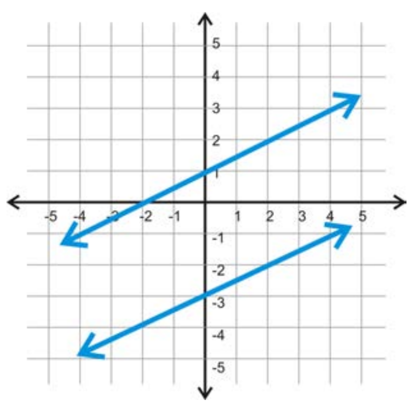
Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/149929

For more interaction with parallel and perpendicular lines, go to http://www.mathwarehouse.com/algebra/linear_equation/parallel-perpendicular-lines.php

Graphing Parallel and Perpendicular Lines

Example 8: Find the equation of the lines below and determine if they are parallel, perpendicular or neither.



Solution: The top line has a $y-$intercept of 1. From there, use "rise over run" to find the slope. From the $y-$intercept of 1, if you go up 1 and over 2, you hit the line again, $m = \frac{1}{2}$. The equation is $y = \frac{1}{2}x + 1$.

For the second line, the $y-$intercept is -3. The "rise" is 1 and the "run" is 2 making the slope $\frac{1}{2}$. The equation of this line is $y = \frac{1}{2}x - 3$.

The lines are parallel because they have the same slope.

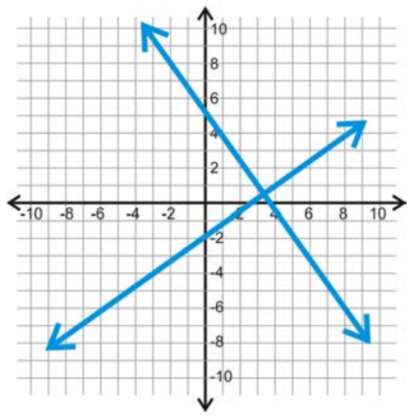
Example 9: Graph $3x - 4y = 8$ and $4x + 3y = 15$. Determine if they are parallel, perpendicular, or neither.

Solution: First, we have to change each equation into slope-intercept form. In other words, we need to solve each equation for y.

TABLE 3.17:

3x−4y−4yy=8=−3x+8=34x−2	4x+3y3yy=15=−4x+15=−43x+5

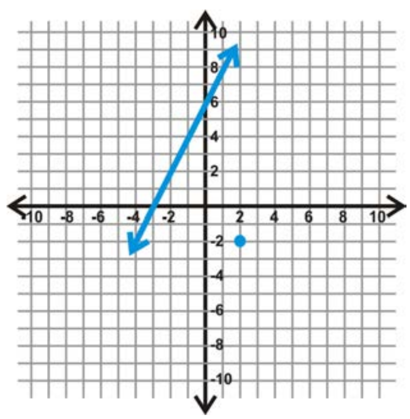
Now that the lines are in slope-intercept form (also called $y-$intercept form), we can tell they are perpendicular because the slopes are opposites signs and reciprocals.



Example 10: Find the equation of the line that is

(a) parallel to the line through the point.

(b) perpendicular to the line through the points.



Solution: First the equation of the line is $y = 2x+6$ and the point is (2, -2). The parallel would have the same slope and pass through (2, -2).

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(-2) &= 2(x-2) \\ y+2 &= 2x-4 \\ y &= 2x-6 \end{aligned}$$

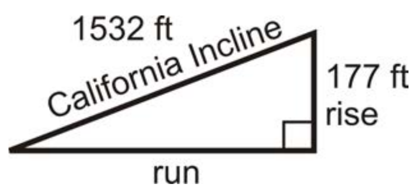
The equation is $y = 2x+-6$

The perpendicular line also goes through (2, -2), but the slope is $-\frac{1}{2}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= -\frac{1}{2}(x - 2) \\ y + 2 &= -\frac{1}{2}(x) + 1 \\ y &= -\frac{1}{2} - 1 \end{aligned}$$

The equation is $y = -\frac{1}{2}x - 1$

Know What? Revisited In order to find the slope, we need to first find the horizontal distance in the triangle to the right. This triangle represents the incline and the elevation. To find the horizontal distance, we need to use the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse.



$$\begin{aligned} 177^2 + run^2 &= 1532^2 \\ 31,329 + run^2 &= 2,347,024 \\ run^2 &= 2,315,695 \\ run &\approx 1521.75 \end{aligned}$$

The slope is then $\frac{177}{1521.75}$, which is roughly $\frac{3}{25}$.

Practice Problems

- Questions 1-6 are similar to Examples 1, 2, 3, and 4.
- Questions 7-14 are similar to Example 6 and 9.
- Questions 15-18 are similar to Example 5.
- Questions 19-22 are similar to Example 7.
- Questions 23-26 are similar to Example 8.
- Questions 27-30 are similar to Example 10.

Find the slope between the two given points.

1. (4, -1) and (-2, -3)
2. (-9, 5) and (-6, 2)
3. (7, 2) and (-7, -2)
4. (-6, 0) and (-1, -10)
5. (1, -2) and (3, 6)
6. (-4, 5) and (-4, -3)

Determine if each pair of lines are parallel, perpendicular, or neither. Then, graph each pair on the same set of axes.

7. $y = -2x + 3$ and $y = \frac{1}{2}x + 3$
8. $y = 4x - 2$ and $y = 4x + 5$
9. $y = -x + 5$ and $y = x + 1$
10. $y = -3x + 1$ and $y = 3x - 1$
11. $2x - 3y = 6$ and $3x + 2y = 6$
12. $5x + 2y = -4$ and $5x + 2y = 8$
13. $x - 3y = -3$ and $x + 3y = 9$
14. $x + y = 6$ and $4x + 4y = -16$

Determine the equation of the line that is ***parallel*** to the given line, through the given point.

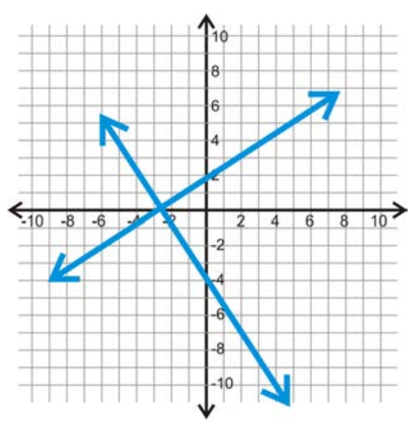
15. $y = -5x + 1;\ (-2, 3)$
16. $y = \frac{2}{3}x - 2;\ (9, 1)$
17. $x - 4y = 12;\ (-16, -2)$
18. $3x + 2y = 10;\ (8, -11)$

Determine the equation of the line that is ***perpendicular*** to the given line, through the given point.

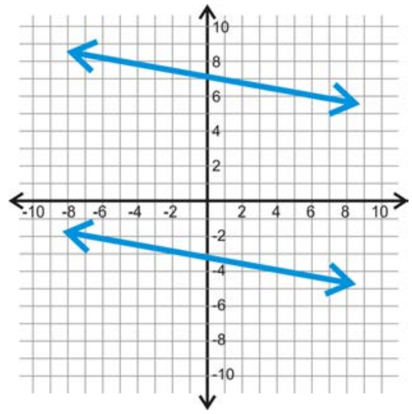
19. $y = x - 1;\ (-6, 2)$
20. $y = 3x + 4;\ (9, -7)$
21. $5x - 2y = 6;\ (5, 5)$
22. $y = 4;\ (-1, 3)$

Find the equation of the two lines in each graph below. Then, determine if the two lines are parallel, perpendicular or neither.

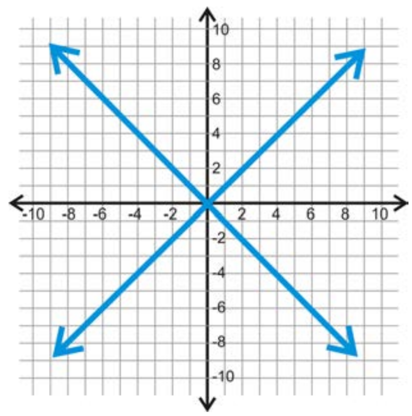
23.



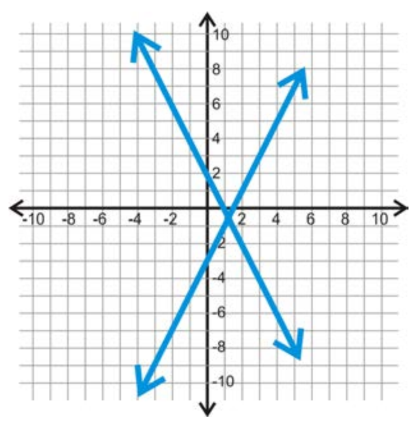
24.



25.



26.

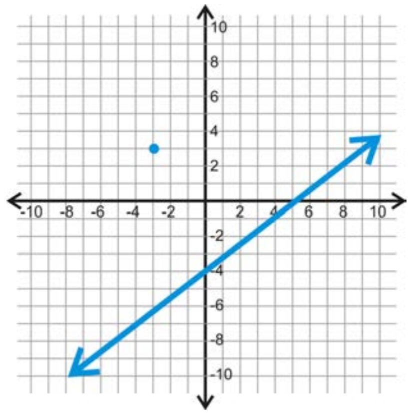


For the line and point below, determine and graph:

(a) A parallel line, through the given point.

(b) A perpendicular line, through the given point.

27.


10
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6
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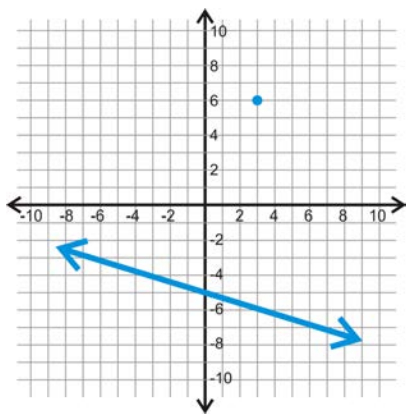
-10 -8 -6 -4 -2 2 4 6 8 10
-2
-4
-6
-8
-10

28.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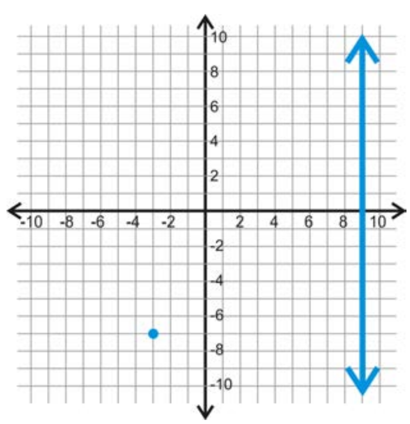
29.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-8
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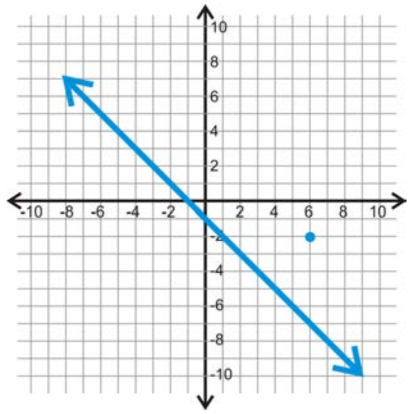
30.

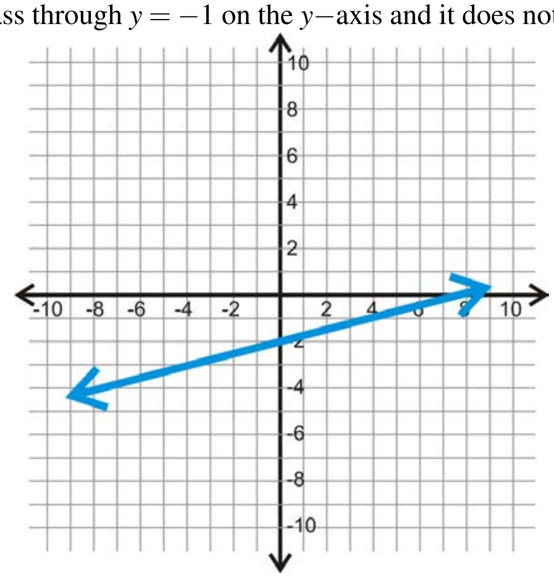


Review and Reflect

31. How are slopes of parallel lines related to slopes of perpendicular lines?

32. Given the equations of 2 lines in standard form ($Ax+By=C$) describe how you can tell if the two lines are parallel, perpendicular, or neither.

Warm-Up Answers

1. $m = \frac{-5-5}{2+3} = \frac{-10}{2} = -5$
2. $m = \frac{2+1}{-2-7} = \frac{3}{-9} = -\frac{1}{3}$
3. Vertical because it has to pass through $x = 3$ on the $x-$axis and doesn't pass through $y-$axis at all.
4. Horizontal because it has to pass through $y = -1$ on the $y-$axis and it does not pass through the $x-$axis at all.
5. 

3.6 The Distance Formula

TEKS G(1)A, G(2)B, G(2)C

Learning Objectives

- Find the distance between two points
- Find the shortest distance between vertical and horizontal lines
- Find the shortest distance between parallel lines with slope of 1 or -1

Warm-Up

1. What is the slope of the line between (-1, 3) and (2, -9)?
2. Find the equation of the line that is *perpendicular* to $y = -2x + 5$ through the point (-4, -5).
3. Find the equation of the line that is *parallel* to $y = \frac{2}{3}x - 7$ through the point (3, 8).

Know What? The shortest distance between two points is a straight line. To the right are distances between cities in the Los Angeles area. What is the longest distance between Los Angeles and Orange? Which distance is the shortest?

The Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) can be defined as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This formula will be derived in Chapter 9.

Example 1: Find the distance between (4, -2) and (-10, 3).

Solution: Plug in (4, -2) for (x_1, y_1) and (-10, 3) for (x_2, y_2) and simplify.

$$
\begin{aligned}
d &= \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}\\
d &= \sqrt{(3-(-2))^2+((-10)-4)^2}\\
d &= \sqrt{5^2+(-14)^2}\\
d &= \sqrt{25+196}\\
d &= \sqrt{221}
\end{aligned}
$$

In Chapter 1 we discussed another way to compute distances using Pythagorean Theorem. When using Pythagorean Theorem to calculate a distance you will need to determine the horizontal and vertical lengths of the triangle formed.

Example 2: Find the distance between (-2, -3) and (3, 9).

Solution: First determine the horizontal distance between -2 and 3. The horizontal distance of 5 will become the horizontal leg our our right triangle. Second, calculate the vertical distance between -3 and 9. The vertical leg of our right triangle will be 12 units. Now use Pythagorean Theorem to determine the distance and simplify.

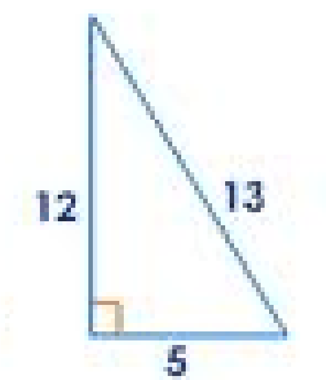


FIGURE 3.12

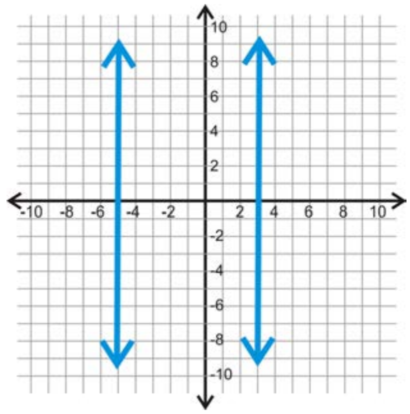
$$
\begin{aligned}
d &= \sqrt{(-2-3)^2+(-3-9)^2}\\
&= \sqrt{(-5)^2+(-12)^2}\\
&= \sqrt{25+144}\\
&= \sqrt{169} = 13\ units
\end{aligned}
$$

Distances are always positive!

Shortest Distance between Vertical and Horizontal Lines

All vertical lines are in the form $x = a$, where a is the $x-$intercept. To find the distance between two vertical lines, count the squares between the two lines.

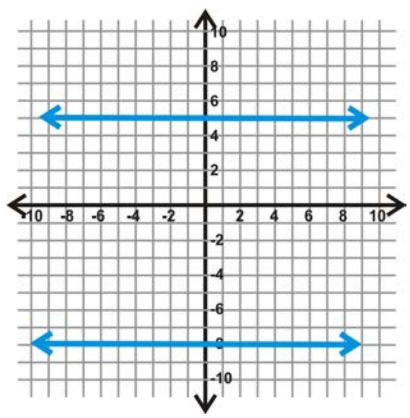
Example 3: Find the distance between $x = 3$ and $x = -5$.



Solution: The two lines are 3 - (-5) units apart, or 8 units apart.

You can use this method for horizontal lines as well. All horizontal lines are in the form $y = b$***, where*** b ***is the*** $y-$***intercept.***

Example 4: Find the distance between $y = 5$ and $y = -8$.

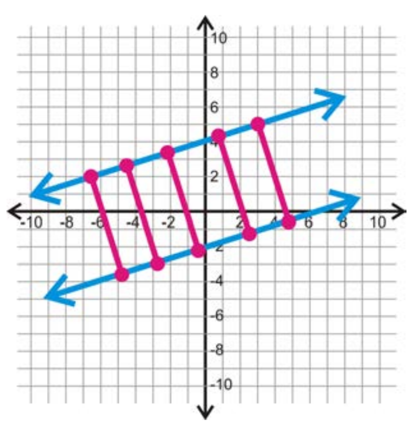


Solution: The two lines are 5 - (-8) units apart, or 13 units.

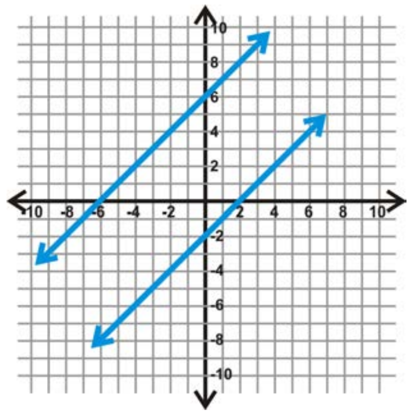
Shortest Distance between Parallel Lines with $m = 1$ **or -1**

The shortest distance between two parallel lines is the perpendicular line between them. There are infinitely many perpendicular lines between two parallel lines.

Notice that all of the pink segments are the same length.



Example 5: Find the distance between $y = x+6$ and $y = x-2$.



Solution:

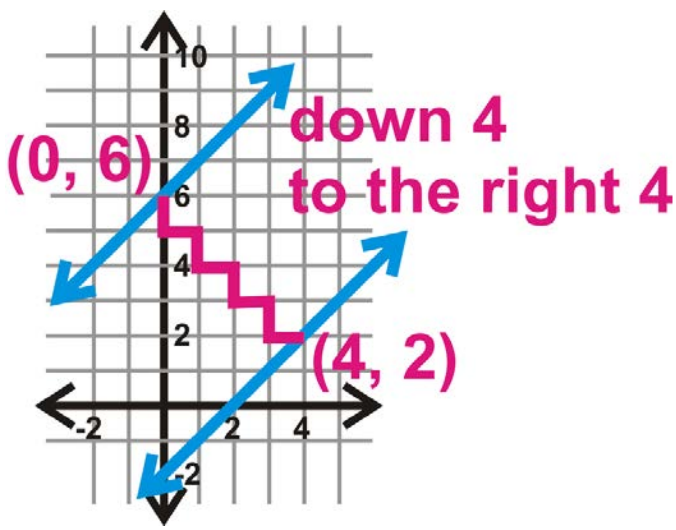
1. Find the perpendicular slope.

$m = 1$, so $m_{\perp} = -1$

2. Find the $y-$intercept of the top line, $y = x+6$. (0, 6)

3. Use the slope and count down 1 and to the right 1 until you hit the bottom line, $y = x-2$.

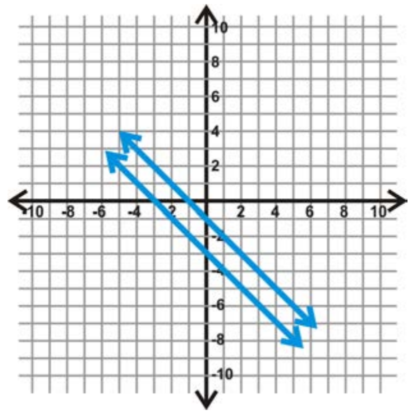
Always rise/run the same amount for $m = 1$ or -1.



4. Use these two points in the distance formula to determine how far apart the lines are.

$$\begin{aligned} d &= \sqrt{(0-4)^2+(6-2)^2} \\ &= \sqrt{(-4)^2+(4)^2} \\ &= \sqrt{16+16} \\ = \sqrt{32} = \sqrt{16} * \sqrt{2} &= 4\sqrt{2} \ units \end{aligned}$$

Example 6: Find the distance between $y = -x-1$ and $y = -x-3$.



Solution:

1. Find the perpendicular slope.

$m = -1$, so $m_{\perp} = 1$

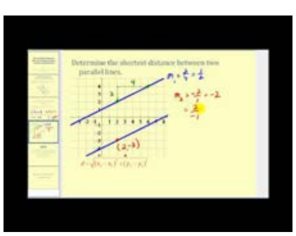
2. Find the $y-$intercept of the top line, $y = -x - 1$. (0, -1)

3. Use the slope and count down 1 and to the *left* 1 until you hit $y = x - 3$.

4. Use these two points to determine the horizontal and vertical lengths of the legs of a right traingle and then use Pythagorean Theorem to find the distance between the the two lines.

$$\begin{aligned}
hyp^2 &= leg^2 + leg^2 \\
hyp^2 &= 1^2 + 1^2 \\
hyp^2 &= 1 + 1 \\
hyp^2 &= 2 \\
hyp &= \sqrt{2} \; units
\end{aligned}$$

The following video is another example of how to find distance between 2 parallel lines.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/149935

Know What? Revisited The shortest distance between Los Angeles and Orange is 26.3 miles along Highway 5. The longest distance is found by adding the distances along the I-10 and 405, or 41.8 miles.

Practice Problems

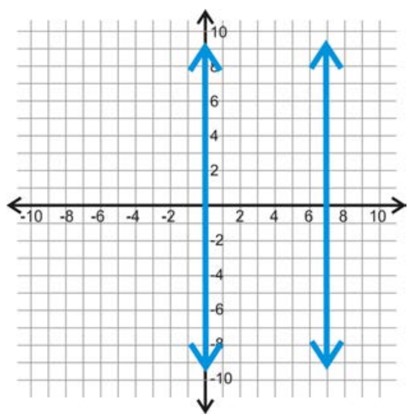
- Questions 1-10 are similar to Examples 1 and 2.
- Questions 11- are similar to Examples 3 and 4.
- Questions are similar to Examples 5 and 6.

Find the distance between each pair of points. Leave your answer in simplest radical form.

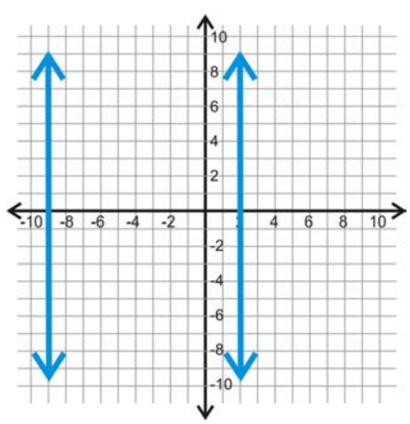
1. (4, 15) and (-2, -1)
2. (-6, 1) and (9, -11)
3. (0, 12) and (-3, 8)
4. (-8, 19) and (3, 5)
5. (3, -25) and (-10, -7)
6. (-1, 2) and (8, -9)
7. (5, -2) and (1, 3)
8. (-30, 6) and (-23, 0)
9. (2, -2) and (2, 5)
10. (-9, -4) and (1, -1)

Use each graph below to determine how far apart each pair of parallel lines is.

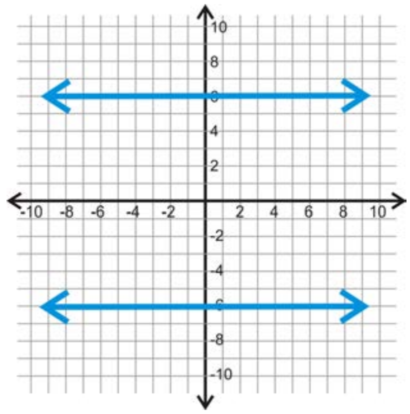
11.



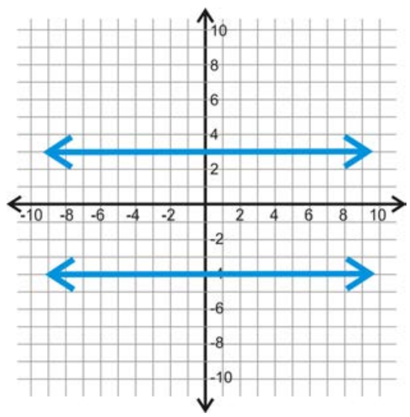
12.



13.



14.



Determine the shortest distance between the each pair of parallel lines. Leave your answer in simplest radical form.

15. $x = 5, x = 1$
16. $y = -6, y = 4$
17. $y = 3, y = 15$
18. $x = -10, x = -1$
19. $x = 8, x = 0$
20. $y = 7, y = -12$
21. What is the slope of the line $y = x + 2$?
22. What is the slope of the line perpendicular to the line from #21?
23. What is the $y-$intercept of the line from #21?
24. What is the equation of the line that is *perpendicular* to $y = x + 2$ through its $y-$intercept?
25. Graph $y = x + 2$ and the line you found in #24. Then, graph $y = x - 4$. Where does the perpendicular line cross $y = x - 4$?
26. Using the answers from #23 and #25 and the distance formula or Pythagorean Theorem, find the distance between $y = x + 2$ and $y = x - 4$.

Find the distance between the parallel lines below. Leave your answer in simplest radical form.

27. $y = x - 3,\ y = x + 11$
28. $y = -x + 4,\ y = -x$
29. $y = -x - 5,\ y = -x + 1$
30. $y = x + 12,\ y = x - 6$

Review and Reflect

31. How might the distance between two parallel lines be used in a real world situation?
32. Describe the way in which you prefer to find the distance between 2 points.

Warm-Up Answers

1. $m = -4$
2. $y = \frac{1}{2}x - 3$
3. $y = \frac{2}{3}x + 6$

3.7 Chapter 3 Review

Symbol Toolbox

|| parallel

$\perp$ perpendicular

Keywords & Theorems

Lines and Angles

- Parallel
- Skew lines
- Parallel Postulate
- Perpendicular Line Postulate
- Transversal
- Corresponding Angles
- Alternate Interior Angles
- Alternate Exterior Angles
- Same Side Interior Angles

Properties of Parallel Lines

- Corresponding Angles Postulate
- Alternate Interior Angles Theorem
- Alternate Exterior Angles Theorem
- Same Side Interior Angles Theorem

Proving Lines Parallel

- Converse of Corresponding Angles Postulate
- Converse of Alternate Interior Angles Theorem
- Converse of the Alternate Exterior Angles Theorem
- Converse of the Same Side Interior Angles Theorem
- Parallel Lines Property

Properties of Perpendicular Lines

- Congruent Linear Pairs
- Theorem 3-1
- Theorem 3-2
- Adjacent Complementary Angles

Parallel and Perpendicular Lines in the Coordinate Plane

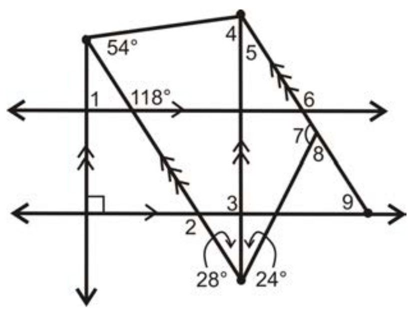
- Slope
- Slope-intercept form ($y-$intercept form)
- Standard Form

Distance Formula

- Distance Formula

Review

Find the value of each of the numbered angles below.



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9688 .

3.8 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1^{st} **Section: Lines and Angles**

Parallel

Skew lines

Parallel Postulate

Perpendicular Line Postulate

Constructing a Perpendicular Line through a Point not on a Line

Constructing a Perpendicular Line through a Point on a Line

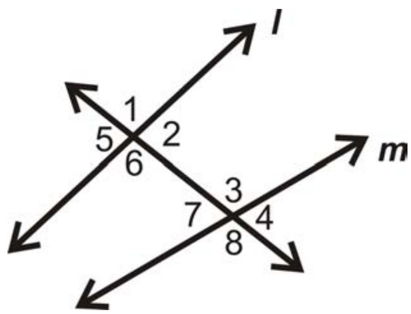
Transversal

Corresponding Angles

Alternate Interior Angles

Alternate Exterior Angles

Same Side Interior Angles



Homework:

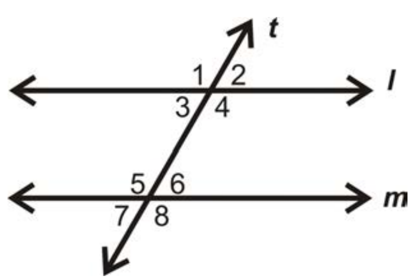
2^{nd} **Section: Properties of Parallel Lines**

Corresponding Angles Postulate

Alternate Interior Angles Theorem

Alternate Exterior Angles Theorem

Same Side Interior Angles Theorem



Homework:

3^{rd} **Section: Proving Lines Parallel**

Converse of Corresponding Angles Postulate

Converse of Alternate Interior Angles Theorem

Converse of the Alternate Exterior Angles Theorem

Converse of the Same Side Interior Angles Theorem

Parallel Lines Property

Homework:

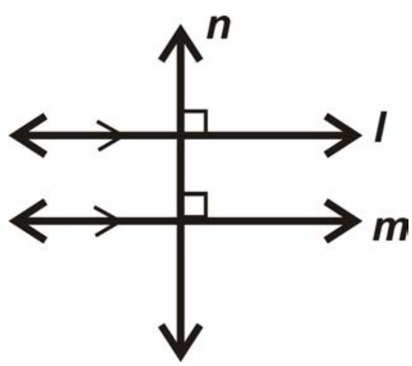
4^{th} **Section: Properties of Perpendicular Lines**

Congruent Linear Pairs

Theorem 3-1

Theorem 3-2

Adjacent Complementary Angles



Homework:

5^{th} **Section: Parallel and Perpendicular Lines in the Coordinate Plane**

Slope

$y-$intercept

Slope-intercept form ($y-$intercept form)

Standard Form

Finding the equation of a Line

Finding the equation of a Parallel Line to a Given Line

Finding the equation of a Perpendicular Line to a Given Line

Homework:

6^{th} **Section: The Distance Formula**

Distance Formula

Distance between Vertical and Horizontal Lines

Distance between Lines with $m = 1$ or -1

Homework:

CHAPTER 4 Triangles and Congruence

Chapter Outline

In this chapter, you will learn all about triangles. First, we will find out how many degrees are in a triangle and other properties of the angles within a triangle. Second, we will use that information to determine if two different triangles are congruent Finally, we will investigate the properties of isosceles and equilateral triangles.

4.1 Triangle Sums

TEKS G(1)B, G(4)D, G(5)A, G(6)D

Learning Objectives

- Understand the Triangle Sum Theorem
- Identify interior and exterior angles in a triangle
- Use the Exterior Angle Theorem

Vocabulary

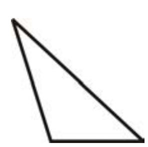
- interior angles
- vertex
- exterior angle
- remote interior angles

Warm-Up

Classify the triangles below by their angles and sides.

1.

2.

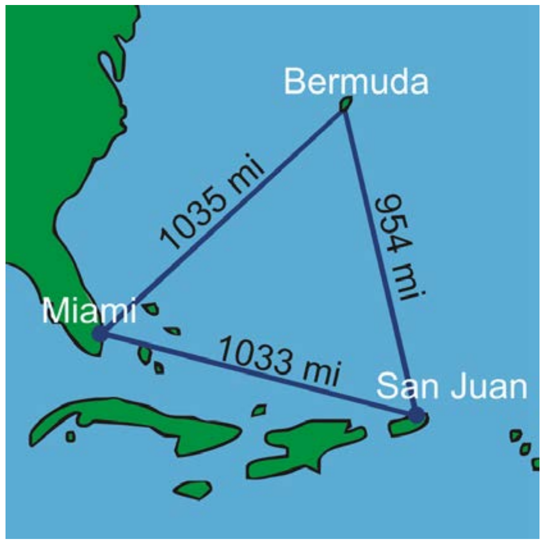


3.

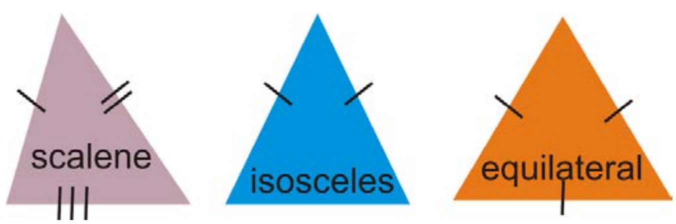
4. Draw and label a straight angle, $\angle ABC$. Which point is the vertex? How many degrees does a straight angle have?

Know What? Below is a map of the Bermuda Triangle. The myth of this triangle is that ships and planes have passed through and mysteriously disappeared.

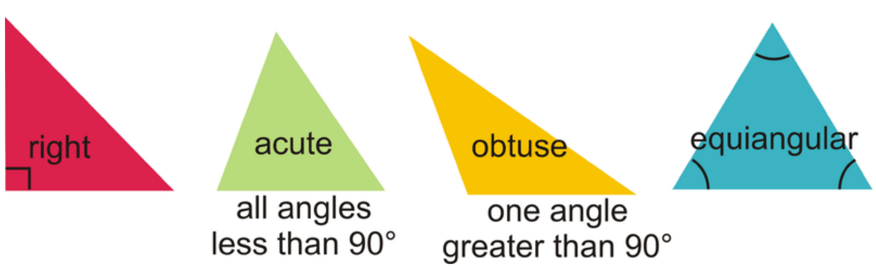
The measurements of the sides of the triangle are in the picture. Classify the Bermuda triangle by its sides and angles. Then, using a protractor, find the measure of each angle. What do they add up to?



Recall that a triangle can be classified by its sides...

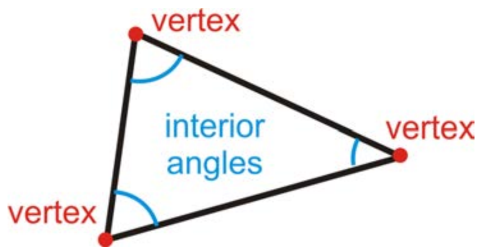


and also classified by its angles...



Interior Angles: The angles inside of a polygon.

Vertex: The point where the sides of a polygon meet.



Triangles have three interior angles, three vertices, and three sides.

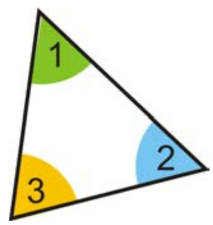
A triangle is labeled by its vertices with a $\triangle$ symbol in front of the 3 vertices. A triangle with 3 vertices labeled, A, B, and C can be labeled as $\triangle ABC$, $\triangle ACB$, $\triangle BCA$, $\triangle BAC$, $\triangle CBA$ or $\triangle CAB$.

Triangle Sum Theorem The interior angles in a polygon are measured in degrees. How many degrees are there in a triangle?

Investigation 4-1: Triangle Tear-Up

Tools Needed: paper, ruler, pencil, colored pencils

1. Draw a triangle on a piece of paper. Make all three angles different sizes. Color the three interior angles three different colors and label each one, $\angle 1$, $\angle 2$, and $\angle 3$.

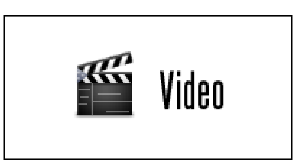


2. Tear off the three colored angles, so you have three separate angles.

3. Line up the angles so the vertices points all match up. What happens? What measure do the three angles add up to?

This investigation shows us that the sum of the angles in a triangle is 180° because the three angles fit together to form a straight angle where all the vertices meet.

The previous investigations can be seen below.



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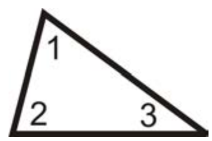
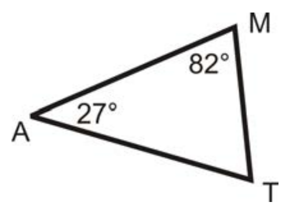
Triangle Sum Theorem

The interior angles of a triangle add up to 180º.



$$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$$

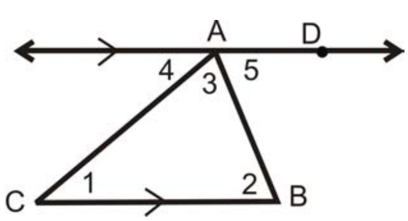
Example 1: What $m\angle T$?



Solution: Set up an equation.

$$\begin{aligned}
m\angle M + m\angle A + m\angle T &= 180^\circ \\
82^\circ + 27^\circ + m\angle T &= 180^\circ \\
109^\circ + m\angle T &= 180^\circ \\
m\angle T &= 71^\circ
\end{aligned}$$

Even thought Investigation 4-1 is a way to show that the angles in a triangle add up to 180°, it is not a proof. Here is the proof of the Triangle Sum Theorem.



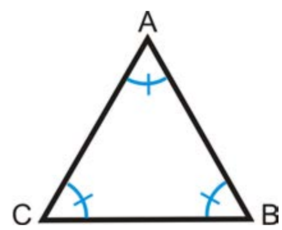
Given: $\triangle ABC$ with $\overleftrightarrow{AD} || \overline{BC}$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$

TABLE 4.1:

Statement	***Reason***
1. $\triangle ABC$ above with $\overleftrightarrow{AD} \| \overline{BC}$	Given
2. $\angle 1 \cong \angle 4,\ \angle 2 \cong \angle 5$	Alternate Interior Angles Theorem
3. $m\angle 1 = m\angle 4,\ m\angle 2 = m\angle 5$	$\cong$ angles have = measures
4. $m\angle 4 + m\angle CAD = 180^\circ$	Linear Pair Postulate
5. $m\angle 3 + m\angle 5 = m\angle CAD$	Angle Addition Postulate
6. $m\angle 4 + m\angle 3 + m\angle 5 = 180^\circ$	Substitution Property of Equality
7. $m\angle 1 + m\angle 3 + m\angle 2 = 180^\circ$	Substitution Property of Equality

Example 2: What is the measure of each angle in an equiangular triangle?



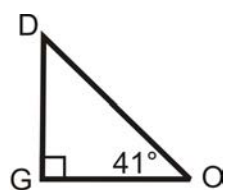
Solution: $\triangle ABC$ is an equiangular triangle, where all three angles are equal. Write an equation.

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= 180^\circ \\ m\angle A + m\angle A + m\angle A &= 180^\circ && \textit{Substitute, all angles are equal.} \\ 3m\angle A &= 180^\circ && \textit{Combine like terms.} \\ m\angle A &= 60^\circ \end{aligned}$$

If $m\angle A = 60^\circ$, then $m\angle B = 60^\circ$ and $m\angle C = 60^\circ$.

Each angle in an equiangular triangle is 60°**.**

Example 3: Find the measure of the missing angle.



Solution: $m\angle O = 41^\circ$ and $m\angle G = 90^\circ$ because it is a right angle.

$$\begin{aligned} m\angle D + m\angle O + m\angle G &= 180^\circ \\ m\angle D + 41^\circ + 90^\circ &= 180^\circ \\ m\angle D + 41^\circ &= 90^\circ \\ m\angle D &= 49^\circ \end{aligned}$$

Notice that $m\angle D + m\angle O = 90^\circ$.

The acute angles in a right triangle are always complementary.

Remember from Chapter 3 that not all geometry is measured on a flat surface. We study Euclidean Geometry which is the study of *flat space*. One type of non Euclidean Geometry is Spherical Geometry which is the study of a 2-dimensional flat surface on a sphere. The following video compares the relationship between the sum of the angles in a triangle on a flat surface, and a triangle on a spherical surface.

MEDIA

Click image to the left or use the URL below.

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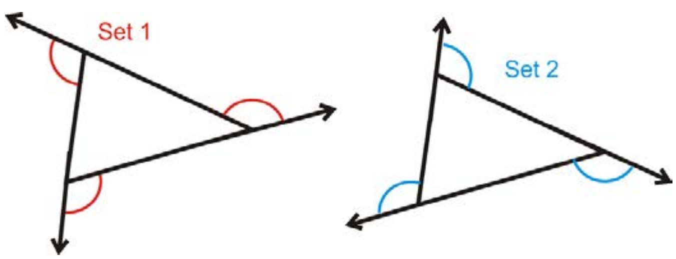
Exterior Angles

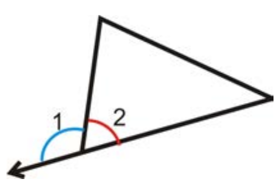
Exterior Angle: The angle formed by one side of a polygon and the extension of the adjacent side.

In all polygons, there are **two** sets of exterior angles, one that goes around clockwise and the other goes around counterclockwise.

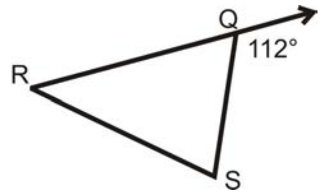


Notice that the interior angle and its adjacent exterior angle form a linear pair and add up to 180°.

$$m\angle 1 + m\angle 2 = 180^\circ$$



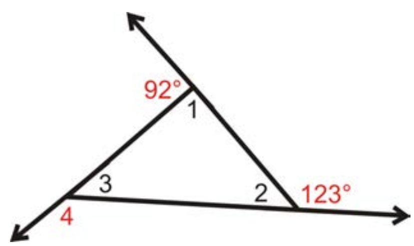
Example 4: Find the measure of $\angle RQS$.



Solution: 112° is an exterior angle of $\triangle RQS$ and is supplementary to $\angle RQS$.

$$\begin{aligned} 112^\circ + m\angle RQS &= 180^\circ \\ m\angle RQS &= 68^\circ \end{aligned}$$

Example 5: Find the measure of the numbered interior and exterior angles in the triangle.



Solution: $m\angle 1 + 92^\circ = 180^\circ$ by the Linear Pair Postulate. $m\angle 1 = 88^\circ$

$m\angle 2 + 123^\circ = 180^\circ$ by the Linear Pair Postulate. $m\angle 2 = 57^\circ$

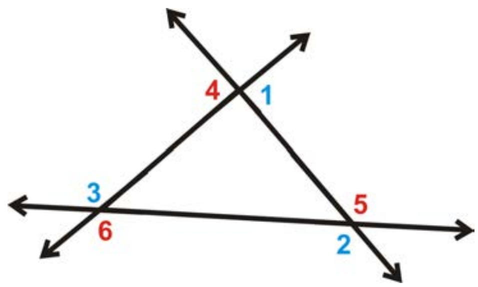
$$\begin{aligned} m\angle 1 + m\angle 2 + m\angle 3 &= 180^\circ && \text{by the Triangle Sum Theorem.} \\ 88^\circ + 57^\circ + m\angle 3 &= 180 \\ m\angle 3 &= 35^\circ \end{aligned}$$

$$\begin{aligned} \text{Lastly, } m\angle 3 + m\angle 4 &= 180^\circ && \text{by the Linear Pair Postulate.} \\ 35^\circ + m\angle 4 &= 180^\circ \\ m\angle 4 &= 145^\circ \end{aligned}$$

In Example 5, the exterior angles are 92°, 123°, and 145°. Adding these angles together, we get $92^\circ + 123^\circ + 145^\circ = 360^\circ$. This is true for any set of exterior angles for any polygon.

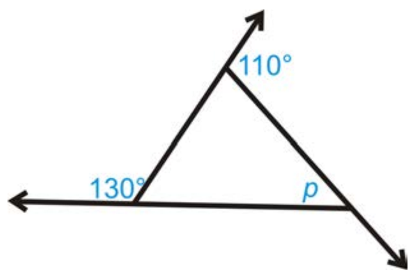
Exterior Angle Sum Theorem

The exterior angles of a polygon add up to 360º.



$$m\angle 1 + m\angle 2 + m\angle 3 = 360^\circ$$
$$m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$$

Example 6: What is the value of p in the triangle below?



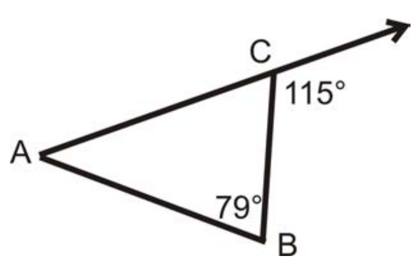
Solution: First, we need to find the missing exterior angle, let's call it x. Set up an equation using the Exterior Angle Sum Theorem.

$$\begin{aligned} 130^\circ + 110^\circ + x &= 360^\circ \\ x &= 360^\circ - 130^\circ - 110^\circ \\ x &= 120^\circ \end{aligned}$$

x and p add up to 180° because they are a linear pair.

$$\begin{aligned} x + p &= 180^\circ \\ 120^\circ + p &= 180^\circ \\ p &= 60^\circ \end{aligned}$$

Example 7: Find $m\angle A$.



Solution:

$$\begin{aligned}
m\angle ACB + 115^\circ &= 180^\circ && \text{because they are a linear pair}\\
m\angle ACB &= 65^\circ\\
m\angle A + 65^\circ + 79^\circ &= 180^\circ && \text{by the Triangle Sum Theorem}\\
m\angle A &= 36^\circ
\end{aligned}$$

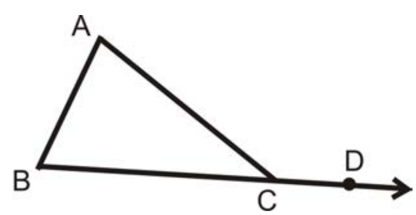
Remote Interior Angles: The two angles in a triangle that are not adjacent to the indicated exterior angle are called the remote interior angles.

In Example 7 above, $\angle A$ and 79° are the remote interior angles relative to 115°.

Exterior Angle Theorem From Example 7, we can find the sum of $m\angle A$ and $m\angle B$, which is $36^\circ + 79^\circ = 115^\circ$. This is equal to the exterior angle at C.

Exterior Angle Theorem

The sum of the remote interior angles is equal to the non-adjacent exterior angle.



$$m\angle A + m\angle B = m\angle ACD$$

Proof of the Exterior Angle Theorem

Given: Triangle with exterior $\angle 4$

Prove: $m\angle 1 + m\angle 2 = m\angle 4$

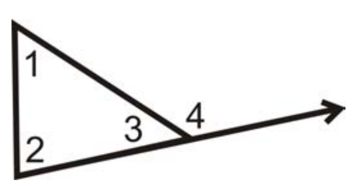
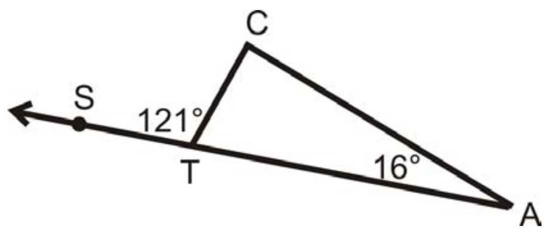


TABLE 4.2:

Statement	***Reason***
1. Triangle with exterior $\angle 4$	Given
2. $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$	Triangle Sum Theorem
3. $m\angle 3 + m\angle 4 = 180^\circ$	Linear Pair Postulate
4. $m\angle 1 + m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$	Transitive Property of Equality
5. $m\angle 1 + m\angle 2 = m\angle 4$	Subtraction Property of Equality

Use the following link to explore exterior angles and the sum of the two remote interior angles of a triangle, https://tube.geogebra.org/student/m60018

Example 8: Find $m\angle C$.

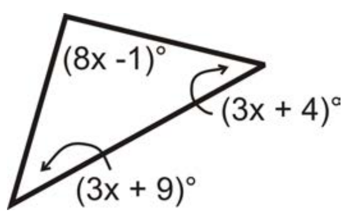


Solution: Using the Exterior Angle Theorem

$$m\angle C + 16^\circ = 121^\circ$$
$$m\angle TCA = 105^\circ$$

If you forget the Exterior Angle Theorem, you can do this problem just like Example 7.

Example 9: ***Algebra Connection*** Find the value of x and the measure of each angle.



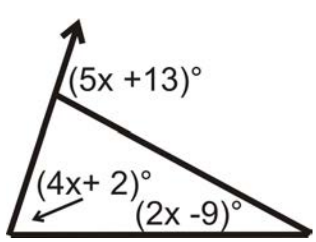
Solution: All the angles add up to 180°.

$$(8x-1)^\circ + (3x+9)^\circ + (3x+4)^\circ = 180^\circ$$
$$(14x+12)^\circ = 180^\circ$$
$$14x = 168^\circ$$
$$x = 12^\circ$$

Substitute in 12° for x to find each angle.

$$3(12^\circ)+9^\circ = 45^\circ \qquad 3(12^\circ)+4^\circ = 40^\circ \qquad 8(12^\circ)-1^\circ = 95^\circ$$

Example 10: ***Algebra Connection*** Find the value of x and the measure of each angle.



Solution: Set up an equation using the Exterior Angle Theorem.

$$\underset{\text{interior angles}}{(4x+2)^\circ + (2x-9)^\circ} = \underset{\text{exterior angle}}{(5x+13)^\circ}$$

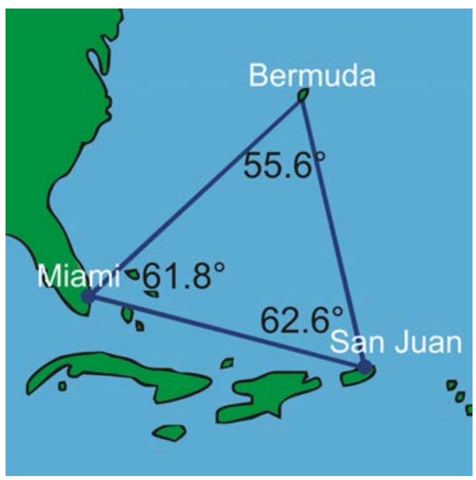
$$(6x-7)^\circ = (5x+13)^\circ$$
$$x = 20^\circ$$

Substitute in 20° for x to find each angle.

$4(20^\circ)+2^\circ = 82^\circ$ $\qquad$ $2(20^\circ)-9^\circ = 31^\circ$ $\qquad$ Exterior angle: $5(20^\circ)+13^\circ = 113^\circ$

Know What? Revisited The Bermuda Triangle is an acute scalene triangle. The angle measures are in the picture below. Your measured angles should be within a degree or two of these measures. The angles should add up to 180°. However, because your measures are estimates using a protractor, they might not exactly add up.

The angle measures in the picture are the measures from a map (which is flat). Because the earth is curved, in real life the measures will be slightly different.

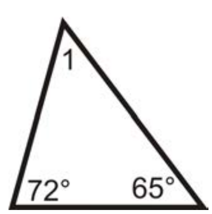


Practice Problems

- Questions 1-16 are similar to Examples 1-8.
- Questions 17 and 18 use the definition of an Exterior Angle and the Exterior Angle Sum Theorem.
- Question 19 is similar to Example 3.
- Questions 20-27 are similar to Examples 9 and 10.

Determine $m\angle 1$.

1.



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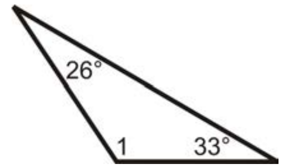

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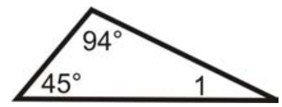

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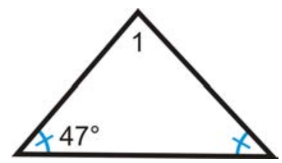

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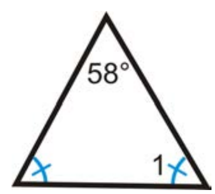

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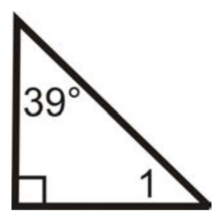

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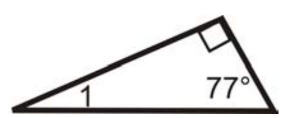

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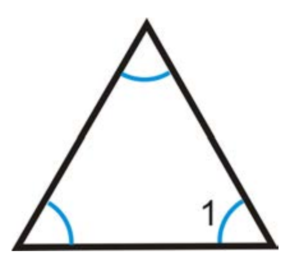

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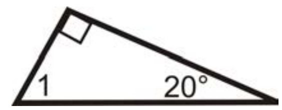

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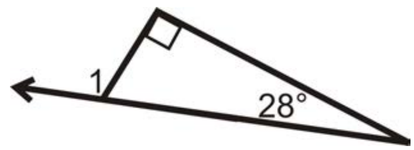

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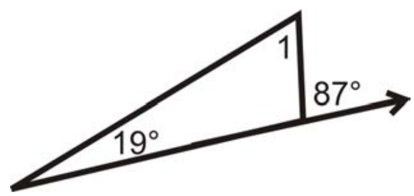

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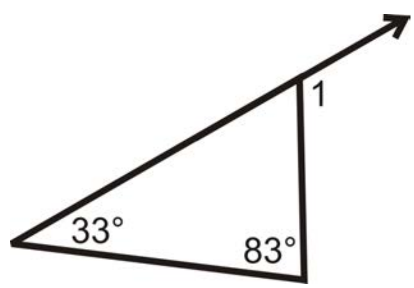

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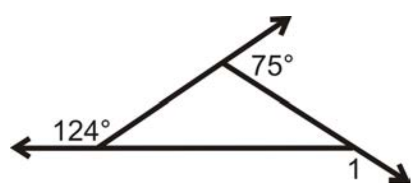
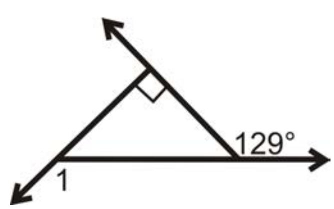
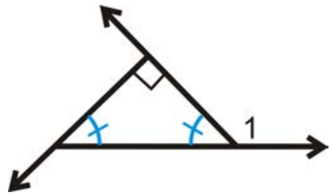


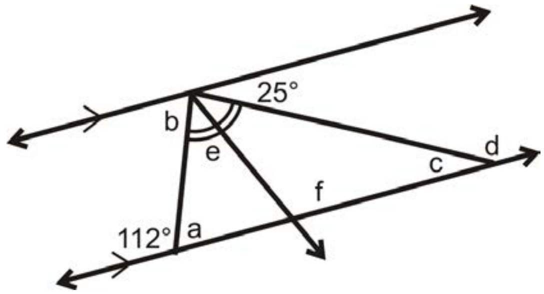
14.



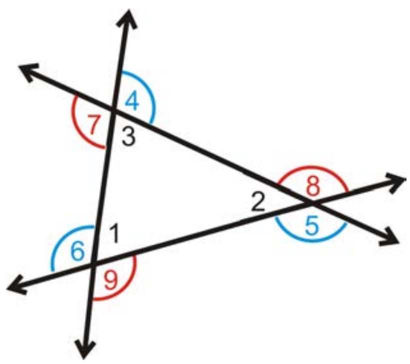
15.



16. Find the lettered angles, $a - f$, in the picture to the below. Note that the two lines are parallel.



17. Using the triangle at the right, both sets of exterior angles are drawn.



a. What is $m\angle 1 + m\angle 2 + m\angle 3$?
b. What is $m\angle 4 + m\angle 5 + m\angle 6$?
c. What is $m\angle 7 + m\angle 8 + m\angle 9$?
d. List all pairs of congruent angles.

18. Fill in the blanks in the proof below. Given: The triangle to the right with interior angles and exterior angles. Prove: $m\angle 4 + m\angle 5 + m\angle 6 = 360°$

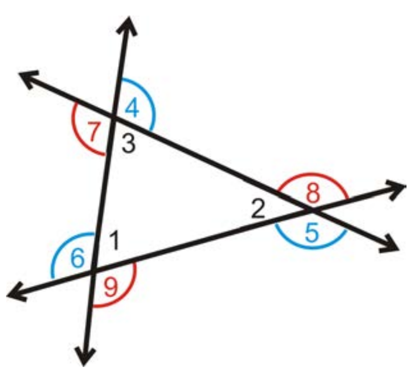


TABLE 4.3:

Statement	***Reason***
1. Triangle with interior and exterior angles.	Given

TABLE 4.3: (continued)

Statement	Reason
2. $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$	
3. $\angle 3$ and $\angle 4$ are a linear pair, $\angle 2$ and $\angle 5$ are a linear pair, and $\angle 1$ and $\angle 6$ are a linear pair	
4.	Linear Pair Postulate (do all 3)
5. $m\angle 1 + m\angle 6 = 180^\circ$	
m$\angle 2 + m\angle 5 = 180^\circ$	
m$\angle 3 + m\angle 4 = 180^\circ$	
6. $m\angle 1 + m\angle 6 + m\angle 2 + m\angle 5 + m\angle 3 + m\angle 4 = 540^\circ$	
7. $m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$	

19. Fill in the blanks in the proof below. Given: $\triangle ABC$ with right angle B. Prove: $\angle A$ and $\angle C$ are complementary.

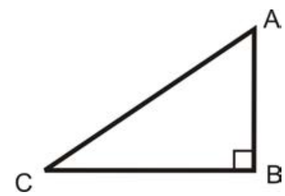
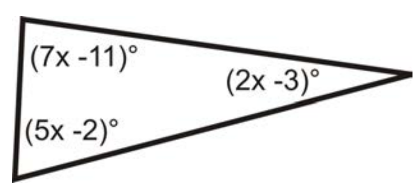


TABLE 4.4:

Statement	Reason
1. $\triangle ABC$ with right angle B.	Given
2.	Definition of a right angle
3. $m\angle A + m\angle B + m\angle C = 180^\circ$	
4. $m\angle A + 90^\circ + m\angle C = 180^\circ$	
5.	
6. $\angle A$ and $\angle C$ are complementary	

Algebra Connection Solve for x and find the measure of each angle.

20.



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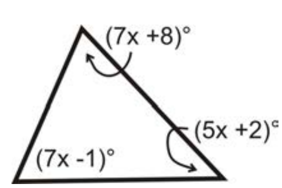

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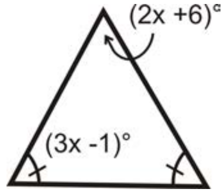

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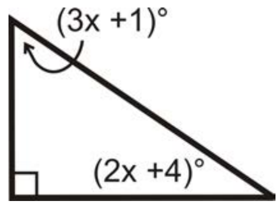

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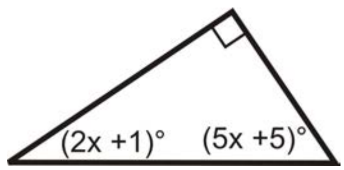

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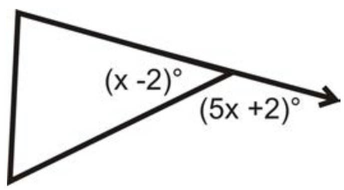

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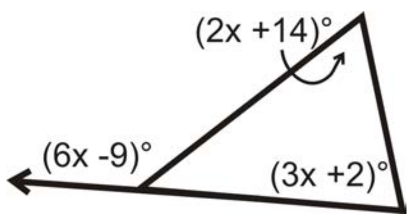
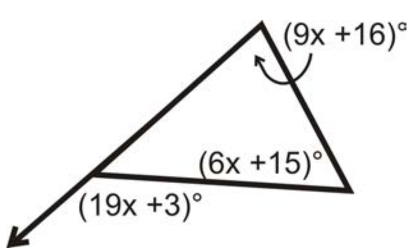


27.



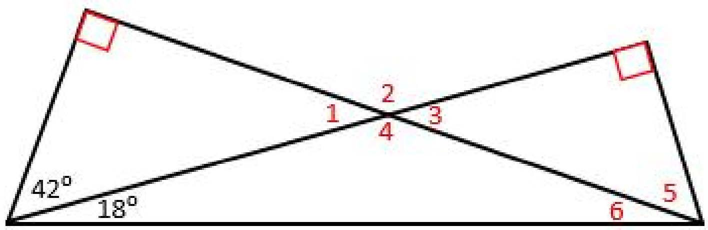


FIGURE 4.1

ANGLE RELATIONSHIPS Find the measure of the numbered angle.

28. $\angle 1$

29. $\angle 2$

30. $\angle 3$

31. $\angle 4$

32. $\angle 5$

33. $\angle 6$

34. In ΔABC, $\angle A \cong \angle C$ and the measure of $\angle B$is half of the measure of $\angle C$. Find the measure of each angle.

35. In ΔDEF, $m\angle E = 2(m\angle F)$ *and* $m\angle D = m\angle E + 30^\circ$. *Find the measure of each angle.*

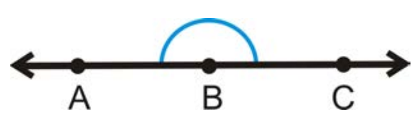
Review and Reflect

36. If the sum of the two remote interior angles is less than 90°, how would you classify the triangle?

37. If two exterior angles of a triangle are given, how can you determine the measures of each interior angle.

Warm-Up Answers

1. acute isosceles
2. obtuse scalene
3. right scalene
4. B is the vertex, 180°,



4.2 Congruent Figures

TEKS G(1)A, G(5)A, G(6)C

Learning Objectives

- Define congruent triangles and use congruence statements
- Understand the Third Angle Theorem

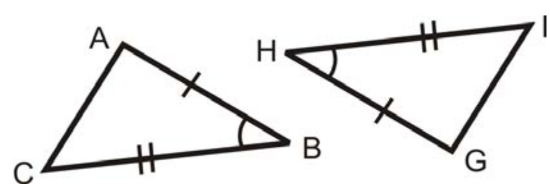
Vocabulary

- congruent triangles

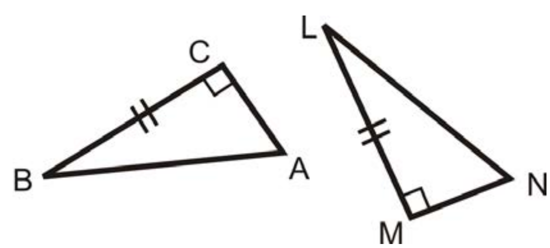
Warm-Up

What part of each pair of triangles are congruent? Write out each congruence statement for the marked congruent sides and angles.

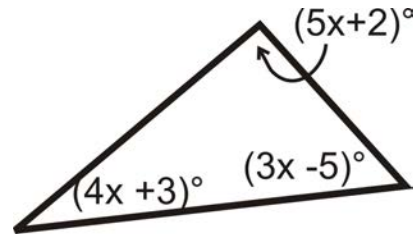
1.



2.



3. Determine the measure of x.

a. 

b. What is the measure of each angle?
c. What type of triangle is this?

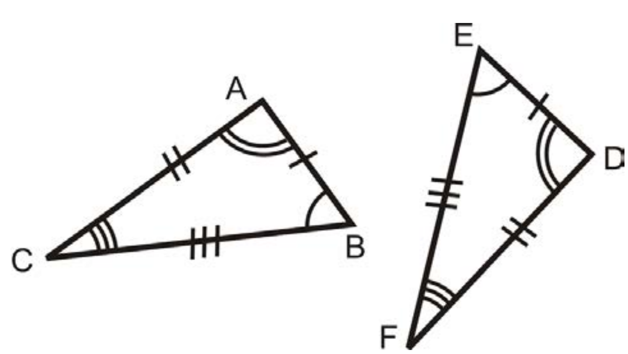
Know What? Quilt patterns are very geometrical. The pattern below is made up of several congruent figures. In order for these patterns to come together, the quilter rotates and flips each block (in this case, a large triangle, smaller triangle, and a smaller square) to get new patterns and arrangements.

How many different sets of colored congruent triangles are there? How many triangles are in each set? How do you know these triangles are congruent?

Congruent Triangles

Two figures are congruent if they have exactly the same size and shape.

Congruent Triangles: Two triangles are congruent if the three corresponding angles and three corresponding sides are also congruent.

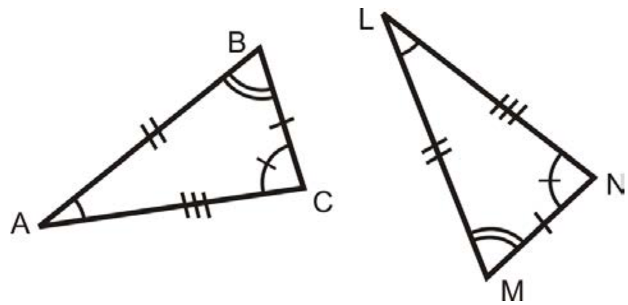


$\triangle ABC$ and $\triangle DEF$ are congruent because

$$\overline{AB} \cong \overline{DE} \qquad \angle A \cong \angle D$$
$$\overline{BC} \cong \overline{EF} \text{ and } \angle B \cong \angle E$$
$$\overline{AC} \cong \overline{DF} \qquad \angle C \cong \angle F$$

When referring to corresponding congruent parts of congruent triangles, it is called **C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent, or **CPCTC**.

Example 1: Are the two triangles below congruent?



Solution: To determine if the triangles are congruent, match up sides with the same number of tic marks: $\overline{BC} \cong \overline{MN}$, $\overline{AB} \cong \overline{LM}$, $\overline{AC} \cong \overline{LN}$.

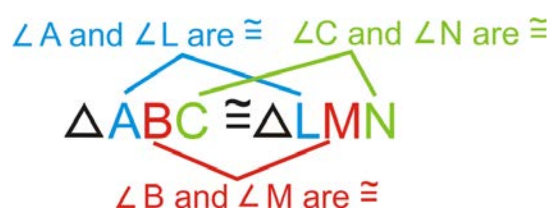
Next match up the angles with the same markings:

$\angle A \cong \angle L$, $\angle B \cong \angle M$, and $\angle C \cong \angle N$.

Lastly, we need to make sure these are *corresponding* parts. To do this, check to see if the congruent angles are opposite congruent sides. Here, $\angle A$ is opposite $\overline{BC}$ and $\angle L$ is opposite $\overline{MN}$. Because $\angle A \cong \angle L$ and $\overline{BC} \cong \overline{MN}$, they are corresponding. Doing this check for the other sides and angles, we see that everything matches up and the two triangles are congruent.

Creating Congruence Statements

In Example 1, we determined that $\triangle ABC$ and $\triangle LMN$ are congruent. When stating that two triangles are congruent, the corresponding parts must be written in the same order. Using Example 1, we would have:



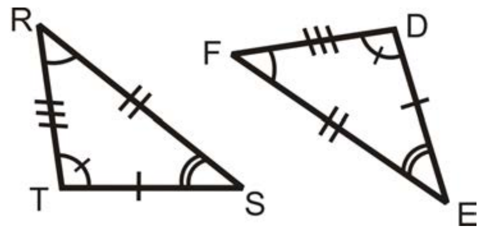
Notice that the congruent sides also line up within the congruence statement.

$\overline{AB} \cong \overline{LM}$, $\overline{BC} \cong \overline{MN}$, $\overline{AC} \cong \overline{LN}$

We can also write this congruence statement five other ways, as long as the congruent angles match up. For example, we can also write $\triangle ABC \cong \triangle LMN$ as:

$\triangle ACB \cong \triangle LNM$	$\triangle BCA \cong \triangle MNL$	$\triangle BAC \cong \triangle MLN$
$\triangle CBA \cong \triangle NML$	$\triangle CAB \cong \triangle NLM$	

Example 2: Write a congruence statement for the two triangles below.



Solution: Line up the corresponding angles in the triangles:

$\angle R \cong \angle F$, $\angle S \cong \angle E$, and $\angle T \cong \angle D$.

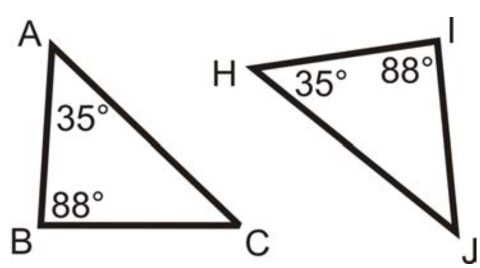
$\triangle RST \cong \angle FED$

Example 3: If $\triangle CAT \cong \triangle DOG$, what else do you know?

Solution: From this congruence statement, we know three pairs of angles and three pairs of sides are congruent.

Third Angle Theorem

Example 4: Find $m\angle C$ and $m\angle J$.



Solution: The sum of the angles in a triangle is 180°.

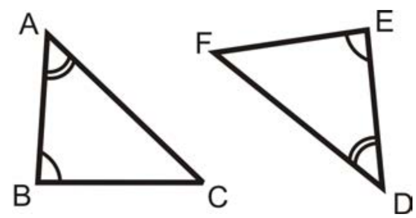
$$\begin{aligned} \triangle ABC: \ 35^\circ + 88^\circ + m\angle C &= 180^\circ \\ m\angle C &= 57^\circ \\ \triangle HIJ: \ 35^\circ + 88^\circ + m\angle J &= 180^\circ \\ m\angle J &= 57^\circ \end{aligned}$$

Notice we were given $m\angle A = m\angle H$ and $m\angle B = m\angle I$ and we found out $m\angle C = m\angle J$. This can be generalized into the Third Angle Theorem.

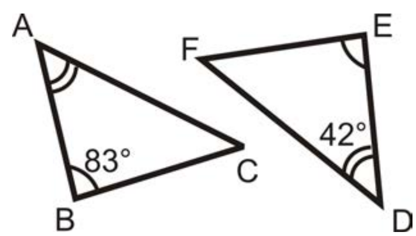
Third Angle Theorem

If two angles in one triangle are congruent to two angles in another triangle, then the third pair of angles must also be congruent.

If $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then by the Third Angle Theorem $\angle C \cong \angle F$.



Example 5: Determine the measure of the missing angles.



Solution: From the Third Angle Theorem, we know $\angle C \cong \angle F$.

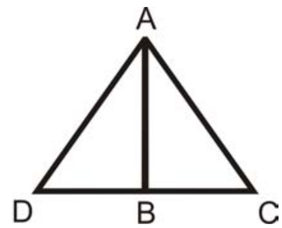
$$\begin{aligned} m\angle A + m\angle B + m\angle C &= 180^\circ \\ m\angle D + m\angle B + m\angle C &= 180^\circ \\ 42^\circ + 83^\circ + m\angle C &= 180^\circ \\ m\angle C &= 55^\circ = m\angle F \end{aligned}$$

Congruence Properties Recall the Properties of Congruence from Chapter 2. They will be very useful in the upcoming sections.

Reflexive Property of Congruence:	$\overline{AB} \cong \overline{AB}$ or $\triangle ABC \cong \triangle ABC$
Symmetric Property of Congruence:	$\angle EFG \cong \angle XYZ$ and $\angle XYZ \cong \angle EFG$
	$\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle ABC$
Transitive Property of Congruence:	$\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$ then
	$\triangle ABC \cong \triangle GHI$

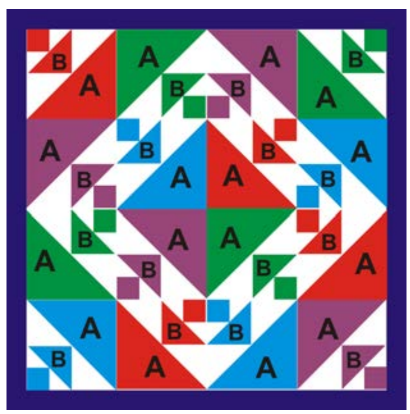
These three properties will be very important when you begin to prove that two triangles are congruent.

Example 6: In order to say that $\triangle ABD \cong \triangle ABC$, you must show the three corresponding angles and sides are congruent. Which pair of sides is congruent by the Reflexive Property?



Solution: The side $\overline{AB}$ is shared by both triangles. In a geometric proof, $\overline{AB} \cong \overline{AB}$ by the Reflexive Property. The Reflexive Property is steadily used when 2 figures share the same side as in ΔABC and ΔABD both share $\overline{AB}$.

Know What? Revisited The 16 "*A*" triangles are congruent. The 16 "*B*" triangles are also congruent. The quilt pattern is made from dividing up the entire square into smaller squares. Both the "*A*" and "*B*" triangles are right triangles.

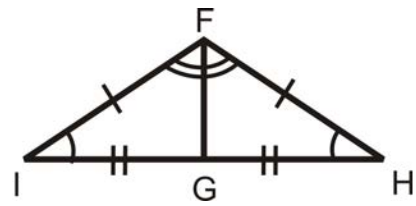


Review Questions

- Questions 1 and 2 are similar to Example 3.
- Questions 3-12 are a review and use the definitions and theorems explained in this section.
- Questions 13-17 are similar to Example 1 and 2.
- Question 18 the definitions and theorems explained in this section.
- Questions 19-22 are similar to Examples 4 and 5.
- Question 23 is a proof of the Third Angle Theorem.
- Questions 24-28 are similar to Example 6.
- Questions 29 and 30 are investigations using congruent triangles, a ruler and a protractor.

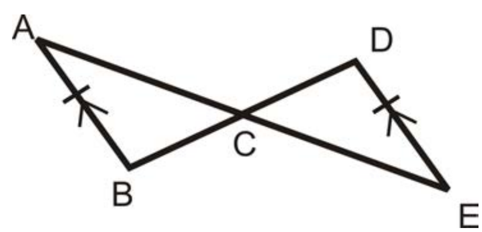
1. If $\triangle RAT \cong \triangle UGH$, name the 3 pairs of corresponding sides and 3 pairs of corresponding angles that are congruent.
2. If $\triangle BIG \cong \triangle TOP$, name the 3 pairs of corresponding sides and 3 pairs of corresponding angles that are congruent.

For questions 3-7, use the picture to the below.



3. What theorem tells us that $\angle FGH \cong \angle FGI$?
4. What is $m\angle FGI$ and $m\angle FGH$? How do you know?
5. What property tells us that the third side of each triangle is congruent?
6. How does $\overline{FG}$ relate to $\angle IFH$?
7. Write the congruence statement for these two triangles.

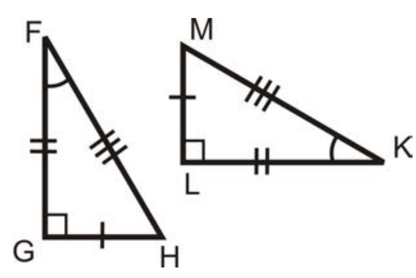
For questions 8-12, use the picture to the right.



8. $\overline{AB} || \overline{DE}$, what angles are congruent? How do you know?
9. Why is $\angle ACB \cong \angle ECD$? It is not the same reason as #8.
10. Are the two triangles congruent with the information you currently have? Why or why not?
11. If you are told that C is the midpoint of $\overline{AE}$ and $\overline{BD}$, what segments are congruent?
12. Write a congruence statement.

For questions 13-16, determine if the triangles are congruent. If they are, write the congruence statement.

13.



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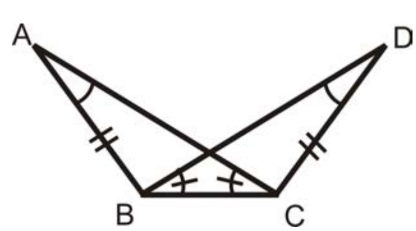

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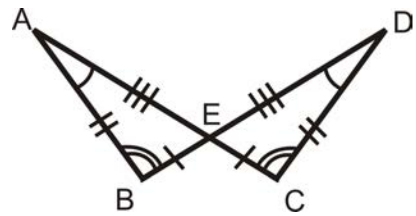

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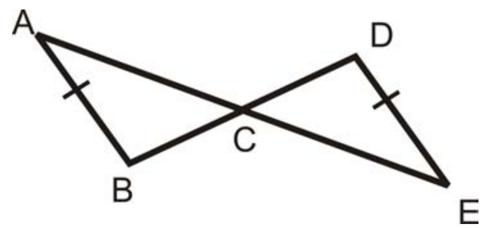
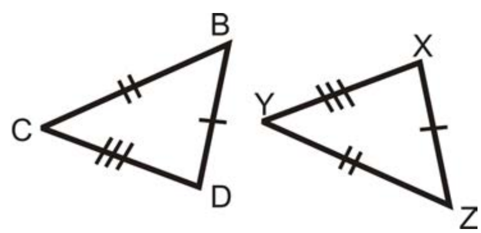


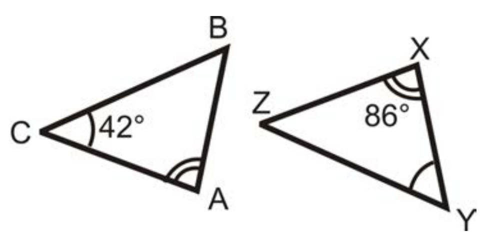
17. Suppose the two triangles below are congruent. Write a congruence statement for these triangles.



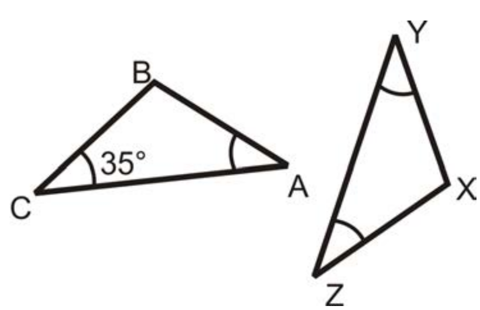
18. Explain how we know that if the two triangles are congruent, then $\angle B \cong \angle Z$.

For questions 19-22, determine the measure of all the angles in the each triangle.

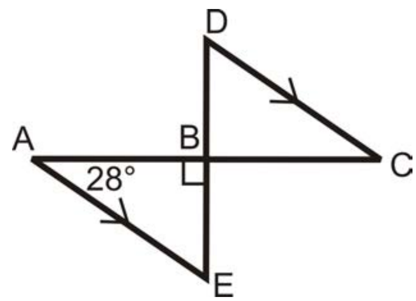
19.



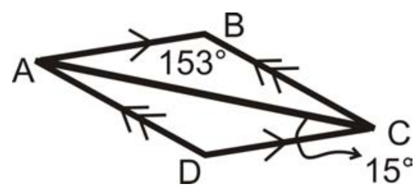
20.



21.



22.



23. Fill in the blanks in the Third Angle Theorem proof below. Given: $\angle A \cong \angle D,\ \angle B \cong \angle E$Prove: $\angle C \cong \angle F$

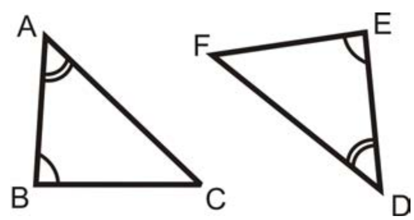


TABLE 4.5:

Statement	***Reason***
1. $\angle A \cong \angle D,\ \angle B \cong \angle E$	
2.	$\cong$ angles have = measures
3. $m\angle A + m\angle B + m\angle C = 180^\circ$ m$\angle D + m\angle E + m\angle F = 180^\circ$	
4.	Substitution Property of Equality
5.	Substitution Property of Equality
6. $m\angle C = m\angle F$	
7. $\angle C \cong \angle F$	

For each of the following questions, determine if the Reflexive, Symmetric or Transitive Properties of Congruence is used.

24. $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$
25. $\overline{AB} \cong \overline{AB}$
26. $\triangle XYZ \cong \triangle LMN$ and $\triangle LMN \cong \triangle XYZ$
27. $\triangle ABC \cong \triangle BAC$
28. What type of triangle is $\triangle ABC$ in #27? How do you know?

ALGEBRA Find the values of *x* and *y*.

29.

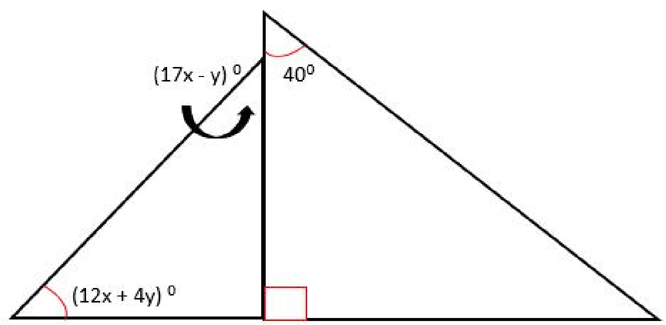


FIGURE 4.2

30.

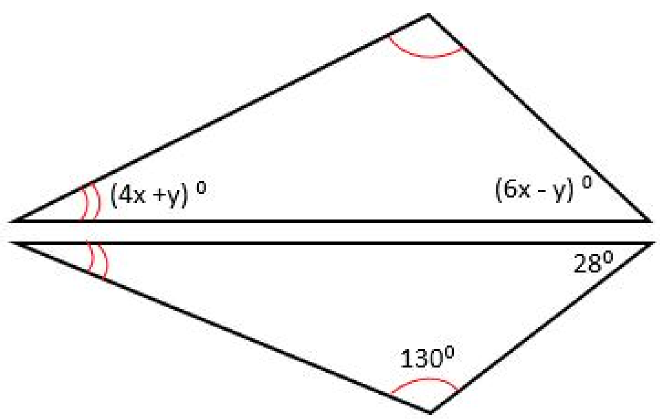


FIGURE 4.3

Review and Reflect

31. Name two real world objects that are congruent and explain how.
32. Why is the order of naming the vertices of congruent figures important?

Warm-Up Answers

1. $\angle B \cong \angle H, \overline{AB} \cong \overline{GH}, \overline{BC} \cong \overline{HI}$
2. $\angle C \cong \angle M, \overline{BC} \cong \overline{LM}$

3. The angles add up to 180°
   a. $$\begin{aligned}(5x+2)^\circ+(4x+3)^\circ+(3x-5)^\circ &= 180^\circ\\ 12x &= 180^\circ\\ x &= 15^\circ\end{aligned}$$
   b. $77^\circ, 63^\circ, 40^\circ$
   c. acute scalene

4.3 Triangle Congruence using SSS and SAS

TEKS G(1)D, G(4)C, G(5)A, G(5)C, G(6)B

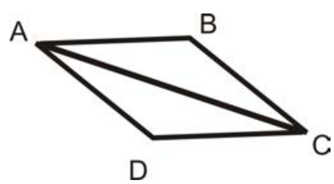
Learning Objectives

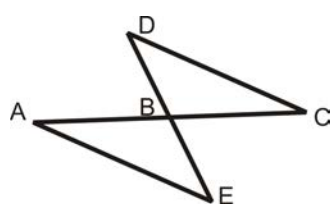
- Use the distance formula to analyze triangle on the x-y plane
- Apply the SSS and SAS Postulate to show two triangles are congruent

Vocabulary

- included angle

Warm-Up

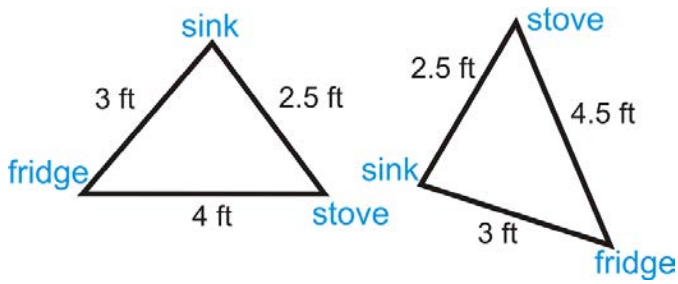
1. Use the distance formula, $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$to find the distance between the two points.
   a. (-1, 5) and (4, 12)
   b. (-6, -15) and (-3, 8)
2. 

   a. If we know that $\overline{AB}||\overline{CD}$, $\overline{AD}||\overline{BC}$, what angles are congruent? By which theorem?
   b. Which side is congruent by the Reflexive Property?
   c. Is this enough to say $\triangle ADC \cong \triangle CBA$?
3. 

   a. If we know that B is the midpoint of $\overline{AC}$ and $\overline{DE}$, what segments are congruent?
   b. Are there any angles that are congruent by looking at the picture? Which ones and why?
   c. Is this enough to say $\triangle ABE \cong \triangle CBD$?

Know What?

The “ideal” measurements in a kitchen from the sink, refrigerator and oven are as close to an equilateral triangle as possible. Your parents are remodeling theirs to be as close to this as possible and the measurements are in the picture below. Your neighbor's kitchen has the measurements on the right. Are the two triangles congruent? Why or why not?



SSS Postulate of Triangle Congruence

Consider the question: If I have three lengths: 3 in, 4 in, and 5 in, can I construct more than one triangle?

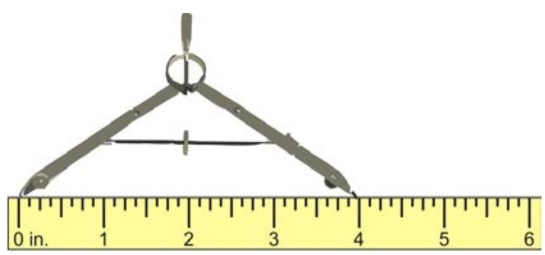
Investigation 4-2: Constructing a Triangle Given Three Sides

Tools Needed: compass, pencil, ruler, and paper

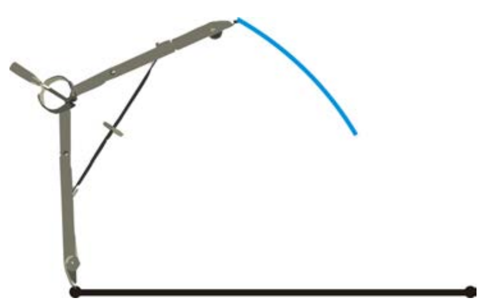
1. Draw the longest side (5 in) horizontally, halfway down the page.

The drawings in this investigation are to scale.

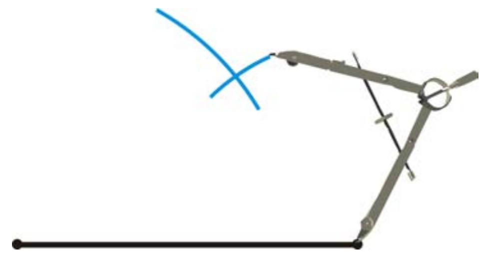
2. Take the compass and, using the ruler, widen the compass to measure 4 in, the second side.



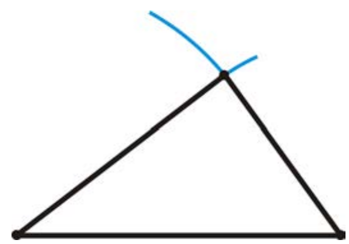
3. Using the measurement from Step 2, place the pointer of the compass on the left endpoint of the side drawn in Step 1. Draw an arc mark above the line segment.



4. Repeat Step 2 with the third measurement, 3 in. Then, like Step 3, place the pointer of the compass on the ***right*** endpoint of the side drawn in Step 1. Draw an arc mark above the line segment. Make sure it intersects the arc mark drawn in Step 3.



5. Draw lines from each endpoint to the arc intersections. These segments are the other two sides of the triangle.

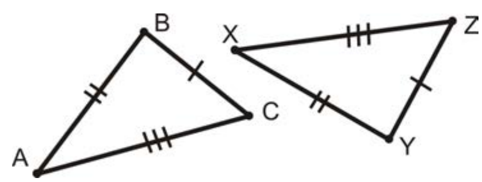


An animation of this construction can be found at: http://www.mathsisfun.com/geometry/construct-ruler-compass-1.html

Can another triangle be drawn with these measurements that look different? NO. ***Only one triangle can be created from any given three lengths. You can rotate, flip, or move this triangle but it will still be the same size.***

Side-Side-Side (SSS) Triangle Congruence Postulate

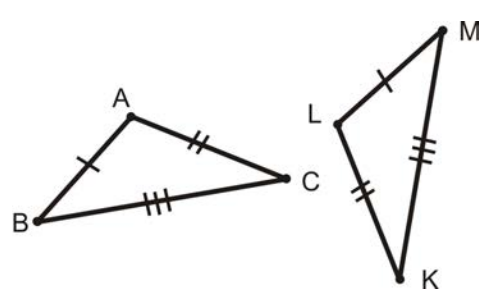
If 3 sides in one triangle are congruent to 3 sides in another triangle, then the triangles are congruent.



If $\overline{BC} \cong \overline{YZ}$, $\overline{AB} \cong \overline{XY}$, and $\overline{AC} \cong \overline{XZ}$, then $\triangle ABC \cong \triangle XYZ$.

The SSS Postulate is a shortcut. Before, you had to show **3 sides and 3 angles** in one triangle were congruent to **3 sides and 3 angles** in another triangle. Now you only have to show **3 sides** in one triangle are congruent to **3 sides** in another.

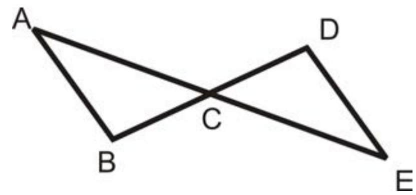
Example 1: Write a triangle congruence statement based on the picture below:



Solution: From the tic marks, we know $\overline{AB} \cong \overline{LM}$, $\overline{AC} \cong \overline{LK}$, $\overline{BC} \cong \overline{MK}$. From the SSS Postulate, the triangles are congruent. Lining up the corresponding sides, we have $\triangle ABC \cong \triangle LMK$.

Don't forget ORDER MATTERS when writing congruence statements. Line up the sides with the same number of tic marks.

Example 2: Write a two-column proof to show that the two triangles are congruent.



Given: $\overline{AB} \cong \overline{DE}$

C is the midpoint of $\overline{AE}$ and $\overline{DB}$.

Prove: $\triangle ACB \cong \triangle ECD$

Solution:

TABLE 4.6:

Statement	***Reason***
1. $\overline{AB} \cong \overline{DE}$	Given
C is the midpoint of $\overline{AE}$ and $\overline{DB}$	
2. $\overline{AC} \cong \overline{CE}$, $\overline{BC} \cong \overline{CD}$	Definition of a midpoint
3. $\triangle ACB \cong \triangle ECD$	SSS Postulate

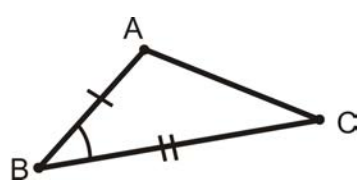
Prove Move: You must clearly state the three sets of sides are congruent BEFORE stating the triangles are congruent.

Prove Move: Mark the picture with the information you are given as well as information that you see in the picture (vertical angles, information from parallel lines, midpoints, angle bisectors, right angles). This information may be used in a proof.

SAS Triangle Congruence Postulate

SAS refers to Side-Angle-Side. The placement of the word Angle is important because it indicates that the angle you are given is *between* the two sides.

Included Angle: When an angle is between two given sides of a polygon.



$\angle B$ would be the included angle for sides $\overline{AB}$ and $\overline{BC}$.

Consider the question: If I have two sides of length 2 in and 5 in and the angle between them is 45°, can I construct one triangle?

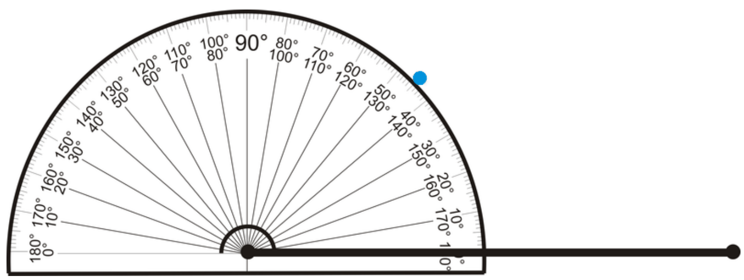
Investigation 4-3: Constructing a Triangle Given Two Sides and Included Angle

Tools Needed: protractor, pencil, ruler, and paper

1. Draw the longest side (5 in) horizontally, halfway down the page.

The drawings in this investigation are to scale.

2. At the left endpoint of your line segment, use the protractor to measure a 45° angle. Mark this measurement.



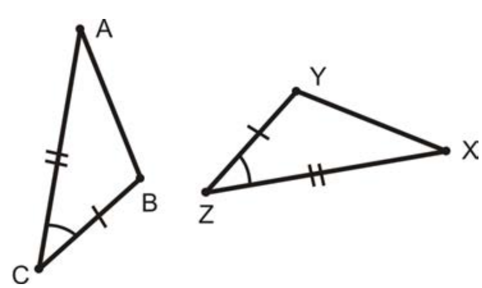
3. Connect your mark from Step 2 with the left endpoint. Make your line 2 in long, the length of the second side.

4. Connect the two endpoints to draw the third side.

Can you draw another triangle, with these measurements that looks different? NO. ***Only one triangle can be created from any two lengths and the INCLUDED angle.***

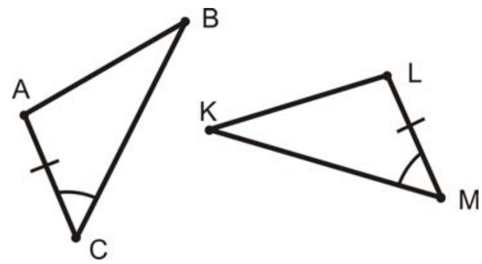
Side-Angle-Side (SAS) Triangle Congruence Postulate

If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.



If $\overline{AC} \cong \overline{XZ}$, $\overline{BC} \cong \overline{YZ}$, and $\angle C \cong \angle Z$, then $\triangle ABC \cong \triangle XYZ$.

Example 3: Which additional piece of information do you need to show that these two triangles are congruent using the SAS Postulate?



a) $\angle ABC \cong \angle LKM$

b) $\overline{AB} \cong \overline{LK}$

c) $\overline{BC} \cong \overline{KM}$

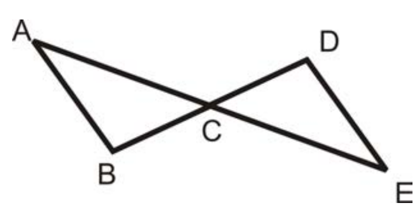
d) $\angle BAC \cong \angle KLM$

Solution: For the SAS Postulate, you need the side on the other side of the angle. In $\triangle ABC$, that is $\overline{BC}$ and in $\triangle LKM$ that is $\overline{KM}$. The answer is c.

Example 4: Write a two-column proof to show that the two triangles are congruent.

Given: C is the midpoint of $\overline{AE}$ and $\overline{DB}$

Prove: $\triangle ACB \cong \triangle ECD$



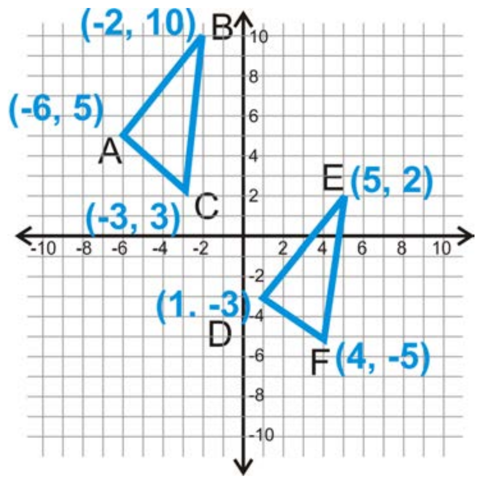
Solution:

TABLE 4.7:

Statement	***Reason***
1. C is the midpoint of $\overline{AE}$ and $\overline{DB}$	Given
2. $\overline{AC} \cong \overline{CE}$, $\overline{BC} \cong \overline{CD}$	Definition of a midpoint
3. $\angle ACB \cong \angle DCE$	Vertical Angles Postulate
4. $\triangle ACB \cong \triangle ECD$	SAS Postulate

SSS in the Coordinate Plane

The only way we will show two triangles are congruent in an $x-y$ plane is using SSS. To do this, you need to use the distance formula or Pythagorean Theorem.



Example 5: Determine if the triangles above are congruent by finding the distances of all the line segments from both triangles.

Solution: Begin with $\triangle ABC$ and its sides. Find the distances with the distance formula.

$$\begin{aligned} AB &= \sqrt{(-6-(-2))^2+(5-10)^2} \\ &= \sqrt{(-4)^2+(-5)^2} \\ &= \sqrt{16+25} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-2-(-3))^2+(10-3)^2} \\ &= \sqrt{(1)^2+(7)^2} \\ &= \sqrt{1+49} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-6-(-3))^2+(5-3)^2} \\ &= \sqrt{(-3)^2+(2)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \end{aligned}$$

Now, find the distances of all the sides in $\triangle DEF$.

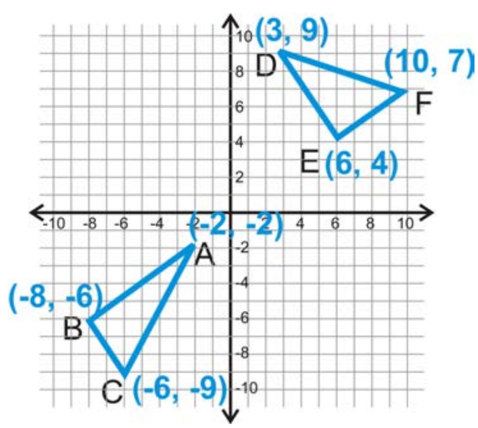
$$\begin{aligned} DE &= \sqrt{(1-5)^2+(-3-2)^2} \\ &= \sqrt{(-4)^2+(-5)^2} \\ &= \sqrt{16+25} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} EF &= \sqrt{(5-4)^2+(2-(-5))^2} \\ &= \sqrt{(1)^2+(7)^2} \\ &= \sqrt{1+49} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} DF &= \sqrt{(1-4)^2+(-3-(-5))^2} \\ &= \sqrt{(-3)^2+(2)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \end{aligned}$$

$AB = DE$, $BC = EF$, and $AC = DF$, so two triangles are congruent by SSS.

Example 6: Determine if the two triangles below are congruent.



Solution: Start with $\triangle ABC$. We are using the distance formula but you may also use Pythagorean Theorem to find the lengths of the sides of the triangle.

$$\begin{aligned} AB &= \sqrt{(-2-(-8))^2+(-2-(-6))^2} \\ &= \sqrt{(6)^2+(4)^2} \\ &= \sqrt{36+16} \\ &= \sqrt{52} = 2\sqrt{13} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-8-(-6))^2+(-6-(-9))^2} \\ &= \sqrt{(-2)^2+(3)^2} \\ &= \sqrt{4+9} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-2-(-6))^2+(-2-(-9))^2} \\ &= \sqrt{(4)^2+(7)^2} \\ &= \sqrt{16+49} \\ &= \sqrt{65} \end{aligned}$$

Now find the sides of $\triangle DEF$.

$$\begin{aligned} DE &= \sqrt{(3-6)^2+(9-4)^2} \\ &= \sqrt{(-3)^2+(5)^2} \\ &= \sqrt{9+25} \\ &= \sqrt{34} \end{aligned}$$

$$\begin{aligned} EF &= \sqrt{(6-10)^2+(4-7)^2} \\ &= \sqrt{(-4)^2+(-3)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25}=5 \end{aligned}$$

$$\begin{aligned} DF &= \sqrt{(3-10)^2+(9-7)^2} \\ &= \sqrt{(-7)^2+(2)^2} \\ &= \sqrt{49+4} \\ &= \sqrt{53} \end{aligned}$$

Notice that no sides have equal measures, so the triangles are not congruent.

Use the following link to investigate congruence by selecting sides and angles and creating congruent triangles.

http://illuminations.nctm.org/Activity.aspx?id=3504

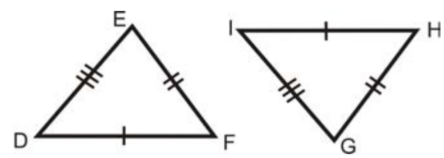
Know What? Revisited From what we have learned in this section, the two triangles are not congruent because the distance from the fridge to the stove in your house is 4 feet and in your neighbor's it is 4.5 ft. The SSS Postulate tells us that all three sides have to be congruent in order for the triangles to be congruent.

Practice Problems

- Questions 1-10 are similar to Example 1.
- Questions 11-16 are similar to Example 3.
- Questions 17-23 are similar to Examples 2 and 4.
- Questions 24-27 are similar to Examples 5 and 6.

Are the pairs of triangles congruent? If so, write the congruence statement and why.

1.



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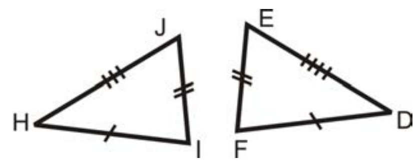

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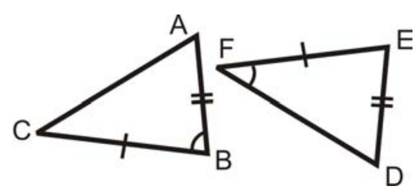

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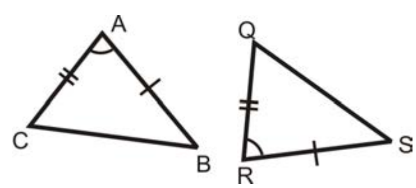

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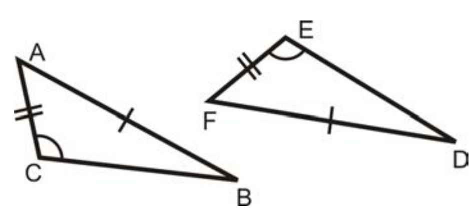

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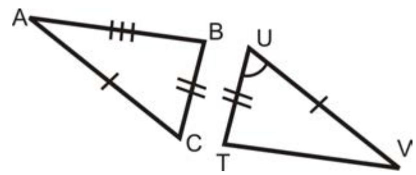

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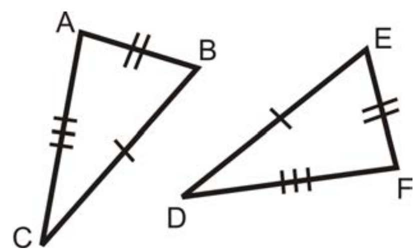
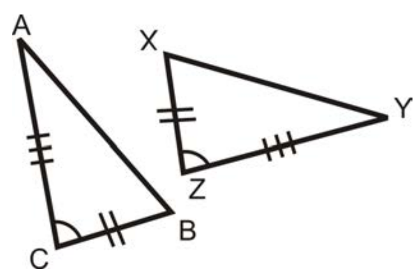


8.



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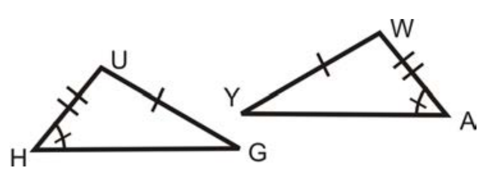

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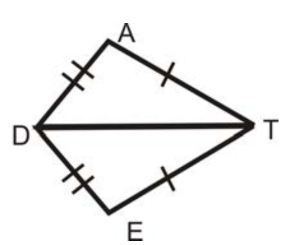
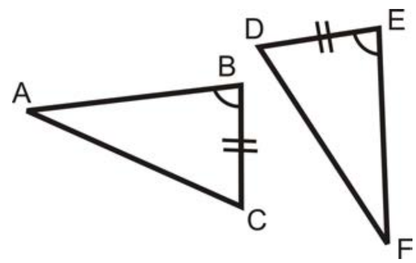
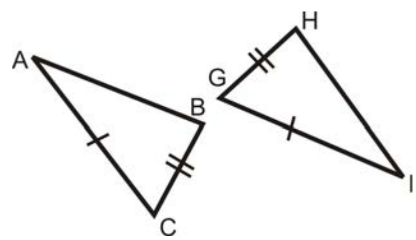


State the additional piece of information needed to show that each pair of triangles is congruent.

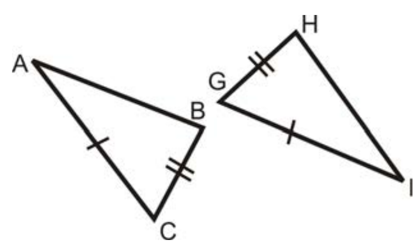
11. Use SAS



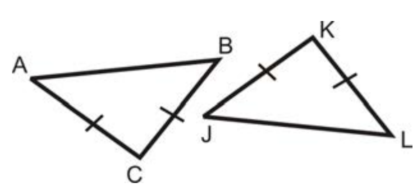
12. Use SSS



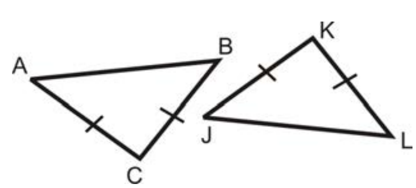
13. Use SAS



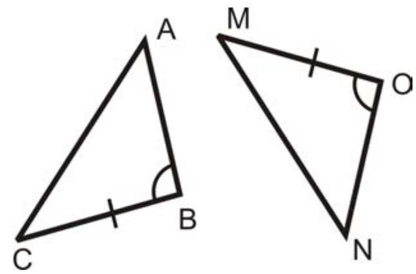
14. Use SAS



15. Use SSS



16. Use SAS



Fill in the blanks in the proofs below.

17. Given: $\overline{AB} \cong \overline{DC}$, $\overline{BE} \cong \overline{CE}$Prove: $\triangle ABE \cong \triangle ACE$

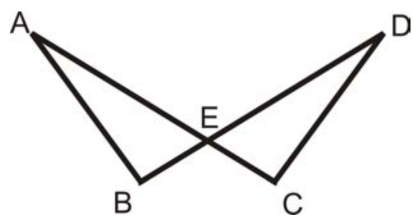


TABLE 4.8:

Statement	***Reason***
1.	1.
2. $\angle AEB \cong \angle DEC$	2.
3. $\triangle ABE \cong \triangle ACE$	3.

18. Given: $\overline{AB} \cong \overline{DC}$, $\overline{AC} \cong \overline{DB}$Prove: $\triangle ABC \cong \triangle DCB$

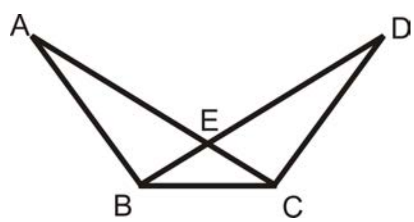


TABLE 4.9:

Statement	***Reason***
1.	1.
2.	2. Reflexive Property of Congruence
3. $\triangle ABC \cong \triangle DCB$	3.

19. Given: B is a midpoint of $\overline{DC}\overline{AB} \perp \overline{DC}$Prove: $\triangle ABD \cong \triangle ABC$

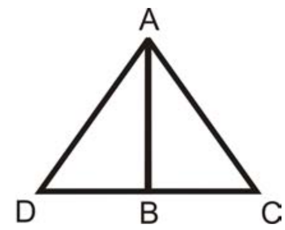


TABLE 4.10:

Statement	Reason
1. B is a midpoint of $\overline{DC}, \overline{AB} \perp \overline{DC}$	1.
2.	2. Definition of a midpoint
3. $\angle ABD$ and $\angle ABC$ are right angles	3.
4.	4. All right angles are $\cong$
5.	5.
6. $\triangle ABD \cong \triangle ABC$	6.

20. Given: $\overline{AB}$ is an angle bisector of $\angle DAC\overline{AD} \cong \overline{AC}$Prove: $\triangle ABD \cong \triangle ABC$

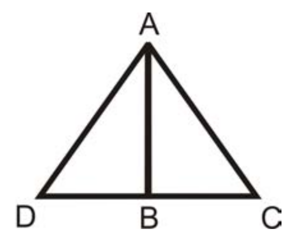


TABLE 4.11:

Statement	Reason
1.	
2. $\angle DAB \cong \angle BAC$	
3.	Reflexive Property of Congruence
4. $\triangle ABD \cong \triangle ABC$	

21. Given: B is the midpoint of $\overline{DC}\overline{AD} \cong \overline{AC}$Prove: $\triangle ABD \cong \triangle ABC$

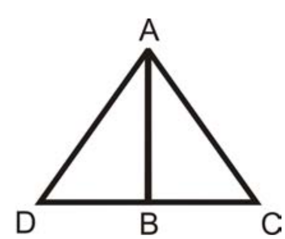


TABLE 4.12:

Statement	Reason
1.	
2.	Definition of a Midpoint
3.	Reflexive Property of Congruence
4. $\triangle ABD \cong \triangle ABC$	

22. Given: B is the midpoint of $\overline{DE}$ and $\overline{AC}\angle ABE$ is a right angle Prove: $\triangle ABE \cong \triangle CBD$

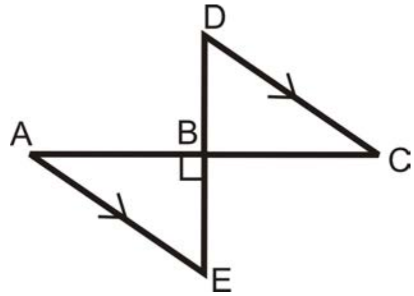


TABLE 4.13:

Statement	Reason
1.	Given
2. $\overline{DB} \cong \overline{BE}$, $\overline{AB} \cong \overline{BC}$	
3.	Definition of a Right Angle
4.	Vertical Angle Theorem
5. $\triangle ABE \cong \triangle CBD$	

23. Given: $\overline{DB}$ is the angle bisector of $\angle ADC\overline{AD} \cong \overline{DC}$Prove: $\triangle ABD \cong \triangle CBD$

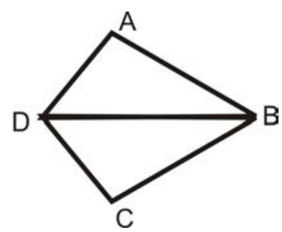
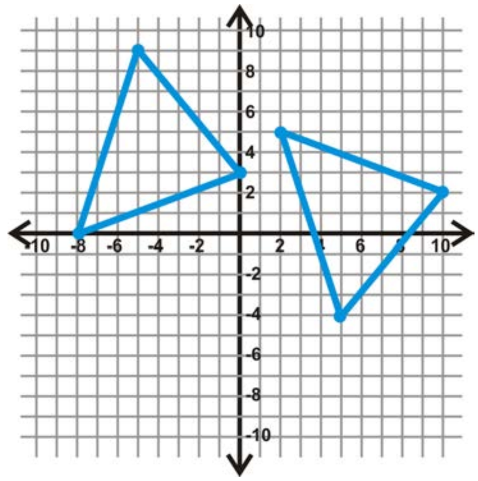


TABLE 4.14:

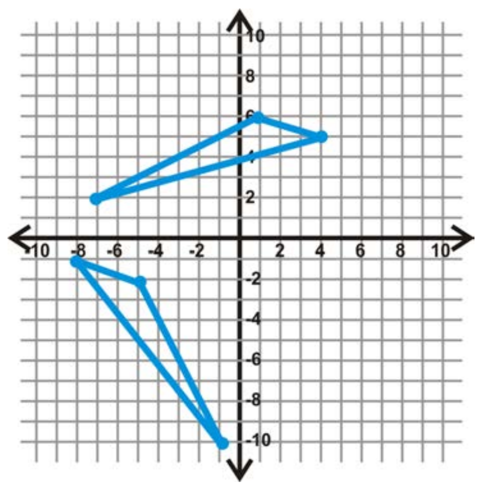
Statement	Reason
1.	
2. $\angle ADB \cong \angle BDC$	
3.	
4. $\triangle ABD \cong \triangle CBD$	

Find the lengths of the sides of each triangle to see if the two triangles are congruent. Leave your answers in simplest radical form

24.



25.



26. $\triangle ABC: A(-1,5), B(-4,2), C(2,-2)$ and $\triangle DEF: D(7,-5), E(4,2), F(8,-9)$
27. $\triangle ABC: A(-8,-3), B(-2,-4), C(-5,-9)$ and $\triangle DEF: D(-7,2), E(-1,3), F(-4,8)$

USING DIAGRAMS Decide whether the figure is stable. *Explain*.

28.

29.

30.

CHALLENGE Find all values of x that make the triangles congruent. *Explain*.

31.

USING ISOSCELES TRIANGLES Suppose $\Delta ABC$and $\Delta DBC$are isoscels triangles with $\overline{AB} \cong \overline{BC}$and $\overline{DB} \cong \overline{BC}$, and $\overline{CB}$bisects $\angle ABD$. Is there enough information to prove that $\Delta ABC \cong \Delta DBC$?

32.

Review and Reflect

33. Compare and contrast the SSS (side-side-side) Postulate and the SAS (side-angle-side) Postulate.

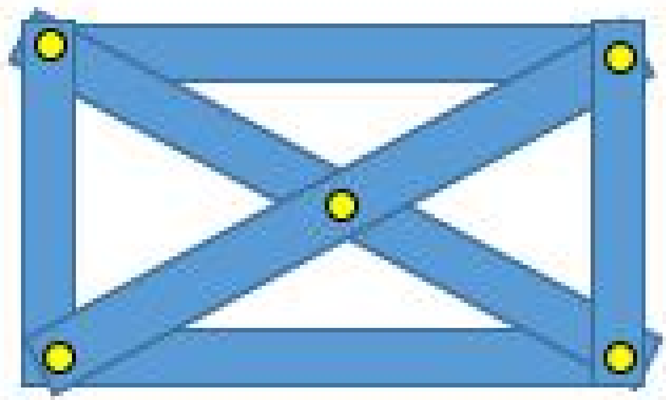


FIGURE 4.4

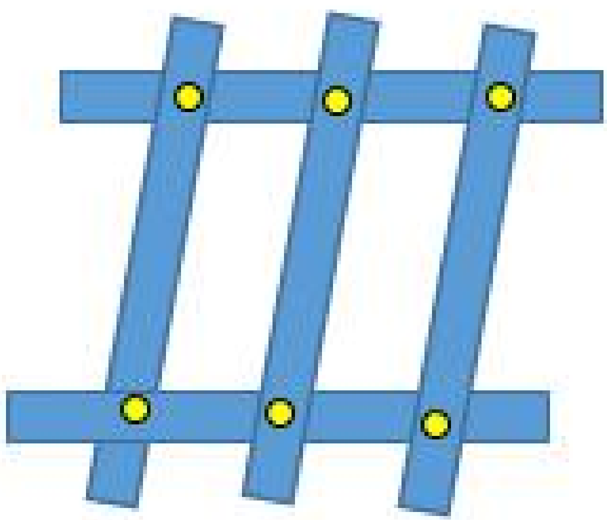


FIGURE 4.5

34. Can two triangles be proven congruent by SSA, where the angle is non-included between the sides? Give a reason or a counter example to disprove your answer.

Warm-Up Answers

1.

a. $\sqrt{74}$
b. $\sqrt{538}$

2.

a. $\angle BAC \cong \angle DCA, \angle DAC \cong \angle BCA$ by the Alternate Interior Angles Theorem.
b. $\overline{AC} \cong \overline{AC}$

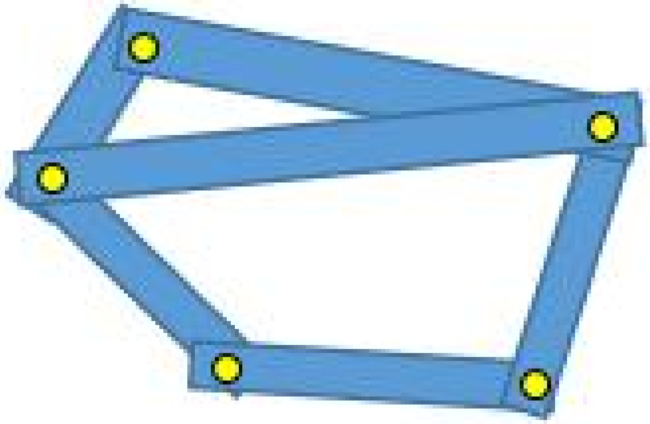


FIGURE 4.6

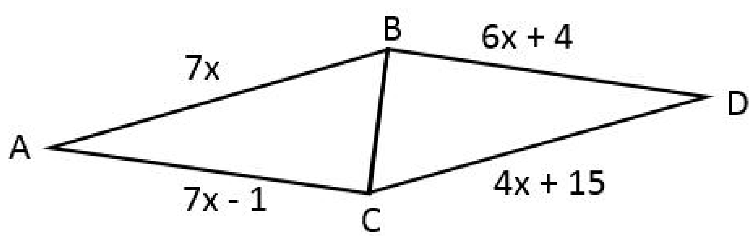


FIGURE 4.7

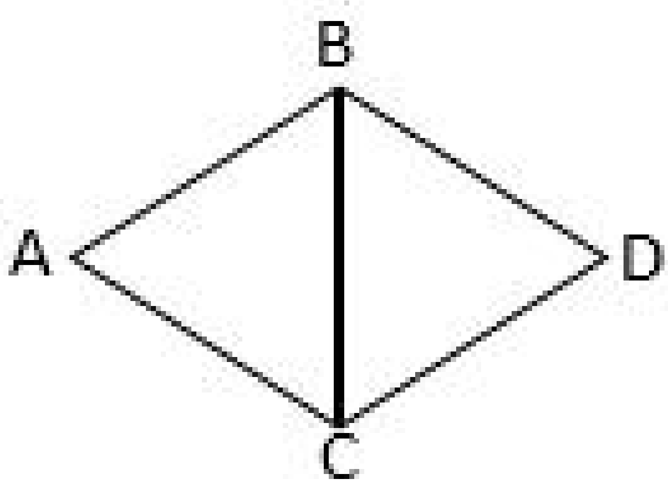


FIGURE 4.8

c. Not yet, this would be ASA.

3.

a. $\overline{DB} \cong \overline{BE}, \overline{AB} \cong \overline{BC}$
b. $\angle DBC \cong \angle ABE$ by the Vertical Angles Theorem.
c. By the end of this section, yes, we will be able to show that these two triangles are congruent by SAS.

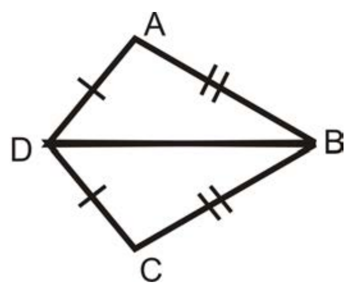
4.4 Triangle Congruence using ASA, AAS, and HL

TEKS G(1)D, G(5)A, G(6)B, G(6)C

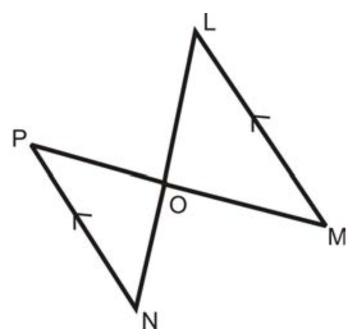
Learning Objectives

- Use and understand the ASA, AAS, and HL Congruence Postulate
- Complete two-column proofs using SSS, SAS, ASA, and AAS.

Warm-Up

1. 

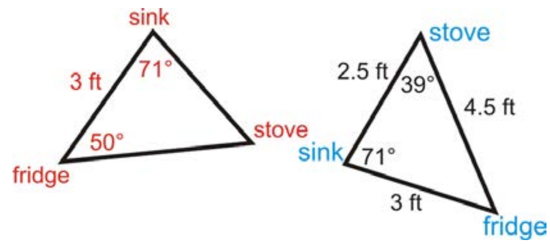
   a. What sides are marked congruent?
   b. Is third side congruent? Why?
   c. Write the congruence statement for the two triangles. Why are they congruent?

2. 

   a. From the parallel lines, what angles are congruent?
   b. How do you know the third angle is congruent?
   c. Are any sides congruent? How do you know?
   d. Are the two triangles congruent? Why or why not?

3. If $\triangle DEF \cong \triangle PQR$, can it be assumed that:
   a. $\angle F \cong \angle R$? Why or why not?
   b. $\overline{EF} \cong \overline{PR}$? Why or why not?

Know What? Your parents changed their minds at the last second about their kitchen layout. Now, the measurements are in the triangle below. Your neighbor's kitchen is in blue on the right. Are the kitchen triangles congruent now?



ASA Congruence

ASA refers to Angle-Side-Angle. The placement of the word Side is important because it indicates that the side that you are given is between the two angles.

Consider the question: If I have two angles that are 45° and 60° and the side between them is 5 in, can I construct only one triangle?

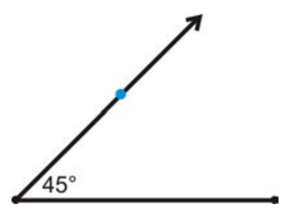
Investigation 4-4: Constructing a Triangle Given Two Angles and Included Side

Tools Needed: protractor, pencil, ruler, and paper

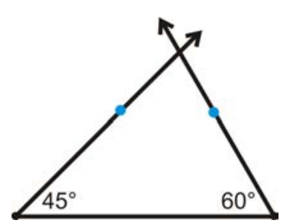
1. Draw the side (5 in) horizontally, about halfway down the page.

The drawings in this investigation are to scale.

2. At the left endpoint of your line segment, use the protractor to measure the 45° angle. Mark this measurement and draw a ray from the left endpoint through the 45° mark.



3. At the right endpoint of your line segment, use the protractor to measure the 60° angle. Mark this measurement and draw a ray from the left endpoint through the 60° mark. Extend this ray so that it crosses through the ray from Step 2.



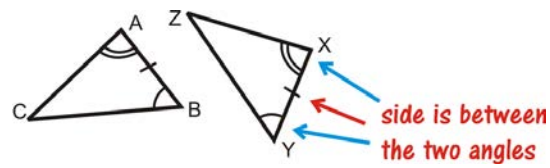
4. Erase the extra parts of the rays from Steps 2 and 3 to leave only the triangle.

Can you draw another triangle, with these measurements that looks different? NO. ***Only one triangle can be created from any given two angle measures and the INCLUDED side.***

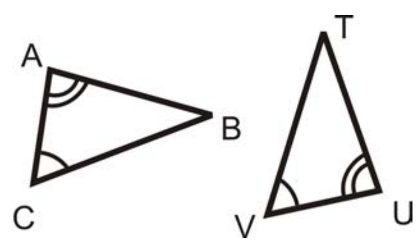
Angle-Side-Angle (ASA) Congruence Postulate

If two angles and the included side in one triangle are congruent to two angles and included side in another triangle, then the two triangles are congruent.

If $\angle A \cong \angle X$, $\angle B \cong \angle Y$, and $\overline{AB} \cong \overline{XY}$, then $\triangle ABC \cong \triangle XYZ$.



Example 1: Which of the following pieces of information do you need to prove that these two triangles are congruent using the ASA Postulate?



a) $\overline{AB} \cong \overline{UT}$

b) $\overline{AC} \cong \overline{UV}$

c) $\overline{BC} \cong \overline{TV}$

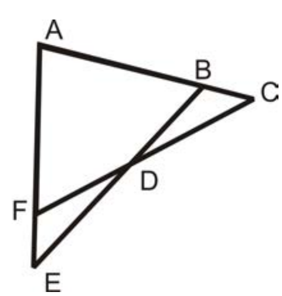
d) $\angle B \cong \angle T$

Solution: For ASA, we need the side between the two given angles, which is $\overline{AC}$ and $\overline{UV}$. The answer is b.

Example 2: Write a 2-column proof.

Given: $\angle C \cong \angle E$, $\overline{AC} \cong \overline{AE}$

Prove: $\triangle ACF \cong \triangle AEB$



Solution:

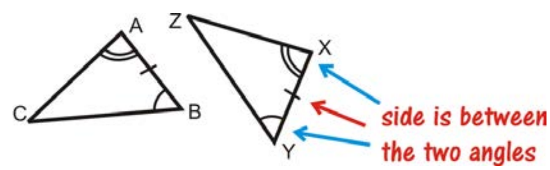
TABLE 4.15:

Statement	Reason
1. $\angle C \cong \angle E$, $\overline{AC} \cong \overline{AE}$	Given
2. $\angle A \cong \angle A$	Reflexive Property of Congruence
3. $\triangle ACF \cong \triangle AEB$	ASA

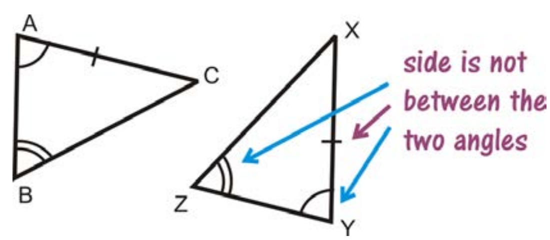
AAS Congruence

A variation on ASA is AAS, which is Angle-Angle-Side. For ASA you need two angles and the side ***between*** them. But, if you know two pairs of angles are congruent, the third pair will also be congruent by the 3^{rd} Angle Theorem. This means you can prove two triangles are congruent when you have any two pairs of ***corresponding*** angles and a pair of sides.

ASA



AAS



Angle-Angle-Side (AAS) Congruence Theorem

If two angles and a non-included side in one triangle are congruent to two angles and the corresponding non-included side in another triangle, then the triangles are congruent.

Proof of AAS Theorem

Given: $\angle A \cong \angle Y$, $\angle B \cong \angle Z$, $\overline{AC} \cong \overline{XY}$

Prove: $\triangle ABC \cong \triangle YZX$

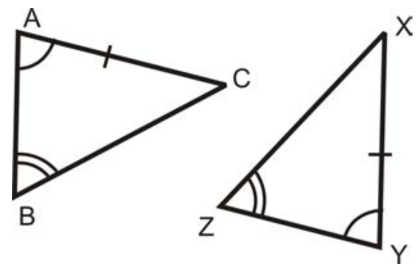


TABLE 4.16:

Statement	**Reason**
1. $\angle A \cong \angle Y,\ \angle B \cong \angle Z,\ \overline{AC} \cong \overline{XY}$	Given
2. $\angle C \cong \angle X$	3^{rd} Angle Theorem
3. $\triangle ABC \cong \triangle YZX$	ASA

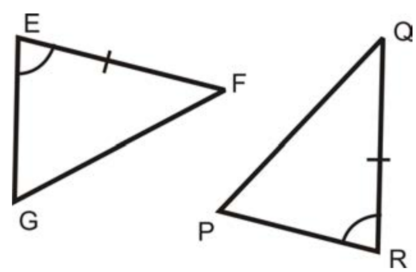
By proving $\triangle ABC \cong \triangle YZX$ with ASA, we have also proved that the AAS Theorem is true.

Example 3: What information do you need to prove that these two triangles are congruent using:

a) ASA?

b) AAS?

c) SAS?



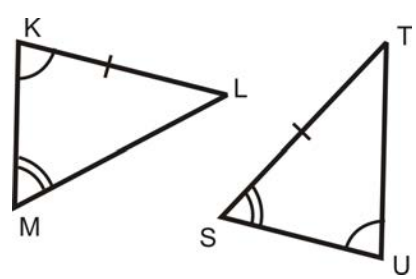
Solution:

a) For ASA, we need the angles on the other side of $\overline{EF}$ and $\overline{QR}$. $\angle F \cong \angle Q$

b) For AAS, we would need the other angle. $\angle G \cong \angle P$

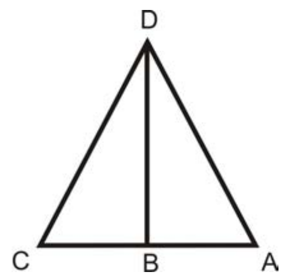
c) For SAS, we need the side on the other side of $\angle E$ and $\angle R$. $\overline{EG} \cong \overline{RP}$

Example 4: Can you prove that the following triangles are congruent? Why or why not?



Solution: We cannot show the triangles are congruent because $\overline{KL}$ and $\overline{ST}$ are *not corresponding*, even though they are congruent. To determine if $\overline{KL}$ and $\overline{ST}$ are corresponding, look at the angles around them, $\angle K$ and $\angle L$ and $\angle S$ and $\angle T$. $\angle K$ has **one** arc and $\angle L$ is unmarked. $\angle S$ has **two** arcs and $\angle T$ is unmarked. In order to use AAS, $\angle S$ needs to be congruent to $\angle K$.

Example 5: Write a 2-column proof.



Given: $\overline{BD}$ is an angle bisector of $\angle CDA$, $\angle C \cong \angle A$

Prove: $\triangle CBD \cong \angle ABD$

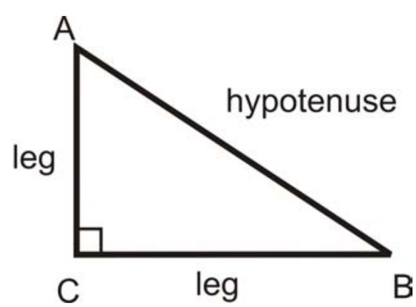
Solution:

TABLE 4.17:

Statement	***Reason***
1. $\overline{BD}$ is an angle bisector of $\angle CDA$, $\angle C \cong \angle A$	Given
2. $\angle CDB \cong \angle ADB$	Definition of an Angle Bisector
3. $\overline{DB} \cong \overline{DB}$	Reflexive Property of Congruence
4. $\triangle CBD \cong \triangle ABD$	AAS

Hypotenuse-Leg

So far, the congruence postulates we have used will work for any triangle. The last congruence theorem can only be used on ***right*** triangles. A right triangle has exactly one right angle. The two sides adjacent to the right angle are called legs and the side opposite the right angle is called the hypotenuse.



You should be familiar with the Pythagorean Theorem, which says, for any ***right*** triangle, this equation is true:

$$(leg)^2 + (leg)^2 = (hypotenuse)^2$$

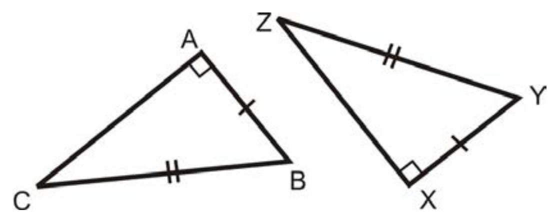
What this means is that if you are given two sides of a right triangle, you can always find the third. Therefore, if you have two sides of a right triangle are congruent to two sides of another right triangle; you can conclude that third sides are also congruent.

The Hypotenuse-Leg (HL) Congruence Theorem is a shortcut of this process.

HL Congruence Theorem

If the hypotenuse and leg in one **right** triangle are congruent to the hypotenuse and leg in another **right** triangle, then the two triangles are congruent.

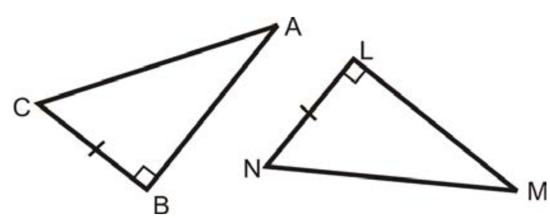
If $\triangle ABC$ and $\triangle XYZ$ are both right triangles and $\overline{AB} \cong \overline{XY}$ and $\overline{BC} \cong \overline{YZ}$ then $\triangle ABC \cong \triangle XYZ$.



Example 6: What information would you need to prove that these two triangles were congruent using the:

a) HL Theorem?

b) SAS Theorem?



Solution:

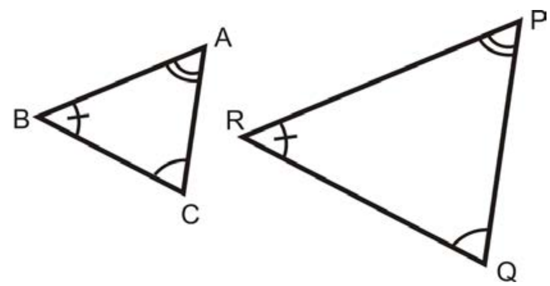
a) For HL, you need the hypotenuses to be congruent. $\overline{AC} \cong \overline{MN}$.

b) To use SAS, we would need the other legs to be congruent. $\overline{AB} \cong \overline{ML}$.

The following video shows how to use the HL THeorem.

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/153955

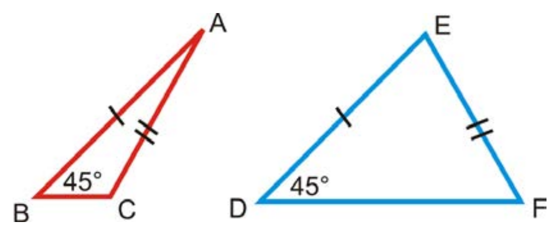
AAA and SSA Relationships There are two other side-angle relationships that we have not discussed: AAA and SSA.



AAA implies that all the angles are congruent.

As you can see, $\triangle ABC$ and $\triangle PRQ$ are not congruent, even though all the angles are.

SSA relationships do not prove congruence either. See $\triangle ABC$ and $\triangle DEF$ below.



Because $\angle B$ and $\angle D$ are **not** the included angles between the congruent sides, we cannot prove that these two triangles are congruent.

Recap

TABLE 4.18:

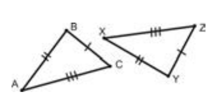
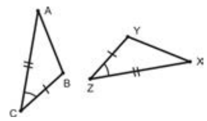
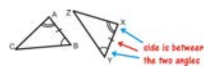
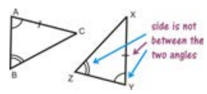
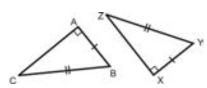
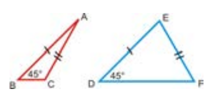
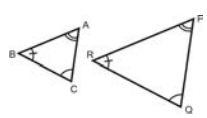
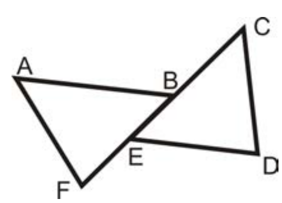
Side-Angle Relationship	*Picture*	*Determine Congruence?*
SSS		**Yes** $\triangle ABC \cong \triangle XYZ$

TABLE 4.18: (continued)

Side-Angle Relationship	*Picture*	*Determine Congruence?*
SAS		**Yes** $\triangle ABC \cong \triangle XYZ$
ASA		**Yes** $\triangle ABC \cong \triangle XYZ$
AAS (or SAA)		**Yes** $\triangle ABC \cong \triangle YZX$
HL		**Yes, Right Triangles Only** $\triangle ABC \cong \triangle XYZ$
SSA		**NO**
AAA		**NO**

Example 7: Write a 2-column proof.



Given: $\overline{AB} || \overline{ED}$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{ED}$

Prove: $\overline{AF} \cong \overline{CD}$

Solution:

TABLE 4.19:

Statement	Reason
1. $\overline{AB}\|\|\overline{ED}$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{ED}$	Given
2. $\angle ABE \cong \angle DEB$	Alternate Interior Angles Theorem
3. $\triangle ABF \cong \triangle DEC$	ASA
4. $\overline{AF} \cong \overline{CD}$	CPCTC

Prove Move: At the beginning of this chapter we introduced CPCTC. Now, it can be used in a proof once two triangles are proved congruent. It is used to prove the parts of congruent triangles are congruent.

Review triangle congruence by visiting the following link, http://www.mathsisfun.com/geometry/triangles-congruent-finding.html

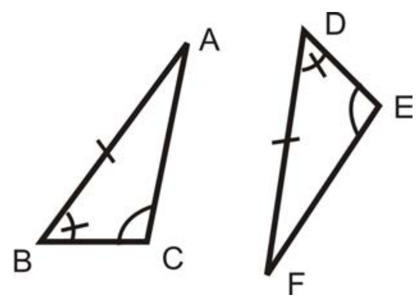
Know What? Revisited Even though we do not know all of the angle measures in the two triangles, we can find the missing angles by using the Third Angle Theorem. In your parents' kitchen, the missing angle is 39°. The missing angle in your neighbor's kitchen is 50°. From this, we can conclude that the two kitchens are now congruent, either by ASA or AAS.

Review Questions

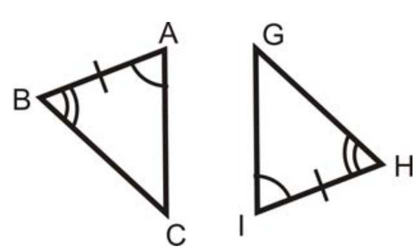
- Questions 1-10 are similar to Examples 1, 3, 4, and 6.
- Questions 11-20 are review and use the definitions and theorems explained in this section.
- Question 21-26 are similar to Examples 1, 3, 4 and 6.
- Questions 27 and 28 are similar to Examples 2 and 5.
- Questions 29-31 are similar to Example 4 and Investigation 4-4.

For questions 1-10, determine if the triangles are congruent. If they are, write the congruence statement and which congruence postulate or theorem you used.

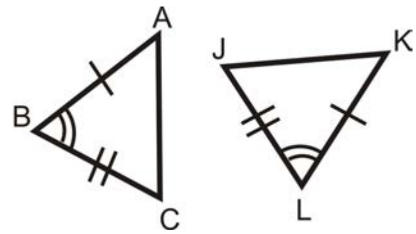
1.



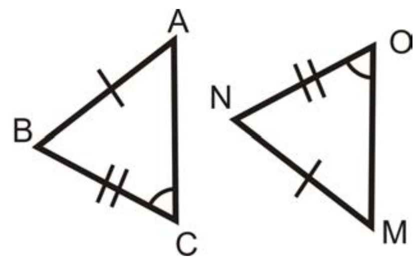
2.



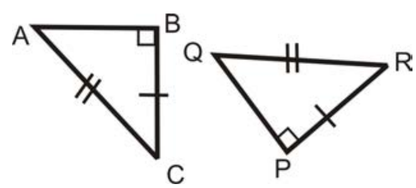
3.



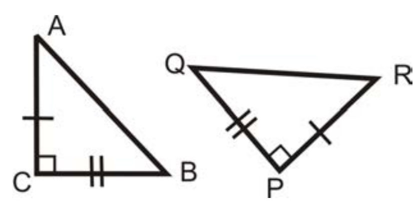
4.



5.



6,.



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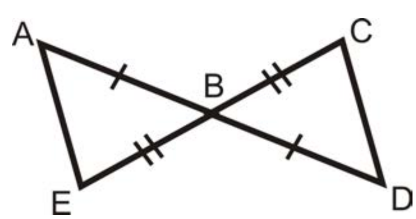

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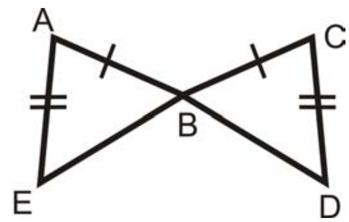
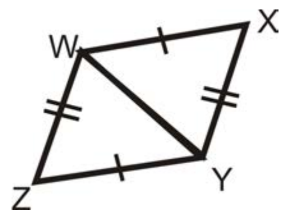
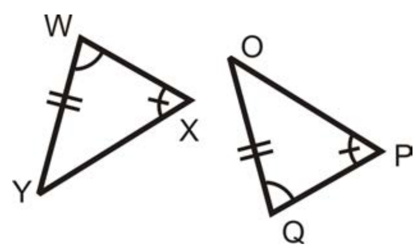


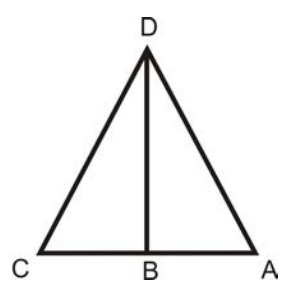
9\.



10\.



For questions 11-15, use the picture to the right and the given information below.

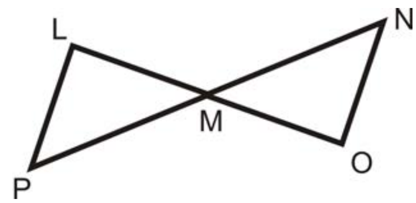


<u>Given</u>: $\overline{DB} \perp \overline{AC}$, $\overline{DB}$ is the angle bisector of $\angle CDA$

11. From $\overline{DB} \perp \overline{AC}$, which angles are congruent and why?
12. Because $\overline{DB}$ is the angle bisector of $\angle CDA$, what two angles are congruent?
13. From looking at the picture, what additional piece of information are you given? Is this enough to prove the two triangles are congruent?
14. Write a 2-column proof to prove $\triangle CDB \cong \triangle ADB$, using #11-13.
15. What would be your reason for $\angle C \cong \angle A$?

For questions 16-20, use the picture to the right and the given information.

<u>Given</u>: $\overline{LP} || \overline{NO}$, $\overline{LP} \cong \overline{NO}$



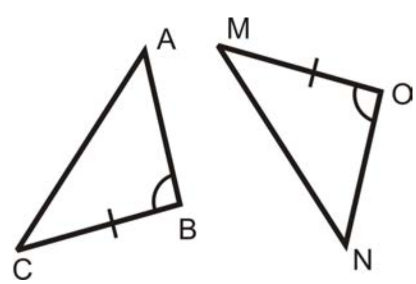
16. From $\overline{LP}||\overline{NO}$, which angles are congruent and why?
17. From looking at the picture, what additional piece of information can you conclude?
18. Write a 2-column proof to prove $\triangle LMP \cong \triangle OMN$.
19. What would be your reason for $\overline{LM} \cong \overline{MO}$?
20. Fill in the blanks for the proof below. Use the given from above. Prove: M is the midpoint of $\overline{PN}$.

TABLE 4.20:

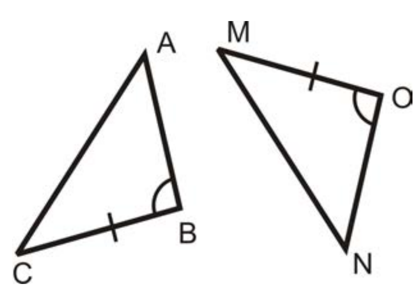
Statement	***Reason***
1. $\overline{LP}\|\|\overline{NO}$, $\overline{LP} \cong \overline{NO}$	Given
2.	Alternate Interior Angles
3.	ASA
4. $\overline{LM} \cong \overline{MO}$	
5. M is the midpoint of $\overline{PN}$.	

Determine the additional piece of information needed to show the two triangles are congruent by the given postulate.

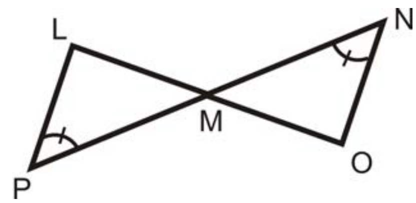
21. AAS



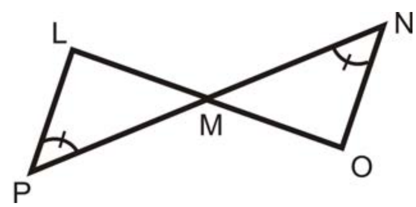
22. ASA



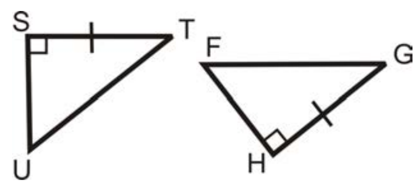
23. ASA



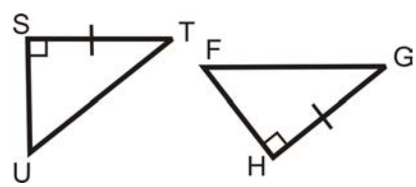
24. AAS



25. HL



26. SAS



Fill in the blanks in the proofs below.

27. Given: $\overline{SV} \perp \overline{WU} T$ is the midpoint of $\overline{SV}$ and $\overline{WU}$Prove: $\overline{WS} \cong \overline{UV}$

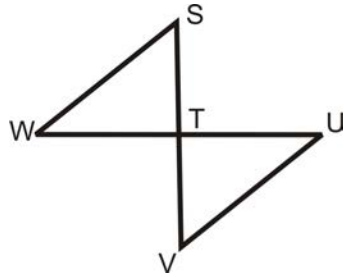


TABLE 4.21:

Statement	***Reason***
1.	
2. $\angle STW$ and $\angle UTV$ are right angles	
3.	
4. $\overline{ST} \cong \overline{TV}$, $\overline{WT} \cong \overline{TU}$	
5. $\triangle STW \cong \triangle UTV$	
6. $\overline{WS} \cong \overline{UV}$	

28. Given: $\angle K \cong \angle T$, $\overline{EI}$ is the angle bisector of $\angle KET$ Prove: $\overline{EI}$ is the angle bisector of $\angle KIT$

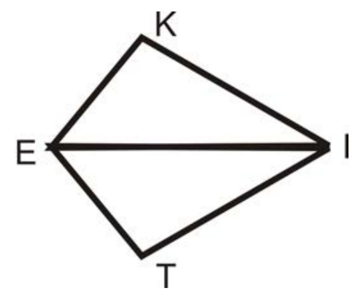
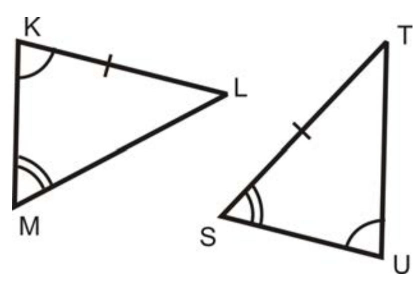


TABLE 4.22:

Statement	***Reason***
1.	
2.	Definition of an angle bisector
3. $\overline{EI} \cong \overline{EI}$	
4. $\triangle KEI \cong \triangle TEI$	
5. $\angle KIE \cong \angle TIE$	
6. $\overline{EI}$ is the angle bisector of $\angle KIT$	

Construction Let's see if we can construct two different triangles like $\triangle KLM$ and $\triangle STU$ from Example 4.



29. Look at $\triangle KLM$.
    a. If $m\angle K = 70^\circ$ and $m\angle M = 60^\circ$, what is $m\angle L$?
    b. If $KL = 2\ in$, construct $\triangle KLM$ using $\angle L$, $\angle K$, $\overline{KL}$ and Investigation 4-4 (ASA Triangle construction).
30. Look at $\triangle STU$.
    a. If $m\angle S = 60^\circ$ and $m\angle U = 70^\circ$, what is $m\angle T$?
    b. If $ST = 2\ in$, construct $\triangle STU$ using $\angle S$, $\angle T$, $\overline{ST}$ and Investigation 4-4 (ASA Triangle construction).
31. Are the two triangles congruent?

OVERLAPPING TRIANGLES *Explain* how you can prove that the indicated triangles are congruent using the given postulate or theorem.

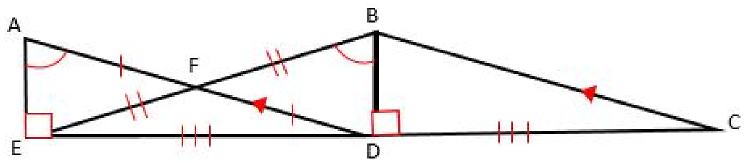


FIGURE 4.9

32. $\Delta AFE \cong \Delta DFB$ by SAS

33. $\Delta AED \cong \Delta BDE$ by AAS

34. $\Delta AED \cong \Delta BDC$by ASA

DETERMINING CONGRUENCE Tell whether you can use the given information to determine whether $\Delta ABC \cong \Delta XYZ$. *Explain* your reasoning.

35. $\angle A \cong \angle X, \overline{AB} \cong \overline{XY}$, $\overline{AC} \cong \overline{XZ}$

36. $\angle A \cong \angle X$, $\angle B \cong \angle Y$, $\angle C \cong \angle Z$

37. $\angle B \cong \angle Y$, $\angle C \cong \angle Z, \overline{AC} \cong \overline{XY}$

Review and Reflect

38. Is it possible to prove two triangles congruent by AAA? If not, give a counter example.

39. When using HL as a justification for proving triangles congruent, why is it only necessary to show two corresponding parts are congruent.

Warm-Up Answers

1.
   a. $\overline{AD} \cong \overline{DC}, \overline{AB} \cong \overline{BC}$
   b. Yes, by the Reflexive Property
   c. $\triangle DAB \cong \triangle DCB$ by SSS
2.
   a. $\angle L \cong \angle N$ and $\angle M \cong \angle P$ by the Alternate Interior Angles Theorem
   b. $\angle PON \cong \angle LOM$ by Vertical Angles or the 3^{rd} Angle Theorem
   c. No, no markings or midpoints
   d. No, no congruent sides.
3.
   a. Yes, CPCTC
   b. No, these sides do not line up in the congruence statement.

4.5 Isosceles and Equilateral Triangles

G 4(B) Identify and determine the validity of the converse, inverse, and contrapositive of a conditional statement and recognize the connection between a biconditional statement and a true conditional statement with a true converse.

G 5(A). Investigate patterns to make conjectures about geometric relationships, including angles formed by parallel lines cut by a transversal, criteria required for triangle congruence, special segments of triangles, diagonals of quadrilaterals, interior and exterior angles of polygons, and special segments and angles of circles choosing from a variety of tools.

G 6(D). Verify theorems about the relationships in triangles, including proof of the Pythagorean Theorem, the sum of interior angles, base angles of isosceles triangles, midsegments, and medians, and apply these relationships to solve problems.

G 1(G). Display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication.

TEKS G(1)G, G(4)B, G(5)A, G(6)D

Learning Objectives

- Understand the properties of isosceles and equilateral triangles.
- Use the Base Angles Theorem and its converse.
- Understand that an equilateral triangle is also equiangular.

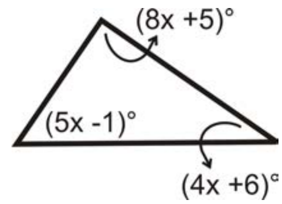
Vocabulary

- isosceles triangle
- base
- base angles
- vertex
- equilateral triangle
- equiangular triangle

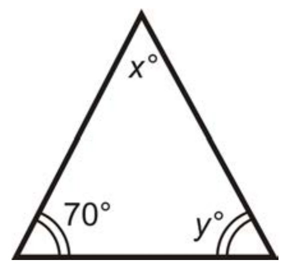
Warm-Up

Find the value of x and/or y.

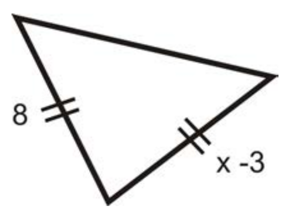
1.



2.

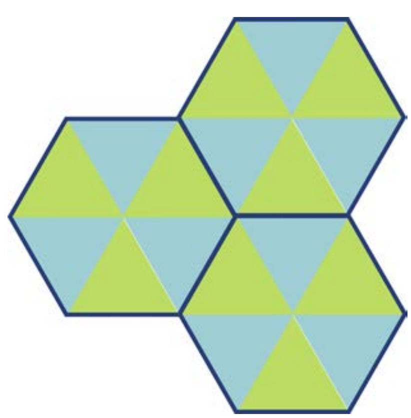


3.

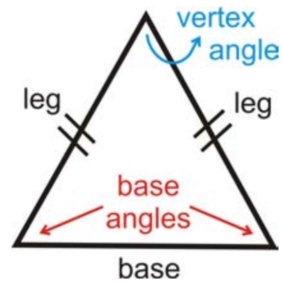


4. If a triangle is equiangular, what is the measure of each angle?

Know What? Your parents now want to redo the bathroom. To the right are 3 of the tiles they would like to place in the shower. Each blue and green triangle is an equilateral triangle. What shape is each dark blue polygon? Find the number of degrees in each of these figures?



Isosceles Triangle Properties

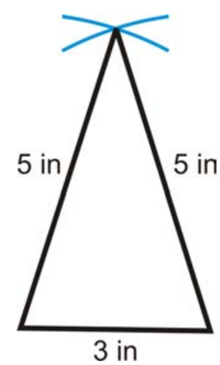


An isosceles triangle is a triangle that has ***at least*** two congruent sides. The congruent sides of the isosceles triangle are called the **legs**.The other side is called the **base**. The angles between the base and the legs are called **base angles**. The angle made by the two legs is called the **vertex angle**.

Investigation 4-5: Isosceles Triangle Construction

Tools Needed: pencil, paper, compass, ruler, protractor

1. Refer back to Investigation 4-2. Using your compass and ruler, draw an isosceles triangle with sides of 3 in, 5 in and 5 in. Draw the 3 in side (the base) horizontally at least 6 inches down the page.



2. Now that you have an isosceles triangle, use your protractor to measure the base angles and the vertex angle.

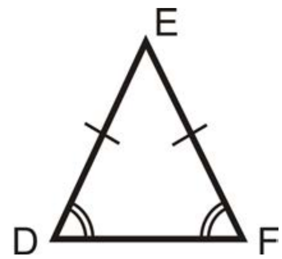
The base angles should each be 72.5° *and the vertex angle should be* 35°.

We can generalize this investigation for all isosceles triangles.

Base Angle Theorem

If two sides of a triangle are congruent, then the angles opposite them are congruent.

For $\triangle DEF$, if $\overline{DE} \cong \overline{EF}$, then $\angle D \cong \angle F$.

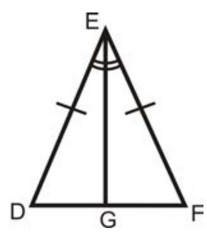


To prove the Base Angles Theorem, we need to draw the angle bisector (Investigation 1-5) of $\angle E$.

Given: Isosceles triangle $\triangle DEF$ above, with $\overline{DE} \cong \overline{EF}$.

Prove: $\angle D \cong \angle F$

TABLE 4.23:

Statement	***Reason***
1. Isosceles triangle $\triangle DEF$ with $\overline{DE} \cong \overline{EF}$	Given
2. Construct angle bisector $\overline{EG}$ of $\angle E$ 	Every angle has one angle bisector
3. $\angle DEG \cong \angle FEG$	Definition of an angle bisector
4. $\overline{EG} \cong \overline{EG}$	Reflexive Property of Congruence
5. $\triangle DEG \cong \triangle FEG$	SAS
6. $\angle D \cong \angle F$	CPCTC

Let's take a further look at the picture from step 2 of our proof.

Because $\triangle DEG \cong \triangle FEG$, we know $\angle EGD \cong \angle EGF$ by CPCTC. These two angles are also a linear pair, so 90° each and $\overline{EG} \perp \overline{DF}$.

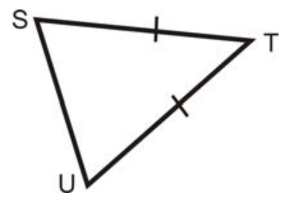
Additionally, $\overline{DG} \cong \overline{GF}$ by CPCTC, so G is the midpoint of $\overline{DF}$. This means that $\overline{EG}$ is the **perpendicular bisector** of $\overline{DF}$.

Isosceles Triangles Theorem

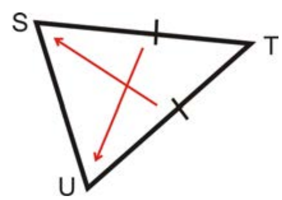
The angles bisector of the vertex angle in an isosceles triangles is also the perpendicular bisector of the base.

Note this is ONLY true of the vertex angle. We will prove this theorem in the review questions.

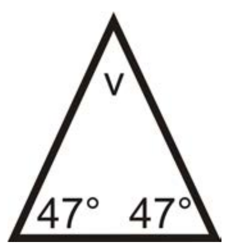
Example 1: Which two angles are congruent?



Solution: This is an isosceles triangle. The congruent angles are opposite the congruent sides. From the arrows we see that $\angle S \cong \angle U$.



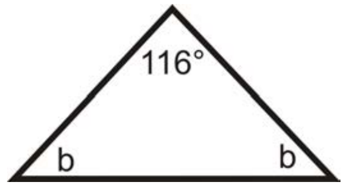
Example 2: If an isosceles triangle has base angles with measures of 47°, what is the measure of the vertex angle?



Solution: Draw a picture and set up an equation to solve for the vertex angle, v.

$$\begin{aligned} 47^\circ + 47^\circ + v &= 180^\circ \\ v &= 180^\circ - 47^\circ - 47^\circ \\ v &= 86^\circ \end{aligned}$$

Example 3: If an isosceles triangle has a vertex angle with a measure of 116°, what is the measure of each base angle?



Solution: Draw a picture and set up and equation to solve for the base angles, b.

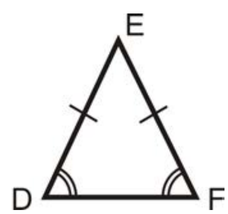
$$\begin{aligned} 116^\circ + b + b &= 180^\circ \\ 2b &= 64^\circ \\ b &= 32^\circ \end{aligned}$$

The converses of the Base Angles Theorem and the Isosceles Triangle Theorem are both true.

Base Angles Theorem Converse

If two angles in a triangle are congruent, then the opposite sides are also congruent

For $\triangle DEF$, if $\angle D \cong \angle F$, then $\overline{DE} \cong \overline{EF}$.



Isosceles Triangle Theorem Converse

The perpendicular bisector of the base of an isosceles triangles is also the angle bisector of the vertex angle.

For isosceles $\triangle DEF$, if $\overline{EG} \perp \overline{DF}$ and $\overline{DG} \cong \overline{GF}$, then $\angle DEG \cong \angle FEG$.

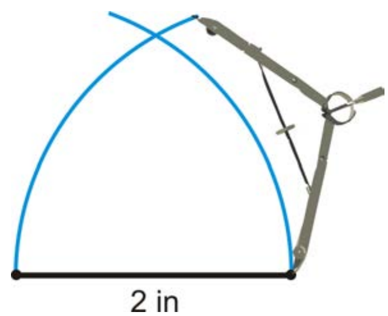
Equilateral Triangles By definition, all sides in an equilateral triangle have the same length.

Investigation 4-6: Constructing an Equilateral Triangle

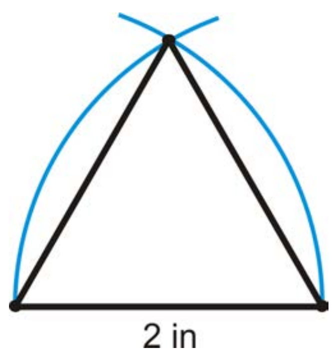
Tools Needed: pencil, paper, compass, ruler, protractor

1. Because all the sides of an equilateral triangle are equal, pick one length to be all the sides of the triangle. Measure this length and draw it horizontally on you paper.

2. Put the pointer of your compass on the left endpoint of the line you drew in Step 1. Open the compass to be the same width as this line. Make an arc above the line. Repeat Step 2 on the right endpoint.



4. Connect each endpoint with the arc intersections to make the equilateral triangle.

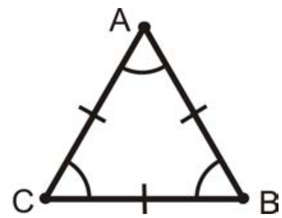


Use the protractor to measure each angle of your constructed equilateral triangle. What do you notice?

From the Base Angles Theorem, the angles opposite congruent sides in an isosceles triangle are congruent. So, if all three sides of the triangle are congruent, then all of the angles are congruent, 60° each.

Equilateral Triangle Theorem

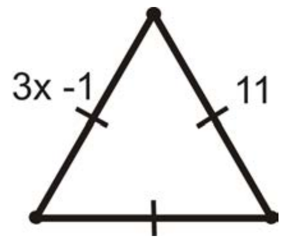
All equilateral triangles are also equiangular. Also, all equiangular triangles are also equilateral..



If $\overline{AB} \cong \overline{BC} \cong \overline{AC}$, then $\angle A \cong \angle B \cong \angle C$.

If $\angle A \cong \angle B \cong \angle C$, then $\overline{AB} \cong \overline{BC} \cong \overline{AC}$.

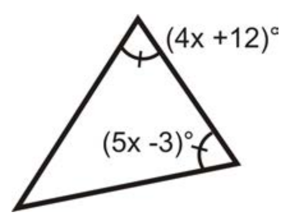
Example 4: *Algebra Connection* Find the value of x.



Solution: Because this is an equilateral triangle $3x - 1 = 11$. Solve for x.

$$\begin{aligned} 3x - 1 &= 11 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Example 5: *Algebra Connection* Find the value of $x$and the measure of each angle.

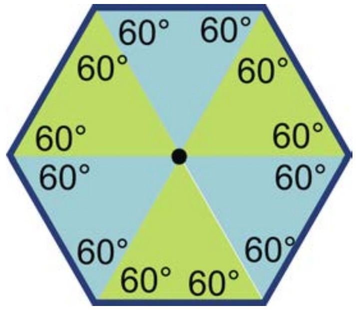


Solution: Similar to Example 4, the two angles are equal, so set them equal to each other and solve for x.

$$\begin{aligned} (4x + 12)^\circ &= (5x - 3)^\circ \\ 15^\circ &= x \end{aligned}$$

Substitute $x = 15^\circ$; the base angles are $4(15^\circ) + 12$, or 72°. The vertex angle is $180^\circ - 72^\circ - 72^\circ = 36^\circ$.

Know What? Revisited Let's focus on one tile. First, these triangles are all equilateral, so this is an equilateral hexagon (6 sides). Second, we now know that every equilateral triangle is also equiangular, so every triangle within this tile has 3 - 60° angles. This makes our equilateral hexagon also equiangular, with each angle measuring 120°. Because there are 6 angles, the sum of the angles in a hexagon are $6 \cdot 120^\circ$ or 720°.



Practice Problems

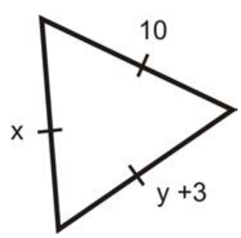
- Questions 1-5 are similar to Investigations 4-5 and 4-6.
- Questions 6-14 are similar to Examples 2-5.
- Question 15 uses the definition of an equilateral triangle.
- Questions 16-20 use the definition of an isosceles triangle.
- Question 21 is similar to Examples 2 and 3.
- Questions 22-25 are proofs and use definitions and theorems learned in this section.
- Questions 26-30 use the distance formula.

Constructions For questions 1-5, use your compass and ruler to:

1. Draw an isosceles triangle with sides 3.5 in, 3.5 in, and 6 in.
2. Draw an isosceles triangle that has a vertex angle of 100° and legs with length of 4 cm. (you will also need your protractor for this one)
3. Draw an equilateral triangle with sides of length 7 cm.
4. Using what you know about constructing an equilateral triangle, construct (without a protractor) a 60° angle.
5. Draw an isosceles right triangle. What is the measure of the base angles?

For questions 6-14, find the measure of x and/or y.

6.



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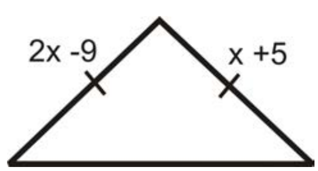

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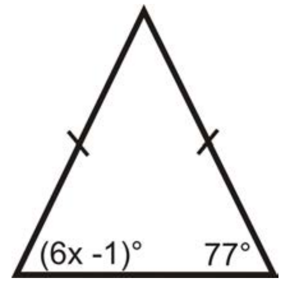
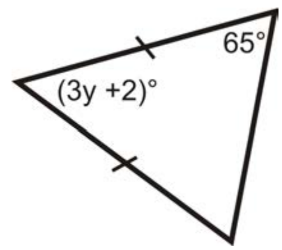


9.



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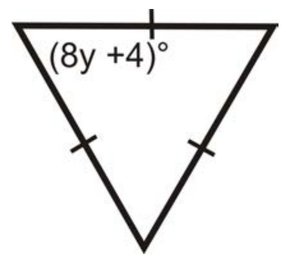
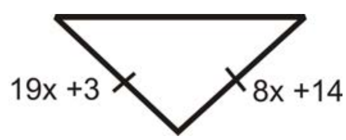
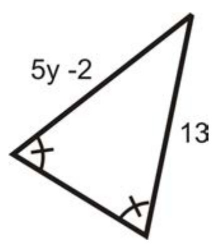


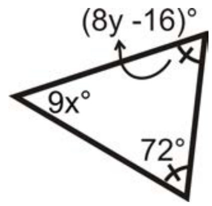
11.



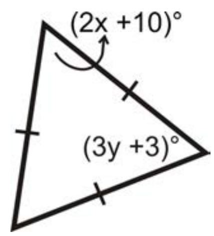
12.



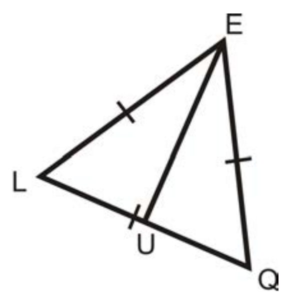
13.



14.



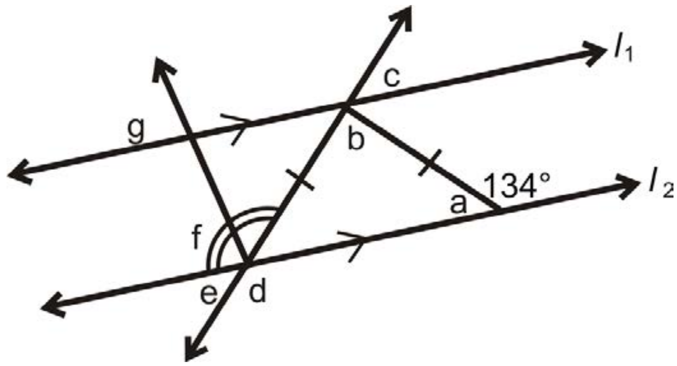
15. $\triangle EQG$ is an equilateral triangle. If $\overline{EU}$ bisects $\angle LEQ$, find:



a. $m\angle EUL$
b. $m\angle UEL$
c. $m\angle ELQ$
d. If $EQ = 4$,find LU.

Determine if the following statements are true or false.

16. Base angles of an isosceles triangle are congruent.
17. Base angles of an isosceles triangle are complementary.
18. Base angles of an isosceles triangle can be equal to the vertex angle.
19. Base angles of an isosceles triangle can be right angles.
20. Base angles of an isosceles triangle are acute.
21. In the diagram below, $l_1 || l_2$. Find all of the lettered angles.



Fill in the blanks in the proofs below.

22. Given: Isosceles $\triangle CIS$, with base angles $\angle C$ and $\angle S\overline{IO}$ is the angle bisector of $\angle CIS$ Prove: $\overline{IO}$is the perpendicular bisector of $\overline{CS}$

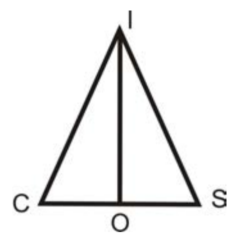


TABLE 4.24:

Statement	Reason
1.	Given
2.	Base Angles Theorem
3. $\angle CIO \cong \angle SIO$	
4.	Reflexive Property of Congruence
5. $\triangle CIO \cong \triangle SIO$	
6. $\overline{CO} \cong \overline{OS}$	
7.	CPCTC
8. $\angle IOC$ and $\angle IOS$ are supplementary	
9.	Congruent Supplements Theorem
10. $\overline{IO}$ is the perpendicular bisector of $\overline{CS}$	

23. Given: Equilateral $\triangle RST$ with $\overline{RT} \cong \overline{ST} \cong \overline{RS}$Prove: $\triangle RST$ is equiangular

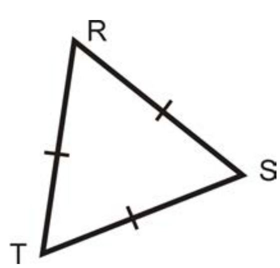


TABLE 4.25:

Statement	Reason
1.	Given

TABLE 4.25: (continued)

Statement	Reason
2.	Base Angles Theorem
3.	Base Angles Theorem
4.	Transitive Property of Congruence
5. $\triangle RST$ is equiangular	

24. Given: Isosceles $\triangle ICS$ with $\angle C$ and $\angle S\overline{IO}$ is the perpendicular bisector of $\overline{CS}$Prove: $\overline{IO}$ is the angle bisector of $\angle CIS$

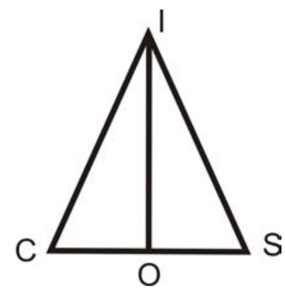


TABLE 4.26:

Statement	Reason
1.	
2. $\angle C \cong \angle S$	
3. $\overline{CO} \cong \overline{OS}$	
4. $m\angle IOC = m\angle IOS = 90^\circ$	
5.	
6.	CPCTC
7. $\overline{IO}$ is the angle bisector of $\angle CIS$	

25. Given: Isosceles $\triangle ABC$ with base angles $\angle B$ and $\angle C$ Isosceles $\triangle XYZ$ with base angles $\angle Y$ and $\angle Z \angle C \cong \angle Z$, $\overline{BC} \cong \overline{YZ}$Prove: $\triangle ABC \cong \triangle XYZ$

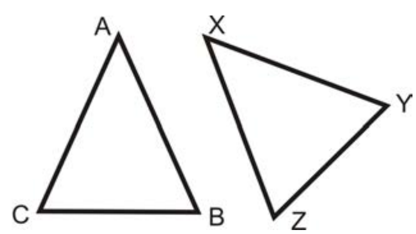


TABLE 4.27:

Statement	Reason
1.	
2. $\angle B \cong \angle C$, $\angle Y \cong \angle Z$	
3. $\angle B \cong \angle Y$	
4. $\triangle ABC \cong \triangle XYZ$	

Coordinate Plane Geometry On the $x-y$ plane, plot the coordinates and determine if the given three points make a scalene or isosceles triangle.

26. (-2, 1), (1, -2), (-5, -2)
27. (-2, 5), (2, 4), (0, -1)
28. (6, 9), (12, 3), (3, -6)
29. (-10, -5), (-8, 5), (2, 3)
30. (-1, 2), (7, 2), (3, 9)

Review and Reflect

31. Using the definitions of equilateral and equiangular triangular, write a new definition as a biconditional statement.

32. Are all isosceles triangles equilateral? Why or why not?

Warm-Up Answers

1. $$\begin{aligned}(5x-1)^\circ+(8x+5)^\circ+(4x+6)^\circ &= 180^\circ\\ 17x+10 &= 180^\circ\\ 17x &= 170^\circ\\ x &= 10^\circ\end{aligned}$$
2. $x = 40^\circ, y = 70^\circ$
3. $x-3=8$
   $x=5$
4. Each angle is $\frac{180^\circ}{3}$, or 60°

4.6 Chapter 4 Review

Symbols Toolbox

Congruent Triangles and their corresponding parts

Definitions, Postulates, and Theorems

Triangle Sums

- Interior Angles
- Vertex
- Triangle Sum Theorem
- Exterior Angle
- Exterior Angle Sum Theorem
- Remote Interior Angles
- Exterior Angle Theorem

Congruent Figures

- Congruent Triangles
- Congruence Statements
- Third Angle Theorem
- Reflexive Property of Congruence
- Symmetric Property of Congruence
- Transitive Property of Congruence

Triangle Congruence using SSS and SAS

- Side-Side-Side (SSS) Triangle Congruence Postulate
- Included Angle
- Side-Angle-Side (SAS) Triangle Congruence Postulate
- Distance Formula

Triangle Congruence using ASA, AAS, and HL

- Angle-Side-Angle (ASA) Congruence Postulate
- Angle-Angle-Side (AAS) Congruence Theorem
- Hypotenuse

- Legs (of a right triangle)
- HL Congruence Theorem

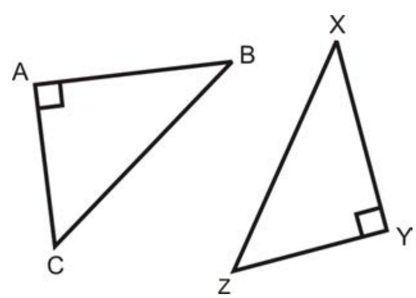
Isosceles and Equilateral Triangles

- Base
- Base Angles
- Vertex Angle
- Legs (of an isosceles triangle)
- Base Angles Theorem
- Isosceles Triangle Theorem
- Base Angles Theorem Converse
- Isosceles Triangle Theorem Converse
- Equilateral Triangles Theorem

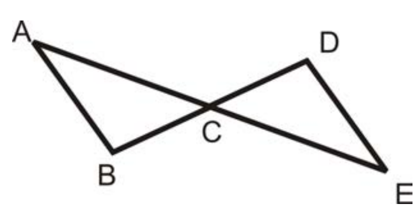
Review

For each pair of triangles, write what needs to be congruent in order for the triangles to be congruent. Then, write the congruence statement for the triangles.

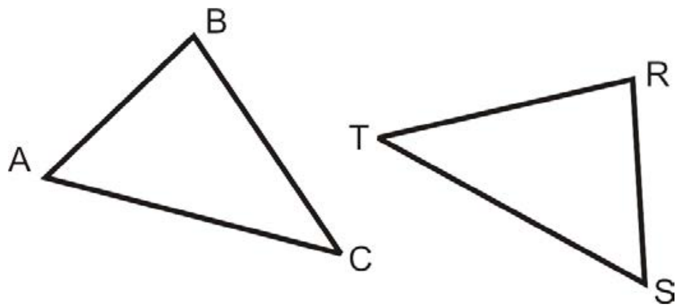
1. HL



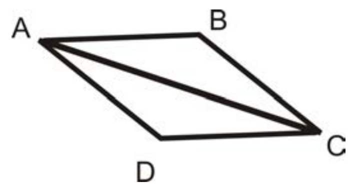
2. ASA



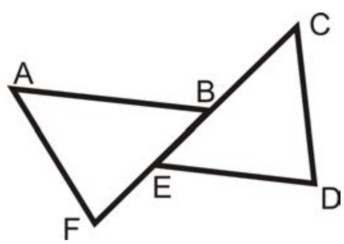
3. AAS



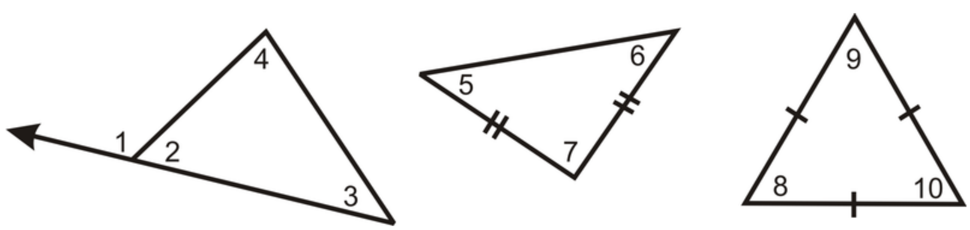
4. SSS



5. SAS



Using the pictures below, determine which theorem, postulate or definition that supports each statement below.



6. $m\angle 1 + m\angle 2 = 180^\circ$
7. $\angle 5 \cong \angle 6$
8. $m\angle 1 + m\angle 4 + m\angle 3$
9. $m\angle 8 = 60^\circ$
10. $m\angle 5 + m\angle 6 + m\angle 7 = 180^\circ$
11. $\angle 8 \cong \angle 9 \cong \angle 10$
12. If $m\angle 7 = 90^\circ$, then $m\angle 5 = m\angle 6 = 45^\circ$

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9689 .

4.7 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1^{st} **Section: Triangle Sums**

Interior Angles

Vertex

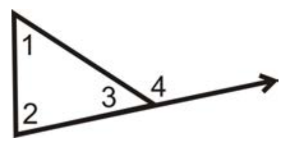
Triangle Sum Theorem

Exterior Angle

Exterior Angle Sum Theorem

Remote Interior Angles

Exterior Angle Theorem



Homework:

2^{nd} **Section: Congruent Figures**

Congruent Triangles

Congruence Statements

Third Angle Theorem

Reflexive Property of Congruence

Symmetric Property of Congruence

Transitive Property of Congruence

Homework:

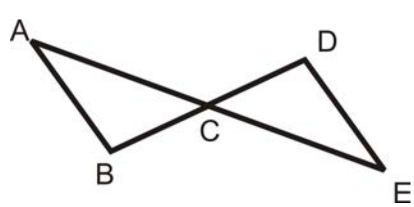
3^{rd} **Section: Triangle Congruence using SSS and SAS**

Side-Side-Side (SSS) Triangle Congruence Postulate

Included Angle

Side-Angle-Side (SAS) Triangle Congruence Postulate

Distance Formula



Homework:

4th **Section: Triangle Congruence using ASA, AAS, and HL**

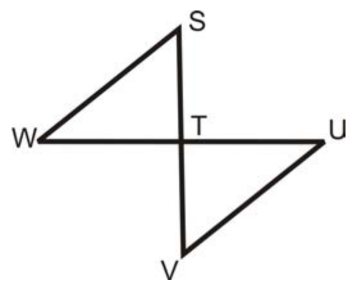
Angle-Side-Angle (ASA) Congruence Postulate

Angle-Angle-Side (AAS) Congruence Theorem

Hypotenuse

Legs (of a right triangle)

HL Congruence Theorem



Homework:

5th **Section: Isosceles and Equilateral Triangles**

Base

Base Angles

Vertex Angle

Legs (of an isosceles triangle)

Base Angles Theorem

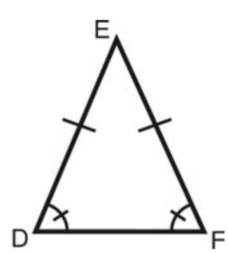
Isosceles Triangle Theorem

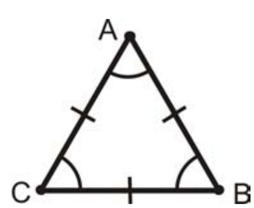
Base Angles Theorem Converse

Isosceles Triangle Theorem Converse

Equilateral Triangle Theorem

Homework:





CHAPTER 5 Relationships with Triangles

Chapter Outline

In this chapter we will explore the properties of midsegments, perpendicular bisectors, angle bisectors, medians, and altitudes. next, we will look at the relationship of the sides of a triangles and how the sides of one triangle can compare to another.

5.1 Midsegments

TEKS G(1)B, G(2)B, G(5)A, G(6)D

Learning Objectives

- define midsegment
- Use the Midsegment Theorem

Vocabulary

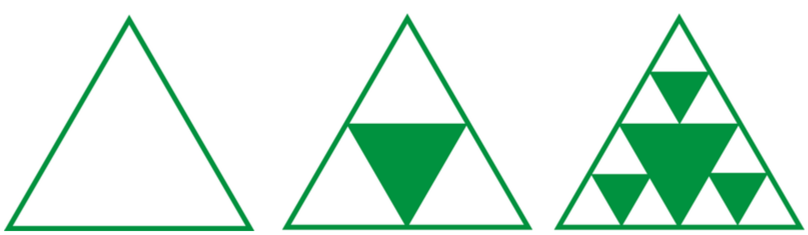
- midsegment

Warm-UP

Find the midpoint between the given points.

1. (-4, 1) and (6, 7)
2. (5, -3) and (11, 5)
3. Find the equation of the line between (-2, -3) and (-1, 1).
4. Find the equation of the line that is parallel to the line from #3 through (2, -7).

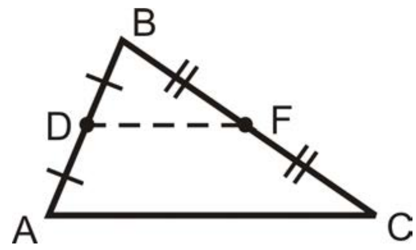
Know What? A fractal is a repeated design using the same shape (or shapes) of different sizes. Below, is an example of the first few steps of a fractal. Draw the next figure in the pattern.



Defining Midsegment

Midsegment: A line segment that connects two midpoints of the sides of a triangle.

$\overline{DF}$ is the midsegment between $\overline{AB}$ and $\overline{BC}$.

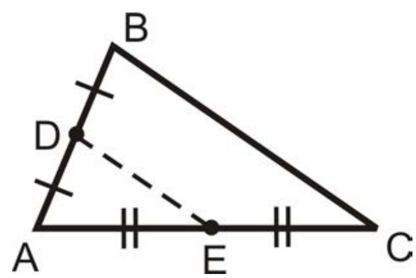


The tic marks show that D and F are midpoints.

$\overline{AD} \cong \overline{DB}$ and $\overline{BF} \cong \overline{FC}$

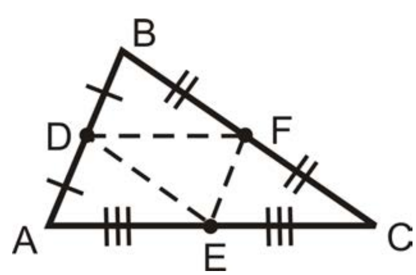
Example 1: Draw the midsegment $\overline{DE}$ between $\overline{AB}$ and $\overline{AC}$ for $\triangle ABC$ above.

Solution: Find the midpoints of $\overline{AB}$ and $\overline{AC}$ using your ruler. Label these points D and E. Connect them to create the midsegment.



Example 2: Draw the midsegment $\overline{DF}$and $\overline{FE}$.

Solution: Find the midpoints of $\overline{BC}$ using your ruler. Label this point F. Connect each of the midpoints to create the midsegment.



For every triangle there are three midsegments.

The following video show how to construct a midsegment.

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/153988

Midsegments in the

Let's transfer what we know about ***midpoints*** in the $x-y$ plane to ***midsegments*** in the $x-y$ plane. We will need to use the midpoint formula, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 3: The vertices of $\triangle LMN$ are $L(4,5)$, $M(-2,-7)$ and $N(-8,3)$. Find the midpoints of all three sides, label them O, P and Q. Then, graph the triangle, plot the midpoints and draw the midsegments.

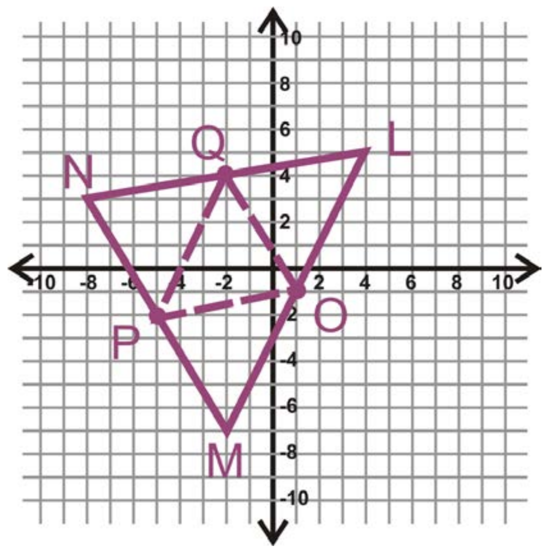
Solution: Use the midpoint formula 3 times to find all the midpoints.

L and $M = \left(\frac{4+(-2)}{2}, \frac{5+(-7)}{2}\right) = (1,-1)$ point O

M and $N = \left(\frac{-2+(-8)}{2}, \frac{-7+3}{2}\right) = (-5,-2)$, point P

L and $N = \left(\frac{4+(-8)}{2}, \frac{5+3}{2}\right) = (-2,4)$, point Q

The graph is to the right.



Example 4: Find the slopes of $\overline{NM}$ and $\overline{QO}$.

Solution: The slope of $\overline{NM}$ is $\frac{-7-3}{-2-(-8)} = \frac{-10}{6} = -\frac{5}{3}$. The slope of $\overline{QO}$ is $\frac{-1-4}{1-(-2)} = -\frac{5}{3}$.

From this we can conclude that $\overline{NM} \| \overline{QO}$. If we were to find the slopes of the other sides and midsegments, we would find $\overline{LM} \| \overline{QP}$ and $\overline{NL} \| \overline{PO}$.

Example 5: Find NM and QO.

Solution: Now, we need to find the lengths of $\overline{NM}$ and $\overline{QO}$. Use the distance formula. Remember you can also use Pythagorean Theorem to find the length of a segment.

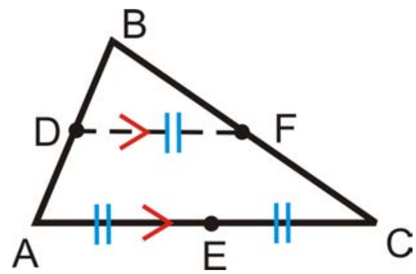
$$NM = \sqrt{(-7-3)^2+(-2-(-8))^2} = \sqrt{(-10)^2+6^2} = \sqrt{100+36} = \sqrt{136} \approx 11.66$$
$$QO = \sqrt{(1-(-2))^2+(-1-4)^2} = \sqrt{3^2+(-5)^2} = \sqrt{9+25} = \sqrt{34} \approx 5.83$$

From this we can conclude that QO is half of NM. If we were to find the lengths of the other sides and midsegments, we would find that OP is **half** of NL and QP is **half** of LM.

The Midsegment Theorem

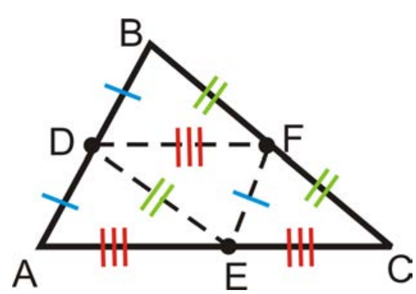
The midsegment of a triangle is half the length of the side it is parallel to.

If $\overline{DF}$ is a midsegment of $\triangle ABC$, then $DF = \frac{1}{2}AC = AE = EC$ and $\overline{DF} \| \overline{AC}$.



Example 6a: Mark all the congruent segments on $\triangle ABC$ with midpoints D, E, and F.

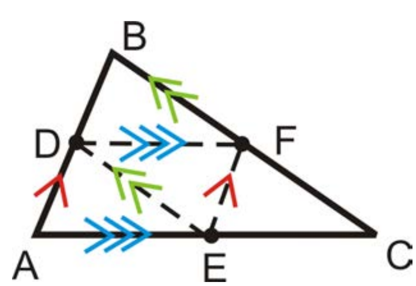
Solution: Drawing in all three midsegments, we have:



Also, this means the four triangles are congruent by SSS.

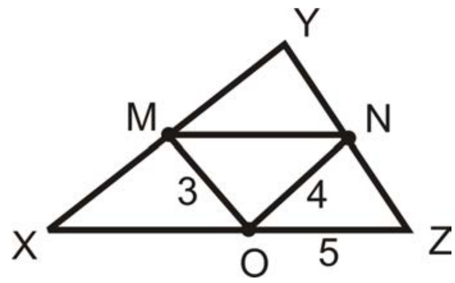
Example 6b: Mark all the parallel lines on $\triangle ABC$, with midpoints D, E, and F.

Solution:



To play with the properties of midsegments, go to http://www.mathopenref.com/trianglemidsegment.html .

Example 7: M, N, and O are the midpoints of the sides of the triangle.



Find

a) MN

b) XY

c) The perimeter of $\triangle XYZ$

Solution: Use the Midsegment Theorem.

a) $MN = OZ = 5$

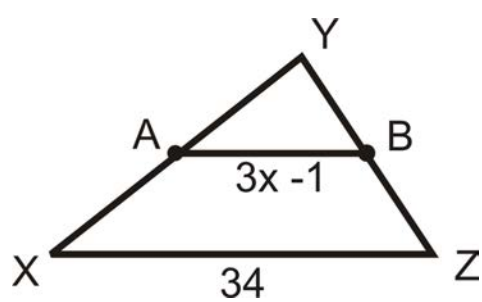
b) $XY = 2(ON) = 2 \cdot 4 = 8$

c) Add up the three sides of $\triangle XYZ$ to find the perimeter.

$$XY + YZ + XZ = 2 \cdot 4 + 2 \cdot 3 + 2 \cdot 5 = 8 + 6 + 10 = 24$$

Remember: The notation of length or distance is no line segment over MN**.**

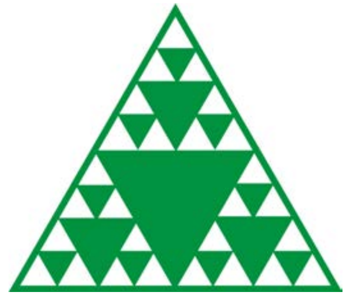
Example 8: *Algebra Connection* Find the value of x and AB. A and B are midpoints.



Solution: $AB = 34 \div 2 = 17$. To find x, set $3x - 1$ equal to 17.

$$\begin{aligned} 3x - 1 &= 17 \\ 3x &= 18 \\ x &= 6 \end{aligned}$$

Know What? Revisited To the left is a picture of the 4^{th} figure in the fractal pattern.



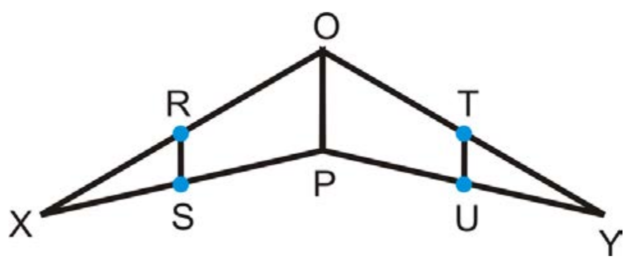
Practice Problems

- Questions 1-5 use the definition of a midsegment and the Midsegment Theorem.
- Questions 6-9 and 18 are similar to Example 7.
- Questions 10-17 are similar to Example 8.
- Questions 19-22 are similar to Example 3.
- Questions 23-30 are similar to Examples 3, 4, and 5.

Determine if each statement is true or false.

1. The endpoints of a midsegment are midpoints.
2. A midsegment is parallel to the side of the triangle that it does not intersect.
3. There are three congruent triangles formed by the midsegments and sides of a triangle.
4. If a line passes through two sides of a triangle and is parallel to the third side, then it is a midsegment.
5. There are three midsegments in every triangle.

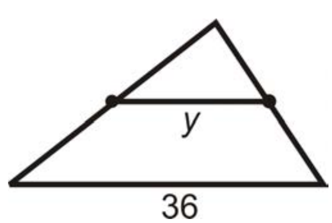
R, S, T, and U are midpoints of the sides of $\triangle XPO$ and $\triangle YPO$.



6. If $OP = 12$, find RS and TU.
7. If $RS = 8$, find TU.
8. If $RS = 2x$, and $OP = 20$, find x and TU.
9. If $OP = 4x$ and $RS = 6x - 8$, find x.

For questions 10-17, find the indicated variable(s). You may assume that all line segments within a triangle are midsegments.

10.



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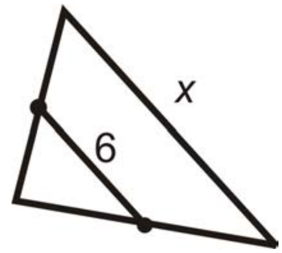

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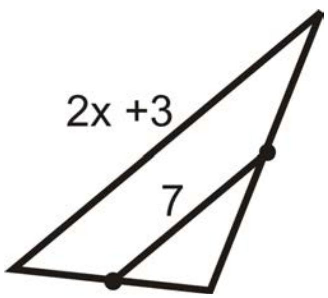

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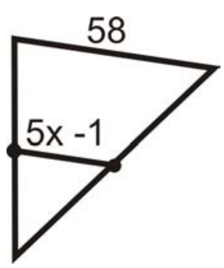

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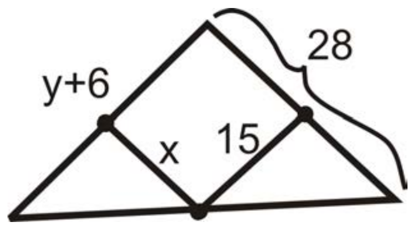

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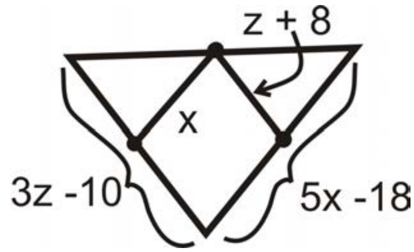

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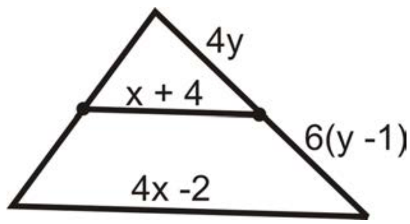
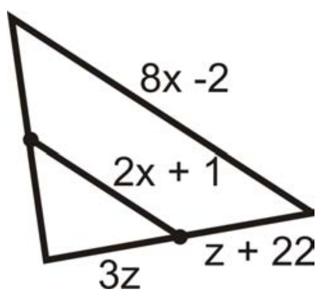


17.

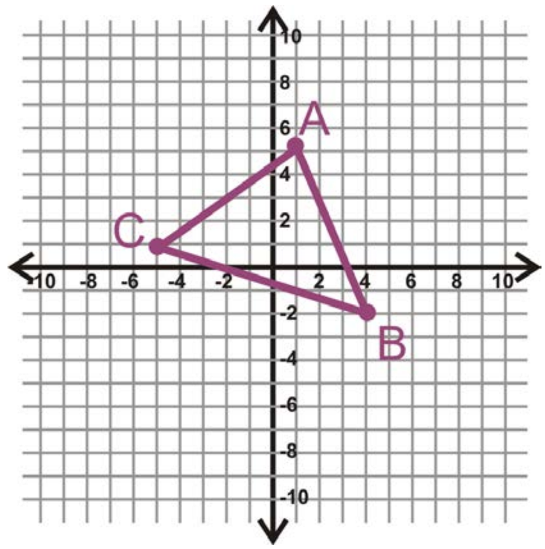


18. The sides of $\triangle XYZ$ are 26, 38, and 42. $\triangle ABC$ is formed by joining the midpoints of $\triangle XYZ$.
    a. What are the lengths of the sides of $\triangle ABC$?
    b. Find the perimeter of $\triangle ABC$.
    c. Find the perimeter of $\triangle XYZ$.
    d. What is the relationship between the perimeter of a triangle and the perimeter of the triangle formed by connecting its midpoints?

Coordinate Geometry Given the vertices of $\triangle ABC$ below find the midpoints of each side.

19. $A(5,-2)$, $B(9,4)$ and $C(-3,8)$
20. $A(-10,1)$, $B(4,11)$ and $C(0,-7)$
21. $A(-1,3)$, $B(5,7)$ and $C(9,-5)$
22. $A(-4,-15)$, $B(2,-1)$ and $C(-20,11)$

Multi-Step Problem The midpoints of the sides of a triangle are $A(1,5)$, $B(4,-2)$, and $C(-5,1)$. Answer the following questions. The graph is below.



23. Find the slope of AB, BC, and AC.
24. The side that passes through A should be parallel to which midsegment? ($\triangle ABC$ are all midsegments of a triangle).
25. Using your answer from #24, take the slope of $\overline{BC}$ and use the "rise over run" in either direction to create a parallel line to $\overline{BC}$ that passes through A. Extend it with a ruler.
26. Repeat #24 and #25 with B and C. What are coordinates of the larger triangle?

Multi-Step Problem The midpoints of the sides of $\triangle RST$ are $G(0,-2)$, $H(9,1)$, and $I(6,-5)$. Answer the following questions.

27. Find the slope of GH, HI, and GI.
28. Plot the three midpoints and connect them to form midsegment triangle, $\triangle GHI$.
29. Using the slopes, find the coordinates of the vertices of $\triangle RST$. (#22 above)
30. Find GH using the distance formula. Then, find the length of the sides it is parallel to. What should happen?

Review and Reflect

31. Why is the length of the midsegment half the length of the side it is parallel to?
32. Explain how you can justify triangle congruence with the 4 triangles formed by the three midsegments.

Warm-Up Answers

1. $\left(\frac{-4+6}{2}, \frac{1+7}{2}\right) = (1,4)$
2. $\left(\frac{5+11}{2}, \frac{-3+5}{2}\right) = (8,1)$
3. $m = \frac{-3-1}{-2-(-1)} = \frac{-4}{-1} = 4$
   $y = mx + b$
   $-3 = 4(-2) + b$
   $b = 5,\ y = 4x + 5$
4. $-7 = 4(2) + b$
   $b = -15,\ y = 4x - 15$

5.2 Perpendicular Bisectors and Angle Bisectors in Triangles

TEKS G(1)D, G(2)B, G(5)B, G(5)C, G(6)A

Learning Objectives

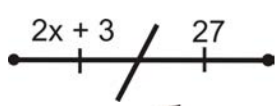
- Apply the Perpendicular Bisector Theorem and its converse
- Apply the Angle Bisector Theorem and its converse
- Analyze properties of perpendicular bisectors and angle bisectors

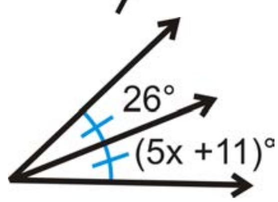
Vocabulary

- perpendicular bisector
- angle bisector

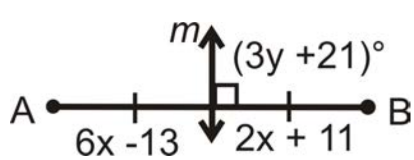
Warm-Up

1. Construct the perpendicular bisector of a 3 inch line. Use Investigation 1-4 from Chapter 1 to help you.
2. Construct the angle bisector of an 80° angle (Investigation 1-5).
3. Find the value of x.

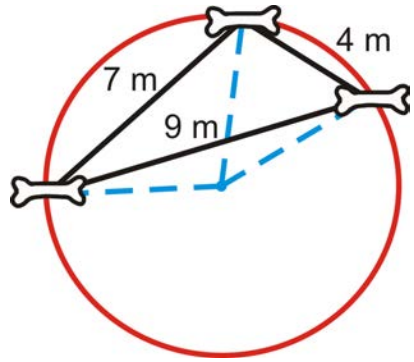
a. 

b. 

4. Find the value of x and y. Is m the perpendicular bisector of AB? How do you know?

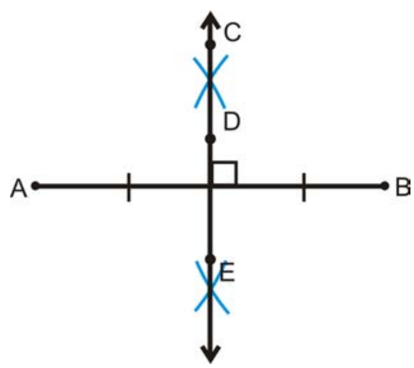


Know What? An archeologist has found three bones in Cairo, Egypt. The bones are 4 meters apart, 7 meters apart and 9 meters apart (to form a triangle). The likelihood that more bones are in this area is very high. If these bones are on the edge of the digging circle, where is the center of the circle?



Perpendicular Bisectors

In Chapter 1, you learned that a perpendicular bisector intersects a line segment at its midpoint and is perpendicular. Let's analyze your construction from #1.



Investigation 5-1: Properties of Perpendicular Bisectors

Tools Needed: #1 from the Review Queue, ruler, pencil

1. Look at your construction (#1 from the Review Queue).

Draw three points on the perpendicular bisector, two above the line and one below it. Label all the points like the picture on the right.

2. Measure the following distances with a ruler: AD, DB, AC, CB, AE, and EB. Record them on your paper.

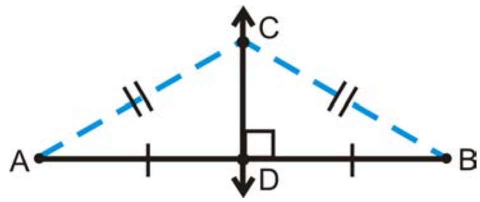
What do you notice about these distances?

From the investigation, you should notice that $AD = DB$, $AC = CB$, and $AE = EB$. This means that C, D, and E are *equidistant* from A and B.

Perpendicular Bisector Theorem

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If $\overleftrightarrow{CD} \perp \overline{AB}$ and $AD = DB$, then $AC = CB$.



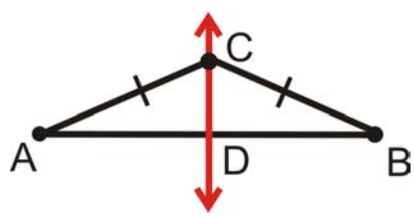
The proof of the Perpendicular Bisector Theorem is in the exercises for this section. In addition to the Perpendicular Bisector Theorem, the converse is also true.

Perpendicular Bisector Theorem Converse

If a point is equidistant from the endpoints of a segment, then the point is on the perpendicular bisector of the segment.

Using the picture above: If $AC = CB$, then $\overleftrightarrow{CD} \perp \overline{AB}$ and $AD = DB$.

Proof of the Perpendicular Bisector Theorem Converse

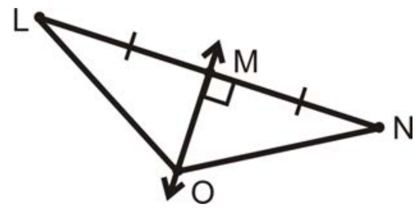


Given: $\overline{AC} \cong \overline{CB}$

Prove: $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$

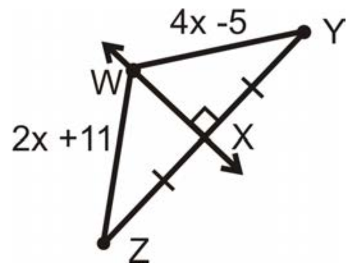
	Statement	*Reason*
	1. $\overline{AC} \cong \overline{CB}$	Given
	2. ΔACB is an isosceles triangle	Definition of an isosceles triangle
	3. $\angle A \cong \angle B$	Isosceles Triangle Theorem
	4. Draw point D, such that D is the midpoint of $\overline{AB}$.	Every line segment has exactly one midpoint
D bisects $\overline{AB}$	5. $\overline{AD} \cong \overline{DB}$	Definition of a midpoint
	6. ΔACD $\approx$ ΔBCD	SAS
	7. $\angle CDA \cong \angle CDB$	CPCTC
	8. $m\angle CDA = m\angle CDB = 90^\circ$	Congruent Supplements Theorem
$\overleftrightarrow{CD}$ is perpendicular to $\overline{AB}$	9. $\overleftrightarrow{CD} \perp \overline{AB}$	Definition of perpendicular lines
	10. $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$	Definition of perpendicular bisector

Example 1: If $\overleftrightarrow{MO}$ is the perpendicular bisector of $\overline{LN}$ and $LO = 8$, what is ON?



Solution: By the Perpendicular Bisector Theorem, $LO = ON$. So, $ON = 8$.

Example 2: ***Algebra Connection*** Find x and the length of each segment.

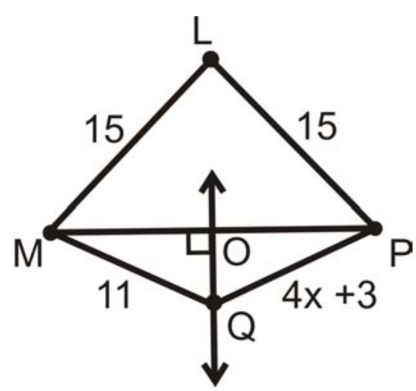


Solution: $\overleftrightarrow{WX}$ is the perpendicular bisector of $\overline{XY}$ and from the Perpendicular Bisector Theorem $WZ = WY$.

$$\begin{aligned}2x + 11 &= 4x - 5\\ 16 &= 2x\\ 8 &= x\end{aligned}$$

$WZ = WY = 2(8) + 11 = 16 + 11 = 27$.

Example 3: $\overleftrightarrow{OQ}$ is the perpendicular bisector of $\overline{MP}$.



a) Which line segments are equal?

b) Find x.

c) Is L on $\overleftrightarrow{OQ}$? How do you know?

Solution:

a) $ML = LP$, $MO = OP$, and $MQ = QP$.

b) $4x+3=11$
$4x=8$
$x=2$

c) Yes, L is on $\overleftrightarrow{OQ}$ because $ML=LP$ (the Perpendicular Bisector Theorem Converse).

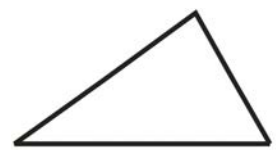
Perpendicular Bisectors and Triangles

Let's investigate what happens when we construct perpendicular lines for the sides of a triangle.

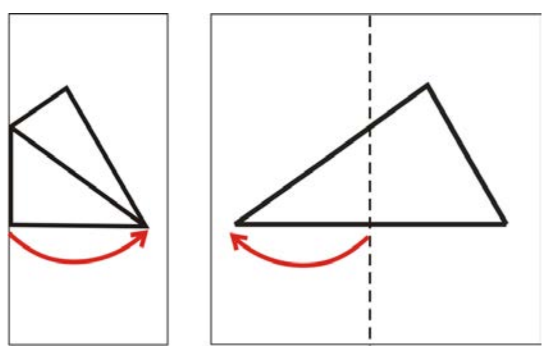
Investigation 5-2: Constructing the Perpendicular Bisectors of the Sides of a Triangle

Tools Needed: patty paper, pencil, ruler, compass

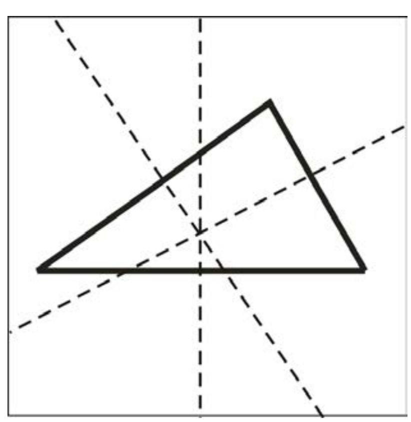
1. Draw a scalene triangle on your patty paper.



2. Fold one vertex over to meet one of the other vertices. Make sure one side perfectly overlaps itself. Crease and open.

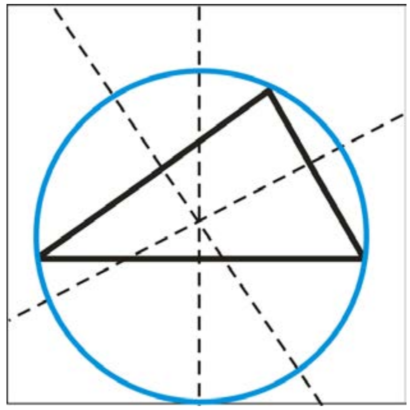


3. Repeat this process for the other two sides. Each crease is the perpendicular bisector of a side. Your paper should look like this:



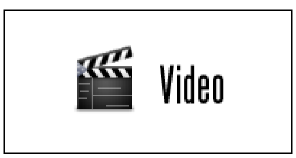
The creases, or perpendicular bisectors, intersect at the same point.

4. This point has an additional property. Put the pointer of your compass on this point of intersection. Open the compass so that the pencil is on one of the vertices. Draw a circle.



The circle you drew passes through all the vertices of the triangle. We say that this circle ***circumscribes*** the triangle or that the triangle is ***inscribed*** in the circle.

To see a construction of a perpendicular bisector, what the following video.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/153991

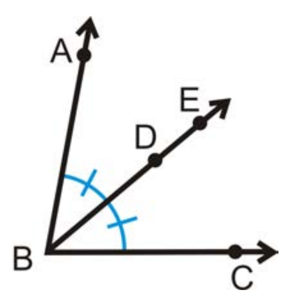
Angle Bisectors

In Chapter 1, you learned that an angle bisector cuts an angle exactly in half.

Investigation 5-3: Properties of an Angle Bisector

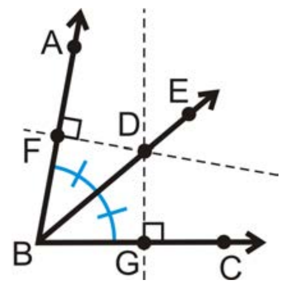
Tools Needed: #2 from your Review Queue, protractor, ruler, pencil

1. Look at #2 from the Review Queue. Label your angle like the one to the right. Place two points, D and E on the angle bisector.



2. Recall the patty paper construct of the perpendicular bisector above (Investigation 5-2). Using this idea, fold a perpendicular line to $\overrightarrow{BC}$ through D. Repeat with D and $\overrightarrow{BA}$. Label the intersections F and G.

3. Measure FD and DG. What do you notice?



4. Repeat #2 and #3 with point E. Do you have the same conclusion?

For #3, you should find that $FD = DG$ and the same thing happens with E.

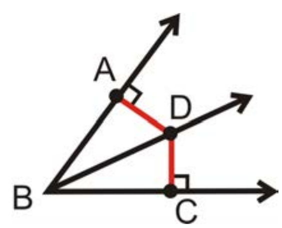
Recall from Chapter 3 that the shortest distance from a point to a line is the perpendicular length between them. $FD = DG$ and are the shortest lengths from D to each side of the angle.

Angle Bisector Theorem

If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

In other words, if $\overrightarrow{BD}$ bisects $\angle ABC$, $\overrightarrow{BA} \perp \overline{FD}$, and, $\overrightarrow{BC} \perp \overline{DG}$ then $FD = DG$.

Proof of the Angle Bisector Theorem



Given: $\overrightarrow{BD}$ bisects $\angle ABC$, $\overrightarrow{BA} \perp \overline{AD}$, and $\overrightarrow{BC} \perp \overline{DC}$

Prove: $\overline{AD} \cong \overline{DC}$

TABLE 5.1:

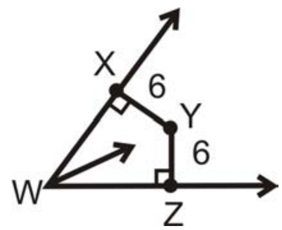
Statement	***Reason***
1. $\overrightarrow{BD}$ bisects $\angle ABC$, $\overrightarrow{BA} \perp \overline{AD}$, $\overrightarrow{BC} \perp \overline{DC}$	Given
2. $\angle ABD \cong \angle DBC$	Definition of an angle bisector
3. $\angle DAB$ and $\angle DCB$ are right angles	Definition of perpendicular lines
4. $\angle DAB \cong \angle DCB$	All right angles are congruent
5. $\overline{BD} \cong \overline{BD}$	Reflexive Property of Congruence
6. $\triangle ABD \cong \triangle CBD$	AAS
7. $\overline{AD} \cong \overline{DC}$	CPCTC

The converse of this theorem is also true. The proof is in the review questions.

Angle Bisector Theorem Converse

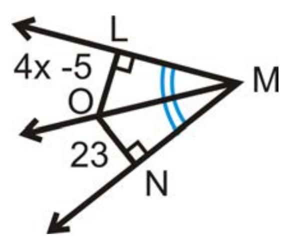
If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

Example 4: Is Y on the angle bisector of $\angle XWZ$?



Solution: If Y is on the angle bisector, then $XY = YZ$ and they need to be perpendicular to the sides of the angle. From the markings we know $\overline{XY} \perp \overrightarrow{WX}$ and $\overline{ZY} \perp \overrightarrow{WZ}$. Second, $XY = YZ = 6$. So, yes, Y is on the angle bisector of $\angle XWZ$.

Example 5: ***Algebra Connection*** $\overrightarrow{MO}$ is the angle bisector of $\angle LMN$. Find the measure of x.



Solution: $LO = ON$ by the Angle Bisector Theorem Converse.

$$\begin{aligned} 4x - 5 &= 23 \\ 4x &= 28 \\ x &= 7 \end{aligned}$$

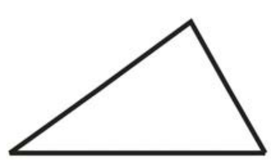
Angle Bisectors in a Triangle

Let's use patty paper to construct the angle bisector of every angle in a triangle.

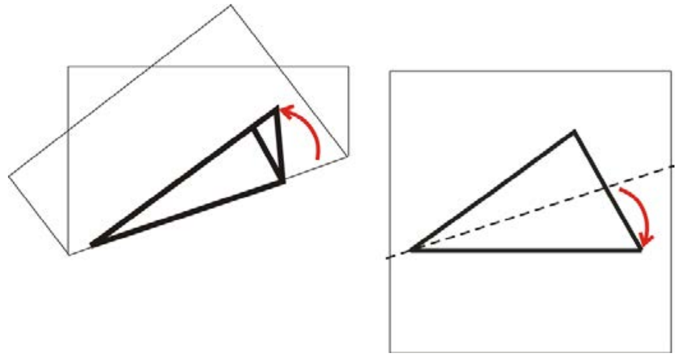
Investigation 5-4: Constructing Angle Bisectors in Triangles

Tools Needed: patty paper, ruler, pencil, compass

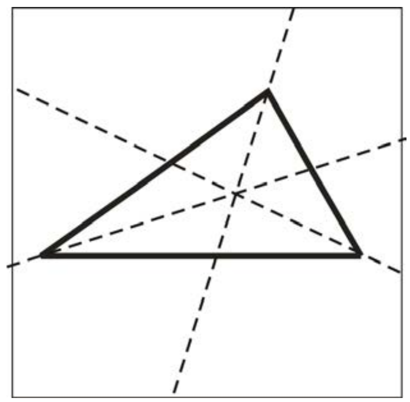
1. Draw a scalene triangle on your patty paper.



2. Fold the patty paper so that two sides of the triangle perfectly overlap and the fold passes through the vertex between these sides. Crease and open.



3. Repeat Step 2 for the other two angles. Your paper should look like:

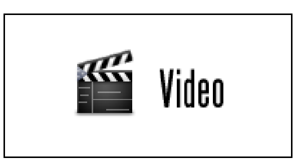


The creases, or angle bisectors, intersect at the same point.

4. This point has an additional property. Place the pointer of the compass on this point. Open the compass "straight down" so that the pencil touches one side of the triangle (the pink line in the picture to the right). Draw a circle.

Notice that the circle touches all three sides of the triangle. We say that this circle is ***inscribed*** in the triangle because it touches all three sides.

The following video shows a construction for angle bisector within a triangle.



MEDIA

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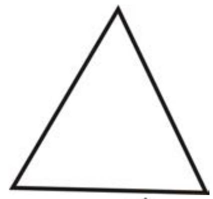
Know What? Revisited If the bones are the vertices of a triangle, then the center of the circle will be the intersection of the perpendicular bisectors. Use Investigation 5-2 to find the perpendicular bisector of at least two sides.

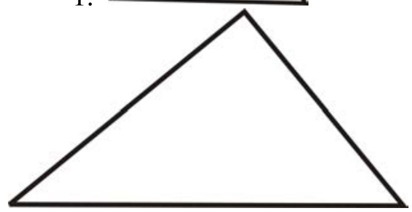
Practice Problems

- Questions 1-3 are similar to Investigation 5-2.
- Questions 4-6 are similar to Investigation 5-4.
- Questions 7-15 are similar to Examples 1-3.
- Questions 16-24 are similar to Examples 4 and 5.

- Question 25-28 are a review of perpendicular lines.
- Questions 29 and 30 are similar to the two proofs in this section.

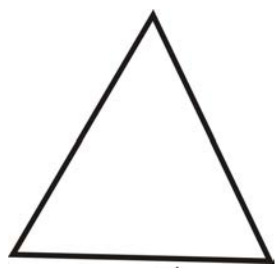
Construction Find the point of intersection of the perpendicular bisectors by tracing each triangle onto a piece of paper (or patty paper) and using Investigation 5-2.

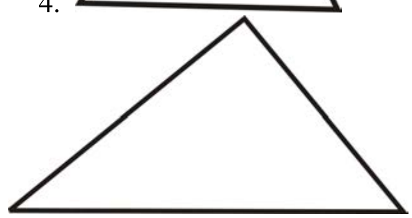
1. 

2. 

3. Construct equilateral triangle $\triangle ABC$ (Investigation 4-6). Construct the perpendicular bisectors of the sides of the triangle. Connect the point of intersection to each vertex. Your original triangle is now divided into six triangles. What can you conclude about the six triangles?

Construction Find the point of intersection of the angle bisectors by tracing each triangle onto a piece of paper (or patty paper) and using Investigation 5-4. Construct the inscribed circle.

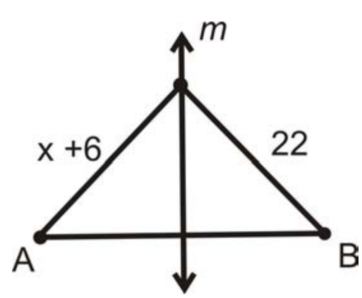
4. 

5. 

6. Refer back to #3, if you were to construct the angle bisectors of an equilateral triangle, what do you think would happen? Would the result of #3 be any different?

Algebra Connection For questions 7-12, find the value of x. m is the perpendicular bisector of AB.

7.



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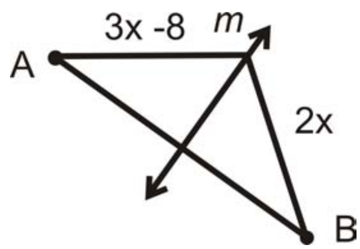

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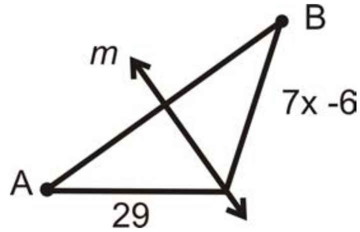

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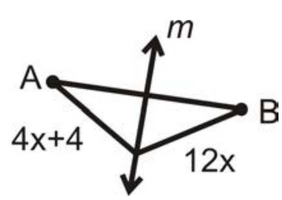

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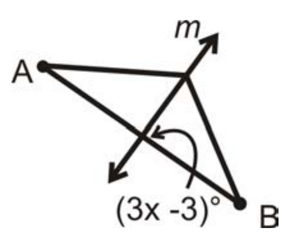

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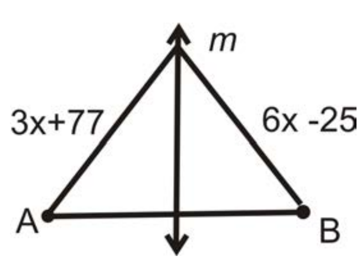
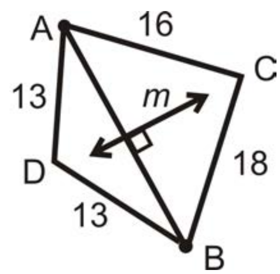
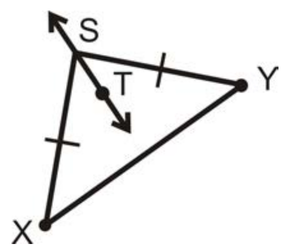


13. m is the perpendicular bisector of $\overline{AB}$.
    a. List all the congruent segments.
    b. Is C on $\overline{AB}$? Why or why not?
    c. Is D on $\overline{AB}$? Why or why not?

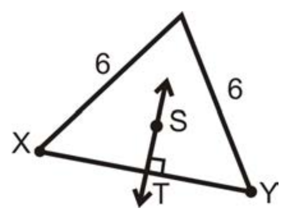


For Questions 14 and 15, determine if $\overleftrightarrow{ST}$ is the perpendicular bisector of $\overline{XY}$. Explain why or why not.

14.

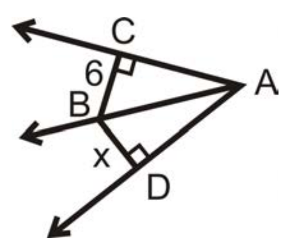


15.



For questions 6-11, $\overrightarrow{AB}$ (is the angle bisector of $\angle CAD$. Solve for the missing variable.

16.



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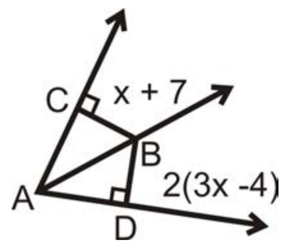

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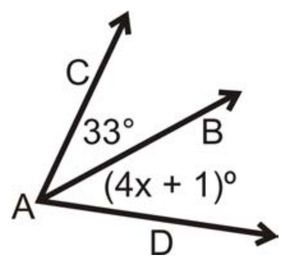

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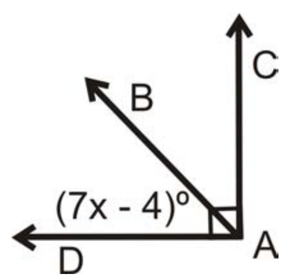

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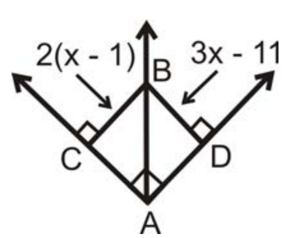

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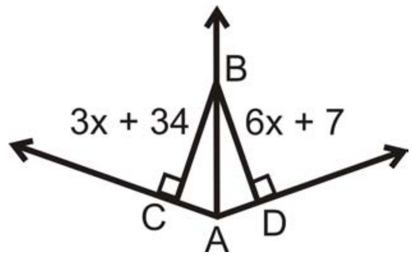
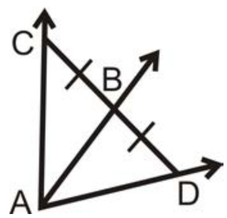


Is there enough information to determine if $\overrightarrow{AB}$ is the angle bisector of $\angle CAD$? Why or why not?

22.



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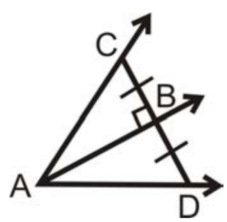

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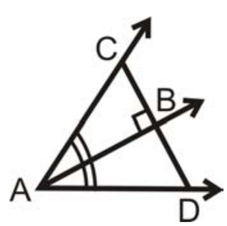
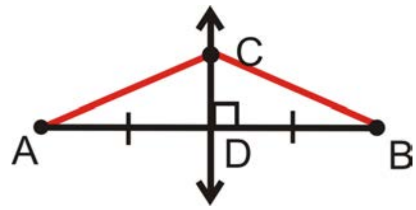


For Questions 25-28, consider line segment $\overline{AB}$ with endpoints $A(2,1)$ and $B(6,3)$.

25. Find the slope of AB.
26. Find the midpoint of AB.
27. What is the slope of the perpendicular line to AB?
28. Find the equation of the line with the slope from #27 and through the midpoint from #26. This is the perpendicular bisector of $\overline{AB}$.
29. Fill in the blanks of the proof of the Perpendicular Bisector Theorem.

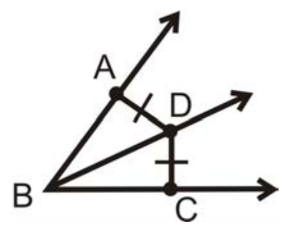


Given: $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$Prove: $\overline{AC} \cong \overline{CB}$

TABLE 5.2:

Statement	Reason
1.	
2. D is the midpoint of $\overline{AB}$	
3.	Definition of a midpoint
4. $\angle CDA$ and $\angle CDB$ are right angles	
5. $\angle CDA \cong \angle CDB$	
6.	Reflexive Property of Congruence
7. $\triangle CDA \cong \triangle CDB$	
8. $\overline{AC} \cong \overline{CB}$	

30. Fill in the blanks in the Angle Bisector Theorem Converse.



Given: $\overline{AD} \cong \overline{DC}$, such that AD and DC are the shortest distances to $\overrightarrow{BA}$ and $\overrightarrow{BC}$Prove: $\overrightarrow{BD}$ bisects $\angle ABC$

TABLE 5.3:

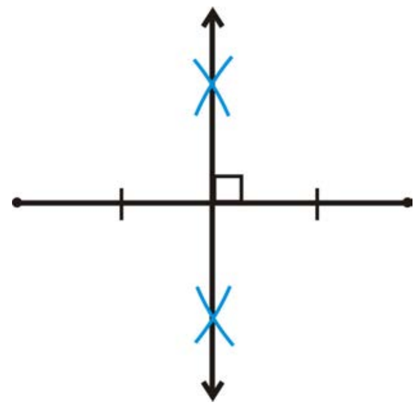
Statement	Reason
1.	
2.	The shortest distance from a point to a line is perpendicular.
3. $\angle DAB$ and $\angle DCB$ are right angles	
4. $\angle DAB \cong \angle DCB$	
5. $\overline{BD} \cong \overline{BD}$	
6. $\triangle ABD \cong \triangle CBD$	
7.	CPCTC
8. $\overrightarrow{BD}$ bisects $\angle ABC$	

Review and Reflect

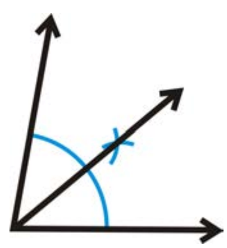
31. Compare and contrast angle bisectors and perpendicular bisectors.
32. How does the slope of a midsegment compare to the slope of the perpendicular bisector?

Warm-Up Answers

1.



2.



3. $2x+3=27$

(a) $2x=24$
$x=12$

(b) $5x+11=26$
$5x=15$
$x=3$

4. $6x-13=2x+11$ $\quad$ $3y+21=90^\circ$
$4x=24$ $\quad$ $3y=69^\circ$
$x=6$ $\quad$ $y=23^\circ$

Yes, m is the perpendicular bisector of AB because it is perpendicular to AB and passes through the midpoint.

5.3 Medians and Altitudes in Triangles

TEKS G(1)B, G(5)A, G(5)B, G(6)D

Learning Objectives

- Define median and find the properties of the centroid
- Apply medians to the coordinate plane
- Construct the altitude of a triangle

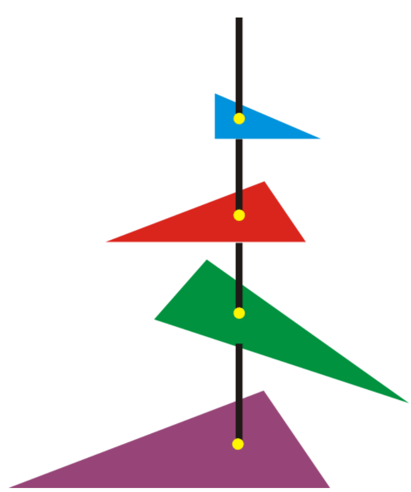
Vocabulary

- median
- centroid
- altitude

Warm-Up

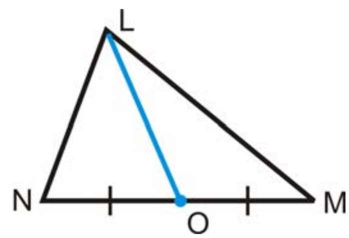
1. Find the midpoint between (9, -1) and (1, 15).
2. Find the slope between (9, -1) and (1, 15). Then find the equation of the line.
3. Find the equation of the line that is perpendicular to the line from #2 through (-6, 2).

Know What? Triangles are frequently used in art. Your art teacher assigns an art project involving triangles. You decide to make a series of hanging triangles of all different sizes from one long piece of wire. Where should you hang the triangles from so that they balance horizontally?



Medians

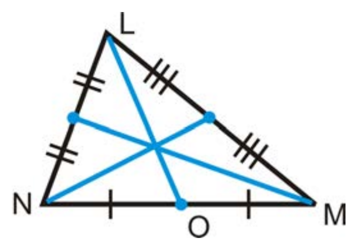
Median: The line segment that joins a vertex and the midpoint of the opposite side (of a triangle).



$\overline{LO}$ is the median from L to the midpoint of $\overline{NM}$.

Example 1: Find the other two medians of $\triangle LMN$.

Solution: Find the midpoints of sides $\overline{LN}$ and $\overline{LM}$, using a ruler. Be sure to always include the appropriate tick marks for the midpoints.

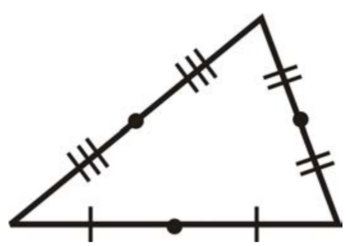


Centroid: The point of intersection for the medians of a triangle.

Investigation 5-5: Properties of the Centroid

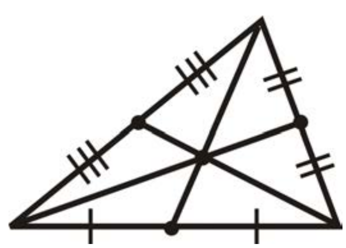
Tools Needed: pencil, paper, ruler, compass

1. Construct a scalene triangle with sides of length 6 cm, 10 cm, and 12 cm (Investigation 4-2). Use the ruler to measure each side and mark the midpoint.



2. Draw in the medians and mark the centroid.

Measure the length of each median. Then, measure the length from each vertex to the centroid and from the centroid to the midpoint. Do you notice anything?



3. Cut out the triangle. Place the centroid on either the tip of the pencil or the pointer of the compass. What happens?

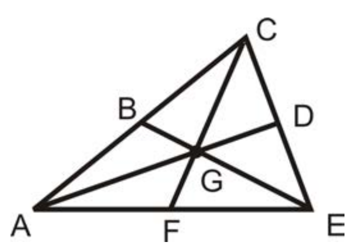
The properties discovered are summarized below.

Median Theorem

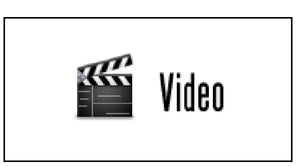
The medians of a triangle intersect at a point that is two-thirds of the distance from the vertices to the midpoint of the opposite side. The centroid is the "balancing point" of a triangle.

If G is the centroid, then:

$$\begin{aligned} & AG = \frac{2}{3}AD,\ CG = \frac{2}{3}CF,\ EG = \frac{2}{3}BE \\ & DG = \frac{1}{3}AD,\ FG = \frac{1}{3}CF,\ BG = \frac{1}{3}BE \\ \text{And}: \quad & DG = \frac{1}{2}AG,\ FG = \frac{1}{2}CG,\ BG = \frac{1}{2}EG \end{aligned}$$



The following video show a construction of a median.



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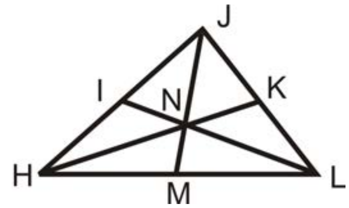
Example 2: I, K, and M are midpoints of the sides of $\triangle HJL$.



a) If $JM = 18$, find JN and NM.

b) If $HN = 14$, find NK and HK.

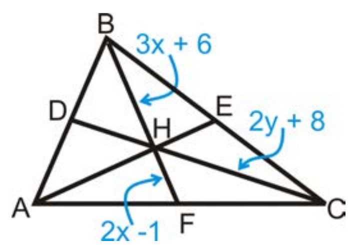
Solution:

a) $JN = \frac{2}{3} \cdot 18 = 12$. $NM = JM - JN = 18 - 12$. $NM = 6$.

b) $14 = \frac{2}{3} \cdot HK$

$14 \cdot \frac{3}{2} = HK = 21$. NK is a third of 21, $NK = 7$.

Example 3: ***Algebra Connection*** H is the centroid of $\triangle ABC$ and $DC = 5y - 16$. Find x and y.



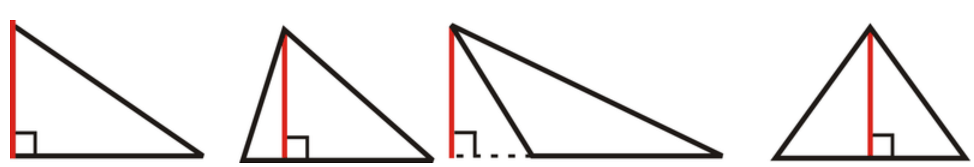
Solution:

$\frac{1}{2}BH = HF \longrightarrow BH = 2HF$
$3x + 6 = 2(2x - 1)$
$3x + 6 = 4x - 2$
$x = 8 \qquad HC = \frac{2}{3}DC \longrightarrow \frac{3}{2}HC = DC$
$\frac{3}{2}(2y + 8) = 5y - 16$
$3y + 12 = 5y - 16$
$28 = 2y$
$y = 14$

Altitudes

The last line segment within a triangle is an altitude. It is also called the height of a triangle.

Altitude: A line segment from a vertex and perpendicular to the opposite side. The red lines below are all altitudes.

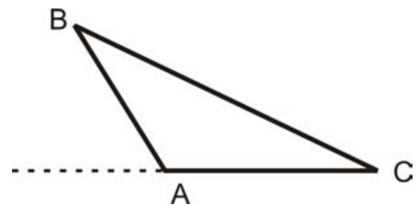


When a triangle is a right triangle, the altitude, or height, is the leg. If the triangle is obtuse, then the altitude will be outside of the triangle.

Investigation 5-6: Constructing an Altitude for an Obtuse Triangle

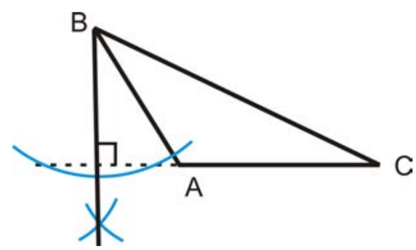
Tools Needed: pencil, paper, compass, ruler

1. Draw an obtuse triangle. Label it $\triangle ABC$, like the picture to the right. Extend side $\overline{AC}$, beyond point A.



2. Using Investigation 3-2, construct a perpendicular line to $\overline{AC}$, through B.

The altitude does not have to extend past side $\overline{AC}$, as it does in the picture. Technically the height is only the vertical distance from the highest vertex to the opposite side.



If you are constructing an altitude for an acute triangle, you may skip Step 1 of this investigation.

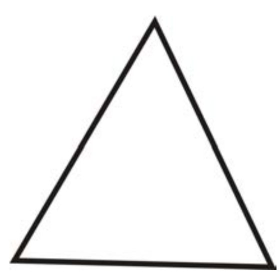
Each of these special segments inside a triangle, meet at a point of concurrency. The point of concurrency has a different name depending on the special segment drawn. There are many centers of a triangle, but the following link will show you the four most popular points of concurrency. http://www.mathsisfun.com/geometry/triangle-centers.html

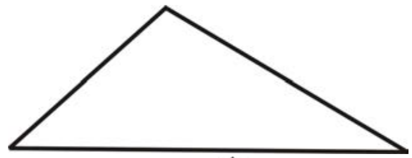
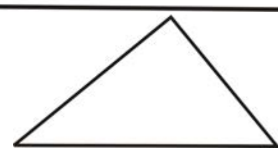
Know What? Revisited The point that you should put the wire through is the centroid. That way, each triangle will balance.

Practice Problems

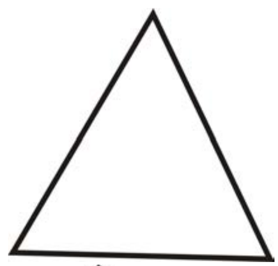
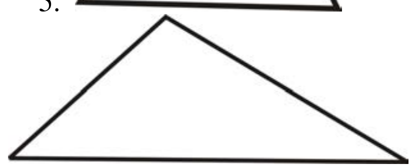
- Questions 1-4 use Investigation 5-5.
- Questions 5-6 use Investigation 3-2 and 5-6.
- Questions 7-18 are similar to Examples 2 and 3.
- Questions 19-26 use review to discover something new.
- Questions 27-34 use the definitions of perpendicular bisector, angle bisector, median and altitude.
- Question 35 is similar to the proofs in the previous section.

Construction Construct the centroid for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-5.

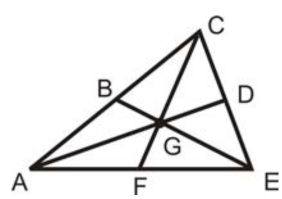
1. 

2. 
3. 
4. Is the centroid always going to be inside of the triangle? Why?

Construction Construct the altitude from the top vertex for the following triangles. Trace each triangle onto a piece of paper and using Investigations 3-2 and 5-6.

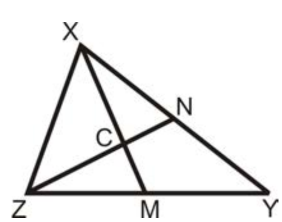
5. 
6. 

For questions 7-11, B, D, and F are the midpoints of each side and G is the centroid. Find the following lengths.



7. If $BG = 5$, find GE and BE
8. If $CG = 16$, find GF and CF
9. If $AD = 30$, find AG and GD
10. If $GF = x$, find GC and CF
11. If $AG = 9x$ and $GD = 5x - 1$, find x and AD.

For questions 12-18, N and M are the midpoints of sides $\overline{XY}$and $\overline{ZY}$.



12. What is point C?
13. If $XN = 5$, find XY.

14. If $XC = 6$, find XM.
15. If $ZN = 45$, find CN.
16. If $CM = 4$, find XC.
17. If $ZM = y$, find ZY.
18. If $ZN = 6x + 15$ and $ZC = 38$, find x and ZN.

Multistep Problem Find the equation of a median in the $x - y$ plane.

19. Plot $\triangle ABC: \ A(-6,4), \ B(-2,4)$ and $C(6,-4)$
20. Find the midpoint of $\overline{AC}$. Label it D.
21. Find the slope of $\overline{BD}$.
22. Find the equation of $\overline{BD}$.
23. Plot $\triangle DEF: \ D(-1,5), \ E(0,-1), \ F(6,3)$
24. Find the midpoint of $\overline{EF}$. Label it G.
25. Find the slope of $\overline{DG}$.
26. Find the equation of $\overline{DG}$.

Determine if the following statements are true or false. If false, give a counter example.

27. The perpendicular bisector goes through the midpoint of a line segment.
28. The perpendicular bisector goes through a vertex.
29. The angle bisector passes through the midpoint.
30. The median bisects the side it intersects.
31. The angle bisectors intersect at one point.
32. The altitude of an obtuse triangle is inside a triangle.
33. The centroid is the balancing point of a triangle.
34. A median and a perpendicular bisector intersect at the midpoint of the side they intersect.

Fill in the blanks in the proof below.

35. Given: Isosceles $\triangle ABC$ with legs $\overline{AB}$ and $\overline{AC}\overline{BD} \perp \overline{DC}$ and $\overline{CE} \perp \overline{BE}$Prove: $\overline{BD} \cong \overline{CE}$

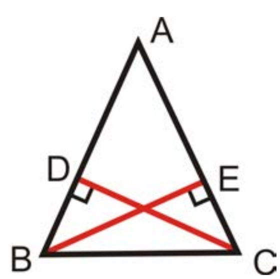


TABLE 5.4:

Statement	***Reason***
1. Isosceles $\triangle ABC$ with legs $\overline{AB}$ and $\overline{AC}$ $\overline{BD} \perp \overline{DC}$ and $\overline{CE} \perp \overline{BE}$	
2. $\angle DBC \cong \angle ECB$	
3.	Definition of perpendicular lines
4. $\angle BEC \cong \angle CEB$	
5.	Reflexive Property of Congruence
6. $\triangle BEC \cong \triangle CDB$	
7. $\overline{BD} \cong \overline{CE}$	

Review and Reflect

36. When performing the Centroid Investigation, why do you think the triangle balances on the end of your pencil?

37. Draw an three triangles; an acute, an obtuse, and a right triangle. Next, draw all the altitudes. Where do the altitude in each triangle meet?

Warm-Up Answers

1. $midpoint = \left(\frac{9+1}{2}, \frac{-1+15}{2}\right) = (5,7)$
2. $m = \frac{15+1}{1-9} = \frac{16}{-8} = -2$ $\quad 15 = -2(1) + b \quad$ $y = -2x + 17$
   $17 = b$
3. $y = \frac{1}{2}x + b$
   $2 = \frac{1}{2}(-6) + b$
   $2 = -3 + b$
   $5 = b$
   $y = \frac{1}{2}x + 5$

5.4 Inequalities in Triangles

TEKS G(1)G, G(5)D

Learning Objectives

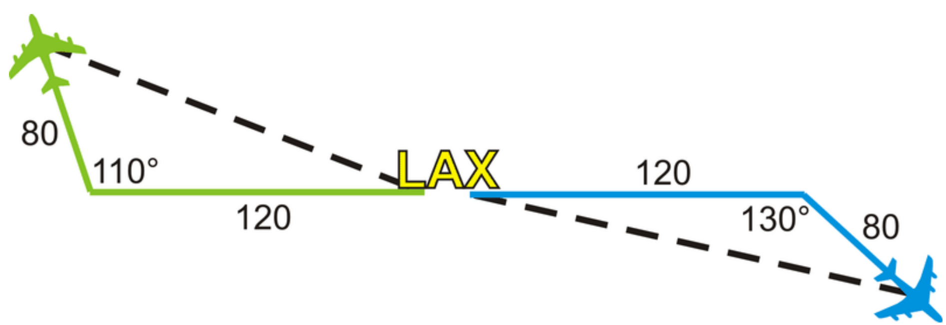
- Determine relationships among the angles and sides of a triangle
- Verify and Use the Triangle Inequality Theorem
- Understand the SAS Inequality Theorem and its converse

Warm-Up

Solve the following inequalities.

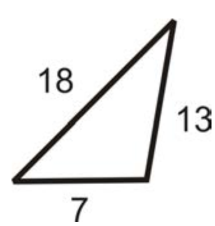
1. $4x-9 \leq 19$
2. $-5 > -2x+13$
3. $\frac{2}{3}x+1 \geq 13$
4. $-7 < 3x-1 < 14$

Know What? Two planes take off from LAX. Their flight patterns are to the right. Both planes travel 200 miles, but which one is further away from LAX?



Comparing Angles and Sides

Look at the triangle to the right. The sides of the triangle are given. Can you determine which angle is the largest? The largest angle will be opposite 18 because it is the longest side. Similarly, the smallest angle will be opposite 7, which is the shortest side.



Theorem 5-9

If one side of a triangle is longer than another side, then the angle opposite the longer side will be larger than the angle opposite the shorter side

Converse of Theorem 5-9

If one angle in a triangle is larger than another angle in a triangle, then the side opposite the larger angle will be longer than the side opposite the smaller angle

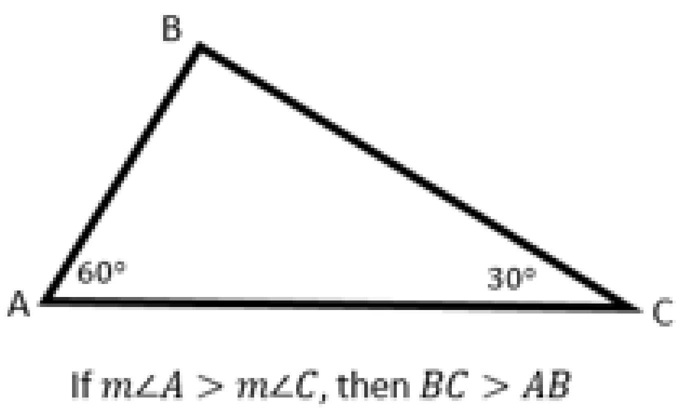
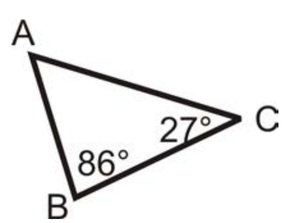


FIGURE 5.1

To prove these theorems, we will do so indirectly. This will be done in the extension at the end of this chapter.

Example 1: List the sides in order, from shortest to longest.

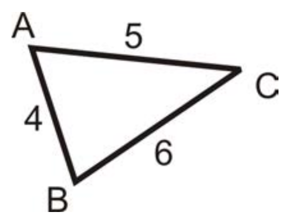


Solution: First, find $m\angle A$. From the Triangle Sum Theorem:

$$\begin{aligned} m\angle A + 86^\circ + 27^\circ &= 180^\circ \\ m\angle A &= 67^\circ \end{aligned}$$

86° is the largest angle, so AC , the side opposite is the longest side. The next angle is 67°, so BC, the side opposite would be the next longest side. 27° is the smallest angle, so AB, the side opposite, is the shortest side. In order, the answer is: AB, BC, AC.

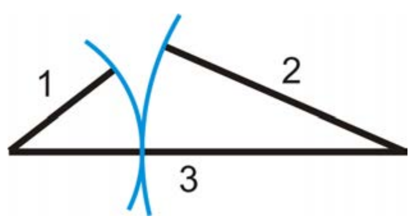
Example 2: List the angles in order, from largest to smallest.



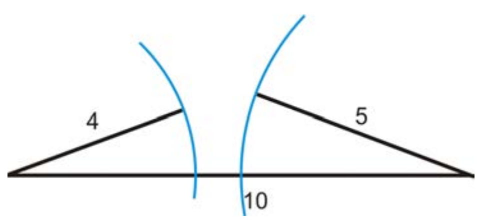
Solution: Just like with the sides, the largest angle is opposite the longest side. The longest side is BC, so the largest angle is $\angle A$. Next would be $\angle B$ and then $\angle A$.

Triangle Inequality Theorem

Can any three lengths make a triangle? The answer is no. For example, the lengths 1, 2, 3 cannot make a triangle because $1+2=3$, so they would all lie on the same line. The lengths 4, 5, 10 also cannot make a triangle because $4+5=9$.



The arc marks show that the two sides would never meet to form a triangle.



Check this out! **https://tube.geogebra.org/student/m577031**

Triangle Inequality Theorem

Any two sides of a triangle must have a sum greater than the third side

Example 3: Do the lengths below make a triangle?

a) 4.1, 3.5, 7.5

b) 4, 4, 8

c) 6, 7, 8

Solution: To verify the Triangle Inequality Theorem, you can add up the two shorter sides and they must be greater than the third.

a) $4.1+3.5 > 7.5$ Yes, $7.6 > 7.5$

b) $4+4 = 8$ No, not a triangle. Two lengths ***cannot equal*** the third.

c) $6+7 > 8$ Yes, $13 > 8$

If you are given two lengths of a triangle and are trying to come up with a possible length of the third side, remember the sum of any two sides has to be greater than the length of the third so. In other words, ***If two sides are lengths*** a ***and*** b***, then the third side,*** s***, has the range*** $|a-b| < s < a+b$.

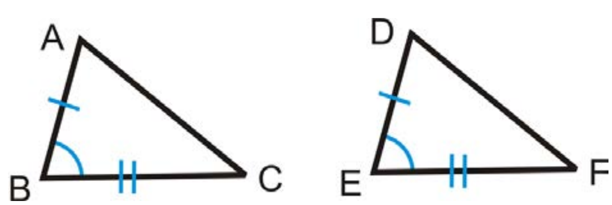
Example 4: Find the length of the third side of a triangle if the other two sides are 10 and 6.

Solution: The Triangle Inequality Theorem can also help you find the range of the third side. The two given sides are 6 and 10. The third side, s, must be between $10-6 = 4$ and $10+6 = 16$. In other words, the range of values for s is $4 < s < 16$.

Notice the range is no less than 4, and ***not equal*** to 4. The third side could be 4.1 because $4.1+6 > 10$. For the same reason, s cannot be greater than 16, but it could 15.9, $10+6 > 15.9$.

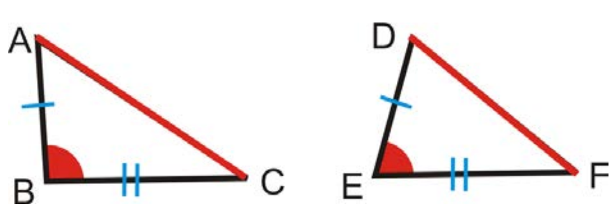
The SAS Inequality Theorem

The SAS Theorem compares two triangles. If we have two congruent triangles $\triangle ABC$ and $\triangle DEF$, marked below:



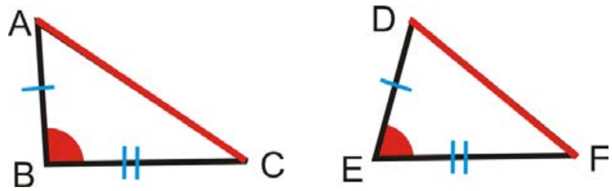
Therefore, if $AB = DE$ and $BC = EF$ and $m\angle B = m\angle E$, then $AC = DF$.

Now, let's make $m\angle B > m\angle E$. Would that make $AC > DF$? Yes.



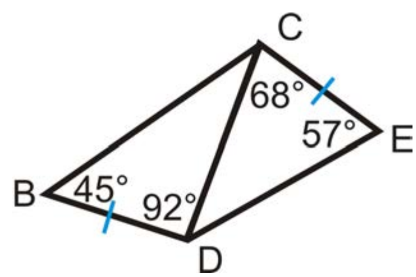
The SAS Inequality Theorem

If two sides of a triangle are congruent to two sides of another triangle, but the included angle of one triangle has a greater measure than the included angle of the other triangle, then the third side of the first triangle is longer than the third side of the second triangle.



If $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$ and $m\angle B > m\angle E$, then $\overline{AC} > \overline{DF}$.

Example 5: List the sides in order, from least to greatest.



Solution: Let's start with $\triangle DCE$. The missing angle is 55°. By Theorem 5-9, the sides, in order are CE, CD, and DE.

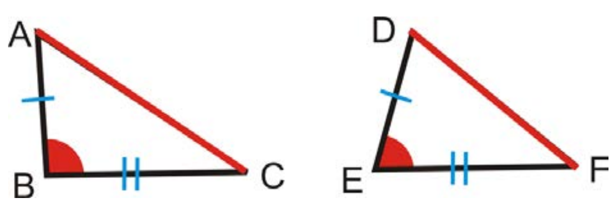
For $\triangle BCD$, the missing angle is 43°. Again, by Theorem 5-9, the order of the sides is BD, CD, and BC.

By the SAS Inequality Theorem, we know that $BC > DE$, so the order of all the sides would be: $BD = CE$, CD, DE, BC.

SSS Inequality Theorem

SSS Inequality Theorem

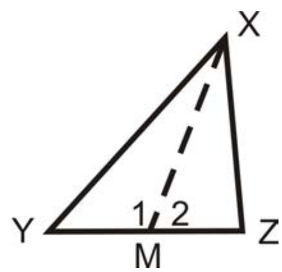
IF two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle



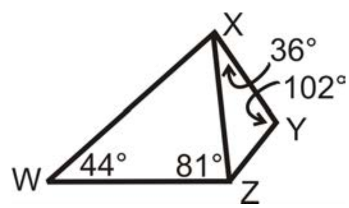
If $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{AC} > \overline{DF}$, then $m\angle B > m\angle E$.

Example 6: If $\overline{XM}$ is a median of $\triangle XYZ$ and $XY > XZ$, what can we say about $m\angle 1$ and $m\angle 2$?

Solution: M is the midpoint of $\overline{YZ}$, so $YM = MZ$. $MX = MX$ by the Reflexive Property and we know $XY > XZ$. We can use the SSS Inequality Theorem to say $m\angle 1 > m\angle 2$.



Example 7: List the sides of the two triangles in order, from least to greatest.



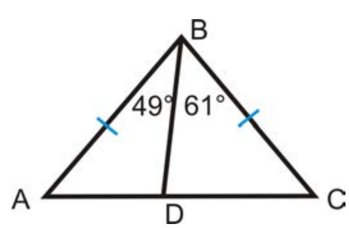
Solution: There are no congruent sides or angles. Look at each triangle separately.

$\triangle XYZ$: The missing angle is 42°. By Theorem 5-9, the order of the sides is YZ, XY, and XZ.

$\triangle WXZ$: The missing angle is 55°. The order is XZ, WZ, and WX.

Because the longest side in $\triangle XYZ$ is the shortest side in $\triangle WXZ$, we can put all the sides together in one list: YZ, XY, XZ, WZ, WX.

Example 8: Below is isosceles triangle $\triangle ABC$. List everything you can about the triangle and why.



Solution:

$AB = BC$ because it is given.

$m\angle A = m\angle C$ by the Base Angle Theorem.

$AD < DC$ because $m\angle ABD < m\angle CBD$ and the SAS Triangle Inequality Theorem.

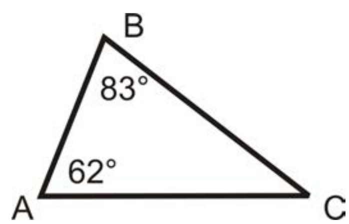
Know What? Revisited The blue plane is further away from LAX because $110° < 130°$. (SAS Inequality Theorem)

Practice Problems

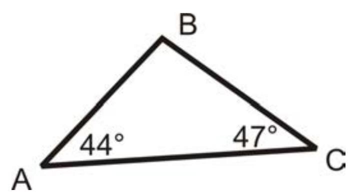
- Questions 1-9 are similar to Examples 1 and 2.
- Questions 10-18 are similar to Example 3.
- Questions 19-26 are similar to Example 4.
- Questions 27 and 28 are similar to Examples 5 and 6.
- Question 29 is similar to Example 7.
- Question 30 is similar to Example 8.

For questions 1-3, list the sides in order from shortest to longest.

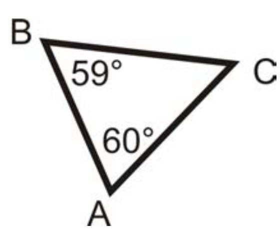
1.



2.

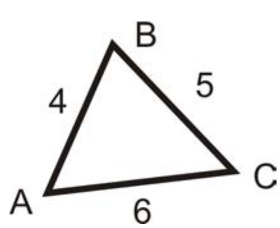


3.

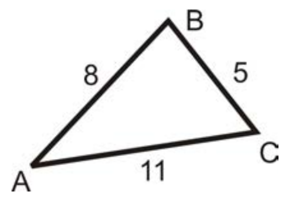


For questions 4-6, list the angles from largest to smallest.

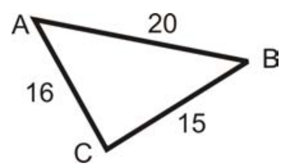
4.



5.



6.



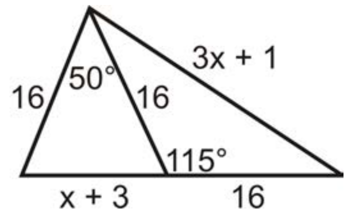
4. Draw a triangle with sides 3 cm, 4 cm, and 5 cm. The angle measures are 90°, 53°, and 37°. Place the angle measures in the appropriate spots.
5. Draw a triangle with angle measures 56°, 54° and the included side is 8 cm. What is the longest side of this triangle?
6. Draw a triangle with sides 6 cm, 7 cm, and 8 cm. The angle measures are 75.5°, 58°, and 46.5°. Place the angle measures in the appropriate spots.

Determine if the sets of lengths below can make a triangle. If not, state why.

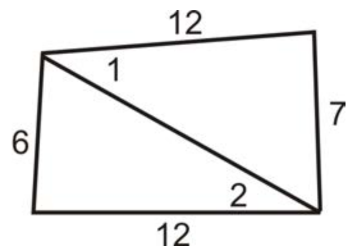
10. 6, 6, 13
11. 1, 2, 3
12. 7, 8, 10
13. 5, 4, 3
14. 23, 56, 85
15. 30, 40, 50
16. 7, 8, 14
17. 7, 8, 15
18. 7, 8, 14.99

If two lengths of the sides of a triangle are given, determine the range of the length of the third side.

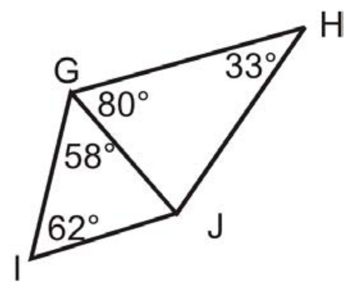
19. 8 and 9
20. 4 and 15
21. 20 and 32
22. 2 and 5
23. 10 and 8
24. x and $2x$
25. The base of an isosceles triangle has length 24. What can you say about the length of each leg?
26. The legs of an isosceles triangle have a length of 12 each. What can you say about the length of the base?
27. What conclusions can you draw about x?



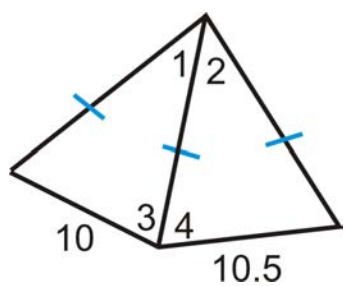
28. Compare $m\angle 1$ and $m\angle 2$.



29. List the sides from shortest to longest.



30. Compare $m\angle 1$ and $m\angle 2$. What can you say about $m\angle 3$ and $m\angle 4$?



Review and Reflect

31. Explain why the difference between and two sides of a triangle is less than the third side.
32. How does the Triangle Inequality Theorem compare to Pythagorean Theorem?

Warm-Up Answers

1. $$\begin{aligned} 4x - 9 &\leq 19 \\ 4x &\leq 28 \\ x &\leq 7 \end{aligned}$$
2. $$\begin{aligned} -5 &> -2x + 13 \\ -18 &> -2x \\ 9 &< x \end{aligned}$$

3. $\frac{2}{3}x+1 \geq 13$
   $\frac{2}{3}x \geq 12$
   $x \geq 18$
4. $-7 < 3x-1 < 14$
   $-6 < 3x < 15$
   $-2 < x < 5$

5.5 Indirect Proof

TEKS G(1)B, G(4)B

Learning Objectives

- Reason indirectly to develop proofs

Vocabulary

- Indirect Proof
- Contradiction

Until now, we have proved theorems true by direct reasoning, where conclusions are drawn from a series of facts and previously proven theorems. Indirect proof is another option.

Indirect Proof: When the conclusion from a hypothesis is assumed false (or opposite of what it states) and then a contradiction is reached from the given or deduced statements.

The easiest way to understand indirect proofs is by example.

Indirect Proofs in Algebra

Example 1: If $x = 2$, then $3x - 5 \neq 10$. Prove this statement is true by contradiction.

Solution: In an indirect proof the first thing you do is assume the conclusion of the statement is ***false.*** In this case, we will assume the ***opposite*** of $3x - 5 \neq 10$

If $x = 2$, then $3x - 5 = 10$

Take this statement as true and solve for x.

$$\begin{aligned} 3x - 5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

$x = 5$***contradicts*** the given statement that $x = 2$. Hence, our ***assumption is incorrect*** and $3x - 5 \neq 10$ is ***true***.

Example 2: If n is an integer and n^2 is odd, then n is odd. Prove this is true indirectly.

Solution: First, assume the ***opposite*** of "n is odd."

n is even.

Now, square n and see what happens.

If n is even, then $n = 2a$, where a is any integer.

$$n^2 = (2a)^2 = 4a^2$$

This means that n^2 is a multiple of 4. No odd number can be divided evenly by an even number, so this ***contradicts our assumption*** that n is even. Therefore, n must be odd if n^2 is odd.

Indirect Proofs in Geometry

Example 3: If $\triangle ABC$ is isosceles, then the measure of the base angles cannot be 92°. Prove this indirectly.

Solution: Assume the ***opposite*** of the conclusion.

The measure of the base angles are 92°.

If the base angles are 92°, then they add up to 184°. This ***contradicts*** the Triangle Sum Theorem that says all triangles add up to 180°. Therefore, the base angles cannot be 92°.

Example 4: Prove the SSS Inequality Theorem is true by contradiction.

Solution: The SSS Inequality Theorem says: "If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle." First, assume the opposite of the conclusion.

The included angle of the first triangle is less than or equal to the included angle of the second triangle.

If the included angles are equal then the two triangles would be congruent by SAS and the third sides would be congruent by CPCTC. This contradicts the hypothesis of the original statement "the third side of the first triangle is longer than the third side of the second." Therefore, the included angle of the first triangle must be larger than the included angle of the second.

To summarize:

- Assume the ***opposite*** of the conclusion (second half) of the statement.
- Proceed as if this assumption is true to find the ***contradiction.***
- Once there is a contradiction, the original statement is true.
- ***DO NOT use specific examples.*** Use variables so that the contradiction can be generalized.

Practice Problems

Prove the following statements true indirectly.

1. If n is an integer and n^2 is even, then n is even.
2. If $m\angle A \neq m\angle B$ in $\triangle ABC$, then $\triangle ABC$ is not equilateral.
3. If $x > 3$, then $x^2 > 9$.
4. The base angles of an isosceles triangle are congruent.
5. If x is even and y is odd, then $x + y$ is odd.
6. In $\triangle ABE$, if $\angle A$ is a right angle, then $\angle B$ cannot be obtuse.
7. If $A,\ B$, and C are collinear, then $AB + BC = AC$ (Segment Addition Postulate).
8. ***Challenge*** Prove the SAS Inequality Theorem is true using indirect proofs.

5.6 Chapter 5 Review

Keywords, Theorems and Postulates

Midsegments in Triangles

- Midsegment
- Midsegment Theorem

Perpendicular Bisectors and Angle Bisectors in Triangles

- Perpendicular Bisector Theorem
- Perpendicular Bisector Theorem Converse
- Inscribe
- Circumscribe
- Angle Bisector Theorem
- Angle Bisector Theorem Converse

Medians and Altitudes in Triangles

- Median
- Centroid
- Median Theorem
- Altitude

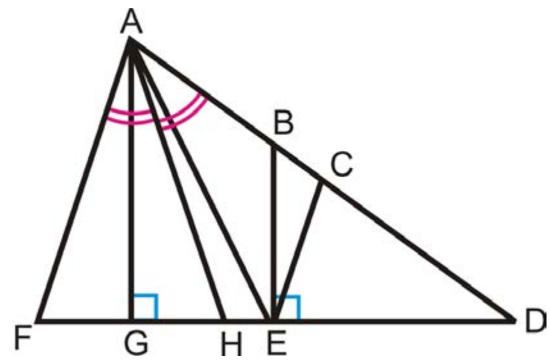
Inequalities in Triangles

- Theorem 5-9
- Converse of Theorem 5-9
- Triangle Inequality Theorem
- SAS Inequality Theorem
- SSS Inequality Theorem

Extension: Indirect Proof

Review

If C and E are the midpoints of the sides they lie on, find:



1. The perpendicular bisector of $\overline{FD}$.
2. The median of $\overline{FD}$.
3. The angle bisector of $\angle FAD$.
4. A midsegment.
5. An altitude.
6. A triangle has sides with length $x+6$ and $2x-1$. Find the range of the third side.

Fill in the blanks.

7. A midsegment connects the __________ of two sides of a triangle.
8. The height of a triangle is also called the __________.
9. The point of intersection for all the medians of a triangle is the __________.
10. The longest side is opposite the __________ angle in a triangle.
11. A point on the __________ bisector is __________ to the endpoints.
12. A point on the __________ bisector is __________ to the sides.
13. A circle is __________ when it touches all the sides of a triangle.
14. An __________ proof is also called a proof by contradiction.
15. For $\triangle ABC$ and $\triangle DEF$: $AB = DE$, $BC = EF$, and $m\angle B > m\angle E$, then __________.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9690 .

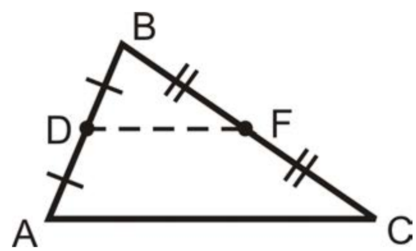
5.7 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1^{st} **Section: Midsegments**

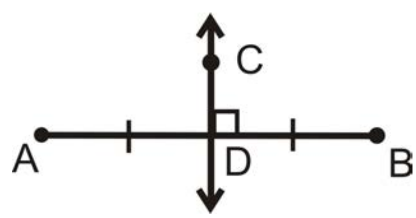
Midsegment

Midsegment Theorem



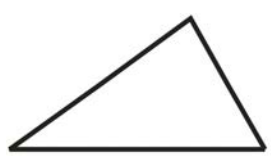
Homework:

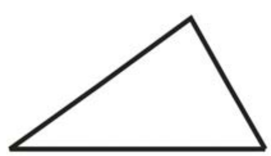
2^{nd} **Section: Perpendicular Bisectors and Angle Bisectors in Triangles**



Perpendicular Bisector Theorem

Perpendicular Bisector Theorem Converse



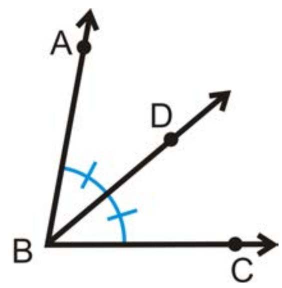


Inscribe (use the triangle)

Circumscribe (use the triangle)

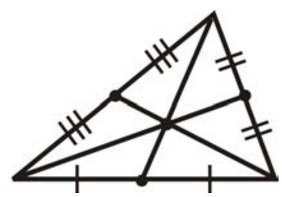
Angle Bisector Theorem

Angle Bisector Theorem Converse



Homework:

3rd **Section: Medians and Altitudes in Triangles**

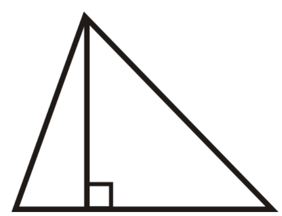


Median

Centroid

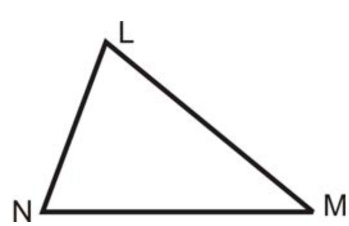
Median Theorem

Altitude



Homework:

4th **Section: Inequalities in Triangles**



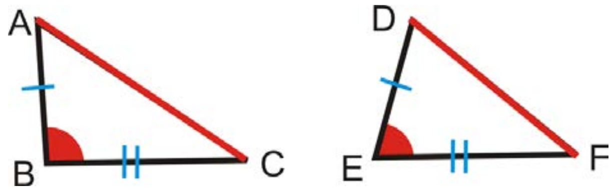
Theorem 5-9

Converse of Theorem 5-9

Triangle Inequality Theorem

SAS Inequality Theorem

SSS Inequality Theorem



Homework:

Extension: Indirect Proof

Homework:

CHAPTER 6 Similarity

Chapter Outline

In this chapter, we will start with a review of ratios and proportions. Second, we will introduce the concept of similarity and apply it to polygons, quadrilaterals and triangles. Then, we will extend proportionality to parallel lines and dilations. Finally, we conclude with an extension about fractals.

6.1 Ratios and Proportions

TEKS G(1)A

Learning Objectives

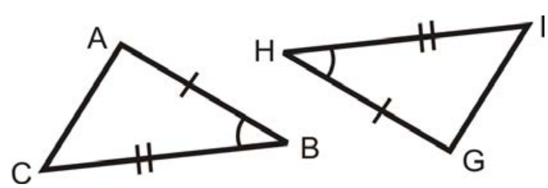
- Write and solve ratios and proportions
- Use ratios and proportions in problem solving

Vocabulary

- ratio
- equivalent ratio
- proportion
- corllary

Warm-Up

1. Are the two triangles congruent? If so, how do you know?



2. If $AC = 5$, what is GI? What is the reason?
3. How many inches are in a:
   a. foot?
   b. yard?
   c. 3 yards?
   d. 5 feet?

Know What? You want to make a scale drawing of your room and furniture for a little redecorating. Your room measures 12 feet by 12 feet. In your room are a twin bed (36 in by 75 in) and a desk (4 feet by 2 feet). You scale down your room to 8 in by 8 in, so it fits on a piece of paper. What size are the bed and desk in the drawing?

Using Ratios

Ratio: A way to compare two numbers. Ratios can be written: $\frac{a}{b}$, $a : b$, and a to b.

Example 1: There are 14 girls and 18 boys in your math class. What is the ratio of girls to boys?

Solution: The ratio would be 14:18. This can be simplified to 7:9.

Example 2: The total bagel sales at a bagel shop for Monday is in the table below. What is the ratio of ***cinnamon raisin*** bagels to ***plain*** bagels?

TABLE 6.1:

Type of Bagel	***Number Sold***
Plain	80
Cinnamon Raisin	30
Sesame	25
Jalapeno Cheddar	20
Everything	45
Honey Wheat	50

Solution: The ratio is 30:80. Reducing the ratio by 10, we get 3:8.

Reduce a ratio just like a fraction. Always reduce ratios.

Example 3: What is the ratio of ***honey wheat*** bagels to ***total bagels*** sold?

Solution: Order matters. Honey wheat is listed first, so that number comes first in the ratio (or on the top of the fraction). Find the total number of bagels sold, $80+30+25+20+45+50=250$.

The ratio is $\frac{50}{250}=\frac{1}{5}$.

Equivalent Ratios: When two or more ratios reduce to the same ratio.

50:250 and 2:10 are ***equivalent*** because they both reduce to 1:5.

Example 4: What is the ratio of ***cinnamon raisin*** bagels to ***sesame*** bagels to ***jalapeno cheddar*** bagels?

Solution: 30:25:20, which reduces to 6:5:4.

Converting Measurements

How many feet are in 2 miles? How many inches are in 4 feet? Ratios are used to convert these measurements.

Example 5: Simplify the following ratios.

a) $\frac{7\ ft}{14\ in}$

b) 9m:900cm

c) $\frac{4\ gal}{16\ gal}$

Solution: Change these so that they are in the same units. There are 12 inches in a foot.

a) $\frac{7\ \cancel{ft}}{14\ \cancel{in}}\cdot\frac{12\ \cancel{in}}{1\ \cancel{ft}}=\frac{84}{14}=\frac{6}{1}$

The inches cancel each other out. ***Simplified ratios do not have units.***

b) It is easier to simplify a ratio when written as a fraction.

$\frac{9\ \cancel{m}}{900\ \cancel{cm}}\cdot\frac{100\ \cancel{cm}}{1\ \cancel{m}}=\frac{900}{900}=\frac{1}{1}$

c) $\frac{4\ \cancel{gal}}{16\ \cancel{gal}}=\frac{1}{4}$

Example 6: A talent show has dancers and singers. The ratio of dances to singers is 3:2. There are 30 performers total, how many of each are there?

Solution: 3:2 is a reduced ratio, so there is a number, n, that we can multiply both by to find the total number in each group.

$$\begin{aligned} \text{dancers} = 3n,\ \text{singers} = 2n \ \longrightarrow\ 3n+2n &= 30 \\ 5n &= 30 \\ n &= 6 \end{aligned}$$

There are $3 \cdot 6 = 18$ dancers and $2 \cdot 6 = 12$ singers.

Solving Proportions

Proportion: Two ratios that are set equal to each other.

Example 7: Solve the proportions.

a) $\frac{4}{5} = \frac{x}{30}$

b) $\frac{y+1}{8} = \frac{5}{20}$

c) $\frac{6}{5} = \frac{2x+5}{x-2}$

Solution: To solve a proportion, you need to ***cross-multiply.***

a)

$$\begin{aligned} \frac{4}{5} &= \frac{x}{30} \\ 4 \cdot 30 &= 5 \cdot x \\ 120 &= 5x \\ 24 &= x \end{aligned}$$

b)

$$\begin{aligned} \frac{y+1}{8} &= \frac{5}{20} \\ (y+1) \cdot 20 &= 5 \cdot 8 \\ 20y + 20 &= 40 \\ 20y &= 20 \\ y &= 1 \end{aligned}$$

c)

$$\begin{aligned} \frac{6}{5} &= \frac{2x+4}{x-2} \\ 6 \cdot (x-2) &= 5 \cdot (2x+4) \\ 6x-12 &= 10x+20 \\ -32 &= 4x \\ -8 &= x \end{aligned}$$

Test your proportion solving skills with this fun game http://www.mathplayground.com/ASB_DirtBikeProportions.html

Cross-Multiplication Theorem

a, b, c, and d are real numbers, with $b \neq 0$ and $d \neq 0$. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$

The proof of the Cross-Multiplication Theorem is an algebraic proof. Recall that multiplying by the same number over itself is 1 $(b \div b = 1)$.

Proof of the Cross-Multiplication Theorem

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} && \text{Multiply the left side by } \frac{d}{d} \text{ and the right side by } \frac{b}{b}. \\ \frac{a}{b} \cdot \frac{d}{d} &= \frac{c}{d} \cdot \frac{b}{b} \\ \frac{ad}{bd} &= \frac{bc}{bd} && \text{The denominators are the same, so the tops are equal.} \\ ad &= bc \end{aligned}$$

Example 8: Your parents have an architect's drawing of their home. On the paper, the house's dimensions are 36 in by 30 in. If the shorter length of the house is actually 50 feet, what is the longer length?

Solution: Set up a proportion. If the shorter length is 50 feet, then it lines up with 30 in, the shorter length of the paper dimensions.

$$\frac{30}{36} = \frac{50}{x} \longrightarrow 1800 = 30x$$
$$60 = x \qquad \text{The longer length is 60 feet.}$$

Properties of Proportions

The Cross-Multiplication Theorem has several sub-theorems, called ***corollaries.***

Corollary: A theorem that follows directly from another theorem.

Corollaries from the Cross Multiplication Theorem

Corollary 6-1: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$. ***Switch*** b ***and*** c.
Corollary 6-2: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$. ***Switch*** a ***and*** d.
Corollary 6-3: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{c}{d}$. ***Flip each ratio upside down.***
In each corollary, you will still end up with $ad = bc$ after cross-multiplying.

Example 9: Suppose we have the proportion $\frac{2}{5} = \frac{14}{35}$. Write three true proportions that follow.

Solution: First of all, we know this is a true proportion because you would multiply $\frac{2}{5}$ by $\frac{7}{7}$ to get $\frac{14}{35}$. Using the three corollaries:

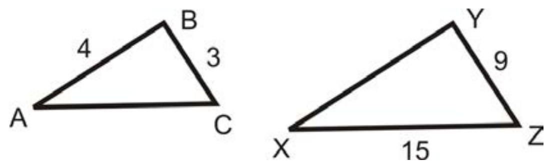
1. $\frac{2}{14} = \frac{5}{35}$
2. $\frac{35}{5} = \frac{14}{2}$
3. $\frac{5}{2} = \frac{35}{14}$

Corollary 6-4: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Corollary 6-5: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Example 10: In the picture, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

Find the measures of AC and XY.



Solution: Plug in the lengths of the sides we know.

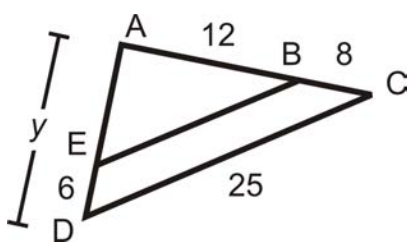
$$\frac{4}{XY} = \frac{3}{9} = \frac{AC}{15}$$

$$\frac{4}{XY} = \frac{3}{9} \qquad\qquad \frac{3}{9} = \frac{AC}{15}$$

$$36 = 3(XY) \qquad\qquad 9(AC) = 45$$

$$XY = 12 \qquad\qquad AC = 5$$

Example 11: In the picture, $\frac{ED}{AD} = \frac{BC}{AC}$. Find y.



Solution: Substitute in the lengths of the sides we know.

$$\frac{6}{y} = \frac{8}{12+8} \longrightarrow 8y = 6(20)$$
$$y = 15$$

Example 12: If $\frac{AB}{BE} = \frac{AC}{CD}$ in the picture above, find BE.

Solution:

$$\frac{12}{BE} = \frac{20}{25} \longrightarrow 20(BE) = 12(25)$$
$$BE = 15$$

Know What? Revisited Everything needs to be scaled down by a factor of $\frac{1}{18}(144\ in \div 8\ in)$. Change everything into inches and then multiply by the scale factor.

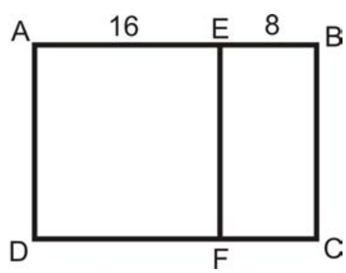
$\underline{\text{Bed}}: \text{ 36 in by 75 in } \longrightarrow \text{ 2 in by 4.167 in}$
$\underline{\text{Desk}}: \text{ 48 in by 24 in } \longrightarrow \text{2.67 in by 1.33 in}$

Practice Problems

- Questions 1 and 2 are similar to Examples 1-4.
- Questions 7-13 are similar to Example 5.
- Questions 14-19, 26, and 27 are similar to Example 6 and 8.
- Questions 20-25 are similar to Example 7.
- Questions 28-31 are similar to Example 9
- Questions 32-35 are similar to Examples 10-12.

1. The votes for president in a club election were: $\text{Smith} : 24 \qquad \text{Munoz} : 32 \qquad \text{Park} : 20$Find the following ratios and write in simplest form.
   a. Votes for Munoz to Smith
   b. Votes for Park to Munoz
   c. Votes for Smith to total votes
   d. Votes for Smith to Munoz to Park

Use the picture to write the following ratios for questions 2-6.



$AEFD$ is a square $\qquad ABCD$ is a rectangle

2. $AE : EF$
3. $EB : AB$
4. $DF : FC$
5. $EF : BC$
6. Perimeter $ABCD$:Perimeter $AEFD$:Perimeter $EBCF$

Convert the following measurements.

7. 16 cups to gallons
8. 8 yards to feet
9. 6 meters to centimeters

Simplify the following ratios.

10. $\frac{25\ in}{5\ ft}$
11. $\frac{8\ pt}{2\ gal}$

12. $\frac{9\ ft}{3\ yd}$
13. $\frac{95\ cm}{1.5\ m}$

14. The measures of the angles of a triangle are have the ratio 3:3:4. What are the measures of the angles?
15. The length and width of a rectangle are in a 3:5 ratio. The perimeter of the rectangle is 64. What are the length and width?
16. The length and width of a rectangle are in a 4:7 ratio. The perimeter of the rectangle is 352. What are the length and width?
17. A math class has 36 students. The ratio of boys to girls is 4:5. How many girls are in the class?
18. The senior class has 450 students in it. The ratio of boys to girls is 8:7. How many boys are in the senior class?
19. The varsity football team has 50 players. The ratio of seniors to juniors is 3:2. How many seniors are on the team?

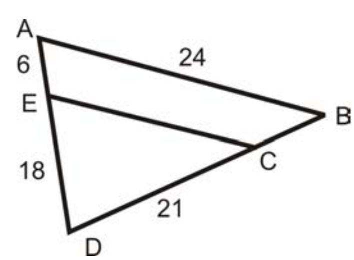
Solve each proportion.

20. $\frac{x}{10} = \frac{42}{35}$
21. $\frac{x}{x-2} = \frac{5}{7}$
22. $\frac{6}{9} = \frac{y}{24}$
23. $\frac{x}{9} = \frac{16}{x}$
24. $\frac{y-3}{8} = \frac{y+6}{5}$
25. $\frac{20}{z+5} = \frac{16}{7}$
26. Shawna drove 245 miles and used 8.2 gallons of gas. At the same rate, if she drove 416 miles, how many gallons of gas will she need? Round to the nearest tenth.
27. The president, vice-president, and financial officer of a company divide the profits is a 4:3:2 ratio. If the company made $1,800,000 last year, how much did each person receive?

Given the true proportion, $\frac{10}{6} = \frac{15}{d} = \frac{x}{y}$ and d, x, and y are nonzero, determine if the following proportions are also true.

28. $\frac{10}{y} = \frac{x}{6}$
29. $\frac{15}{10} = \frac{d}{6}$
30. $\frac{6+10}{10} = \frac{y+x}{x}$
31. $\frac{15}{x} = \frac{y}{d}$

For questions 32-35, $\frac{AE}{ED} = \frac{BC}{CD}$ and $\frac{ED}{AD} = \frac{CD}{DB} = \frac{EC}{AB}$.



32. Find DB.
33. Find EC.
34. Find CB.
35. Find AD.

Review and Reflect

36. What is the difference between a ratio and a proportion?
37. How can you use a scale factor on a map to determine the distance in miles between 2 cities?

Warm-Up Answers

1. Yes, they are congruent by SAS.
2. $GI = 5$ by CPCTC
3.
   a. 12 in = 1 ft
   b. 36 in = 3 ft
   c. 108 in = 3 yd
   d. 60 in = 5 ft.

6.2 Similar Polygons

TEKS G(1)F , G(7)A

Learning Objectives

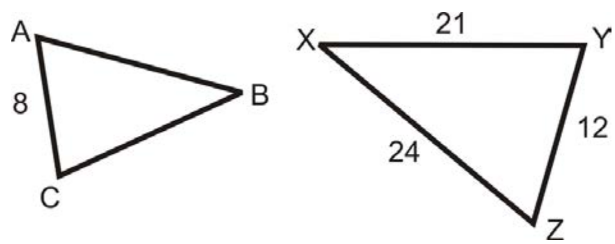
- Recognize similar polygons
- Identify corresponding angles and sides of similar polygons from a similarity statement
- Use scale factors

Vocabulary

- similar polygons
- scale factor

Warm-Up

1. Solve the proportions.
   a. $\frac{6}{x} = \frac{10}{15}$
   b. $\frac{4}{7} = \frac{2x+1}{42}$
   c. $\frac{5}{8} = \frac{x-2}{2x}$
2. In the picture, $\frac{AB}{XZ} = \frac{BC}{XY} = \frac{AC}{YZ}$.
   a. Find AB.
   b. Find BC.



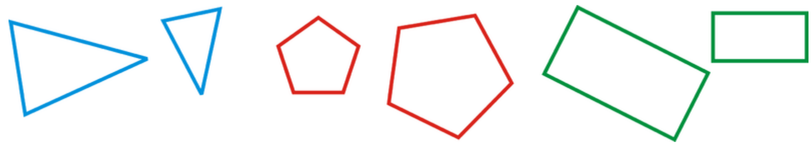
   c. What is $AB : XZ$?

Know What? A baseball diamond is a square with 90 foot sides. A softball diamond is a square with 60 foot sides. Are the two diamonds similar? If so, what is the scale factor?

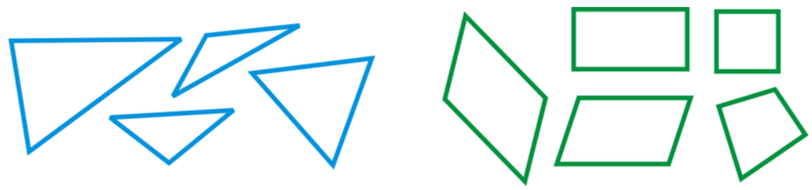
Similar Polygons

Think about similar polygons as enlarging or shrinking the same shape. The symbol $\sim$ is used to represent similar.

Similar Polygons: Two polygons with the same shape, but not the same size. The corresponding angles are *congruent,* and the corresponding sides are *proportional.*



These polygons are not similar:

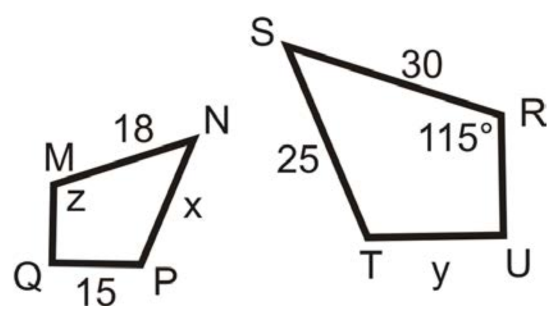


Example 1: Suppose $\triangle ABC \sim \triangle JKL$. Based on the similarity statement, which angles are congruent and which sides are proportional?

Solution: Just like a congruence statement, the congruent angles line up within the statement. So, $\angle A \cong \angle J$, $\angle B \cong \angle K$, and $\angle C \cong \angle L$. Write the sides in a proportion, $\frac{AB}{JK} = \frac{BC}{KL} = \frac{AC}{JL}$.

Because of the corollaries we learned in the last section, the proportions in Example 1 could be written several different ways. For example, $\frac{AB}{BC} = \frac{JK}{KL}$ is also true.

Example 2: $MNPQ \sim RSTU$. What are the values of x, y and z?

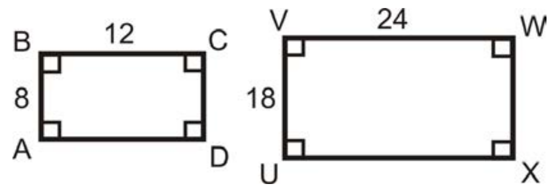


Solution: In the similarity statement, $\angle M \cong \angle R$, so $z = 115^\circ$. For x and y, set up a proportion because corresponding sides of similar figures are proportional.

$$\begin{aligned} \frac{18}{30} &= \frac{x}{25} \\ 450 &= 30x \\ x &= 15 \end{aligned} \qquad\qquad \begin{aligned} \frac{18}{30} &= \frac{15}{y} \\ 450 &= 18y \\ y &= 25 \end{aligned}$$

Specific types of triangles, quadrilaterals, and polygons will always be similar. For example, ***all equilateral triangles are similar*** and ***all squares are similar.***

Example 3: $ABCD$ and $UVWX$ are below. Are these two rectangles similar?



Solution: All the corresponding angles are congruent because the shapes are rectangles.

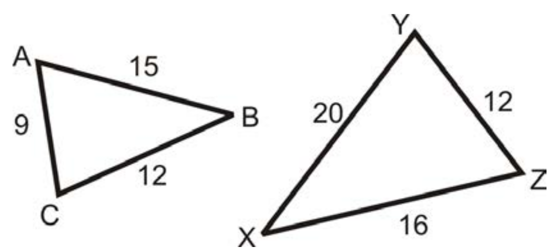
Let's see if the sides are proportional. $\frac{8}{12}=\frac{2}{3}$ and $\frac{18}{24}=\frac{3}{4}$. $\frac{2}{3}\neq\frac{3}{4}$, so the sides are ***not*** in the same proportion, so the rectangles are ***not*** similar.

Scale Factors

If two polygons are similar, we know the lengths of corresponding sides are proportional.

Scale Factor: In similar polygons, the ratio of one side of a polygon to the corresponding side of the other.

Example 4: What is the scale factor of $\triangle ABC$ to $\triangle XYZ$? Write the similarity statement.

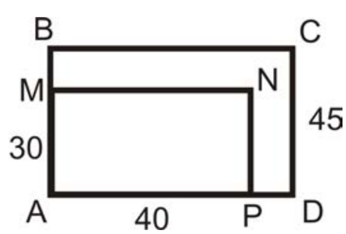


Solution: All the sides are in the same ratio. Pick the two largest (or smallest) sides to find the ratio.

$$\frac{15}{20}=\frac{3}{4}$$

For the similarity statement, line up the proportional sides. $AB\rightarrow XY, BC\rightarrow XZ, AC\rightarrow YZ$, so $\triangle ABC\sim\triangle YXZ$.

Example 5: $ABCD\sim AMNP$. Find the scale factor and the length of BC.



Solution: Line up the corresponding sides. $AB:AM$, so the scale factor is $\frac{30}{45}=\frac{2}{3}$ or $\frac{3}{2}$. Because BC is in the bigger rectangle, we will multiply 40 by $\frac{3}{2}$ because it is greater than 1. $BC=\frac{3}{2}(40)=60$.

Watch this!

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/155961

Example 6: Find the perimeters of $ABCD$ and $AMNP$. Then find the ratio of the perimeters.

Solution: Perimeter of $ABCD = 60+45+60+45 = 210$

Perimeter of $AMNP = 40+30+40+30 = 140$

The ratio of the perimeters is 140:210, which reduces to 2:3.

Theorem 6-2

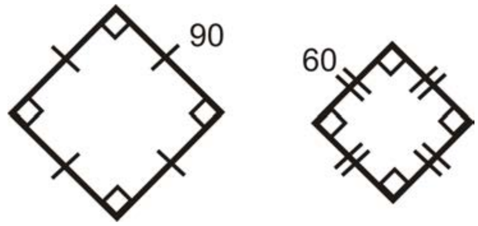
The ratio of the perimeters of two similar polygons is the same as the ratio of the sides

In addition to the perimeter having the same ratio as the sides, ***all parts of a polygon are in the same ratio as the sides.*** This includes diagonals, medians, midsegments, altitudes, and others.

Example 7: $\triangle ABC \sim \triangle MNP$. The perimeter of $\triangle ABC$ is 150, $AB = 32$ and $MN = 48$. Find the perimeter of $\triangle MNP$.

Solution: From the similarity statement, AB and MN are corresponding sides. The scale factor is $\frac{32}{48} = \frac{2}{3}$. $\triangle ABC$ is the smaller triangle, so the perimeter of $\triangle MNP$ is $\frac{3}{2}(150) = 225$.

Know What? Revisited The baseball diamond is on the left and the softball diamond is on the right. All the angles and sides are congruent, so all squares are similar. All of the sides in the baseball diamond are 90 feet long and 60 feet long in the softball diamond. This means the scale factor is $\frac{90}{60} = \frac{3}{2}$.



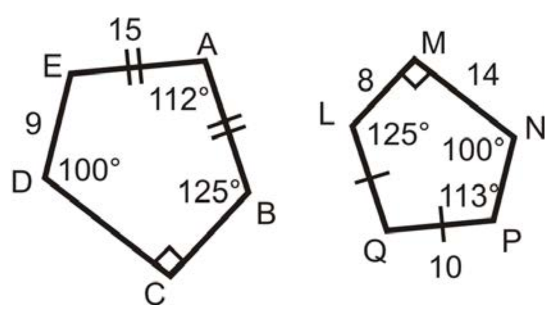
Practice Problems

- Questions 1-8 use the definition of similarity and different types of polygons.
- Questions 9-13 are similar to Examples 1, 5, 6, and 7.
- Questions 14 and 15 are similar to the **Know What?**
- Questions 16-20 are similar to Example 2.
- Questions 21-30 are similar to Examples 3 and 4.

For questions 1-8, determine if the following statements are true or false.

1. All equilateral triangles are similar.
2. All isosceles triangles are similar.
3. All rectangles are similar.
4. All rhombuses are similar.
5. All squares are similar.
6. All congruent polygons are similar.
7. All similar polygons are congruent.
8. All regular pentagons are similar.
9. $\triangle BIG \sim \triangle HAT$. List the congruent angles and proportions for the sides.
10. If $BI = 9$ and $HA = 15$, find the scale factor.
11. If $BG = 21$, find HT.
12. If $AT = 45$, find IG.
13. Find the perimeter of $\triangle BIG$ and $\triangle HAT$. What is the ratio of the perimeters?
14. An NBA basketball court is a rectangle that is 94 feet by 50 feet. A high school basketball court is a rectangle that is 84 feet by 50 feet. Are the two rectangles similar?
15. HD TVs have sides in a ratio of 16:9. Non-HD TVs have sides in a ratio of 4:3. Are these two ratios equivalent?

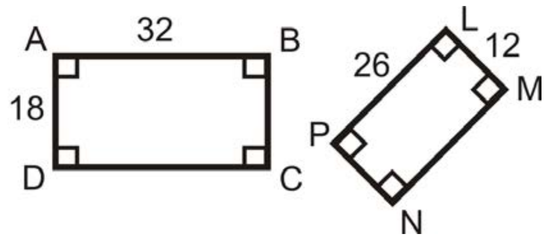
Use the picture to the right to answer questions 16-20.



16. Find $m\angle E$ and $m\angle Q$.
17. $ABCDE \sim QLMNP$, find the scale factor.
18. Find BC.
19. Find CD.
20. Find NP.

Determine if the following triangles and quadrilaterals are similar. If they are, write the similarity statement.

21.



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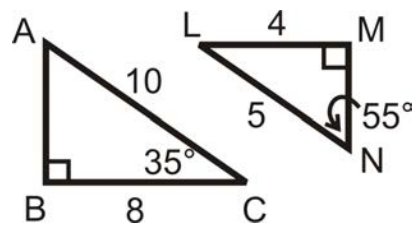

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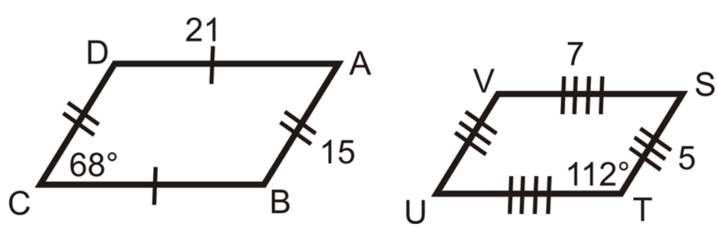

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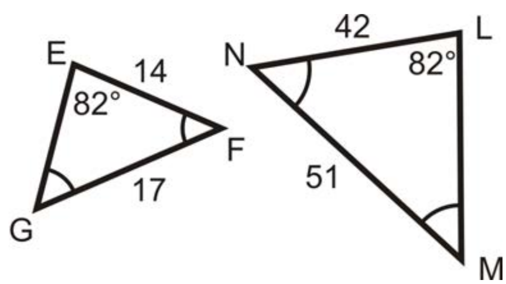

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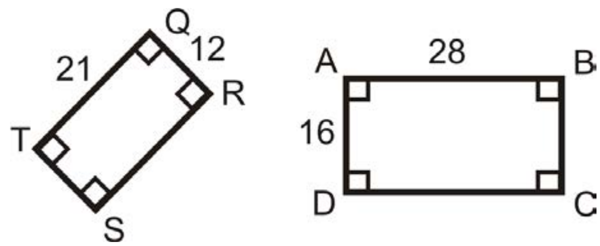
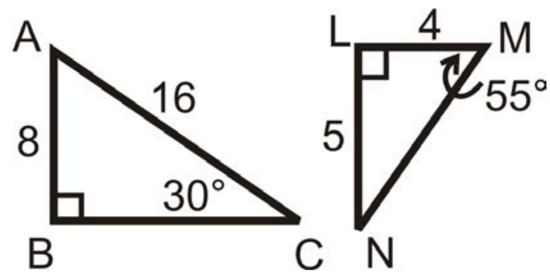

Q
12
21
R
T
S
A
28
B
16
D
C

25.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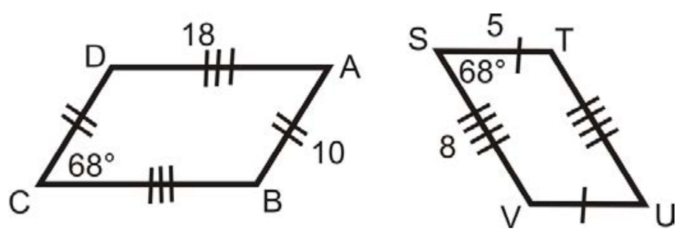
26.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8
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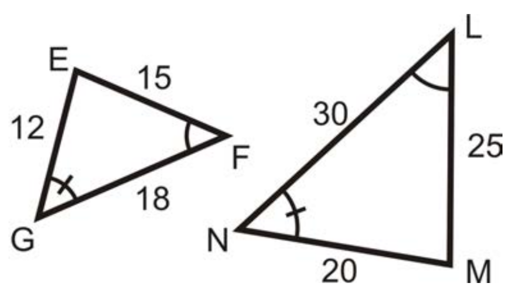
27.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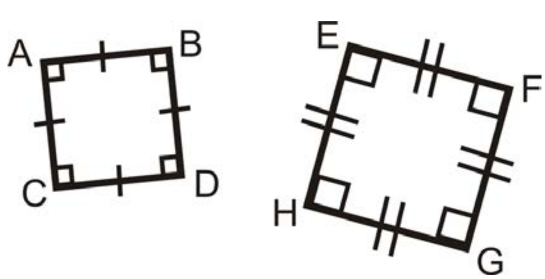
28.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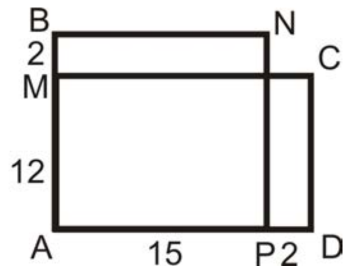


Review and Reflect

30. Describe how to determine if 2 figures are similar, what information is needed?

31. How can you use scale factors in a real world application?

Warm-Up Answers

a. $x = 9$
b. $x = 11.5$
c. $x = 8$

a. $AB = 16$
b. $BC = 14$
c. $\frac{2}{3}$

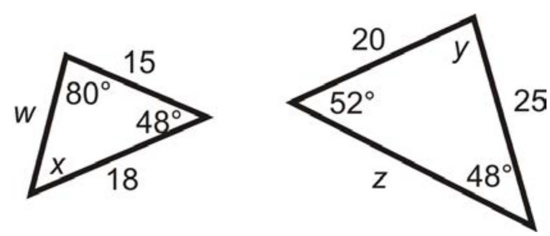
6.3 Similarity by AA

TEKS G(1)D, G(7)A, G(7)B

Learning Objectives

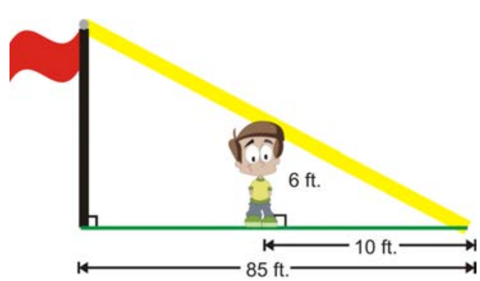
- Determine whether triangles are similar using the AA Postulate
- Solve problems involving similar triangles

Warm-Up



1. a. Find the measures of x and y.
   b. The two triangles are similar. Find w and z.
2. Use the true proportion $\frac{6}{8} = \frac{x}{28} = \frac{27}{y}$ to answer the following questions.
   a. Find x.
   b. Find y.

Know What? George wants to measure the height of a flagpole. He is 6 feet tall and his shadow is 10 feet long. At the same time, the shadow of the flagpole is 85 feet long. How tall is the flagpole?



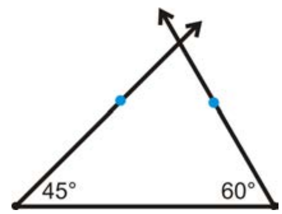
Angles in Similar Triangles

The Third Angle Theorem says if two angles are congruent to two angles in another triangle the third angles are congruent too. Let's see what happens when two different triangles have the same angle measures.

Investigation 7-1: Constructing Similar Triangles

Tools Needed: pencil, paper, protractor, ruler

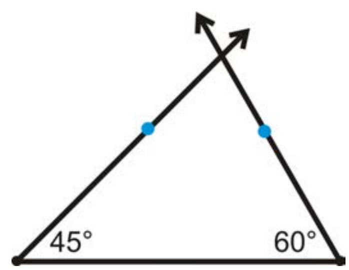
1. Draw a 45° angle. Make the horizontal side ***3 inches*** and draw a 60° angle on the other endpoint.



2. Extend the other sides of the 45° and 60° angles so that they intersect to form a triangle.

Find the measure of the third angle and measure the length of each side.

3. Repeat Steps 1 and 2, but make the horizontal side between the 45° and 60° angle ***4 inches***.

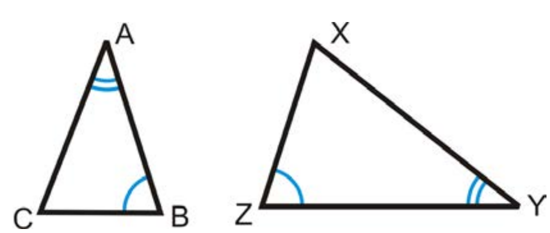


Find the measure of the third angle and measure the length of each side.

4. Find the ratio of the sides. Put the sides opposite the 45° angles over each other, the sides opposite the 60° angles over each other, and the sides opposite the third angles over each other. What happens?

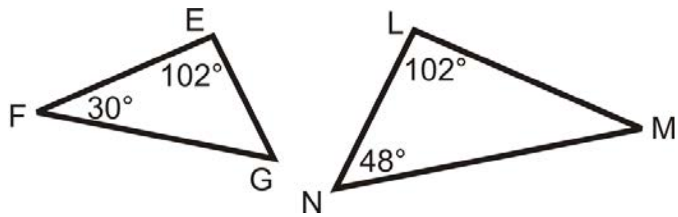
AA Similarity Postulate

If two angles in one triangle are congruent to two angles in another triangle, then the two triangles are similar.



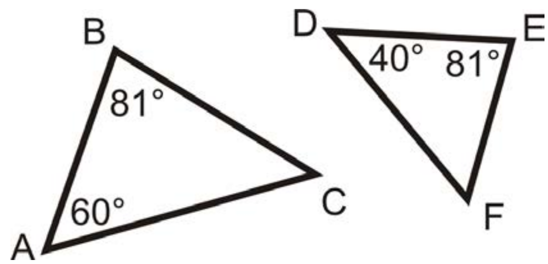
If $\angle A \cong \angle Y$ and $\angle B \cong \angle Z$, then $\triangle ABC \sim \triangle YZX$.

Example 1: Determine if the following two triangles are similar. If so, write the similarity statement.



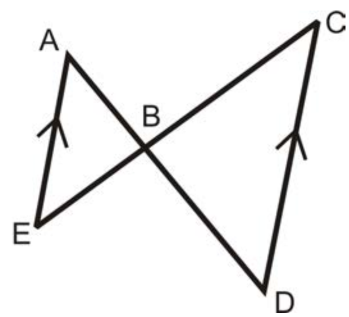
Solution: $m\angle G = 48^\circ$ and $m\angle M = 30^\circ$ So, $\angle F \cong \angle M, \angle E \cong \angle L$ and $\angle G \cong \angle N$ and the triangles are similar. $\triangle FEG \sim \triangle MLN$.

Example 2: Determine if the following two triangles are similar. If so, write the similarity statement.



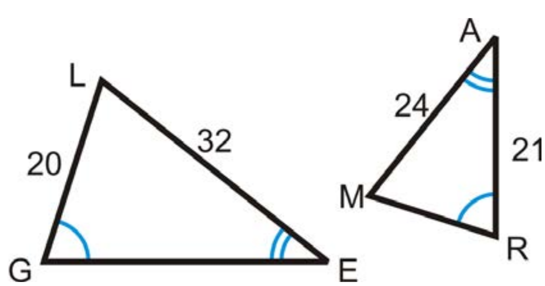
Solution: $m\angle C = 39^\circ$ and $m\angle F = 59^\circ$. $m\angle C \neq m\angle F$, So $\triangle ABC$ and $\triangle DEF$ are not similar.

Example 3: Are the following triangles similar? If so, write the similarity statement.



Solution: Because $\overline{AE} \| \overline{CD}$, $\angle A \cong \angle D$ and $\angle C \cong \angle E$ by the Alternate Interior Angles Theorem. By the AA Similarity Postulate, $\triangle ABE \sim \triangle DBC$.

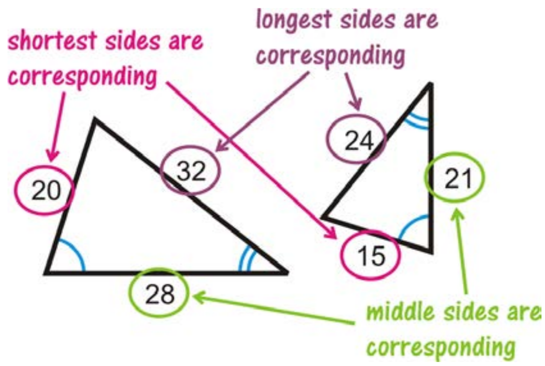
Example 4: $\triangle LEG \sim \triangle MAR$ by AA. Find GE and MR.



Solution: Set up a proportion to find the missing sides.

$$\begin{aligned} \frac{24}{32} &= \frac{MR}{20} \\ 480 &= 32MR \\ 15 &= MR \end{aligned} \qquad \begin{aligned} \frac{24}{32} &= \frac{21}{GE} \\ 24GE &= 672 \\ GE &= 28 \end{aligned}$$

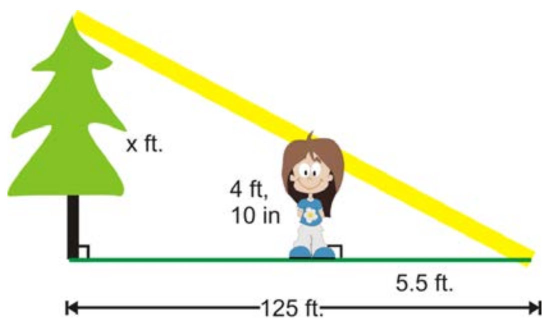
When two triangles are similar, the corresponding sides are proportional. But, what are the corresponding sides? Using the triangles from Example 4, we see how the sides line up in the diagram to the right.



Indirect Measurement

An application of similar triangles is to measure lengths *indirectly.* You can use this method to measure the width of a river or canyon or the height of a tall object.

Example 5: A tree outside Ellie's building casts a 125 foot shadow. At the same time of day, Ellie casts a 5.5 foot shadow. If Ellie is 4 feet 10 inches tall, how tall is the tree?



Solution: Draw a picture. We see that the tree and Ellie are parallel, so the two triangles are similar.

$$\frac{4\ ft, 10\ in}{x\ ft} = \frac{5.5\ ft}{125\ ft}$$

The measurements need to be in the same units. Change everything into inches and then we can cross multiply.

$$\begin{aligned} \frac{58\ in}{x\ ft} &= \frac{66\ in}{1500\ ft} \\ 87000 &= 66x \\ x &\approx 1318.\overline{18}\ in \text{ or } 109.85\ ft \end{aligned}$$

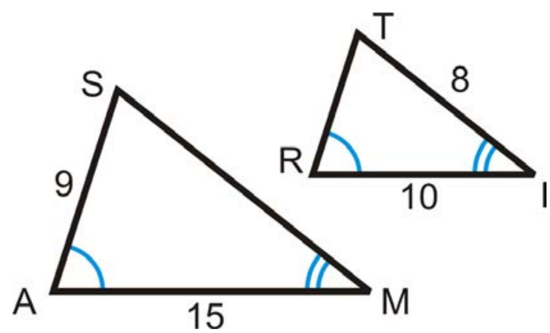
Know What? Revisited It is safe to assume that George and the flagpole stand vertically, making them parallel. This is very similar to Example 4. Set up a proportion.

$$\begin{aligned} \frac{10}{85} = \frac{6}{x} \longrightarrow\ 10x &= 510 \\ x &= 51\ ft. \quad \text{The height of the flagpole is 51 feet.} \end{aligned}$$

Practice Problems

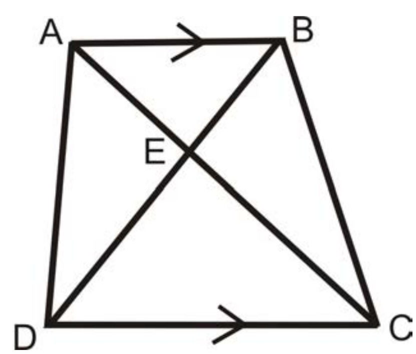
- Questions 1-13 are similar to Examples 1-4 and review.
- Question14 compares the definitions of congruence and similarity.
- Questions 15-23 are similar to Examples 1-3.
- Questions 24-30 are similar to Example 5 and the **Know What?**

Use the diagram to complete each statement in questions 1-5.



1. $\triangle SAM \sim \triangle$ _______
2. $\frac{SA}{?} = \frac{SM}{?} = \frac{?}{RI}$
3. $SM =$ _______
4. $TR =$ _______
5. $\frac{9}{?} = \frac{?}{8}$

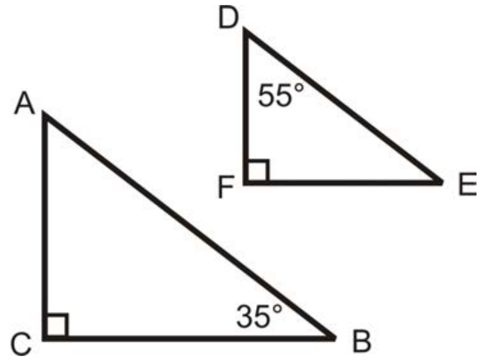
Answer questions 6-9 about trapezoid $ABCD$.



6. Name two similar triangles. How do you know they are similar?
7. Write a true proportion.
8. Name two other triangles that might *not* be similar.
9. If $AB = 10, AE = 7$, and $DC = 22$, find AC. Be careful!

Use the triangles below for questions 10-12.

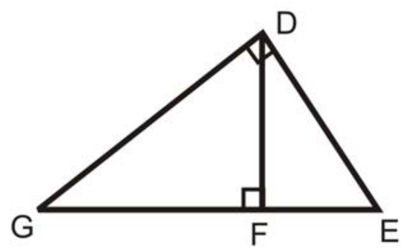
$AB = 20, DE = 15$, and $BC = k$.



10. Are the two triangles similar? How do you know?
11. Write an expression for FE in terms of k.
12. If $FE = 12$,, what is k?
13. Fill in the blanks: If an acute angle of a _______ triangle is congruent to an acute angle in another ________-triangle, then the two triangles are _______.

Are the following triangles similar? If so, write a similarity statement.

14.



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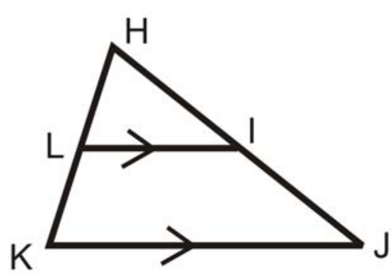

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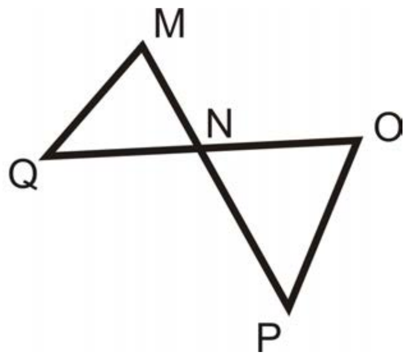

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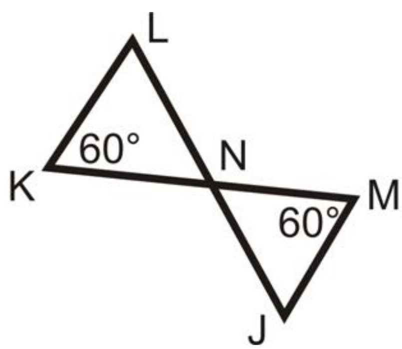

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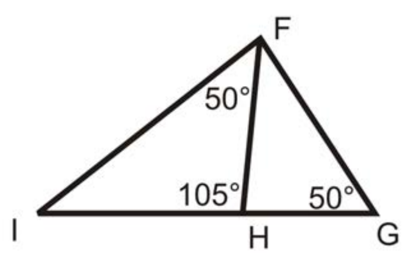

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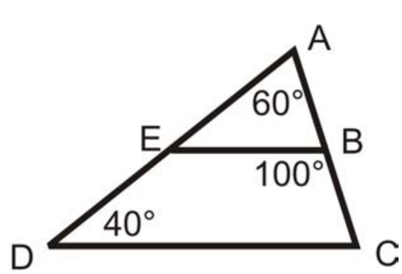

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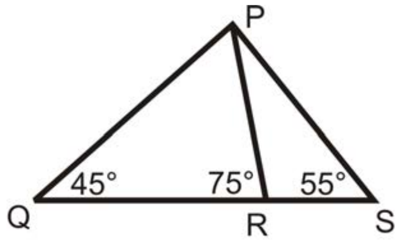

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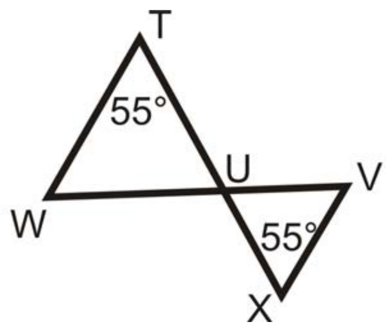

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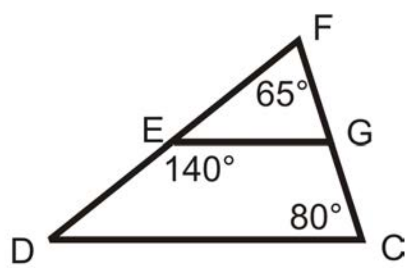
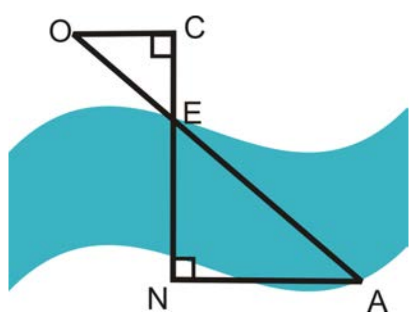


In order to estimate the width of a river, the following technique can be used. Use the diagram below.



Place three markers, O, C, and E on the upper bank of the river. E is on the edge of the river and $\overline{OC} \perp \overline{CE}$. Go across the river and place a marker, N so that it is collinear with C and E. Then, walk along the lower bank of the river and place marker A, so that $\overline{CN} \perp \overline{NA}$. $OC = 50$ feet, $CE = 30$ feet, $NA = 80$ feet.

23. Is $\overline{OC} || \overline{NA}$?How do you know?

24. Is $\triangle OCE \sim \triangle ANE$? How do you know?

25. What is the width of the river? Find EN.
26. Can we find EA? If so, find it. If not, explain.
27. The technique above was used to measure the distance across the Grand Canyon. Using the same set up and marker letters, $OC = 72\ ft, CE = 65\ ft$, and $NA = 14,400\ ft$. Find EN (the distance across the Grand Canyon).
28. Cameron is 5 ft tall and casts a 12 ft shadow. At the same time of day, a nearby building casts a 78 ft shadow. How tall is the building?
29. The Empire State Building is 1250 ft. tall. At 3:00, Pablo stands next to the building and has an 8 ft. shadow. If he is 6 ft tall, how long is the Empire State Building's shadow at 3:00?

Review and Reflect

30. How do congruent triangles and similar triangles differ? How are they the same?

31. Why is it necessary to show two triangles are similar with only having 2 pairs of corresponding angles congruent?

Warm-Up Answers

a. $x = 52°, y = 80°$

b. $\frac{w}{20} = \frac{15}{25}$ $\qquad$ $\frac{15}{25} = \frac{18}{z}$

$$\begin{aligned} 25w &= 15(20) & 25(18) &= 15z \\ 25w &= 300 & 450 &= 15z \\ w &= 12 & 30 &= z \end{aligned}$$

a. $$\begin{aligned} 168 &= 8x & 6y &= 216 \\ x &= 21 & y &= 36 \end{aligned}$$

b. Answers will vary. One possibility: $\frac{28}{8} = \frac{21}{6}$

c. $$\begin{aligned} 28(12) &= 8(6+x) \\ 336 &= 48+8x \\ 288 &= 8x \\ 36 &= x \end{aligned}$$ Because $x \neq 21$, like in part a, this is not a true proportion.

6.4 Similarity by SSS and SAS

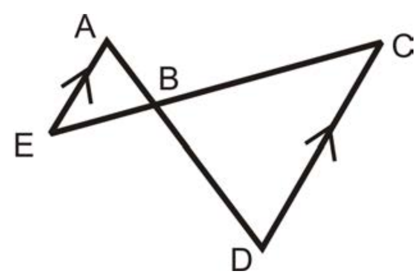
TEKS G(1)C, G(5)C, G(7)A

Learning Objectives

- Use SSS and SAS to determine whether triangles are similar
- Apply SSS and SAS to solve real-world situations

Warm-Up

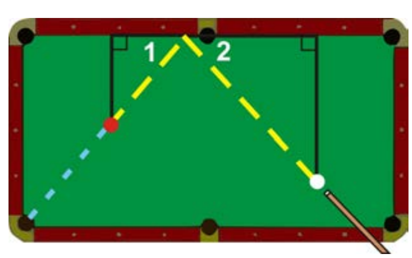
1.
   a. What are the congruent angles? List each pair.



   b. Write the similarity statement.
   c. If $AB = 8, BD = 20$, and $BC = 25$, find BE.

2. Solve the following proportions.
   a. $\frac{6}{8} = \frac{21}{x}$
   b. $\frac{x+2}{6} = \frac{2x-1}{15}$

Know What? Recall from Chapter 2, that the game of pool relies heavily on angles. In Section 2.5, we discovered that $m\angle 1 = m\angle 2$.



You decide to hit the cue ball so it follows the yellow path to the right. Are the two triangles similar?

Link for an interactive game of pool: http://www.coolmath-games.com/0-poolgeometry/index.html

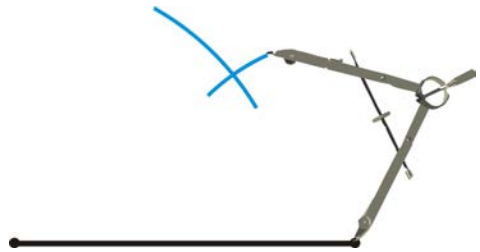
SSS for Similar Triangles

If you do not know any angle measures, can you say two triangles are similar?

Investigation 7-2: SSS Similarity

Tools Needed: ruler, compass, protractor, paper, pencil

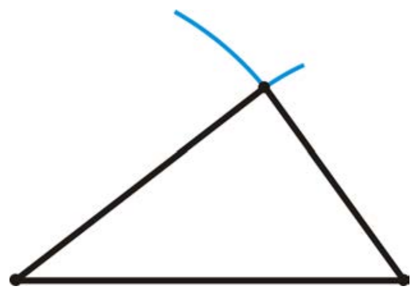
1. Using Investigation 4-2, construct a triangle with sides 6 cm, 8 cm, and 10 cm.



2. Construct a second triangle with sides 9 cm, 12 cm, and 15 cm.

3. Using your protractor, measure the angles in both triangles. What do you notice?

4. Line up the corresponding sides. Write down the ratios of these sides. What happens?

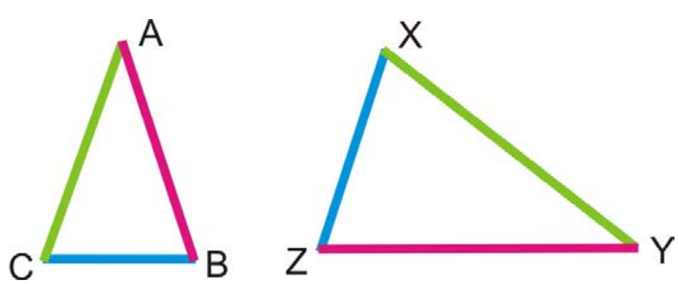


To see an animated construction of this, click: http://www.mathsisfun.com/geometry/construct-ruler-compass-1.html

From #3, you should notice that the angles in the two triangles are equal. Second, the sides are all in the same proportion, $\frac{6}{9} = \frac{8}{12} = \frac{10}{15}$.

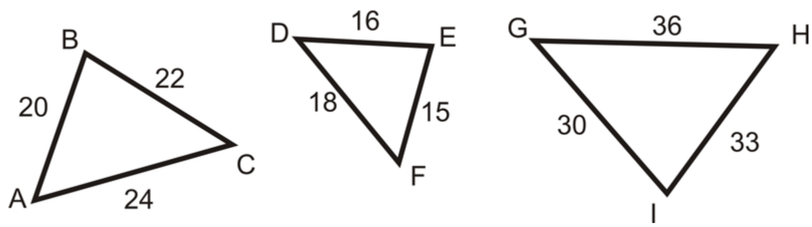
SSS Similarity Theroem

If the corresponding sides of two triangles are proportional, then the two triangles are similar



If $\frac{AB}{YZ} = \frac{BC}{ZX} = \frac{AC}{XY}$, then $\triangle ABC \sim \triangle YZX$.

Example 1: Determine if any of the triangles below are similar.



Solution: Compare two triangles at a time.

$\underline{\triangle ABC \text{ and } \triangle DEF}$: $\frac{20}{15} = \frac{22}{16} = \frac{24}{18}$

Reduce each fraction to see if they are equal. $\frac{20}{15} = \frac{4}{3}, \frac{22}{16} = \frac{11}{8}$, and $\frac{24}{18} = \frac{4}{3}$.

$\frac{4}{3} \neq \frac{11}{8}, \triangle ABC$ and $\triangle DEF$ are **not** similar.

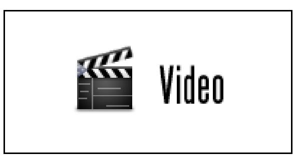
$\underline{\triangle DEF \text{ and } \triangle GHI}$: $\frac{15}{30} = \frac{16}{33} = \frac{18}{36}$

$\frac{15}{30} = \frac{1}{2}, \frac{16}{33} = \frac{16}{33}$, and $\frac{18}{36} = \frac{1}{2}$. $\frac{1}{2} \neq \frac{16}{33}, \triangle DEF$ is **not** similar to $\triangle GHI$.

$\underline{\triangle ABC \text{ and } \triangle GHI}$: $\frac{20}{30} = \frac{22}{33} = \frac{24}{36}$

$\frac{20}{30} = \frac{2}{3}, \frac{22}{33} = \frac{2}{3}$, and $\frac{24}{36} = \frac{2}{3}$. All three ratios reduce to $\frac{2}{3}$, $\triangle ABC \sim \triangle GIH$.

Watch this!



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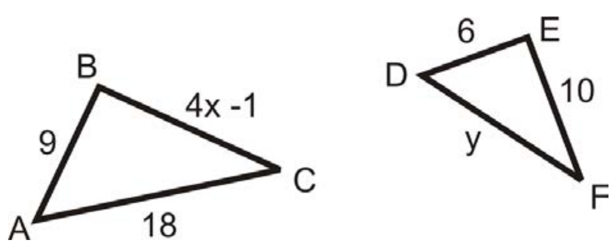
Example 2: ***Algebra Connection*** Find x and y, such that $\triangle ABC \sim \triangle DEF$.



Solution: According to the similarity statement, the corresponding sides are: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. Substituting in what we know, we have $\frac{9}{6} = \frac{4x-1}{10} = \frac{18}{y}$.

$$\begin{aligned} \frac{9}{6} &= \frac{4x-1}{10} \\ 9(10) &= 6(4x-1) \\ 90 &= 24x-6 \\ 96 &= 24x \\ x &= 4 \end{aligned} \qquad\qquad \begin{aligned} \frac{9}{6} &= \frac{18}{y} \\ 9y &= 18(6) \\ 9y &= 108 \\ y &= 12 \end{aligned}$$

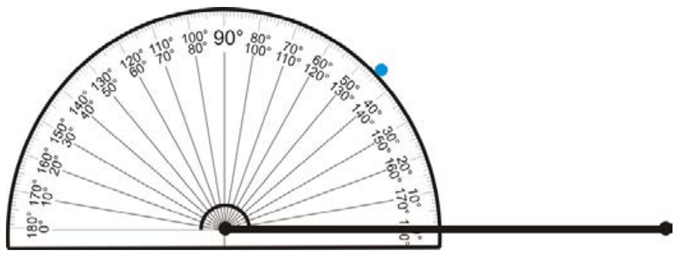
SAS for Similar Triangles

SAS is the last way to show two triangles are similar.

Investigation 7-3: SAS Similarity

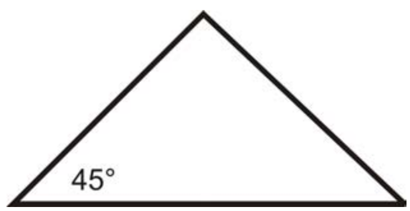
Tools Needed: paper, pencil, ruler, protractor, compass

1. Using Investigation 4-3, construct a triangle with sides 6 cm and 4 cm and the *included* angle is 45°.



2. Repeat Step 1 and construct another triangle with sides 12 cm and 8 cm and the included angle is 45°.

3. Measure the other two angles in both triangles. What do you notice?

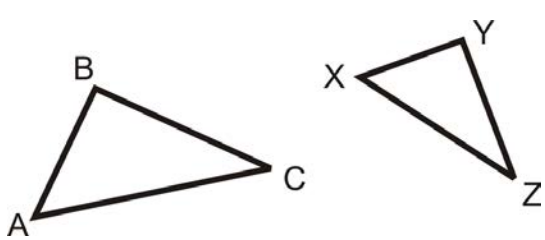


4. Measure the third side in each triangle. Make a ratio. Is this ratio the same as the ratios of the sides you were given?

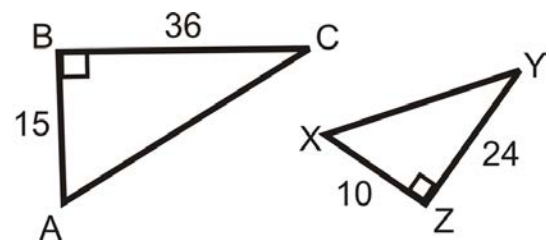
SAS Similarity Theorem

If two sides in one triangle are proportional to two sides in another triangle and the included angle in both are congruent, then the two triangles are similar

If $\frac{AB}{XY} = \frac{AC}{XZ}$ and $\angle A \cong \angle X$, then $\triangle ABC \sim \triangle XYZ$.

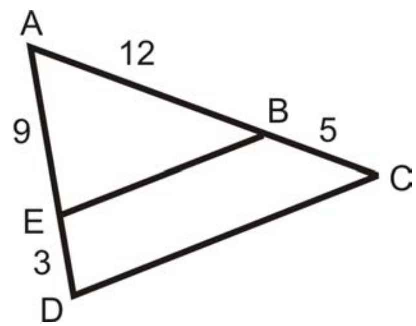


Example 3: Are the two triangles similar? How do you know?



Solution: $\angle B \cong \angle Z$ because they are both right angles and $\frac{10}{15} = \frac{24}{36}$. So, $\frac{AB}{XZ} = \frac{BC}{ZY}$ and $\triangle ABC \sim \triangle XZY$ by SAS.

Example 4: Are there any similar triangles? How do you know?



Solution: $\angle A$ is shared by $\triangle EAB$ and $\triangle DAC$, so it is congruent to itself. Let's see if $\frac{AE}{AD} = \frac{AB}{AC}$.

$$\frac{9}{9+3} = \frac{12}{12+5}$$
$$\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17} \qquad \text{The two triangles are } \textit{not} \text{ similar.}$$

Example 5: From Example 4, what should BC equal for $\triangle EAB \sim \triangle DAC$?

Solution: The proportion we ended up with was $\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17}$. AC needs to equal 16, so that $\frac{12}{16} = \frac{3}{4}$. $AC = AB + BC$ and $16 = 12 + BC$. BC should equal 4.

Test your knowledge of similarity with the following link. **https://www.mathsisfun.com/geometry/triangles-similar-finding.html**

Know What? Revisited Yes, the two triangles are similar because they both have a right angle and, from early in this text learned that $m\angle 1 = m\angle 2$.

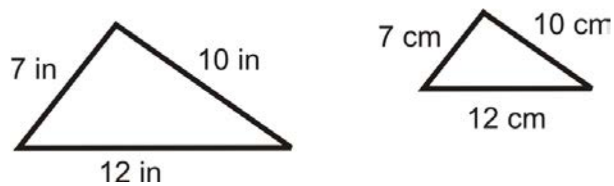
Practice Problems

- Questions 1-5 are vocabulary.
- Questions 6-18 are similar to Examples 1, 3, and 4 and review.
- Questions 19-24 are similar to Examples 3 and 4.
- Questions 25-28 are similar to Example 2.
- Questions 29 and 30 are a review of the last section.

Fill in the blanks.

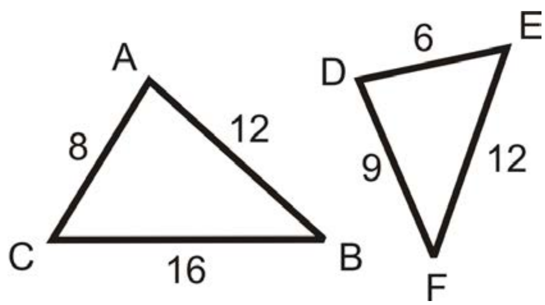
1. Two triangles are similar if two angles in each triangle are _____________.
2. If all three sides in one triangle are __________________ to the three sides in another, then the two triangles are similar.
3. Two triangles are congruent if the corresponding sides are _____________.
4. Two triangles are similar if the corresponding sides are _____________.
5. If two sides in one triangle are _________________ to two sides in another and the ________________ angles are _________________, then the triangles are ______________.

Use the following diagram for questions 6-8. *The diagram is to scale.*



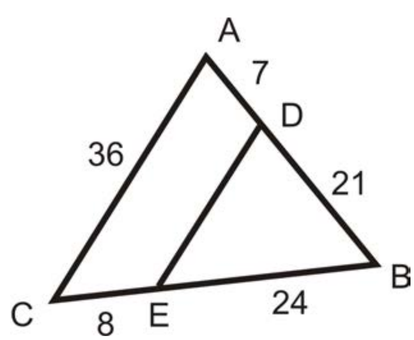
6. Are the two triangles similar? Explain your answer.
7. Are the two triangles congruent? Explain your answer.
8. What is the scale factor for the two triangles?

Fill in the blanks in the statements below. Use the diagram to the left.



9. $\triangle ABC \sim \triangle$_____
10. $\frac{AB}{?} = \frac{BC}{?} = \frac{AC}{?}$
11. If $\triangle ABC$ had an altitude, $AG = 10$, what would be the length of altitude $\overline{DH}$?
12. Find the perimeter of $\triangle ABC$ and $\triangle DEF$. Find the ratio of the perimeters.

Use the diagram to the right for questions 13-18.

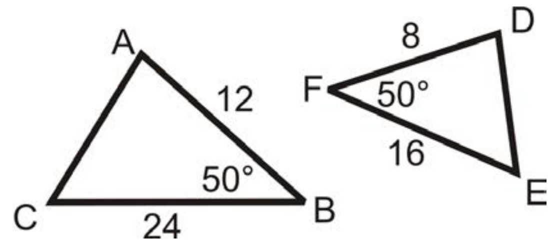


13. $\triangle ABC \sim \triangle$_____

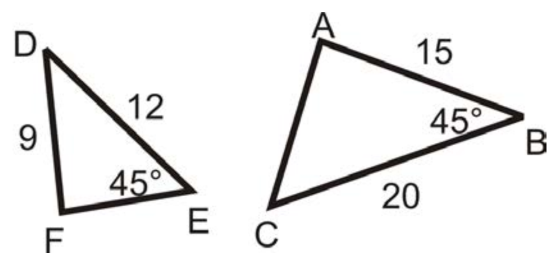
14. Why are the two triangles similar?
15. Find ED.
16. $\frac{BD}{?} = \frac{?}{BC} = \frac{DE}{?}$
17. Is $\frac{AD}{DB} = \frac{CE}{EB}$ true?
18. Is $\frac{AD}{DB} = \frac{AC}{DE}$ true?

Determine if the following triangles are similar. If so, write the similarity theorem and statement.

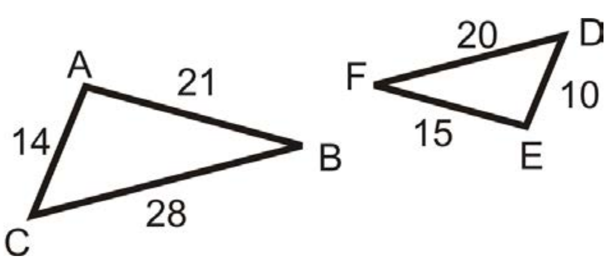
19.



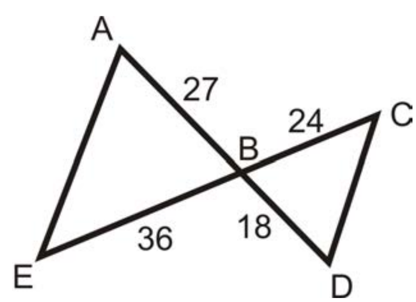
20.



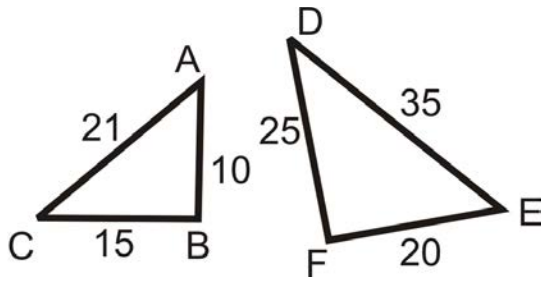
21.



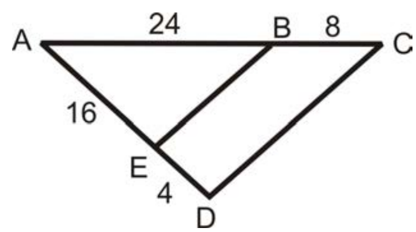
22.



23.

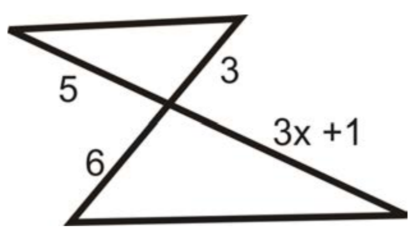


24.

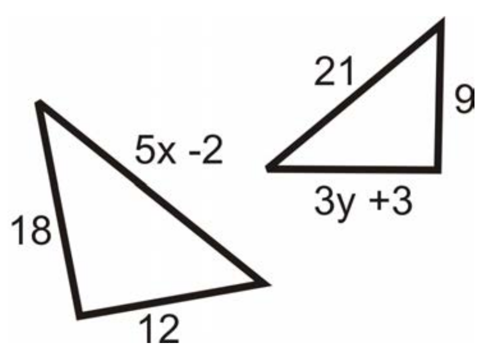


Algebra Connection Find the value of the missing variable(s) that makes the two triangles similar.

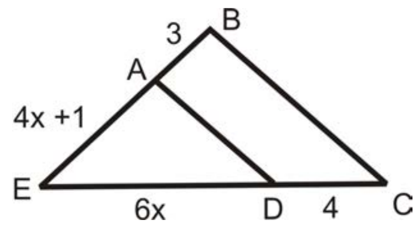
25.



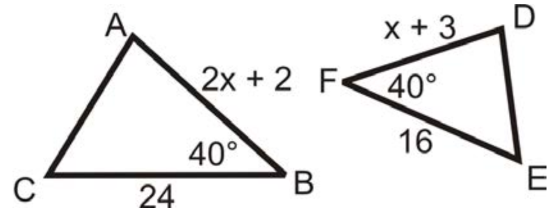
26.



27.



28.



29. At a certain time of day, a building casts a 25 ft shadow. At the same time of day, a 6 ft tall stop sign casts a 15 ft shadow. How tall is the building?
30. A child who is 42 inches tall is standing next to the stop sign in #21. How long is her shadow?

Review and Reflect

31. Describe the differences between using SAS and SSS similarity.

32. Why is ASA similarity not used?

Warm-Up Answers

1.

a. $\angle A \cong \angle D, \angle E \cong \angle C$
b. $\triangle ABE \sim \triangle DBC$
c. $BE = 10$

2.

a. $\frac{6}{8} = \frac{21}{x}, x = 28$
b.
$$\begin{aligned} 15(x+2) &= 6(2x-1) \\ 15x+30 &= 12x-6 \\ 3x &= -36 \\ x &= -12 \end{aligned}$$

6.5 Proportionality Relationships

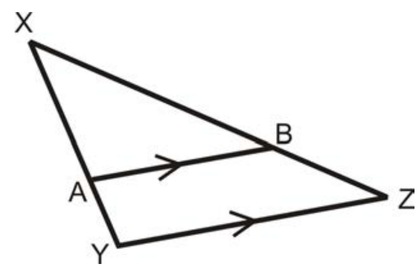
TEKS G(1)D, G(8)A

Learning Objectives

- Identify proportional segments within triangles
- Extend triangle proportionality to parallel lines

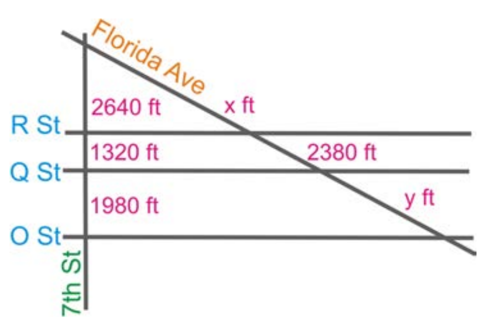
Warm-Up

1. Write a similarity statement for the two triangles in the diagram. Why are they similar?



2. If $XA = 16, XY = 18, XB = 32$, find XZ.
3. If $YZ = 27$, find AB.
4. Find AY and BZ.

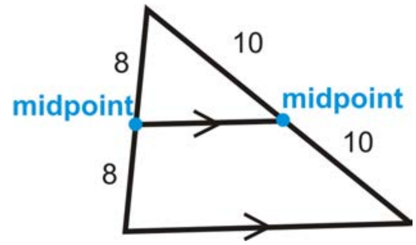
Know What? To the right is a street map of part of Washington DC. R Street, Q Street, and O Street are parallel and 7^{th} Street is perpendicular to all three. All the measurements are given on the map. What are x and y?



Triangle Proportionality

Think about a midsegment of a triangle. A midsegment is parallel to one side of a triangle and divides the other two sides into congruent halves. The midsegment divides those two sides ***proportionally.***

Example 1: A triangle with its midsegment is drawn below. What is the ratio that the midsegment divides the sides into?



Solution: The midsegment splits the sides evenly. The ratio would be 8:8 or 10:10, which both reduce to 1:1.

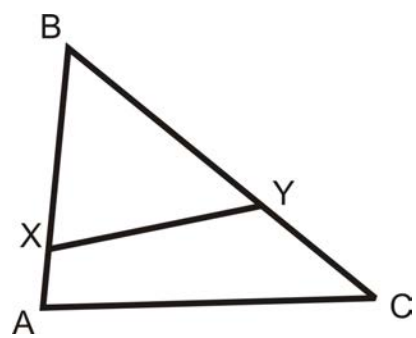
The midsegment divides the two sides of the triangle proportionally, but what about other segments?

Investigation 7-4: Triangle Proportionality

Tools Needed: pencil, paper, ruler

1. Draw $\triangle ABC$. Label the vertices.

2. Draw $\overline{XY}$ so that X is on $\overline{AB}$ [U+0305] and Y is on $\overline{BC}$. X and Y can be ***anywhere*** on these sides.

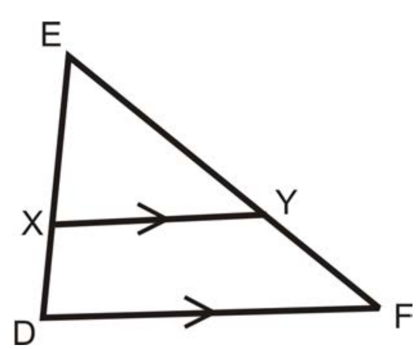


3. Is $\triangle XBY \sim \triangle ABC$? Why or why not? Measure AX, XB, BY, and YC. Then set up the ratios $\frac{AX}{XB}$ and $\frac{YC}{YB}$. Are they equal?

4. Draw a second triangle, $\triangle DEF$. Label the vertices.

5. Draw $\overline{XY}$ so that X is on $\overline{DE}$ and Y is on $\overline{EF}$ AND $\overline{XY} \| \overline{DF}$.

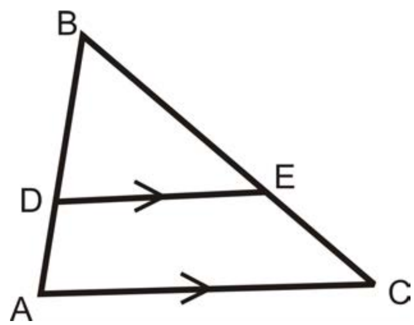
6. Is $\triangle XEY \sim \triangle DEF$? Why or why not? Measure DX, XE, EY, and YF. Then set up the ratios $\frac{DX}{XE}$ and $\frac{FY}{YE}$. Are they equal?



From this investigation, we see that if $\overline{XY} \| \overline{DF}$, then $\overline{XY}$ divides the sides proportionally.

Triangle Proportionality Theorem

If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally



If $\overline{DE} \| \overline{AC}$, then $\frac{BD}{DA} = \frac{BE}{EC}$. ($\frac{DA}{BD} = \frac{EC}{BE}$ *is also a true proportion.*)

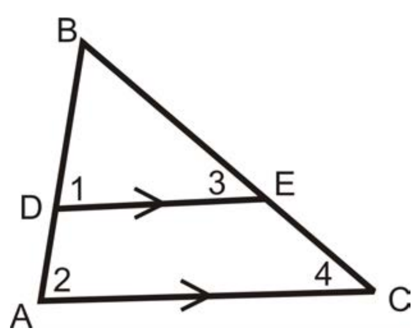
For the converse:

If $\frac{BD}{DA} = \frac{BE}{EC}$, then $\overline{DE} \| \overline{AC}$.

Triangle Proportionality Theorem Converse

If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Proof of the Triangle Proportionality Theorem



Given: $\triangle ABC$ with $\overline{DE} \| \overline{AC}$

Prove: $\frac{AD}{DB} = \frac{CE}{EB}$

TABLE 6.2:

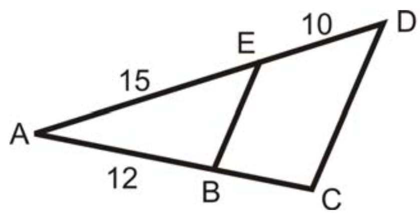
Statement	*Reason*
1. $\overline{DE} \| \overline{AC}$	Given
2. $\angle 1 \cong \angle 2, \angle 3 \cong \angle 4$	Corresponding Angles Postulate
3. $\triangle ABC \sim \triangle DBE$	AA Similarity Postulate
4. $AD + DB = AB, EC + EB = BC$	Segment Addition Postulate

TABLE 6.2: (continued)

Statement	***Reason***
5. $\frac{AB}{BD} = \frac{BC}{BE}$	Corresponding sides in similar triangles are proportional
6. $\frac{AD+DB}{BD} = \frac{EC+EB}{BE}$	Substitution Property of Equality
7. $\frac{AD}{BD} + \frac{DB}{DB} = \frac{EC}{BE} + \frac{BE}{BE}$	Separate the fractions
8. $\frac{AD}{BD} + 1 = \frac{EC}{BE} + 1$	Substitution Property of Equality (something over itself always equals 1)
9. $\frac{AD}{BD} = \frac{EC}{BE}$	Subtraction Property of Equality

We will not prove the converse; it is basically this proof but in the reverse order.

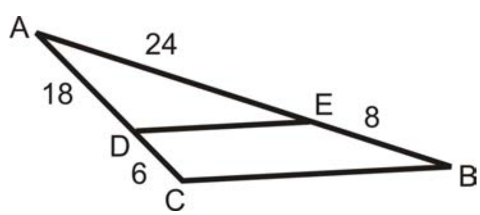
Example 2: In the diagram below, $\overline{EB} \| \overline{BD}$. Find BC.



Solution: Set up a proportion.

$$\frac{10}{15} = \frac{BC}{12} \longrightarrow 15(BC) = 120$$
$$BC = 8$$

Example 3: Is $\overline{DE} \| \overline{CB}$?



Solution: If the ratios are equal, then the lines are parallel.

$$\frac{6}{18} = \frac{8}{24} = \frac{1}{3}$$

Because the ratios are equal, $\overline{DE} \| \overline{CB}$.

Watch this!

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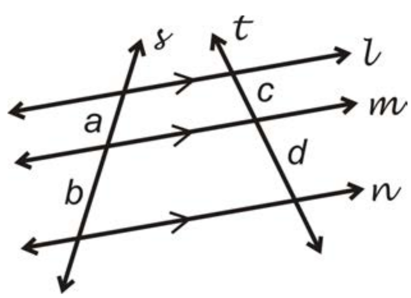
Parallel Lines and Transversals

We can extend the Triangle Proportionality Theorem to multiple parallel lines.

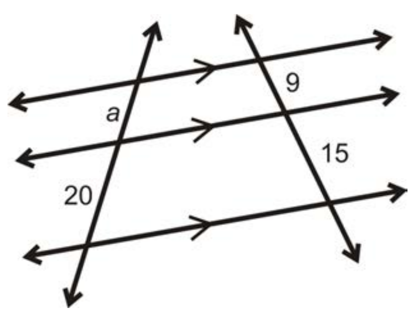
Theorem 6-7

If three parallel lines are cut by two transversals, then they divide the transversals proportionally



If $l \parallel m \parallel n$, then $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{c} = \frac{b}{d}$.

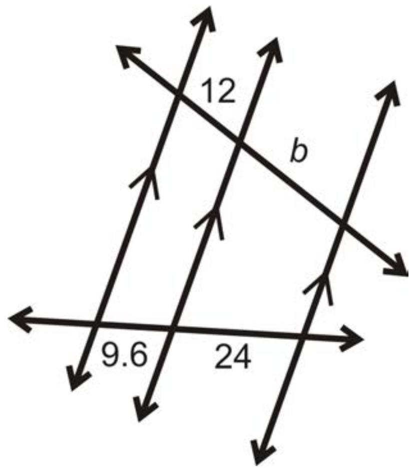
Example 4: Find a.



Solution: The three lines are marked parallel, set up a proportion.

$$\frac{a}{20} = \frac{9}{15}$$
$$180 = 15a$$
$$a = 12$$

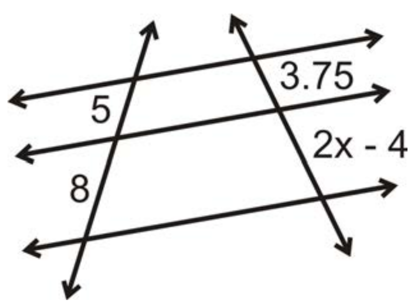
Example 5: Find b.



Solution: Set up a proportion.

$$\frac{12}{9.6} = \frac{b}{24}$$
$$288 = 9.6b$$
$$b = 30$$

Example 6: ***Algebra Connection*** Find the value of x that makes the lines parallel.

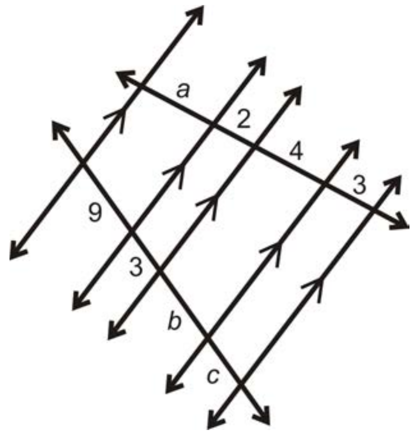


Solution: Set up a proportion and solve for x.

$$\frac{5}{8} = \frac{3.75}{2x-4} \longrightarrow 5(2x-4) = 8(3.75)$$
$$10x - 20 = 30$$
$$10x = 50$$
$$x = 5$$

Theorem 6-7 can be expanded to ***any*** number of parallel lines with ***any*** number of transversals. When this happens all corresponding segments of the transversals are proportional.

Example 7: Find a, b, and c.



Solution: Line up the segments that are opposite each other.

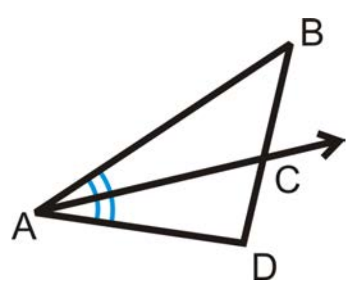
$$\frac{a}{9} = \frac{2}{3} \qquad \frac{2}{3} = \frac{4}{b} \qquad \frac{2}{3} = \frac{3}{c}$$
$$3a = 18 \qquad 2b = 12 \qquad 2c = 9$$
$$a = 6 \qquad b = 6 \qquad c = 4.5$$

Proportions with Angle Bisectors

The last proportional relationship we will explore is how an angle bisector intersects the opposite side of a triangle.

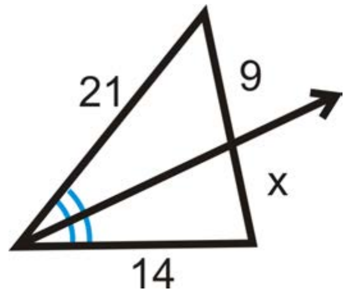
Theorem 6-8

If a ray bisects an angle of a triangle, then it divides the opposite side into segments that are proportional to the lengths of the other two sides



If $\triangle BAC \cong \triangle CAD$, then $\frac{BC}{CD} = \frac{AB}{AD}$.

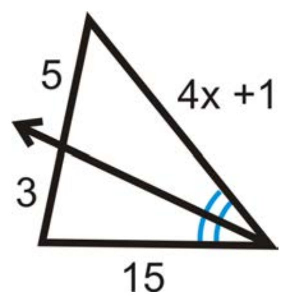
Example 8: Find x.



Solution: The ray is the angle bisector and it splits the opposite side in the same ratio as the sides. The proportion is:

$$\begin{aligned}\frac{9}{x}&=\frac{21}{14}\\21x&=126\\x&=6\end{aligned}$$

Example 9: ***Algebra Connection*** Find the value of x that would make the proportion true.



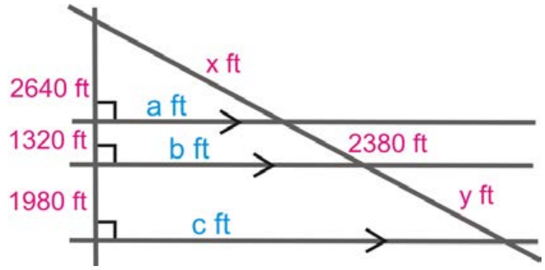
Solution: You can set up this proportion like the previous example.

$$\begin{aligned}\frac{5}{3}&=\frac{4x+1}{15}\\75&=3(4x+1)\\75&=12x+3\\72&=12x\\6&=x\end{aligned}$$

Know What? Revisited To find x and y, you need to set up a proportion using parallel the parallel lines.

$$\frac{2640}{x}=\frac{1320}{2380}=\frac{1980}{y}$$

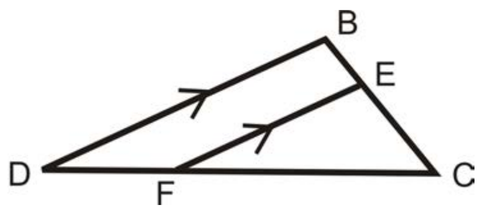
From this, $x=4760\ ft$ and $y=3570\ ft$.



Practice Problems

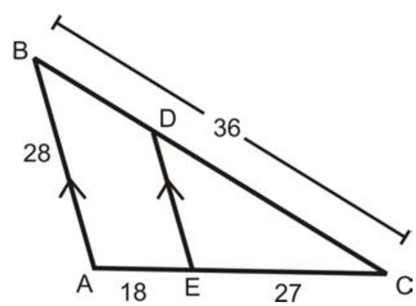
- Questions 1-12 are similar to Examples 1 and 2 and review.
- Questions 13-18 are similar to Example 3.
- Questions 19-24 are similar to Examples 8 and 9.
- Questions 25-30 are similar to Examples 4-7.

Use the diagram to answers questions 1-5. $\overline{DB} \| \overline{FE}$.



1. Name the similar triangles. Write the similarity statement.
2. $\frac{BE}{EC} = \frac{?}{FC}$
3. $\frac{EC}{CB} = \frac{CF}{?}$
4. $\frac{DB}{?} = \frac{BC}{EC}$
5. $\frac{FC+?}{FC} = \frac{?}{FE}$

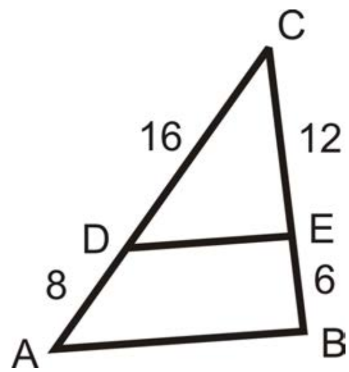
Use the diagram to answer questions 6-12. $\overline{AB} \| \overline{DE}$.



6. Find BD.
7. Find DC.
8. Find DE.
9. Find AC.
10. What is $BD : DC$?
11. What is $DC : BC$?
12. Why $BD : DC \neq DC : BC$?

Use the given lengths to determine if $\overline{AB} \| \overline{DE}$.

13.



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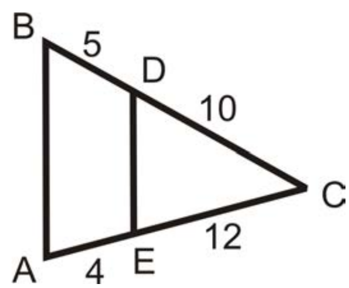

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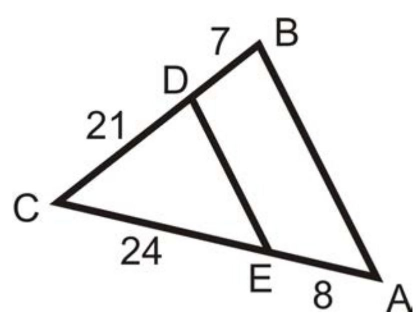

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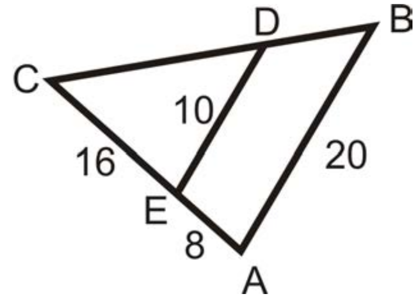

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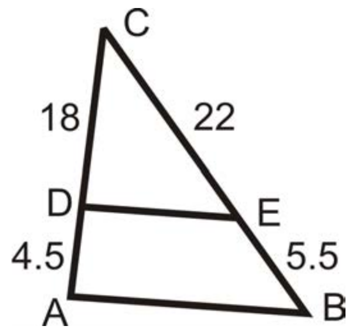

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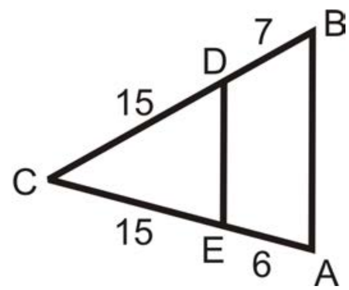
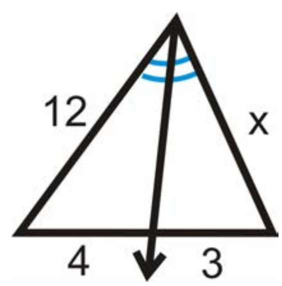


Algebra Connection Find the value of the missing variable(s).

19.



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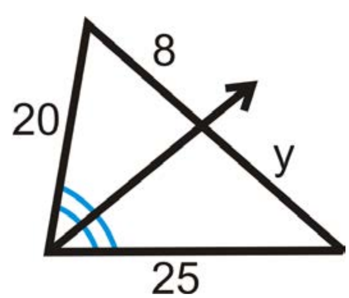

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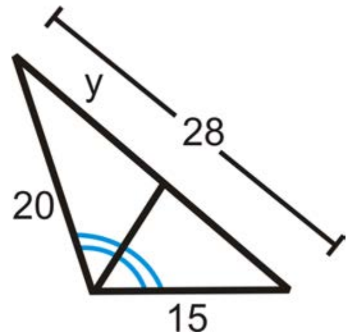

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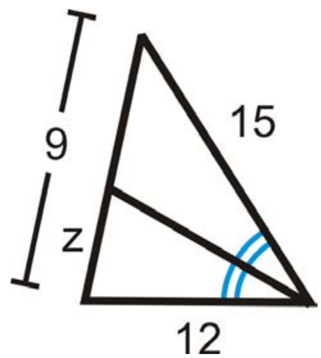

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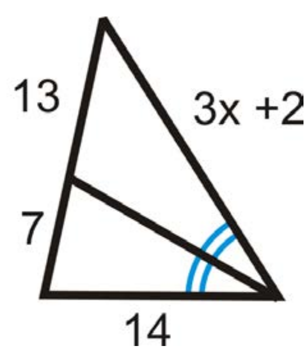

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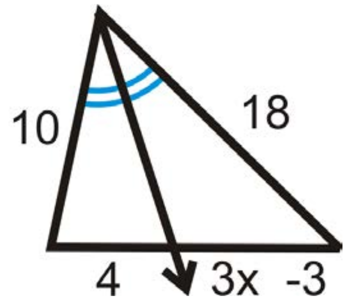
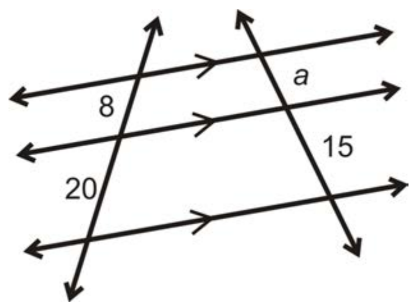
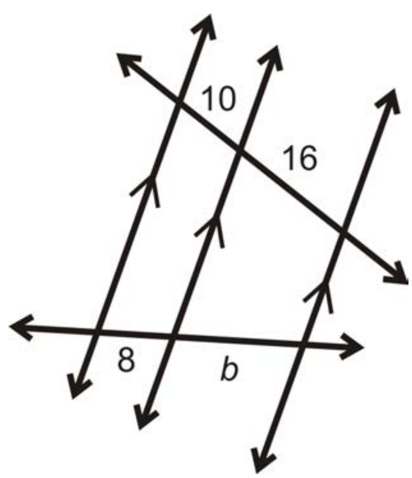


Find the value of each variable in the pictures below.

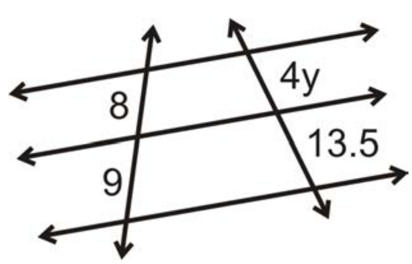
25.



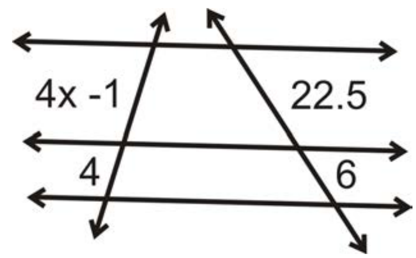
26.



27.



28.



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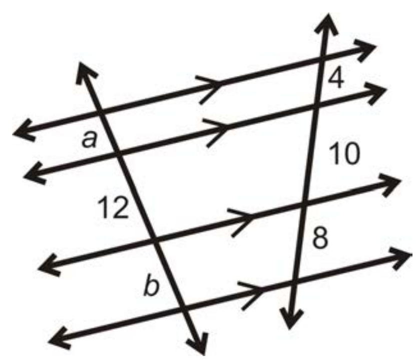
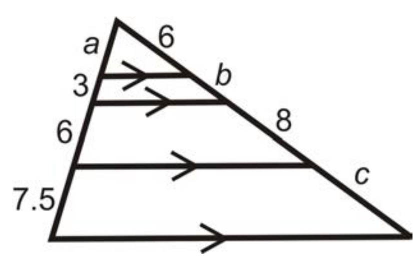


30.



Review and Reflect

31. True or False. If false give a counter example. If a ray bisects an angle of a triangle, then it divides the opposite side into two congruent segments.

32. Compare and contrast the Triangle Proportionality Theorem with the Midsegment Theorem.

Warm-Up Answers

1. $\triangle AXB \sim \triangle YXZ$ by AA Similarity Postulate
2. $\frac{16}{18} = \frac{32}{XZ}, XZ = 36$
3. $\frac{16}{18} = \frac{AB}{27}, AB = 24$
4. $AY = 18 - 16 = 2, BZ = 36 - 32 = 4$

6.6 Similarity Transformations

TEKS G(1)C, G(3)A, G(7)A

Learning Objectives

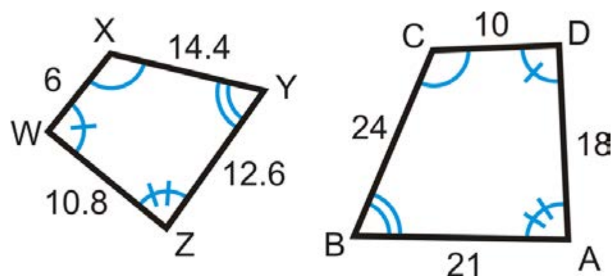
- Draw a dilation of a given figure
- Plot an image when given the center of dilation and a scale factor
- Determine if one figure is a dilation of another

Vocabulary

- dilation
- center
- mapping

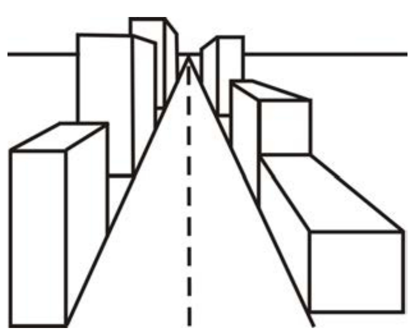
Warm-Up

1. Are the two quadrilaterals similar? How do you know?



2. What is the scale factor from $XYZW$ to $CDAB$? *Leave as a fraction.*

Know What? One practical application of dilations is perspective drawings. These drawings use a *vanishing point* (the point where the road meets the horizon) to trick the eye into thinking the picture is three-dimensional. The picture to the right is called a one-point perspective. They are typically used to draw streets, train tracks, or anything that is linear.



Your task for this **Know What?** is to draw your own perspective drawing with one vanishing point and at least 4 objects (buildings, cars, sidewalk, train tracks, etc).

Dilations

A dilation makes a figure larger or smaller and has the same shape as the original.

Dilation: An enlargement or reduction of a figure that preserves shape but not size. All dilations are similar to the original figure.

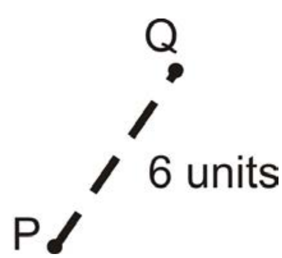
Dilations have a **center** and a **scale factor**. The center is the point of reference for the dilation and scale factor tells us how much the figure stretches or shrinks. A scale factor is labeled k and ***always greater than zero.*** A dilation, or copy, is always followed by *a* '.

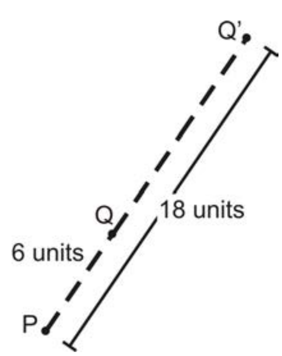
TABLE 6.3:

Label It	***Say It***
'	"prime" (copy of the original)
A'	"a prime" (copy of point A)
A''	"a double prime" (second copy)

Example 1: The center of dilation is P and the scale factor is 3.

Find Q'.

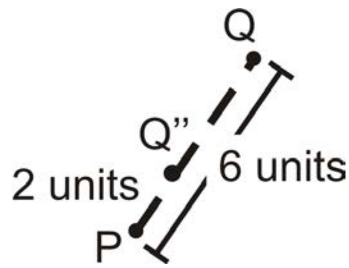




Solution: If the scale factor is 3 and Q is 6 units away from P, then Q' is going to be $6 \times 3 = 18$ units away from P. The dilation will be on the same line as the original and center.

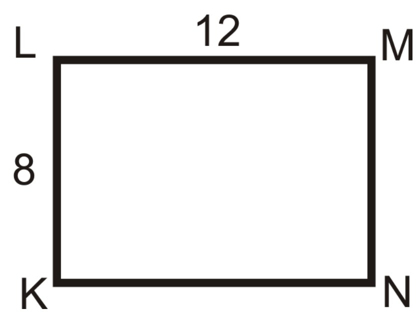
Example 2: Using the picture above, change the scale factor to $\frac{1}{3}$.

Find Q''.

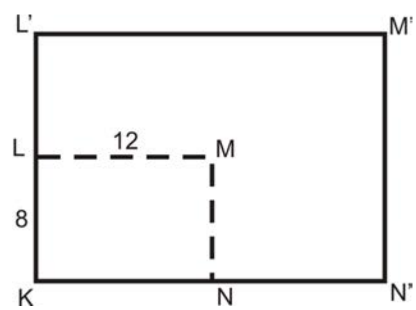


Solution: The scale factor is $\frac{1}{3}$, so Q'' is going to be $6 \times \frac{1}{3} = 2$ units away from P. Q'' will also be collinear with Q and center.

Example 3: $KLMN$ is a rectangle. If the center of dilation is K and $k = 2$, draw $K'L'M'N'$.



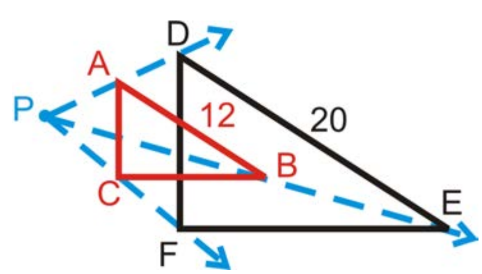
Solution: If K is the center of dilation, then K and K' will be the same point. From there, L' will be 8 units above L and N' will be 12 units to the right of N.



Example 4: Find the perimeters of $KLMN$ and $K'L'M'N'$. Compare this ratio to the scale factor.

Solution: The perimeter of $KLMN = 12 + 8 + 12 + 8 = 40$. The perimeter of $K'L'M'N' = 24 + 16 + 24 + 16 = 80$. The ratio is 80:40, which reduces to 2:1, which is the same as the scale factor.

Example 5: $\triangle ABC$ is a dilation of $\triangle DEF$. If P is the center of dilation, what is the scale factor?



Solution: Because $\triangle ABC$ is a dilation of $\triangle DEF$, then $\triangle ABC \sim \triangle DEF$. The scale factor is the ratio of the sides. Since $\triangle ABC$ is smaller than the original, $\triangle DEF$, the scale factor is going to be less than one, $\frac{12}{20} = \frac{3}{5}$.

If $\triangle DEF$ was the dilated image, the scale factor would have been $\frac{5}{3}$.

If the dilated image is <u>smaller</u> than the original, then $0 < k < 1$.

If the dilated image is <u>larger</u> than the original, then $k > 1$.

Watch this!

MEDIA

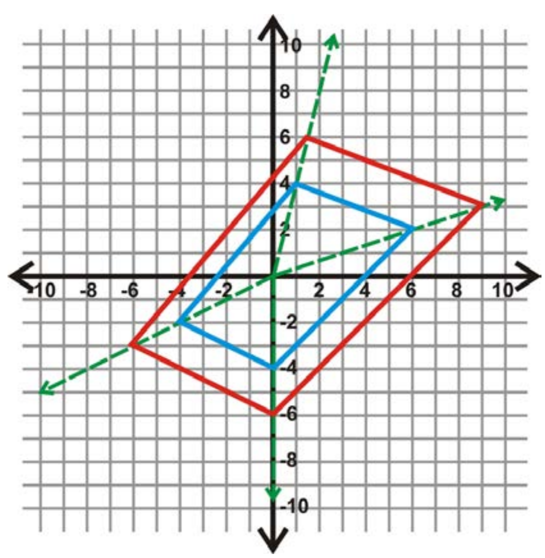
Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/157329

Dilations in the Coordinate Plane

In this text, the center of dilation will always be the origin.

Example 6: Quadrilateral $EFGH$ has vertices $E(-4,-2), F(1,4), G(6,2)$ and $H(0,-4)$. Draw the dilation with a scale factor of 1.5.



Solution: To dilate something in the coordinate plane, multiply each coordinate by the scale factor. This is called ***mapping.***

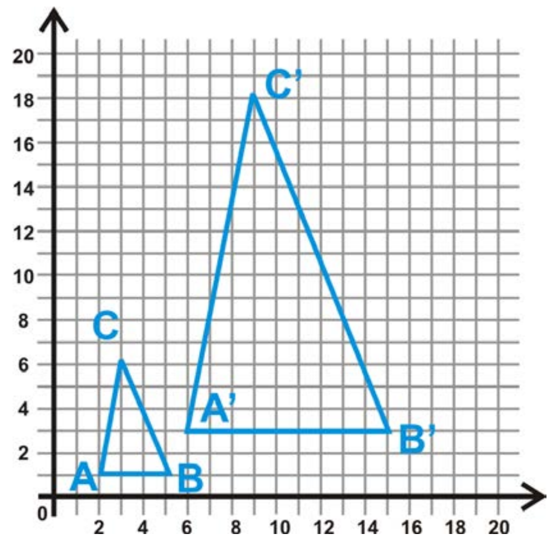
For any dilation the mapping will be $(x,y) \rightarrow (kx,ky)$**.**

For this dilation, the mapping will be $(x,y) \rightarrow (1.5x,1.5y)$.

$$E(-4,-2) \rightarrow (1.5(-4), 1.5(-2)) \rightarrow E'(-6,-3)$$
$$F(1,4) \rightarrow (1.5(1), 1.5(4)) \rightarrow F'(1.5,6)$$
$$G(6,2) \rightarrow (1.5(6), 1.5(2)) \rightarrow G'(9,3)$$
$$H(0,-4) \rightarrow (1.5(0), 1.5(-4)) \rightarrow H'(0,-6)$$

In the graph above, the blue quadrilateral is the original and the red image is the dilation.

Example 7: Determine the coordinates of $\triangle ABC$ and $\triangle A'B'C'$ and find the scale factor.



Solution: The coordinates of $\triangle ABC$ are $A(2,1)$, $B(5,1)$ and $C(3,6)$. The coordinates of $\triangle A'B'C'$ are $A'(6,3)$, $B'(15,3)$ and $C'(9,18)$. Each of the corresponding coordinates are three times the original, so $k = 3$.

Use the following link, to dilate and resize images on a coordinate plane. **http://www.mathsisfun.com/geometry/resizing.html**

Example 8: Show that dilations preserve shape by using the distance formula. Find the lengths of the sides of both triangles in Example 7.

Solution:

$\underline{\triangle ABC}$	$\underline{\triangle A'B'C'}$
$AB = \sqrt{(2-5)^2+(1-1)^2} = \sqrt{9} = 3$	$A'B' = \sqrt{(6-15)^2+(3-3)^2} = \sqrt{81} = 9$
$AC = \sqrt{(2-3)^2+(1-6)^2} = \sqrt{26}$	$A'C' = \sqrt{(6-9)^2+(3-18)^2} = \sqrt{234} = 3\sqrt{26}$
$CB = \sqrt{(3-5)^2+(6-1)^2} = \sqrt{29}$	$C'B' = \sqrt{(9-15)^2+(18-3)^2} = \sqrt{261} = 3\sqrt{29}$

From this, we also see that all the sides of $\triangle A'B'C'$ are three times larger than $\triangle ABC$.

Know What? Revisited Answers to this project will vary depending on what you decide to draw. Make sure that you have at least four objects of detail. If you are having trouble getting started, go to the website: http://www.drawing-and-painting-techniques.com/drawing-perspective.html

Practice Problems

- Questions 1-6 are similar to Examples 1 and 2.
- Questions 7-10 are similar to Example 3.
- Questions 11-18 are similar to Example 5.
- Questions 19-24 are similar to Examples 6 and 7.
- Questions 25-30 are similar to Example 8.

Given A and the scale factor, determine the coordinates of the dilated point, A'. You may assume the center of dilation is the origin.

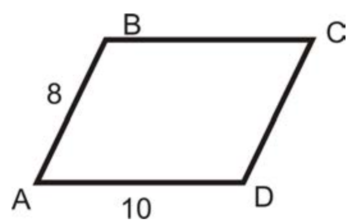
1. $A(3,9), k = \frac{2}{3}$
2. $A(-4,6), k = 2$
3. $A(9,-13), k = \frac{1}{2}$

Given A and A', find the scale factor. You may assume the center of dilation is the origin.

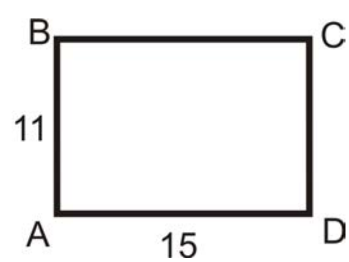
4. $A(8,2), A'(12,3)$
5. $A(-5,-9), A'(-45,-81)$
6. $A(22,-7), A(11,-3.5)$

For the given shapes, draw the dilation, given the scale factor and center.

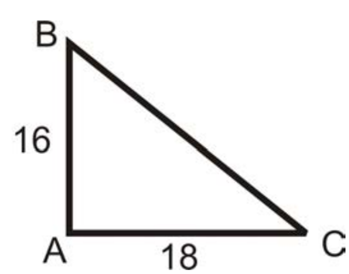
7. $k = 3.5$, center is A



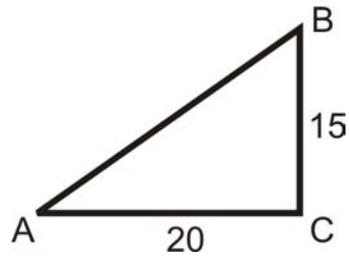
8. $k = 2$, center is D



9. $k = \frac{3}{4}$, center is A

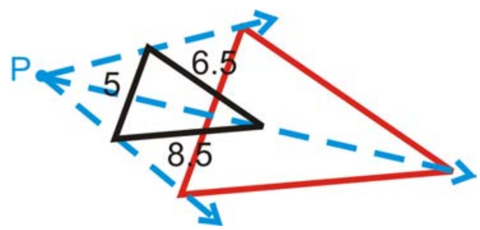


10. $k = \frac{2}{5}$, center is A

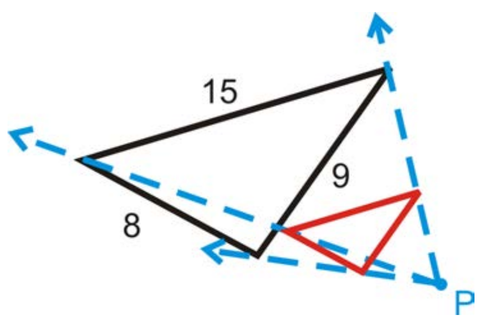


In the four questions below, you are told the scale factor. Determine the dimensions of the dilation. In each diagram, the **black** figure is the original and P is the center of dilation.

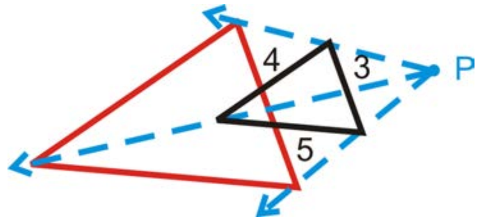
11. $k = 4$



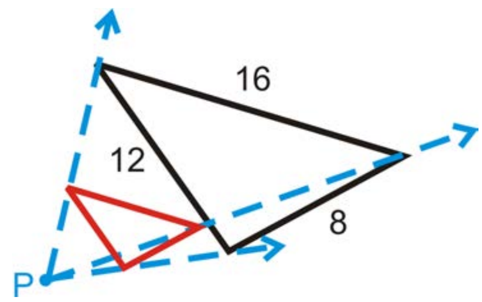
12. $k = \frac{1}{3}$



13. $k = 2.5$

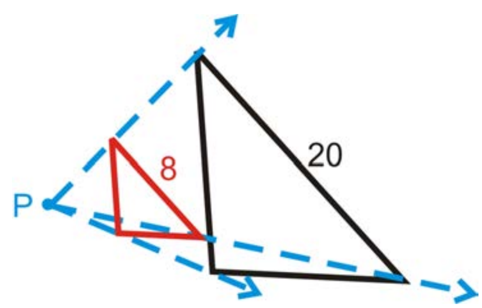


14. $k = \frac{1}{4}$

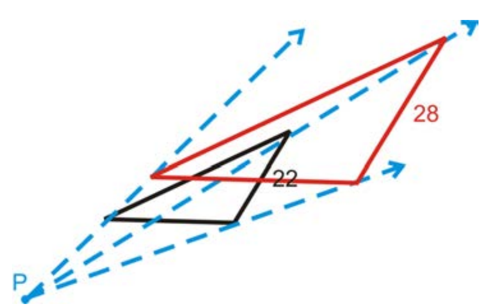


In the four questions below, find the scale factor, given the corresponding sides. In each diagram, the **black** figure is the original and *P* is the center of dilation.

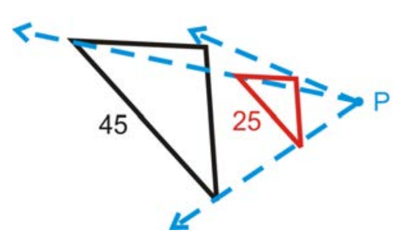
15\.



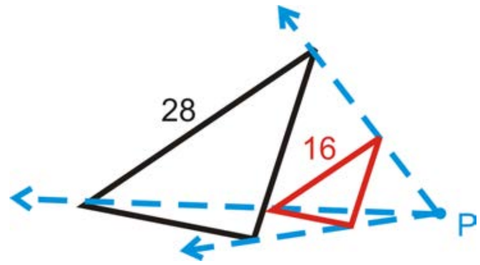
16\.



17\.



18\.



The origin is the center of dilation. Draw the dilation of each figure, given the scale factor.

19. $A(2,4), B(-3,7), C(-1,-2); k = 3$
20. $A(12,8), B(-4,-16), C(0,10); k = \frac{3}{4}$

Multi-Step Problem Questions 21-24 build upon each other.

21. Plot $A(1,2), B(12,4), C(10,10)$. Connect to form a triangle.
22. Make the origin the center of dilation. Draw 4 rays from the origin to each point from #21. Then, plot $A'(2,4), B'(24,8), C'(20,20)$. What is the scale factor?
23. Use $k = 4$, to find $A''B''C''$. Plot these points.
24. What is the scale factor from $A'B'C'$ to $A''B''C''$?

If O is the origin, find the following lengths (using 21-24 above). Round all answers to the nearest hundredth.

25. OA
26. AA'
27. AA''
28. OA'
29. OA''
30. AB
31. $A'B'$
32. $A''B''$
33. Compare the ratios $OA : OA'$ and $AB : A'B'$. What do you notice? Why do you think that is?
34. Compare the ratios $OA : OA''$ and $AB : A''B''$. What do you notice? Why do you think that is?

Review and Reflect

35. If you were given 2 objects with the same shape, describe how you would determine if the objects were similar.

36. How do you dilate a figure in a coordinate plane?

Warm-Up Answers

1. Yes, all the angles are congruent and the corresponding sides are in the same ratio.
2. $\frac{5}{3}$
3. Yes, $LMNO \sim EFGH$ because $LMNO$ is exactly half of $EFGH$.

6.7 Self-Similarity

TEKS G(1)B, G(1)F

Learning Objectives

Understand basic fractals

Vocabulary

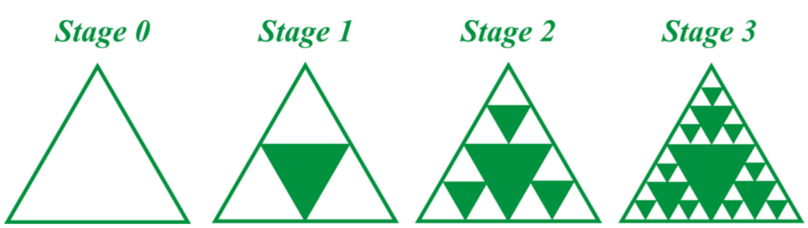
- self similar
- iteration
- fractals

Self-Similar: When one part of an object can be enlarged (or shrunk) to look like the whole object.

To explore self-similarity, we will go through some examples. Typically, each step of repetition is called an ***iteration.*** The first level is called ***Stage 0.***

Sierpinski Triangle

The Sierpinski triangle iterates a triangle by connecting the midpoints of the sides and shading the central triangle (Stage 1). Repeat this process for the unshaded triangles in Stage 1 to get Stage 2.



Example 1: Determine the number of shaded and unshaded triangles in each stage of the Sierpinkski triangles. Determine if there is a pattern.

Solution:

TABLE 6.4:

	Stage 0	*Stage 1*	*Stage 2*	*Stage 3*
Unshaded	1	3	9	27
Shaded	0	1	4	13

The unshaded triangles seem to be powers of $3, 3^0, 3^1, 3^2, 3^3, \ldots$ The shaded triangles are add the previous number of unshaded triangles to the total. For Example, Stage 4 would equal 9 + 13 shaded triangles.

Fractals

A fractal is another self-similar object that is repeated at smaller scales. Below are the first three stages of the Koch snowflake.

Example 2: Determine the number of edges and the perimeter of each snowflake.

TABLE 6.5:

	Stage 0	*Stage 1*	*Stage 2*
Number of Edges	3	12	48
Edge Length	1	$\frac{1}{3}$	$\frac{1}{9}$
Perimeter	3	4	$\frac{48}{9} = 5.\overline{3}$

The Cantor Set

The Cantor set is another fractal that consists of dividing a segment into thirds and then erasing the middle third.

Practice Problems

1. Draw Stage 4 of the Cantor set.
2. Use the Cantor Set to fill in the table below.

TABLE 6.6:

	Number of Segments	*Length of each Segment*	*Total Length of the Segments*
Stage 0	1	1	1
Stage 1	2	$\frac{1}{3}$	$\frac{2}{3}$
Stage 2	4	$\frac{1}{9}$	$\frac{4}{9}$
Stage 3			
Stage 4			
Stage 5			

3. How many segments are in Stage n?
4. Draw Stage 3 of the Koch snowflake.
5. A variation on the Sierpinski triangle is the Sierpinski carpet, which splits a square into 9 equal squares, coloring the middle one only. Then, split the uncolored squares to get the next stage. Draw the first 3 stages of this fractal.
6. How many colored vs. uncolored square are in each stage?
7. Fractals are very common in nature. For example, a fern leaf is a fractal. As the leaves get closer to the end, they get smaller and smaller. Find three other examples of fractals in nature.

6.8 Chapter 6 Review

Keywords and Theorems

Rations Proportions

- Ratio
- Proportion
- Means
- Extremes
- Cross-Multiplication Theorem
- Corollary
- Corollary 7-1
- Corollary 7-2
- Corollary 7-3
- Corollary 7-4
- Corollary 7-5

Similar Polygons

- Similar Polygons
- Scale Factor
- Theorem 7-2

Similarity by AA

- AA Similarity Postulate
- Indirect Measurement

Similarity by SSS and SAS

- SSS Similarity Theorem
- SAS Similarity Theorem

Proportionality Relationships

- Triangle Proportionality Theorem
- Triangle Proportionality Theorem Converse
- Theorem 7-7
- Theorem 7-8

Dilations

- Dilation

Self-Similarity

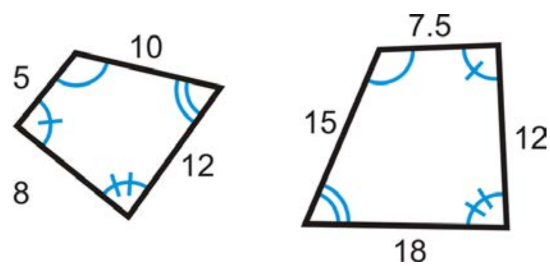
- Self-Similar
- Fractal

Review Questions

1. Solve the following proportions.
   a. $\frac{x+3}{3} = \frac{10}{2}$
   b. $\frac{8}{5} = \frac{2x-1}{x+3}$
2. The extended ratio of the angle in a triangle are 5:6:7. What is the measure of each angle?
3. Rewrite 15 quarts in terms of gallons.

Determine if the following pairs of polygons are similar. If it is two triangles, write *why* they are similar.

4.



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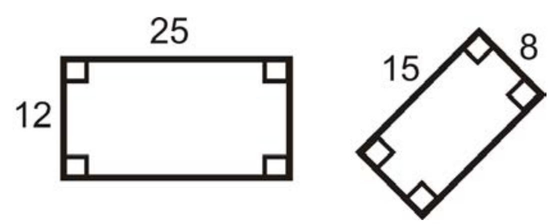

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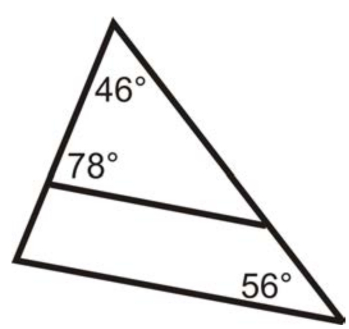

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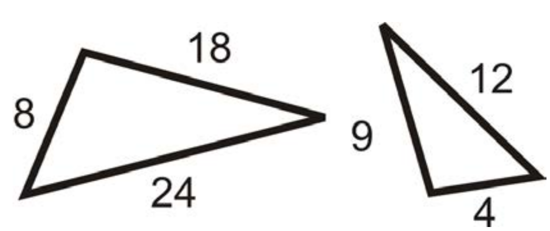

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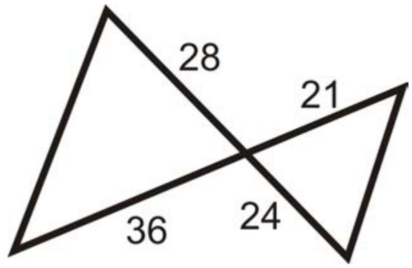

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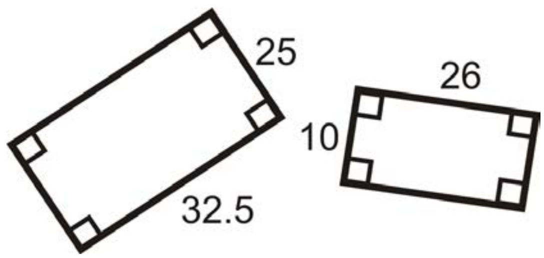
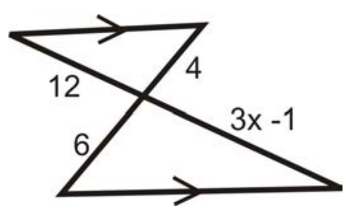


10. Draw a dilation of $A(7,2), B(4,9)$, and $C(-1,4)$ with $k = \frac{3}{2}$.

Algebra Connection Find the value of the missing variable(s).

11.



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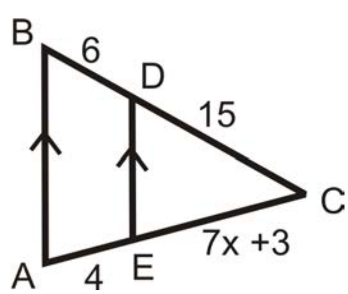

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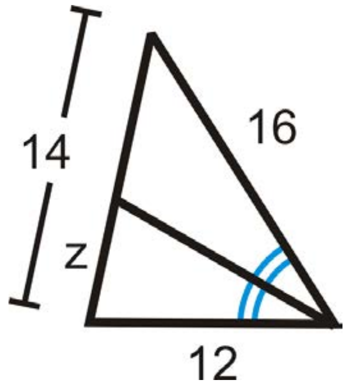

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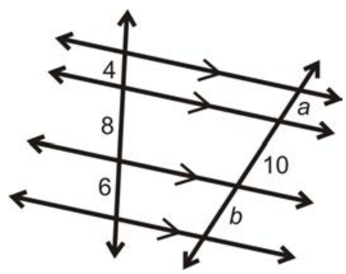


Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9692 .

6.9 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st **Section: Ratios Proportions**

Ratio

Proportion

Means

Extremes

Cross-Multiplication Theorem

Corollary

Corollary 7-1

Corollary 7-2

Corollary 7-3

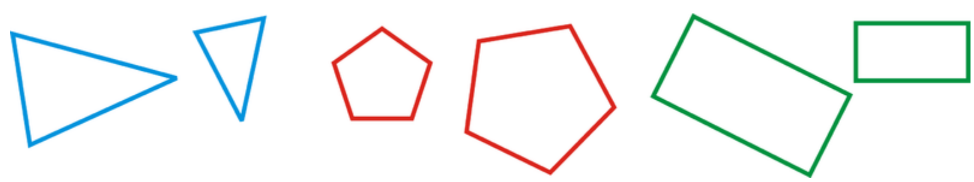
Corollary 7-4

Corollary 7-5

Homework:

2nd **Section: Similar Polygons**

Similar Polygons



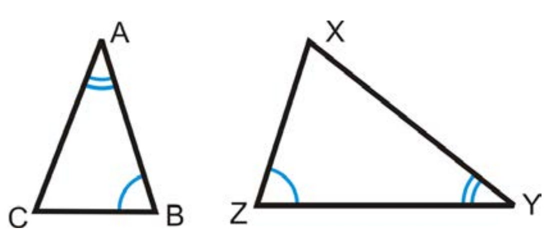
Scale Factor

Theorem 7-2

Homework:

3rd **Section: Similarity by AA**

AA Similarity Postulate

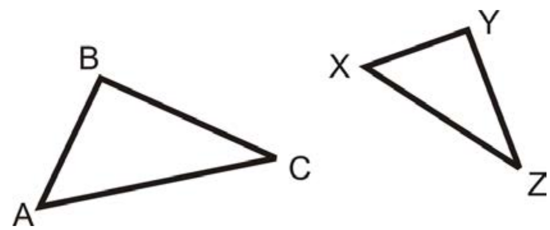


Indirect Measurement

Homework:

4^{th} **Section: Similarity by SSS and SAS**

SSS Similarity Theorem

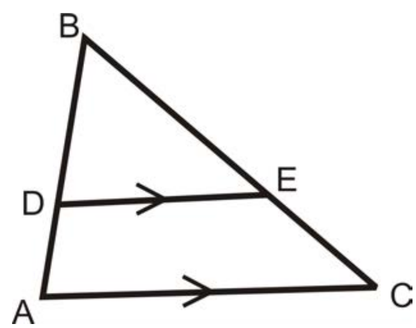


SAS Similarity Theorem

Homework:

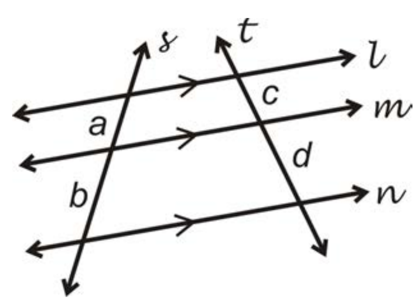
5^{th} **Section: Proportionality Relationships**

Triangle Proportionality Theorem

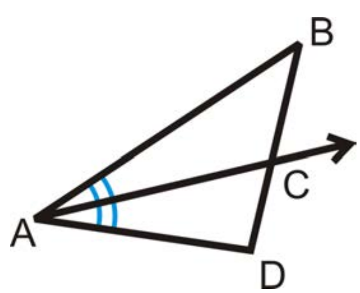


Triangle Proportionality Theorem Converse

Theorem 7-7



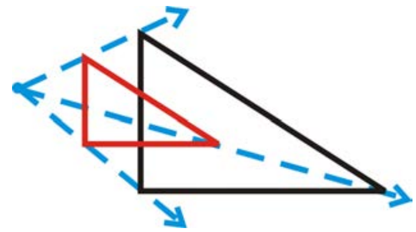
Theorem 7-8



Homework:

6^{th} **Section: Dilations**

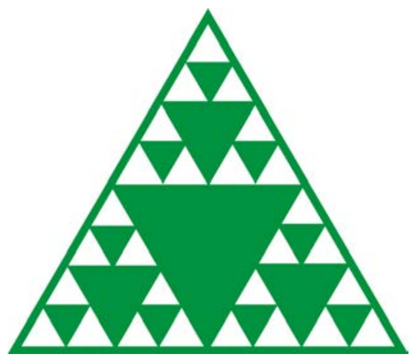
Dilation



Homework:

Extension: Self-Similarity

Self-Similar



Fractal

Homework:

CHAPTER 7 Polygons and Quadrilaterals

Chapter Outline

This chapter starts with the properties of polygons and narrows to focus on quadrilaterals. We will study several different types of quadrilaterals: parallelograms, rhombi, rectangles, squares, kites and trapezoids. Then, we will prove that different types of quadrilaterals are parallelograms.

7.1 Angles in Polygons

TEKS G(1)B, G(1)D, G(5)A, G(6)D

Learning Objectives

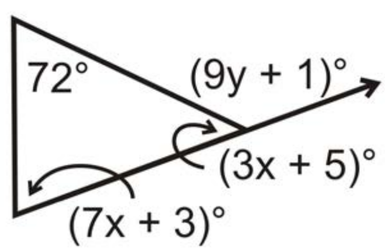
- Extend the concept of interior and exterior angles from triangles to convex polygons
- Find the sums of interior angles in convex polygons

Vocabulary

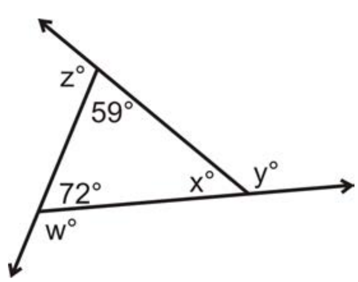
- Diagonal
- Interior Angle
- Exterior Angle

Warm-Up

1. a. What do the angles in a triangle add up to?
   b. Find the measure of x and y.

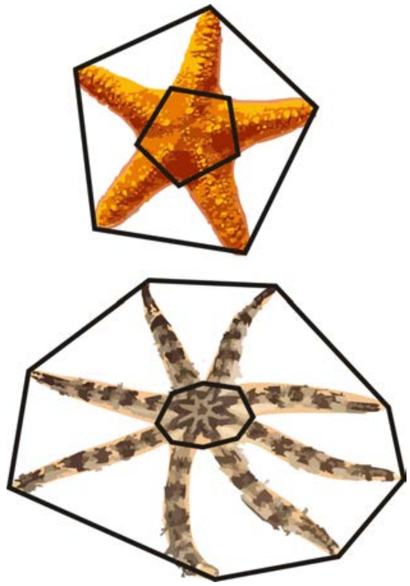


   c. A linear pair adds up to _____.



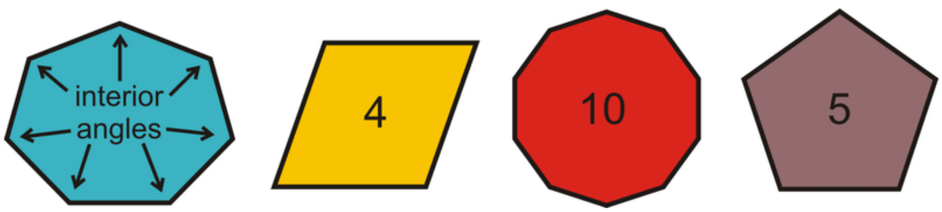
2.
   a. Find $w^\circ, x^\circ, y^\circ$, and z°.
   b. What is $w^\circ + y^\circ + z^\circ$?

Know What? In nature, geometry is all around us. For example, sea stars have geometric symmetry. The common sea star, top, has five arms, but some species have over 20! To the right are two different kinds of sea stars. Name the polygon that is created by joining their arms and determine if either polygon is regular.



Interior Angles in Convex Polygons

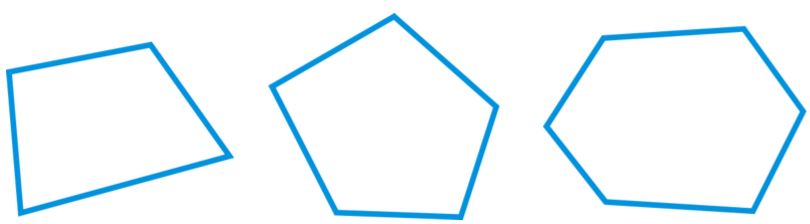
In Chapter 4, you learned that interior angles are the angles inside a triangle and that these angles add up to 180°. This concept will now be extended to any polygon. As you can see in the images below, a polygon has the same number of interior angles as it does sides. But, what do the angles add up to?



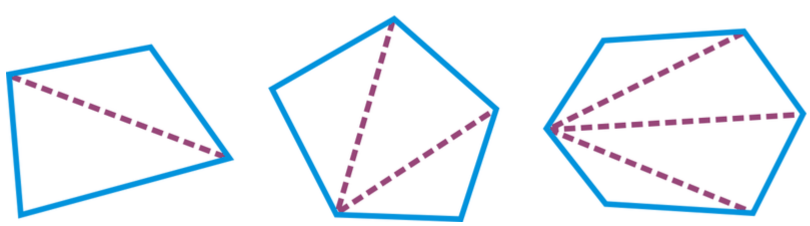
Investigation 6-1: Polygon Sum Formula

Tools Needed: paper, pencil, ruler, colored pencils (optional)

1. Draw a quadrilateral, pentagon, and hexagon.



2. Cut each polygon into triangles by drawing all the diagonals from one vertex. Count the number of triangles.



Make sure none of the triangles overlap.

3. Make a table with the information below.

TABLE 7.1:

Name of Polygon	Number of Sides	Number of $\triangle s$ from one vertex	(Column 3) $\times$ (Degrees in a $\triangle$)	Total Number of Degrees
Quadrilateral	4	2	$2 \times 180^\circ$	360°
Pentagon	5	3	$3 \times 180^\circ$	540°
Hexagon	6	4	$4 \times 180^\circ$	720°

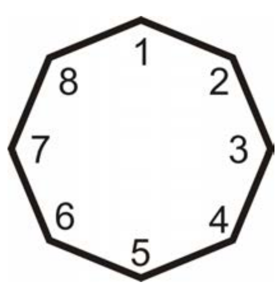
4. Notice that the total number of degrees goes up by 180°. So, if the number sides is n, then the number of triangles from one vertex is $n-2$. Therefore, the formula would be $(n-2) \times 180^\circ$.

Check This Out

Investigate this concept further using the following link.

 http://web.geogebra.org/app/?id=56818

Polygon Sum Formula: For any $n-$gon, the interior angles add up to $(n-2) \times 180^\circ$.



$$\begin{aligned} &\rightarrow n = 8 \\ &(8-2)\times 180^\circ \\ &\quad 6\times 180^\circ \\ &\quad 1080^\circ \end{aligned}$$

Example 1: The interior angles of a polygon add up to 1980°. How many sides does it have?

Solution: Use the Polygon Sum Formula and solve for n.

The polygon has 13 sides.

Regular Polygon: When a polygon is equilateral **and** equiangular.

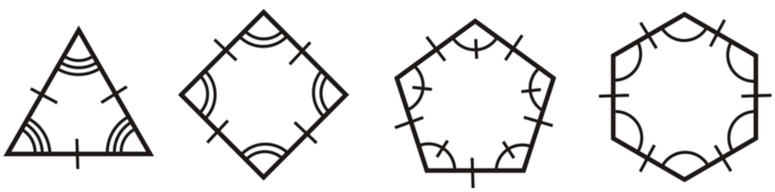


FIGURE 7.1

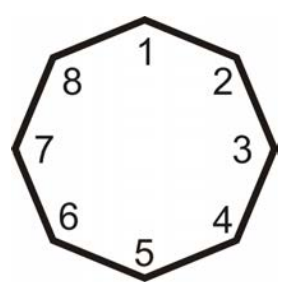
Example 2: How many degrees does ***each angle*** in a regular nonagon have?

Solution: First we need to find the sum of the interior angles, set $n = 9$.

$$(9-2)\times 180^\circ = 7\times 180^\circ = 1260^\circ$$

"Regular" tells us every angle is equal. So, each angle is $\frac{1260^\circ}{9} = 140^\circ$.

Regular Polygon Formula: For any regular $n-$gon, the measure of each angle is $\frac{(n-2)\times 180^\circ}{n}$.



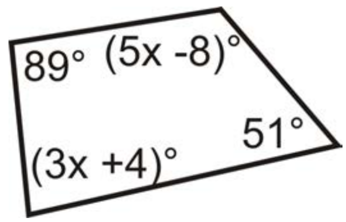
If $m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4 = m\angle 5 = m\angle 6 = m\angle 7 = m\angle 8$, then each angle is $\frac{(8-2)\times 180^\circ}{8} = \frac{6\times 180^\circ}{8} = \frac{1080^\circ}{8} = 135^\circ$

Example 3: An interior angle in a regular polygon is 135°. How many sides does this polygon have?

Solution: Here, we will set the Regular Polygon Formula equal to 135° and solve for n.

$$\begin{aligned}\frac{(n-2)\times 180^\circ}{n} &= 135^\circ \\ 180^\circ n - 360^\circ &= 135^\circ n \\ -360^\circ &= -45^\circ n \\ n &= 8 \qquad \text{The polygon is an octagon.}\end{aligned}$$

Example 4: ***Algebra Connection*** Find the measure of x.



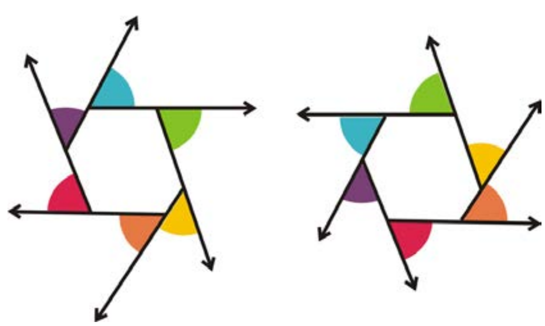
Solution: From our investigation, we found that a quadrilateral has 360°.

Write an equation and solve for x.

$$\begin{aligned}89^\circ + (5x-8)^\circ + (3x+4)^\circ + 51^\circ &= 360^\circ \\ 8x &= 224^\circ \\ x &= 28^\circ\end{aligned}$$

Exterior Angles in Convex Polygons

An exterior angle is an angle that is formed by extending a side of the polygon (Chapter 4).



As you can see, there are two sets of exterior angles for any vertex on a polygon, one going around clockwise (1^{st} hexagon), and the other going around counter-clockwise (2^{nd} hexagon). The angles with the same colors are vertical and congruent.

The Exterior Angle Sum Theorem said the exterior angles of a triangle add up to 360°. Let's extend this theorem to all polygons.

Investigation 6-2: Exterior Angle Tear-Up

Tools Needed: pencil, paper, colored pencils, scissors

1. Draw a hexagon like the ones above. Color in the exterior angles.

2. Cut out each exterior angle.

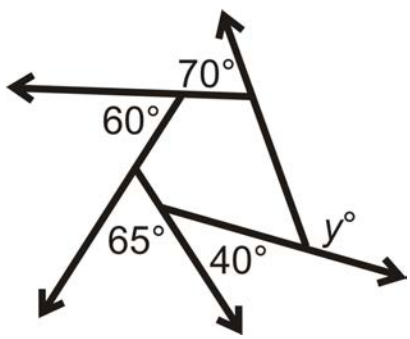
3. Fit the six angles together by putting their vertices together. What happens?

The angles all fit around a point, meaning that the angles add up to 360°.

Exterior Angle Sum Theorem

The sum of the exterior angles of any polygon is 360°

Example 5: What is y?



Solution: y is an exterior angle and all the given angles add up to 360°. Set up an equation.

$$\begin{aligned} 70^\circ + 60^\circ + 65^\circ + 40^\circ + y &= 360^\circ \\ y &= 125^\circ \end{aligned}$$

Example 6: What is the measure of each exterior angle of a regular heptagon?

Solution: Because the polygon is regular, the interior angles are equal. It also means the exterior angles are equal. $\frac{360^\circ}{7} \approx 51.43^\circ$

Know What? Revisited The stars make a pentagon and an octagon. The pentagon looks to be regular, but we cannot tell without angle measurements or lengths.

Practice Problems

- Questions 1-13 are similar to Examples 1-3 and 6.

- Questions 14-30 are similar to Examples 4 and 5.

1. Fill in the table.

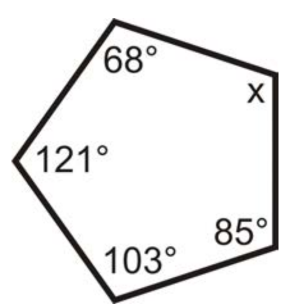
TABLE 7.2:

# of sides	# of $\triangle s$ from one vertex	$\triangle s \times 180°$(sum)	Each angle in a *regular* $n-$gon	Sum of the *exterior* angles
3	1	180°	60°	
4	2	360°	90°	
5	3	540°	108°	
6	4	720°	120°	
7				
8				
9				
10				
11				
12				

2. ***Writing*** Do you think the interior angles of a regular $n-$gon could ever be 180°? Why or why not? What about 179°?
3. What is the sum of the angles in a 15-gon?
4. What is the sum of the angles in a 23-gon?
5. The sum of the interior angles of a polygon is 4320°. How many sides does the polygon have?
6. The sum of the interior angles of a polygon is 3240°. How many sides does the polygon have?
7. What is the measure of each angle in a regular 16-gon?
8. What is the measure of each angle in an equiangular 24-gon?
9. Each interior angle in a regular polygon is 156°. How many sides does it have?
10. Each interior angle in an equiangular polygon is 90°. How many sides does it have?
11. What is the measure of each exterior angle of a dodecagon?
12. What is the measure of each exterior angle of a 36-gon?
13. What is the sum of the exterior angles of a 27-gon?

Algebra Connection For questions 14-26, find the measure of the missing variable(s).

14.



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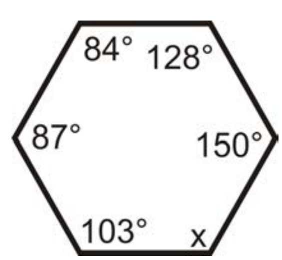

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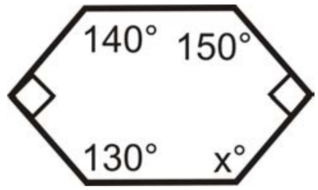

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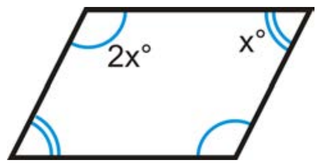

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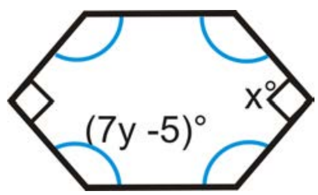

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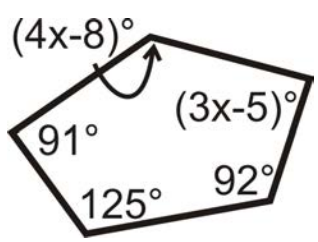

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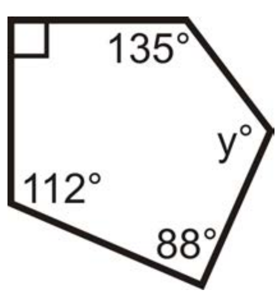

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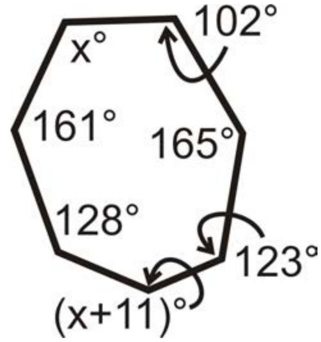

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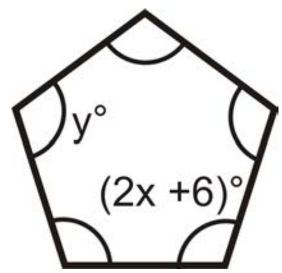

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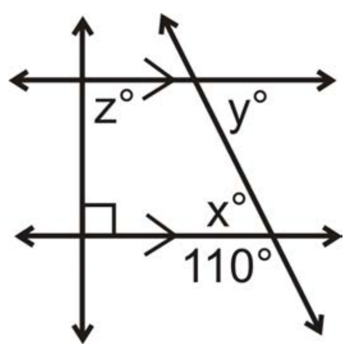

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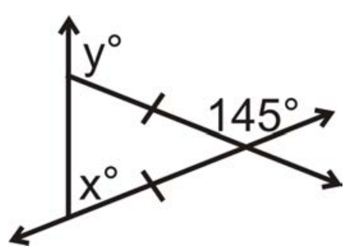

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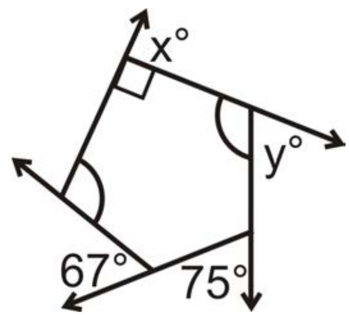

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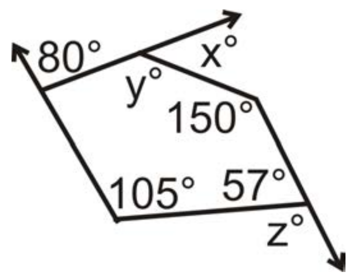
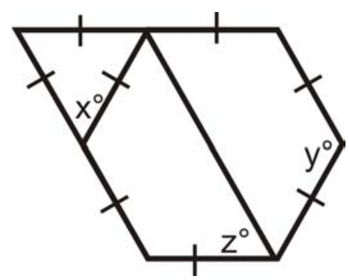


27.



28. The interior angles of a pentagon are $x^\circ, x^\circ, 2x^\circ, 2x^\circ$, and $2x^\circ$. What is x?
29. The exterior angles of a quadrilateral are $x^\circ, 2x^\circ, 3x^\circ$, and $4x^\circ$. What is x?
30. The interior angles of a hexagon are $x^\circ, (x+1)^\circ, (x+2)^\circ, (x+3)^\circ, (x+4)^\circ$, and $(x+5)^\circ$. What is x?

Review And Reflect

31. If you were given one exterior angle of a regular polygon, how can you find the sum of the interior angles?
32. Is there a maximum sum of the interior angles of any *n-gon*?

Warm-Up Answers

1.

a. 180°

b. $$\begin{aligned} 72^\circ + (7x+3)^\circ + (3x+5)^\circ &= 180^\circ \\ 10x + 80^\circ &= 180^\circ \\ 10x &= 100^\circ \\ x &= 10^\circ \end{aligned}$$

c. 180°

2.

a. $w = 108^\circ,\ x = 49^\circ,\ y = 131^\circ,\ z = 121^\circ$
b. 360°

7.2 Properties of Parallelograms

TEKS G(1)D, G(2)C,G(5)A

Learning Objectives

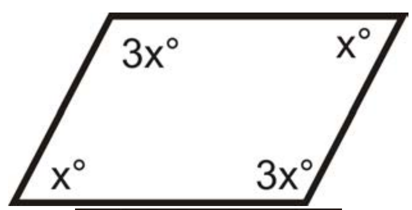
- Define a parallelogram
- Understand the properties of a parallelogram
- Apply theorems about a parallelogram's sides, angles, and diagonals

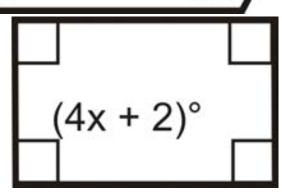
Vocabulary

- Parallelogram

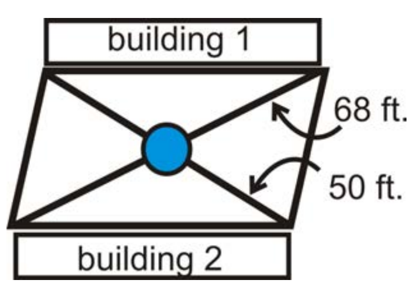
Warm-Up

1. Draw a quadrilateral with one set of parallel sides.
2. Draw a quadrilateral with two sets of parallel sides.
3. Find the measure of the missing angles in the quadrilaterals below.

a.


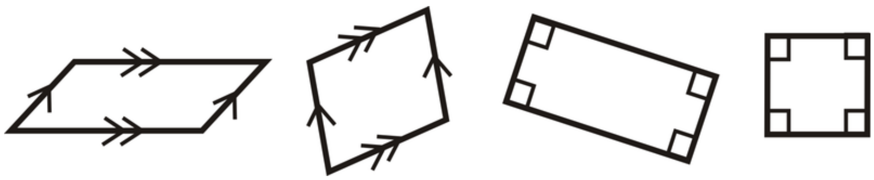
b.


Know What? A college has a parallelogram-shaped courtyard between two buildings. The school wants to build two walkways on the diagonals of the parallelogram and a fountain where they intersect. The walkways are going to be 50 feet and 68 feet long. Where would the fountain be?



What is a Parallelogram?

Parallelogram: A quadrilateral with two pairs of parallel sides.

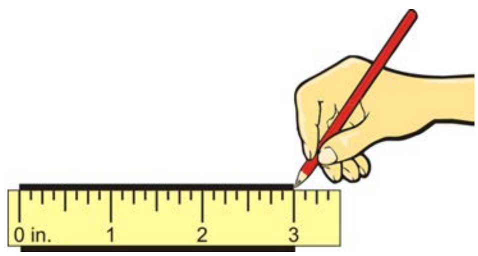


Notice that each pair of sides is marked parallel. Also, recall that two lines are parallel when they are perpendicular to the same line. Parallelograms have a lot of interesting properties.

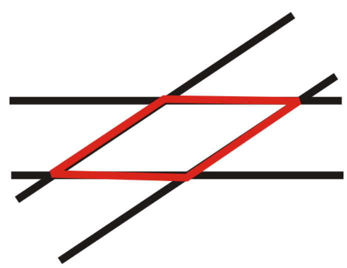
Investigation 6-2: Properties of Parallelograms

Tools Needed: Paper, pencil, ruler, protractor

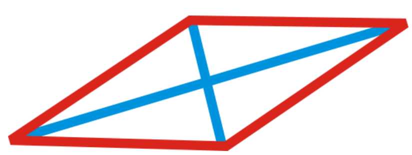
1. Draw a set of parallel lines by drawing a 3 inch line on either side of your ruler.



2. Rotate the ruler and repeat so you have a parallelogram. If you have colored pencils, outline the parallelogram in another color.



3. Measure the four interior angles of the parallelogram as well as the length of each side. What do you notice?
4. Draw the diagonals. Measure each and then measure the lengths from the point of intersection to each vertex.

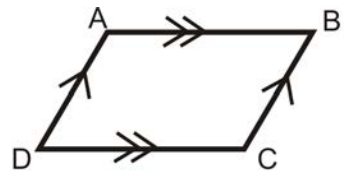


To continue to explore the properties of a parallelogram, see the website: http://www.mathwarehouse.com/geometry/quadrilaterals/parallelograms/interactive-parallelogram.php

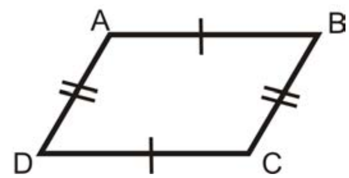
Opposite Sides Theorem

If a quadrilateral is a parallelogram, then the opposite sides are congruent

If



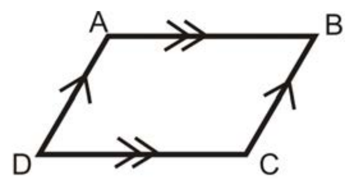
then



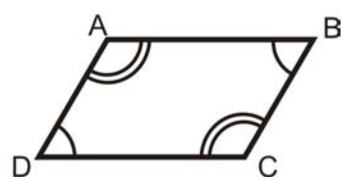
Opposite Angles Theorem

If a quadrilateral is a parallelogram, then the opposite angles are congruent

If



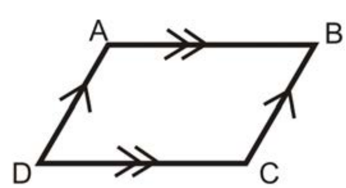
then



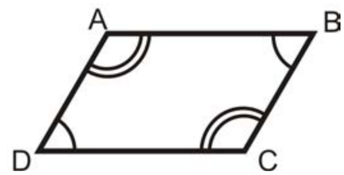
Consecutive Angles Theorem

If a quadrilateral is a parallelogram, then the consecutive angles are supplementary

If



then

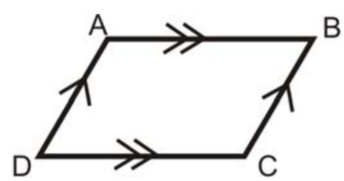


$$m\angle A + m\angle D = 180^\circ$$
$$m\angle A + m\angle B = 180^\circ$$
$$m\angle B + m\angle C = 180^\circ$$
$$m\angle C + m\angle D = 180^\circ$$

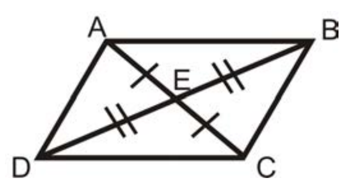
Parallelogram Diagonals Theorem

If a quadrilateral is a parallelogram, then the diagonals bisect each other

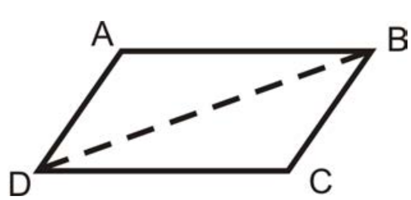
If



then



Proof of Opposite Sides Theorem



Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$

Prove: $\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$

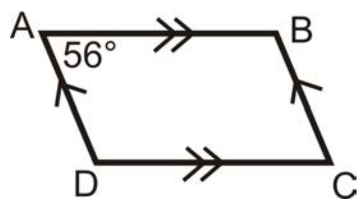
TABLE 7.3:

Statement	Reason
1. $ABCD$ is a parallelogram with diagonal $\overline{BD}$	Given
2. $\overline{AB} \| \overline{DC}, \overline{AD} \| \overline{BC}$	Definition of a parallelogram
3. $\angle ABD \cong \angle BDC, \angle ADB \cong \angle DBC$	Alternate Interior Angles Theorem
4. $\overline{DB} \cong \overline{DB}$	Reflexive Property of Congruence
5. $\triangle ABD \cong \triangle CDB$	ASA
6. $\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$	CPCTC

The proof of the Opposite Angles Theorem is almost identical. For the last step, the *angles* are congruent by CPCTC.

Example 1: $ABCD$ is a parallelogram. If $m\angle A = 56^\circ$, find the measure of the other angles.

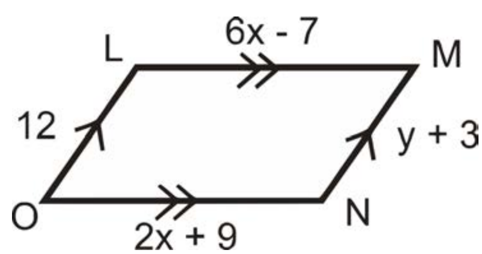
Solution: Draw a picture. When labeling the vertices, the letters are listed, in order, clockwise.



If $m\angle A = 56^\circ$, then $m\angle C = 56^\circ$ by the Opposite Angles Theorem.

$$\begin{aligned} m\angle A + m\angle B &= 180^\circ \quad \text{by the Consecutive Angles Theorem.} \\ 56^\circ + m\angle B &= 180^\circ \\ m\angle B &= 124^\circ \quad m\angle D = 124^\circ \quad \text{because it is opposite angles with } \angle B. \end{aligned}$$

Example 2: ***Algebra Connection*** Find the values of x and y.



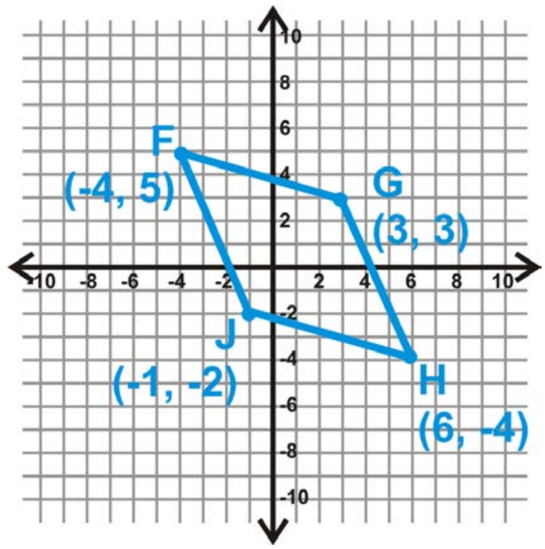
Solution: Opposite sides are congruent.

$$\begin{aligned} 6x - 7 &= 2x + 9 \\ 4x &= 16 \\ x &= 4 \end{aligned} \qquad\qquad \begin{aligned} y + 3 &= 12 \\ y &= 9 \end{aligned}$$

Diagonals in a Parallelogram

From the Parallelogram Diagonals Theorem, we know that the diagonals of a parallelogram bisect each other.

Example 3: Show that the diagonals of $FGHJ$ bisect each other.



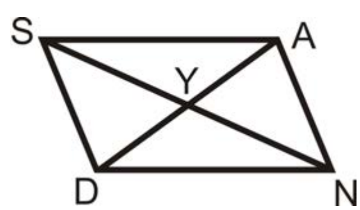
Solution: Find the midpoint of each diagonal.

$$\text{Midpoint of } \overline{FH}: \quad \left(\frac{-4+6}{2}, \frac{5-4}{2}\right) = (1, 0.5)$$

$$\text{Midpoint of } \overline{GJ}: \quad \left(\frac{3-1}{2}, \frac{3-2}{2}\right) = (1, 0.5)$$

Because they are the same point, the diagonals intersect at each other's midpoint. This means they bisect each other. ***This is one way to show a quadrilateral is a parallelogram.***

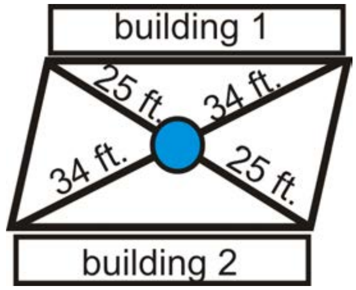
Example 4: ***Algebra Connection*** $SAND$ is a parallelogram and $SY = 4x - 11$ and $YN = x + 10$. Solve for x.



Solution:

$$\begin{aligned} SY &= YN \\ 4x - 11 &= x + 10 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

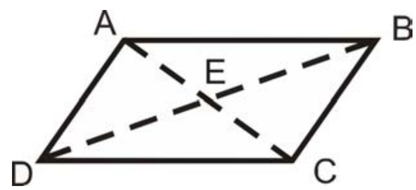
Know What? Revisited The diagonals bisect each other, so the fountain is going to be 34 feet from either endpoint on the 68 foot diagonal and 25 feet from either endpoint on the 50 foot diagonal.



Practice Problems

- Questions 1-6 are similar to Examples 2 and 4.
- Questions 7-10 are similar to Example 1.
- Questions 11-23 are similar to Examples 2 and 4.
- Questions 24-27 are similar to Example 3.
- Questions 28 and 29 are similar to the proof of the Opposite Sides Theorem.
- Question 30 is a challenge. Use the properties of parallelograms.

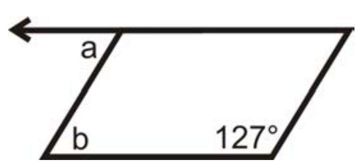
ABCD is a parallelogram. Fill in the blanks below.



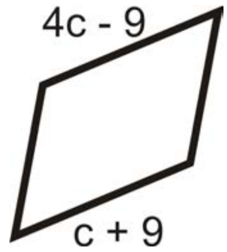
1. If $AB = 6$, then $CD =$ ______.
2. If $AE = 4$, then $AC =$ ______.
3. If $m\angle ADC = 80^\circ, m\angle DAB =$ ______.
4. If $m\angle BAC = 45^\circ, m\angle ACD =$ ______.
5. If $m\angle CBD = 62^\circ, m\angle ADB =$ ______.
6. If $DB = 16$, then $DE =$ ______.
7. If $m\angle B = 72^\circ$ in parallelogram *ABCD*, find the other three angles.
8. If $m\angle S = 143^\circ$ in parallelogram *PQRS*, find the other three angles.
9. If $\overline{AB} \perp \overline{BC}$ in parallelogram *ABCD*, find the measure of all four angles.
10. If $m\angle F = x^\circ$ in parallelogram *EFGH*, find the other three angles.

For questions 11-19, find the measures of the variable(s). All the figures below are parallelograms.

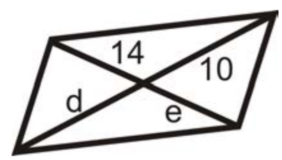
11.



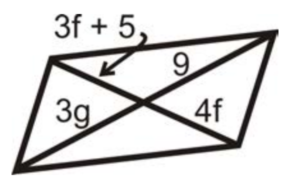
12.



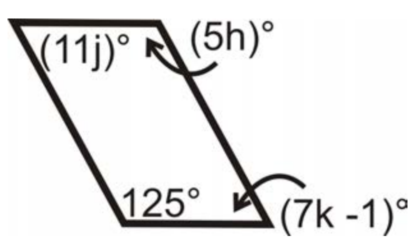
13.



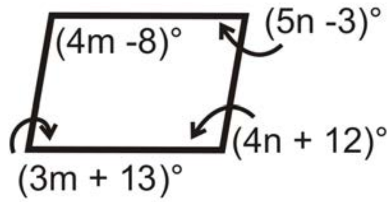
14.



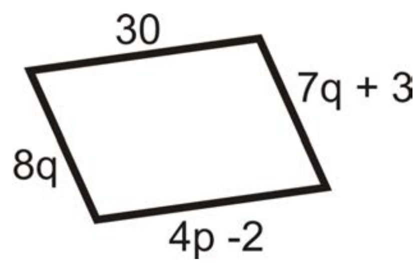
15.



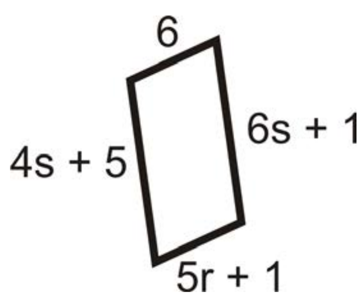
16.



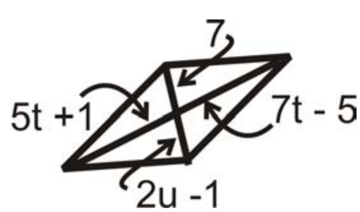
17.



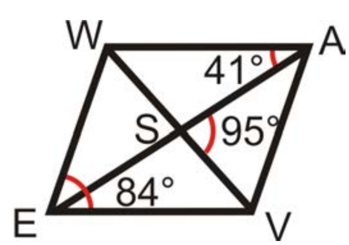
18.



19.



Use the parallelogram $WAVE$ to find:



20. $m\angle AWE$
21. $m\angle ESV$
22. $m\angle WEA$
23. $m\angle AVW$

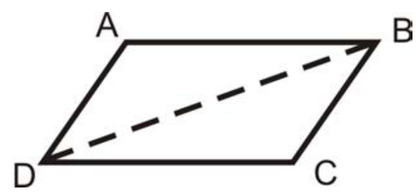
Find the point of intersection of the diagonals to see if $EFGH$ is a parallelogram.

24. $E(-1,3), F(3,4), G(5,-1), H(1,-2)$

25. $E(3,-2), F(7,0), G(9,-4), H(5,-4)$
26. $E(-6,3), F(2,5), G(6,-3), H(-4,-5)$
27. $E(-2,-2), F(-4,-6), G(-6,-4), H(-4,0)$

Fill in the blanks in the proofs below.

28. ***Opposite Angles Theorem***



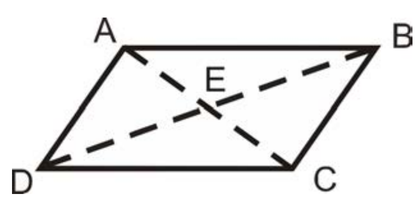
Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$

Prove: $\angle A \cong \angle C$

TABLE 7.4:

Statement	***Reason***
1.	Given
2. $\overline{AB} \parallel \overline{DC}, \overline{AD} \parallel \overline{BC}$	
3.	Alternate Interior Angles Theorem
4.	Reflexive Property of Congruence
5. $\triangle ABD \cong \triangle CDB$	
6. $\angle A \cong \angle C$	

29. ***Parallelogram Diagonals Theorem***



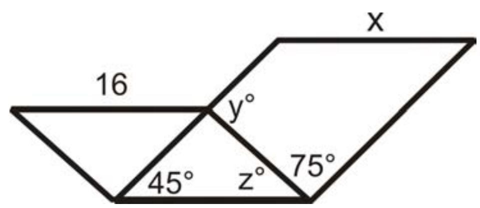
Given: $ABCD$ is a parallelogram with diagonals $\overline{BD}$ and $\overline{AC}$

Prove: $\overline{AE} \cong \overline{EC}, \overline{DE} \cong \overline{EB}$

TABLE 7.5:

Statement	***Reason***
1.	
2.	Definition of a parallelogram
3.	Alternate Interior Angles Theorem
4. $\overline{AB} \cong \overline{DC}$	
5.	
6. $\overline{AE} \cong \overline{EC}, \overline{DE} \cong \overline{EB}$	

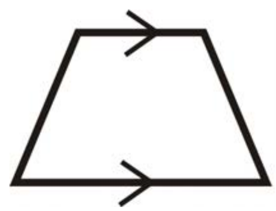
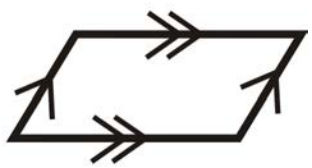
30. ***Challenge*** Find x, y°, and z°. (The two quadrilaterals with the same side are parallelograms.)



Review and Reflect

31. If you are only given one interior angle inside of a parallelogram, describe how you would find the other angles.
32. Describe the similarities and differences of a quadrilateral and a parallelogram.

Warm-Up Answers

1. 
2. 
3. $$\begin{aligned} 3x+x+3x+x &= 360^\circ \\ 8x &= 360^\circ \\ x &= 45^\circ \end{aligned}$$
4. $$\begin{aligned} 4x+2 &= 90^\circ \\ 4x &= 88^\circ \\ x &= 22^\circ \end{aligned}$$

7.3 Proving Quadrilaterals are Parallelograms

TEKS G(1)G, G(2)B, G(4)B, G(6)E

Learning Objectives

- Prove a quadrilateral is a parallelogram
- Show a quadrilateral is a parallelogram in the x-palne

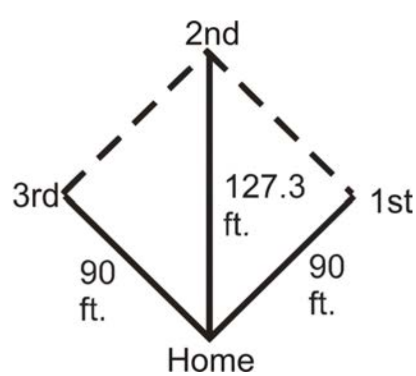
Vocabulary

- Parallelogram

Warm-Up

1. Plot the points $A(2,2), B(4,-2), C(-2,-4)$, and $D(-6,-2)$.
   a. Find the slopes of $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{AD}$. Is $ABCD$ a parallelogram?
   b. Find the point of intersection of the diagonals by finding the midpoint of each.

Know What? You are marking out a baseball diamond and standing at home plate. 3^{rd} base is 90 feet away, 2^{nd} base is 127.3 feet away, and 1^{st} base is also 90 feet away. The angles at all of the bases are 90°. Find the distance from 1^{st} base to 3^{rd} base(using the Pythagorean Theorem) and determine if the baseball diamond is a parallelogram.



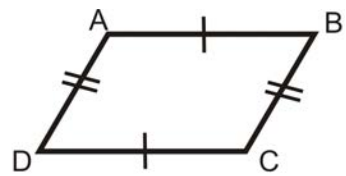
Determining if a Quadrilateral is a Parallelogram

The converses of the theorems in the last section will now be used to see if a quadrilateral is a parallelogram.

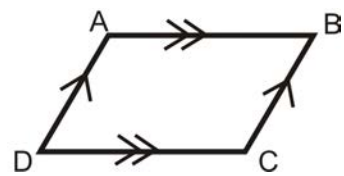
Opposite Sides Theorem Converse

If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram

If



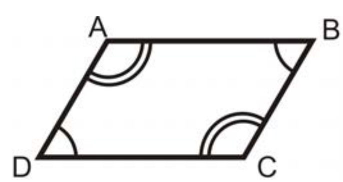
then



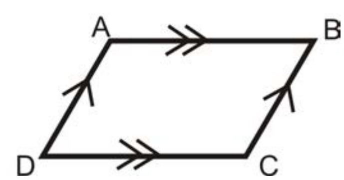
Opposite Angles Theorem Converse

If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram

If



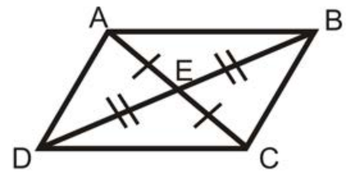
then



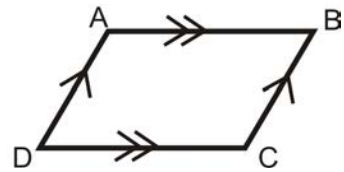
Parallelogram Diagonals Theorem Converse

If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram

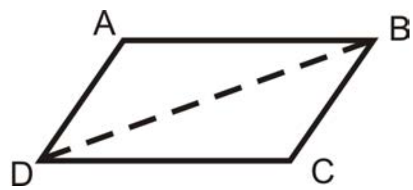
If



then



Proof of the Opposite Sides Theorem Converse



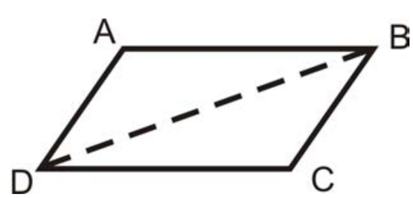
Given: $\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$

Prove: $ABCD$ is a parallelogram

TABLE 7.6:

Statement	***Reason***
1. $\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$	Given
2. $\overline{DB} \cong \overline{DB}$	Reflexive Property of Congruence
3. $\triangle ABD \cong \triangle CDB$	Side-Side-Side (SSS)
4. $\angle ABD \cong \angle BDC, \angle ADB \cong \angle DBC$	Corresponding Parts of Congruent Triangles are Congruent (CPCTC)
5. $\overline{AB} \| \overline{DC}, \overline{AD} \| \overline{BC}$	Alternate Interior Angles Converse
6. $ABCD$ is a parallelogram	Definition of a parallelogram

Example 1: Write a two-column proof.



Given: $\overline{AB} \| \overline{DC}$, and $\overline{AB} \cong \overline{DC}$

Prove: $ABCD$ is a parallelogram

Solution:

TABLE 7.7:

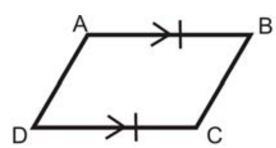
Statement	***Reason***
1. $\overline{AB} \| \overline{DC}$, and $\overline{AB} \cong \overline{DC}$	Given
2. $\angle ABD \cong \angle BDC$	Alternate Interior Angles
3. $\overline{DB} \cong \overline{DB}$	Reflexive Property of Congruence
4. $\triangle ABD \cong \triangle CDB$	Side-Angle-Side (SAS)
5. $\overline{AD} \cong \overline{BC}$	Corresponding Parts of Congruent Triangles are Congruent (CPCTC)
6. $ABCD$ is a parallelogram	Opposite Sides Converse

This is the proof for Theorem 7-10

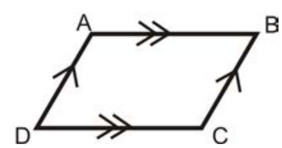
Theorem 7-10

If a quadrilateral has one set of parallel sides and they are also congruent, then it is a parallelogram

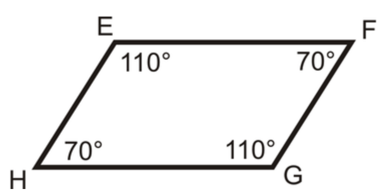
If

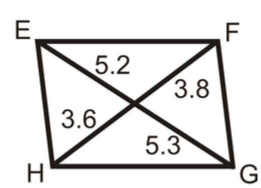


then



Example 2: Are the following quadrilaterals parallelograms? How do you know?



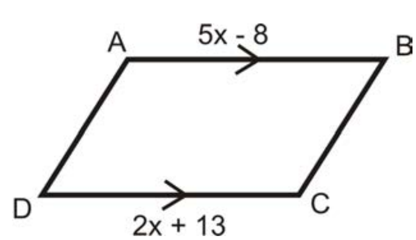


Solution:

a) By the Opposite Angles Theorem Converse, $EFGH$ is a parallelogram.

b) $EFGH$ is not a parallelogram because the diagonals do not bisect each other.

Example 3: ***Algebra Connection*** What value of x would make $ABCD$ a parallelogram?



Solution: $\overline{AB} \| \overline{DC}$. By Theorem 6-10, $ABCD$ would be a parallelogram if $AB = DC$.

$$\begin{aligned} 5x - 8 &= 2x + 13 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

Watch This!!!

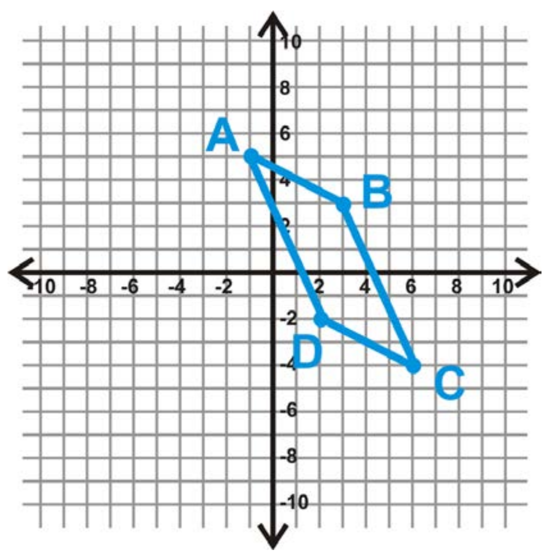
https://www.youtube.com/watch?v=lRoaKjmEJoY

Showing a Quadrilateral is a Parallelogram in the

To show that a quadrilateral is a parallelogram in the $x-y$ plane, you might need:

- The Slope Formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- The Distance Formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The Midpoint Formula, $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 4: Is the quadrilateral $ABCD$ a parallelogram?



Solution: Let's use Theorem 6-10 to see if $ABCD$ is a parallelogram. First, find the length of AB and CD.

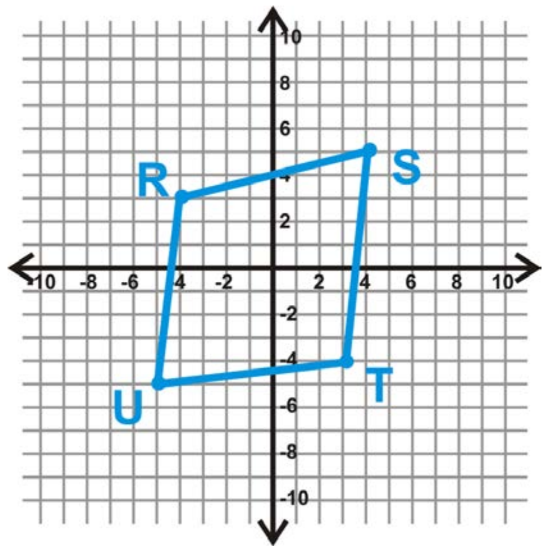
$$\begin{aligned} AB &= \sqrt{(-1-3)^2 + (5-3)^2} \\ &= \sqrt{(-4)^2 + 2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned} \qquad \begin{aligned} CD &= \sqrt{(2-6)^2 + (-2+4)^2} \\ &= \sqrt{(-4)^2 + 2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned}$$

Find the slopes.

Slope $AB = \frac{5-3}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$ Slope $CD = \frac{-2+4}{2-6} = \frac{2}{-4} = -\frac{1}{2}$

$AB = CD$ and the slopes are the same, $ABCD$ is a parallelogram.

Example 5: Is the quadrilateral $RSTU$ a parallelogram?

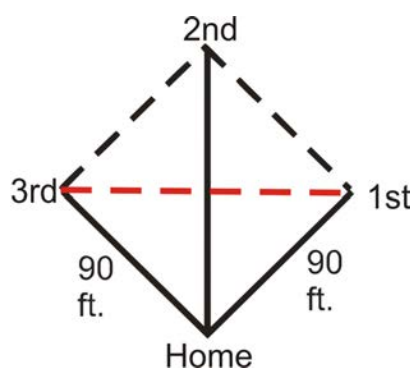


Solution: Let's use the Parallelogram Diagonals Converse to see if $RSTU$ is a parallelogram. Find the midpoint of each diagonal.

$$\text{Midpoint of } RT = \left(\frac{-4+3}{2}, \frac{3-4}{2}\right) = (-0.5, -0.5)$$
$$\text{Midpoint of } SU = \left(\frac{4-5}{2}, \frac{5-5}{2}\right) = (-0.5, 0)$$

$RSTU$ is not a parallelogram because the midpoints are not the same.

Know What? Revisited Use the Pythagorean Theorem to find the length of the second diagonal.



$$\begin{aligned} 90^2 + 90^2 &= d^2 \\ 8100 + 8100 &= d^2 \\ 16200 &= d^2 \\ d &= 127.3 \end{aligned}$$

The diagonals are equal, so the other two sides of the diamond must also be 90 feet. The baseball diamond is a parallelogram, and more specifically, a square.

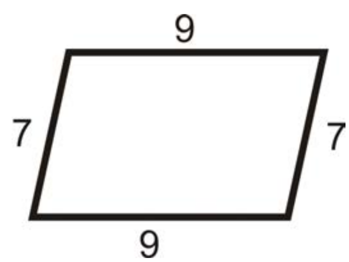
Practice Problems

- Questions 1-12 are similar to Example 2.
- Questions 13-15 are similar to Example 3.
- Questions 16-22 are similar to Examples 4 and 5.
- Questions 23-25 are similar to Example 1 and the proof of the Opposite Sides Converse.

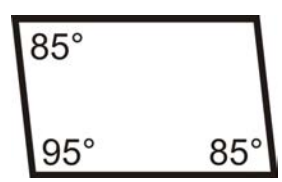
For questions 1-12, determine if the quadrilaterals are parallelograms.

1.

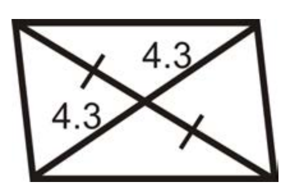
2.



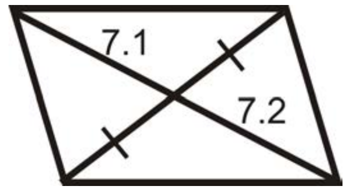
3.



4.

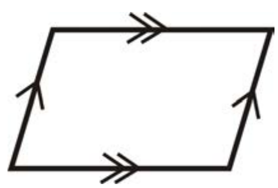


5.

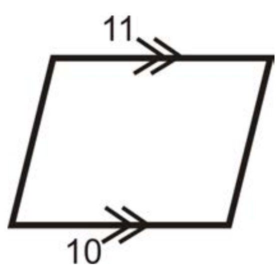


6.

7.



8.



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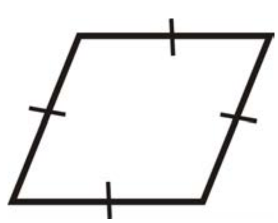

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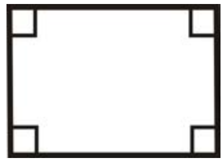
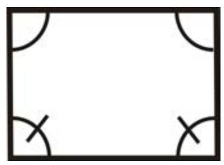


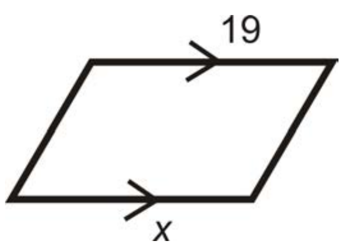
11.

12.



Algebra Connection For questions 13-18, determine the value of x and y that would make the quadrilateral a parallelogram.

13.



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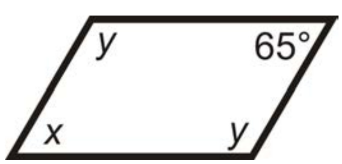
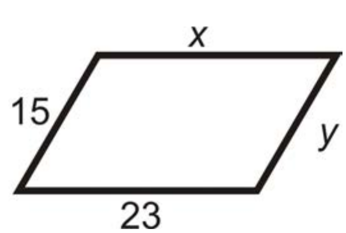
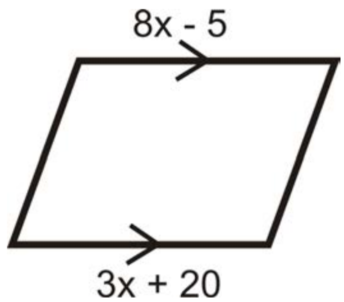


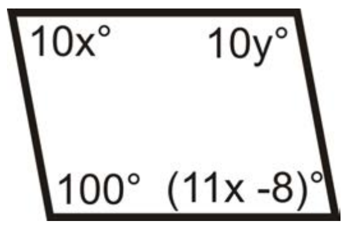
15.



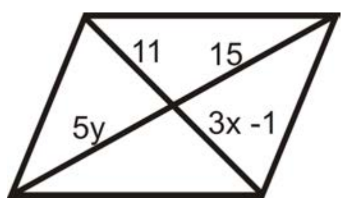
16.



17.



18.

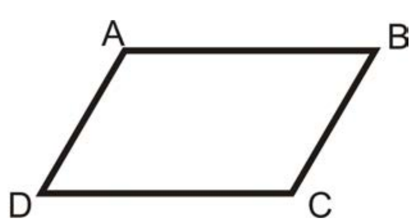


For questions 19-22, determine if *ABCD* is a parallelogram.

19. $A(8,-1), B(6,5), C(-7,2), D(-5,-4)$
20. $A(-5,8), B(-2,9), C(3,4), D(0,3)$
21. $A(-2,6), B(4,-4), C(13,-7), D(4,-10)$
22. $A(-9,-1), B(-7,5), C(3,8), D(1,2)$

Fill in the blanks in the proofs below.

23. ***Opposite Angles Theorem Converse***



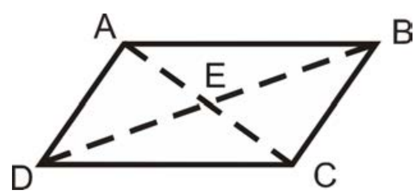
Given: $\angle A \cong \angle C, \angle D \cong \angle B$

Prove: *ABCD* is a parallelogram

TABLE 7.8:

Statement	***Reason***
1.	
2. $m\angle A = m\angle C, m\angle D = m\angle B$	
3.	Definition of a quadrilateral
4. $m\angle A + m\angle A + m\angle B + m\angle B = 360^\circ$	
5.	Combine Like Terms
6.	Division Property of Equality
7. $\angle A$ and $\angle B$ are supplementary $\angle A$ and $\angle D$ are supplementary	
8.	Consecutive Interior Angles Converse
9. $ABCD$ is a parallelogram	

24. ***Parallelogram Diagonals Theorem Converse***



Given: $\overline{AE} \cong \overline{EC}, \overline{DE} \cong \overline{EB}$

Prove: $ABCD$ is a parallelogram

TABLE 7.9:

Statement	***Reason***
1.	
2.	Vertical Angles Theorem
3. $\triangle AED \cong \triangle CEB$ $\triangle AEB \cong \triangle CED$	
4.	
5. $ABCD$ is a parallelogram	

25. Given: $\angle ADB \cong \angle CBD, \overline{AD} \cong \overline{BC}$Prove: $ABCD$ is a parallelogram

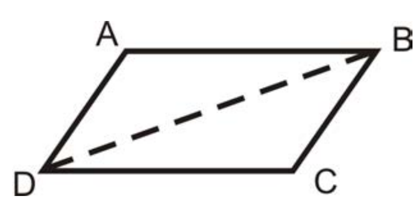


TABLE 7.10:

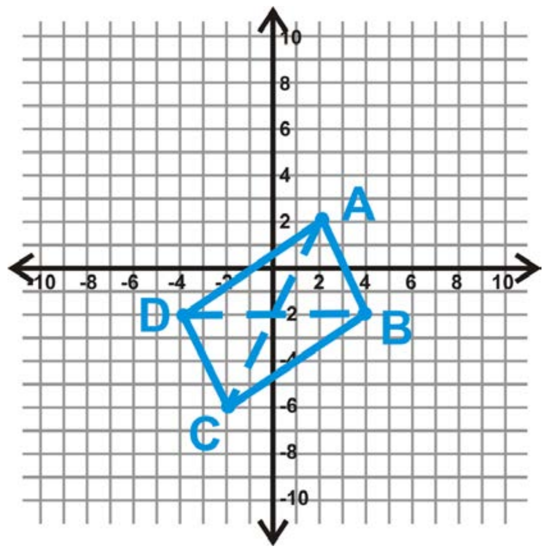
Statement	***Reason***
1.	
2. $\overline{AD} \parallel \overline{BC}$	
3. $ABCD$ is a parallelogram	

Review and Reflect

26. When proving that a quadrilateral in the coordinate plane is a parallelogram, which method do you prefer and why?

Warm-Up Answers

1.



(a) Slope AB = Slope $CD = -\frac{1}{2}$
Slope AD = Slope $BC = \frac{2}{3}$

$ABCD$ is a parallelogram because the opposite sides are parallel.

(b) Midpoint of $BD = (0,-2)$
Midpoint of $AC = (0,-2)$

Yes, the midpoints of the diagonals are the same, so they bisect each other.

7.4 Rectangles, Rhombuses and Squares

TEKS G1)F, G(2)B, G(4)B, G(5)A, G(6)E

Learning Objectives

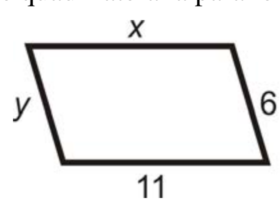
- Define a rectangle, rhombus, and square
- Determine if a parallelogram is a rectangle, rhombus, or square in the x-y plane
- Compare the diagonals of a rectangle, rhombus, and square

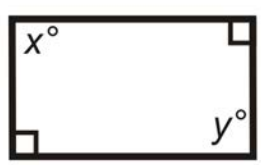
Vocabulary

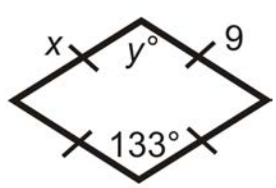
- Rectangle
- Rhombus
- Square

Warm-Up

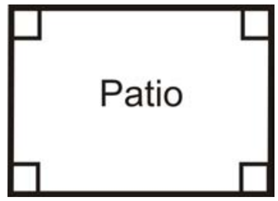
1. List five examples where you might see a square, rectangle, or rhombus in real life.
2. Find the values of x and $y$that would make the quadrilateral a parallelogram.

a. 

b. 

c. 

Know What? You are designing a patio for your backyard and are marking it off with a tape measure. Two sides are ***21 feet*** long and two sides are ***28 feet*** long. Explain how you would only use the tape measure to make your patio a rectangle. (You do not need to find any measurements.)

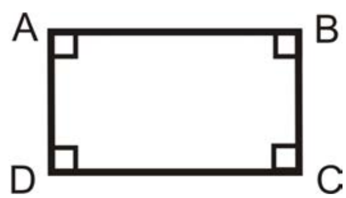


Defining Special Parallelograms

Rectangles, Rhombuses (also called Rhombi) and Squares are all more specific versions of parallelograms, also called special parallelograms. Taking the theorems we learned in the previous two sections, we have three more new theorems.

Rectangle Theorem

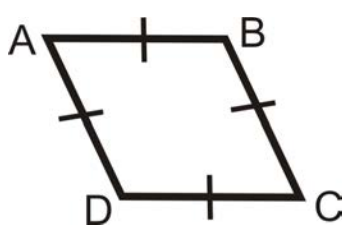
A quadrilateral is a rectangle if and only if it has four right angles



$ABCD$ is a rectangle if and only if $\angle A \cong \angle B \cong \angle C \cong \angle D$.

Rhombus Theorem

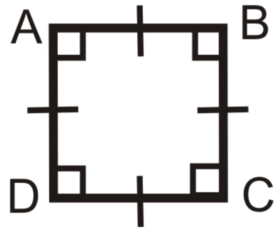
A quadrilateral is a rhombus if and only if it has four congruent sides



$ABCD$ is a rhombus if and only if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.

Square Theorem

A quadrilateral is a square if and only if it has four right angles and four congruent sides

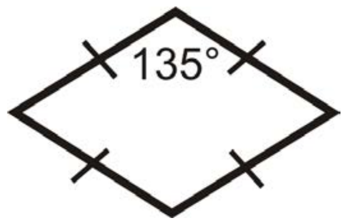


$ABCD$ is a square if and only if $\angle A \cong \angle B \cong \angle C \cong \angle D$**and** $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.

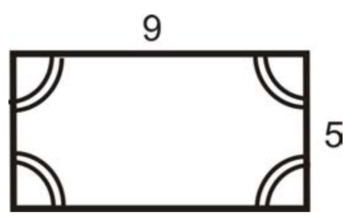
From the Square Theorem, we can also conclude that a ***square is a rectangle and a rhombus.***

Example 1: What type of parallelogram are the ones below?

a)



b)



Solution:

a) All sides are congruent. This means that the quadrilateral is a rhombus, but could also be a square if all the angles are right. Since one angle is 135° this quadrilateral a rhombus and not a square.

b) All four angles are congruent. We know that the angles in a quadrilateral add up to 360°, therefore each angle is 90°. The sides of the quadrilateral are not congruent. This quadrilateral is a rectangle.

Example 2: Is a rhombus SOMETIMES, ALWAYS, or NEVER a square? Explain why.

Solution: A rhombus has four congruent sides and a square has four congruent sides ***and*** angles. Therefore, a rhombus is a square when it has congruent angles. This means a rhombus is SOMETIMES a square.

Example 3: Is a rectangle SOMETIMES, ALWAYS, or NEVER a parallelogram? Explain why.

Solution: A rectangle has two sets of parallel sides, so it is ALWAYS a parallelogram.

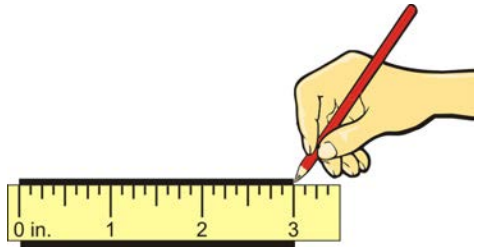
Diagonals in Special Parallelograms

Recall from previous lessons that the ***diagonals in a parallelogram bisect each other***. Therefore, the diagonals of a rectangle, square and rhombus also bisect each other. They also have additional properties.

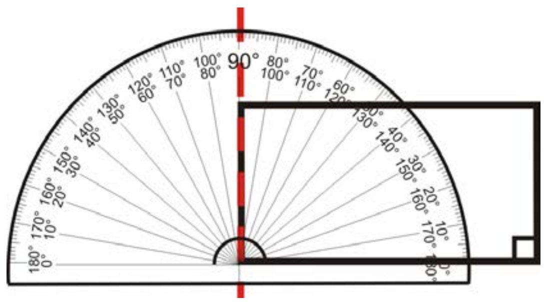
Investigation 6-3: Drawing a Rectangle

Tools Needed: pencil, paper, protractor, ruler

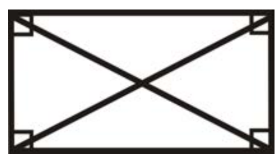
1. Like with Investigation 6-2, draw two lines on either side of your ruler, making them parallel. Make these lines 3 inches long.



2. Using the protractor, mark two 90° angles, 2.5 inches apart on the bottom line from Step 1. Extend the sides to intersect the top line.

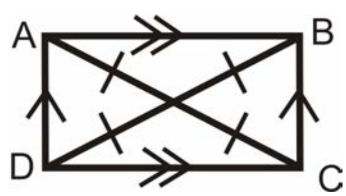


3. Draw in the diagonals and measure. What do you discover?



Theorem 7-14

A parallelogram is a rectangle if the diagonals are congruent



$ABCD$ is parallelogram. If $\overline{AC} \cong \overline{BD}$, then $ABCD$ is also a rectangle.

Check This Out!!!

http://web.geogebra.org/app/?id=21464

Investigation 6-4: Drawing a Rhombus

Tools Needed: pencil, paper, protractor, ruler

1. Like with Investigation 6-2 and 6-3, draw two lines on either side of your ruler, 3 inches long.

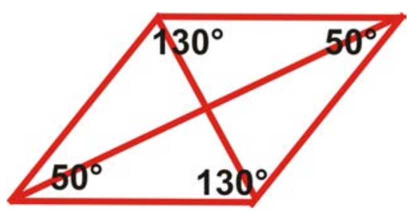
2. Remove the ruler and mark a 50° angle, at the left end of the bottom line drawn in Step 1. Draw the other side of the angle and make sure it intersects the top line. Measure the length of this side.

3. Mark the length found in Step 2 on the bottom line and the top line from the point of intersection with the 50° angle. Draw in the fourth side. It will connect the two endpoints of these lengths.

4. By the way we drew this parallelogram; it is a rhombus because all the sides are equal. Draw in the diagonals.

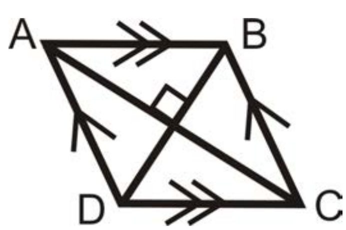
Measure the angles at the point of intersection of the diagonals (4).

Measure the angles created by the sides and each diagonal (8).



Theorem 7-15

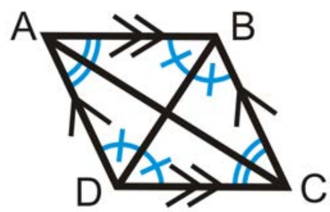
A parallelogram is a rhombus if the diagonals are perpendicular



ABCD is a parallelogram. If $\overline{AC} \perp \overline{BD}$, then *ABCD* is also a rhombus.

Theorem 7-16

A parallelogram is a rhombus if the diagonals bisect each angle



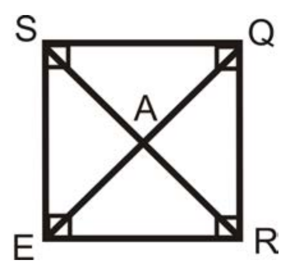
$ABCD$ is a parallelogram. If $\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$**and** $\overline{BD}$ bisects $\angle ABC$ $\angle ADC$, then $ABCD$ is also a rhombus.

Check This Out!!!

http://web.geogebra.org/app/?id=21466

The converses of these three theorems are true. There are no theorems about the diagonals of a square. ***The diagonals of a square have the properties of a rhombus and a rectangle.***

Example 4: List ***everything*** you know about the square $SQRE$.



Solution: A square has all the properties of a parallelogram, rectangle and rhombus.

TABLE 7.11:

Properties of a Parallelogram	***Properties of a Rhombus***	***Properties of a Rectangle***
• $\overline{SQ} \parallel \overline{ER}$	• $\overline{SQ} \cong \overline{ER} \cong \overline{SE} \cong \overline{QR}$	• $m\angle SER = m\angle SQR = m\angle QSE = m\angle QRE = 90^\circ$
• $\overline{SE} \parallel \overline{QR}$	• $\overline{SR} \perp \overline{QE}$	
	• $\angle SEQ \cong \angle QER \cong \angle SQE \cong \angle EQR$	• $\overline{SR} \cong \overline{QE}$
	• $\angle QSR \cong \angle RSE \cong \angle QRS \cong \angle SRE$	• $\overline{SA} \cong \overline{AR} \cong \overline{QA} \cong \overline{AE}$

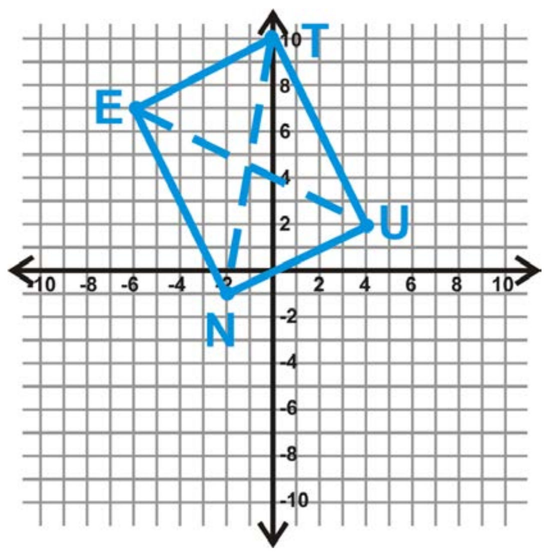
All the bisected angles are 45°.

Check This Out!!!

http://web.geogebra.org/app/?id=21465

Parallelograms in the Coordinate Plane

Example 5: Determine what type of parallelogram $TUNE$ is: $T(0,10), U(4,2), N(-2,-1)$, and $E(-6,7)$.



Solution: Let's see if the diagonals are equal. If they are, then $TUNE$ is a rectangle.

$$\begin{aligned} EU &= \sqrt{(-6-4)^2+(7-2)^2} \\ &= \sqrt{(-10)^2+5^2} \\ &= \sqrt{100+25} \\ &= \sqrt{125} \end{aligned} \qquad \begin{aligned} TN &= \sqrt{(0+2)^2+(10+1)^2} \\ &= \sqrt{2^2+11^2} \\ &= \sqrt{4+121} \\ &= \sqrt{125} \end{aligned}$$

If the diagonals are also perpendicular, then $TUNE$ is a square.

Slope of $EU = \frac{7-2}{-6-4} = -\frac{5}{10} = -\frac{1}{2}$ Slope of $TN = \frac{10-7}{0-(-6)} = \frac{3}{6} = \frac{1}{2}$

The slope of $EU \neq$ slope of TN, so $TUNE$ is a rectangle.

Steps to determine if a quadrilateral is a parallelogram, rectangle, rhombus, or square.

1. Graph the four points on ***graph paper.***

2. See if the ***diagonals bisect each other.*** (midpoint formula)

Yes: ***Parallelogram***, continue to #2. **No:** A ***quadrilateral***, done.

3. See if the ***diagonals are equal.*** (distance formula)

Yes: ***Rectangle,*** skip to #4. **No:** Could be a rhombus, continue to #3.

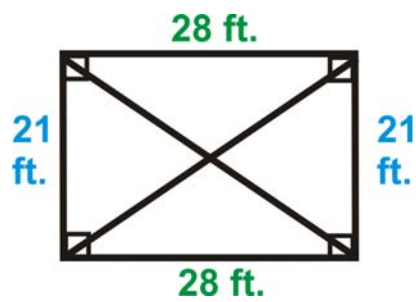
4. See if the ***sides are congruent.*** (distance formula)

Yes: ***Rhombus,*** done. **No:** ***Parallelogram,*** done.

5. See if the ***diagonals are perpendicular.*** (find slopes)

Yes: ***Square,*** done. **No:** ***Rectangle,*** done.

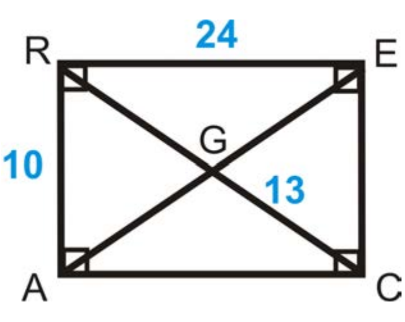
Know What? Revisited In order for the patio to be a rectangle, the opposite sides must be congruent (see picture). To ensure that the parallelogram is a rectangle *without* measuring the angles, the diagonals must be equal.



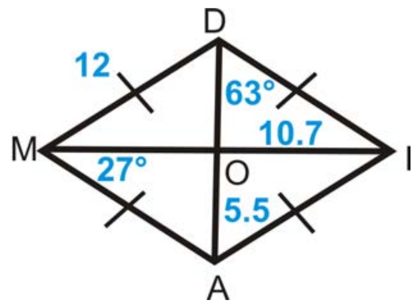
Practice Problems

- Questions 1-3 are similar to #2 in the Review Queue and Example 1.
- Questions 4-15 are similar to Example 1.
- Questions 16-21 are similar to Examples 2 and 3.
- Questions 22-25 are similar to Investigations 6-3 and 6-4.
- Questions 26-29 are similar to Example 4.
- Question 30 is a challenge.

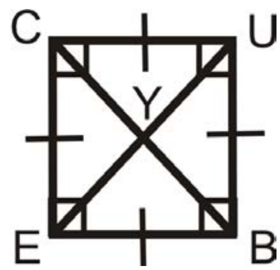
1. $RACE$ is a rectangle. Find:
   a. RG
   b. AE
   c. AC
   d. EC
   e. $m\angle RAC$



2. $DIAM$ is a rhombus. Find:
   a. MA
   b. MI
   c. DA
   d. $m\angle DIA$
   e. $m\angle MOA$

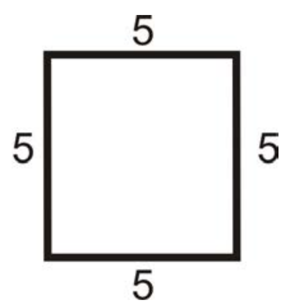


3. $CUBE$is a square. Find:
   a. $m\angle UCE$
   b. $m\angle EYB$
   c. $m\angle UBY$
   d. $m\angle UEB$



For questions 4-15, determine if the quadrilateral is a parallelogram, rectangle, rhombus, square or none.

4.



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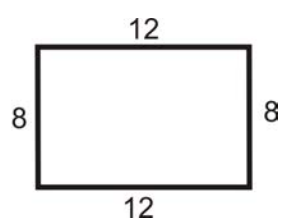

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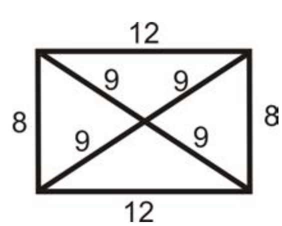

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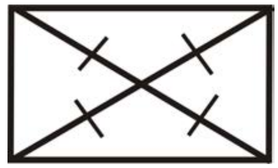

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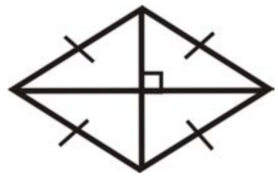

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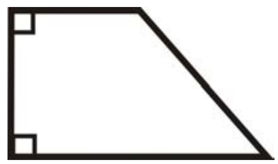

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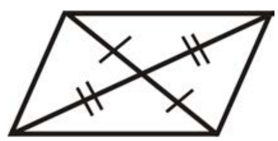
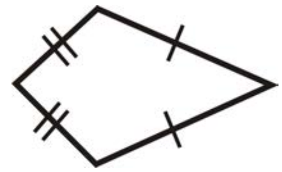


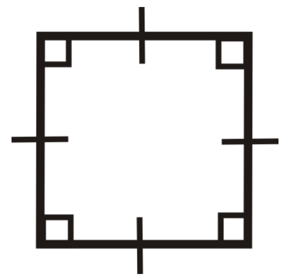
11.

12.

13.



14.



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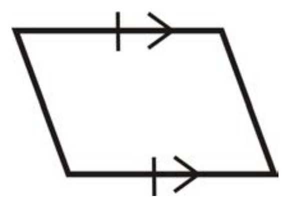


For questions 16-21 determine if the following are ALWAYS, SOMETIME, or NEVER true. Explain your reasoning.

16. A rectangle is a rhombus.
17. A square is a parallelogram.
18. A parallelogram is regular.
19. A square is a rectangle.
20. A rhombus is equiangular.
21. A quadrilateral is a pentagon.

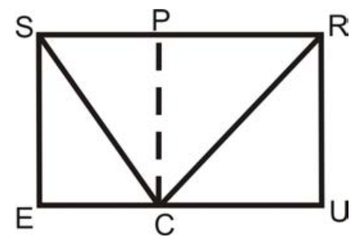
Construction Draw or construct the following quadrilaterals.

22. A quadrilateral with congruent diagonals that is not a rectangle.
23. A quadrilateral with perpendicular diagonals that is not a rhombus or square.
24. A rhombus with a 6 cm diagonal and an 8 cm diagonal.
25. A square with 2 inch sides.

For questions 26-29, determine what type of quadrilateral $ABCD$ is. Use Example 4 and the steps following it to help you.

26. $A(-2,4), B(-1,2), C(-3,1), D(-4,3)$
27. $A(-2,3), B(3,4), C(2,-1), D(-3,-2)$
28. $A(1,-1), B(7,1), C(8,-2), D(2,-4)$
29. $A(10,4), B(8,-2), C(2,2), D(4,8)$
30. ***Challenge*** $SRUE$ is a rectangle and $PRUC$is a square.
    a. What type of quadrilateral is $SPCE$?

b. If $SR = 20$ and $RU = 12$, find CE.
c. Find SC and RC based on the information from part b. Round your answers to the nearest hundredth.



Reflect and Review

For Questions, 31 - 33 explain the differences between the two quadrilaterals.

31. Rectangle - Rhombus

32. Rectangle - Square

33. Rhombus - Square

Warm-Up Answers

1. Possibilities: picture frame, door, baseball diamond, windows, walls, floor tiles, book cover, pages/paper, table/desk top, black/white board, the diamond suit (in a deck of cards).
   a. $x = 11,\ y = 6$
   b. $x = y = 90^\circ$
   c. $x = 9,\ y = 133^\circ$

7.5 Trapezoids and Kites

TEKS G(1)F, G(2)B, G(5)A, G(6)D

Learning Objectives

- Define Trapezoids, isosceles trapezoids, and kites
- Define the midsegments of trapezoids
- Plot trapezoids, isosceles trapezoids, and kites in the x-y plane

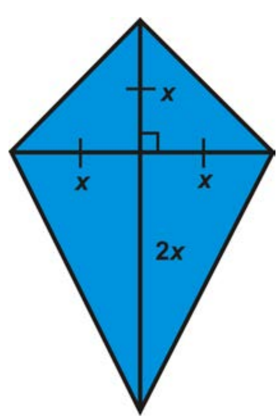
Vocabulary

- Trapezoid
- Isosceles trapezoid
- Base angles
- Midsegment
- Kite
- Vertex Angles

Warm-Up

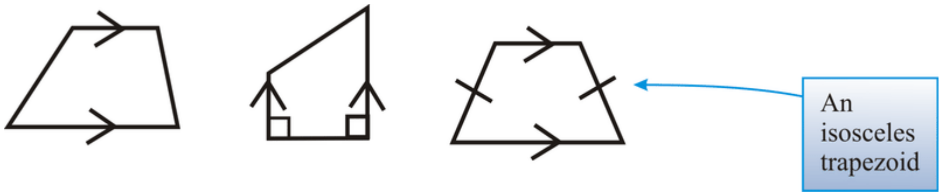
1. Draw a quadrilateral with one set of parallel lines.
2. Draw a quadrilateral with one set of parallel lines and two right angles.
3. Draw a quadrilateral with one set of parallel lines and two congruent sides.
4. Draw a quadrilateral with one set of parallel lines and three congruent sides.

Know What? A kite, seen at the right, is made by placing two pieces of wood perpendicular to each other and one piece of wood is bisected by the other. The typical dimensions are included in the picture. If you have two pieces of wood, 36 inches and 54 inches, determine the values of x and $2x$.



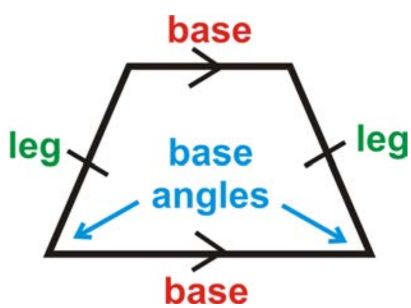
Trapezoids

Trapezoid: A quadrilateral with exactly one pair of parallel sides.



Isosceles Trapezoid: A trapezoid where the non-parallel sides are congruent.

Isosceles Trapezoids

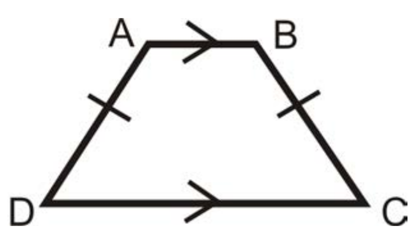


Previously, we introduced the Base Angles Theorem with isosceles triangles, which says, the two base angles are congruent. This property holds true for isosceles trapezoids. ***The two angles along the same base in an isosceles trapezoid are congruent.***

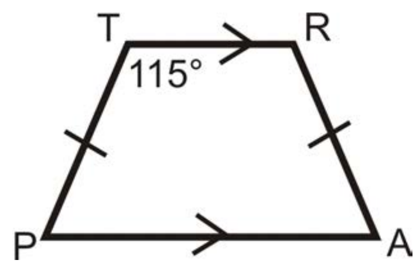
Theorem 7-17

The base angles of an isosceles trapezoid are congruent

If $ABCD$ is an isosceles trapezoid, then $\angle A \cong \angle B$ and $\angle C \cong \angle D$.



Example 1: Look at trapezoid $TRAP$ below. What is $m\angle A$?



Solution: $TRAP$ is an isosceles trapezoid. $m\angle R = 115^\circ$ also.

To find $m\angle A$, set up an equation.

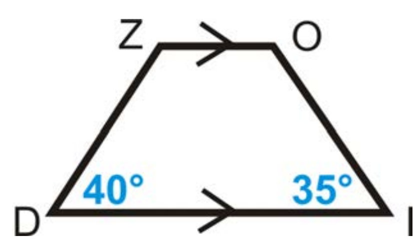
$$\begin{aligned} 115^\circ + 115^\circ + m\angle A + m\angle P &= 360^\circ \\ 230^\circ + 2m\angle A &= 360^\circ \qquad \rightarrow m\angle A = m\angle P \\ 2m\angle A &= 130^\circ \\ m\angle A &= 65^\circ \end{aligned}$$

Notice that $m\angle R + m\angle A = 115^\circ + 65^\circ = 180^\circ$. These angles will always be supplementary because of the Consecutive Interior Angles Theorem from Chapter 3.

Theorem 7-17 Converse

If a trapezoid has congruent base angles, then it is an isosceles trapezoid

Example 2: Is $ZOID$ an isosceles trapezoid? How do you know?

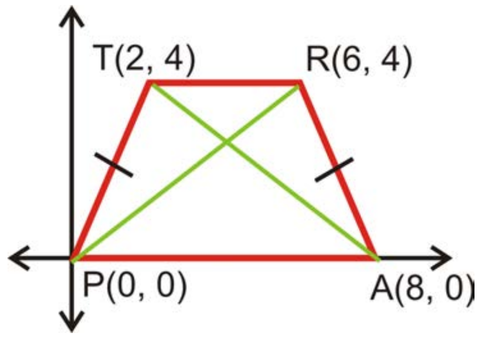


Solution: $40^\circ \neq 35^\circ$, $ZOID$ is not an isosceles trapezoid.

Isosceles Trapezoid Diagonals Theorem

The diagonals of an isosceles trapezoid are congruent

Example 3: Show $TA = RP$.

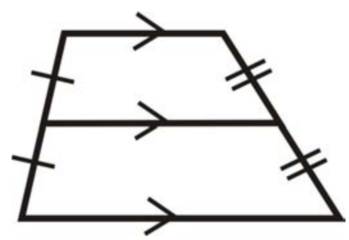


Solution: Use the distance formula to show $TA = RP$.

$$\begin{aligned} TA &= \sqrt{(2-8)^2+(4-0)^2} \\ &= \sqrt{(-6)^2+4^2} \\ &= \sqrt{36+16} = \sqrt{52} \end{aligned} \qquad \begin{aligned} RP &= \sqrt{(6-0)^2+(4-0)^2} \\ &= \sqrt{6^2+4^2} \\ &= \sqrt{36+16} = \sqrt{52} \end{aligned}$$

Midsegment of a Trapezoid

Midsegment (of a trapezoid): A line segment that connects the midpoints of the non-parallel sides.



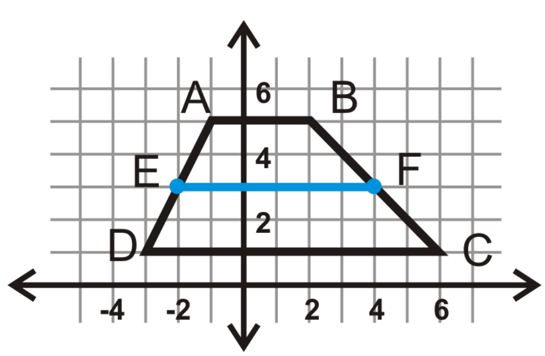
There is only one midsegment in a trapezoid. It will be parallel to the bases because it is located halfway between them.

Investigation 6-5: Midsegment Property

Tools Needed: graph paper, pencil, ruler

1. Plot $A(-1,5), B(2,5), C(6,1)$ and $D(-3,1)$ and connect them. This is NOT an isosceles trapezoid.

2. Find the midpoint of the non-parallel sides by using the midpoint formula. Label them E and F. Connect the midpoints to create the midsegment.

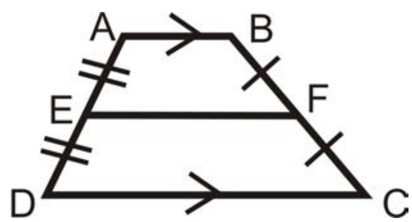


3. Find the lengths of AB, EF, and CD. What do you notice?

Midsegment Theorem

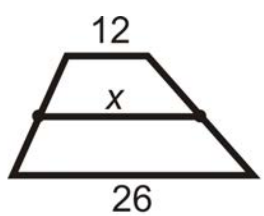
The length of the midsegment of a trapezoid is the average of the lengths of the bases

If $\overline{EF}$ is the midsegment, then $EF = \frac{AB+CD}{2}$.

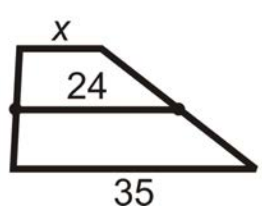


Example 4: ***Algebra Connection*** Find x. All figures are trapezoids with the midsegment.

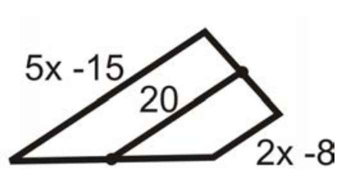
a)



b)



c)



Solution:

a) x is the average of 12 and 26. $\frac{12+26}{2} = \frac{38}{2} = 19$

b) 24 is the average of x and 35.

$$\begin{aligned}\frac{x+35}{2} &= 24\\ x+35 &= 48\\ x &= 13\end{aligned}$$

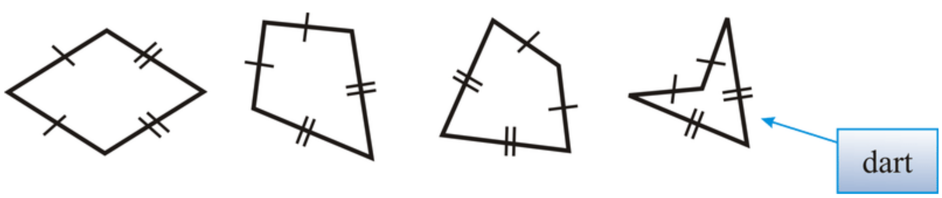
c) 20 is the average of $5x-15$ and $2x-8$.

$$\begin{aligned}\frac{5x-15+2x-8}{2} &= 20\\ 7x-23 &= 40\\ 7x &= 63\\ x &= 9\end{aligned}$$

Kites

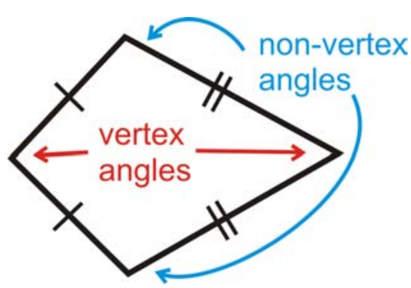
The last quadrilateral to study is a kite. Like you might think, it looks like a kite that flies in the air.

Kite: A quadrilateral with two distinct sets of adjacent congruent sides.



From the definition, a kite could be concave. If a kite is concave, it is called a ***dart.***

The angles between the congruent sides are called ***vertex angles.*** The other angles are called ***non-vertex angles.*** If we draw the diagonal through the vertex angles, we would have two congruent triangles.



Given: $KITE$ with $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$

Prove: $\angle K \cong \angle T$

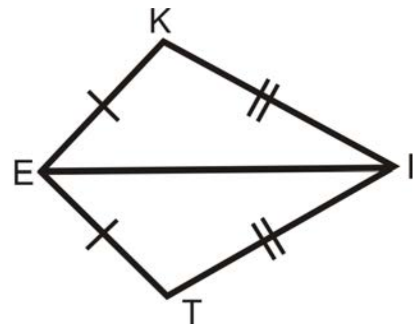
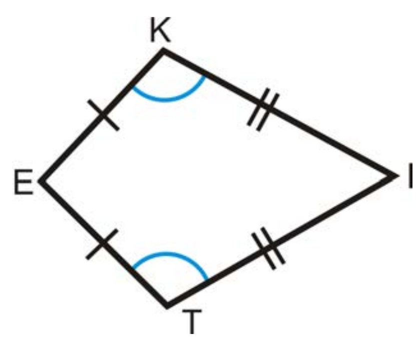


TABLE 7.12:

Statement	***Reason***
1. $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$	Given
2. $\overline{EI} \cong \overline{EI}$	Reflexive Property of Congruence
3. $\triangle EKI \cong \triangle ETI$	Side-Side-Side (SSS)
4. $\angle K \cong \angle T$	Corresponding Parts of Congruent Triangles are Congruent (CPCTC)

Theorem 7-21

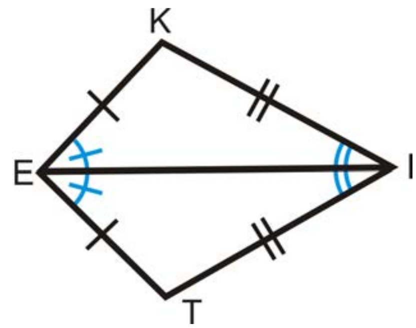
The non-vertex angles of a kite are congruent



If $KITE$ is a kite, then $\angle K \cong \angle T$.

Theorem 7-22

The diagonal through the vertex angles is the angle bisector for both angles



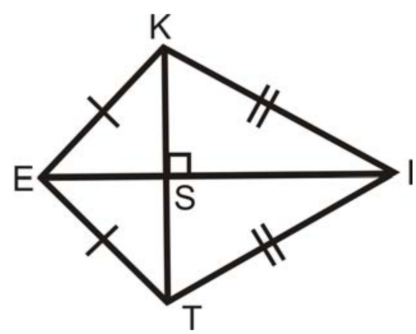
If $KITE$ is a kite, then $\angle KEI \cong \angle IET$ and $\angle KIE \cong \angle EIT$.

The proof of Theorem 6-22 is very similar to the proof above for Theorem 6-21.

Kite Diagonals Theorem

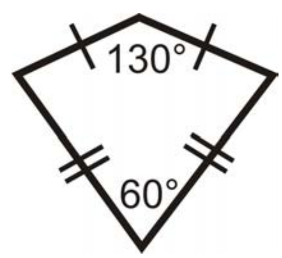
The diagonals of a kite are perpendicular

$\triangle KET$ and $\triangle KIT$ triangles are isosceles triangles, so $\overline{EI}$ is the perpendicular bisector of $\overline{KT}$ (Isosceles Triangle Theorem, Chapter 4).

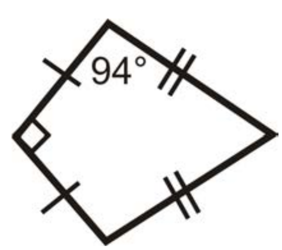


Example 5: Find the missing measures in the kites below.

a)



b)



Solution:

a) The two angles left are the non-vertex angles, which are congruent.

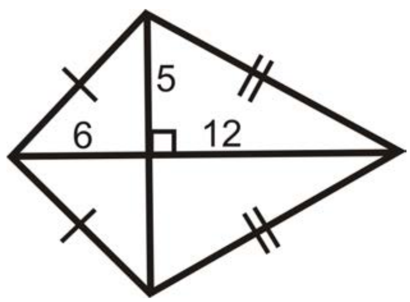
$$\begin{aligned}130^\circ + 60^\circ + x + x &= 360^\circ\\ 2x &= 170^\circ\\ x &= 85^\circ \qquad \text{Both angles are } 85^\circ.\end{aligned}$$

b) The other non-vertex angle is also 94°. To find the fourth angle, subtract the other three angles from 360°.

$$\begin{aligned}90^\circ + 94^\circ + 94^\circ + x &= 360^\circ\\ x &= 82^\circ\end{aligned}$$

Be careful with the definition of a kite. The congruent pairs are distinct, which means that ***a rhombus and square cannot be a kite.***

Example 6: Use the Pythagorean Theorem to find the length of the sides of the kite.



Solution: Recall that the Pythagorean Theorem is $a^2 + b^2 = c^2$, where c is the hypotenuse. In this kite, the sides are the hypotenuses.

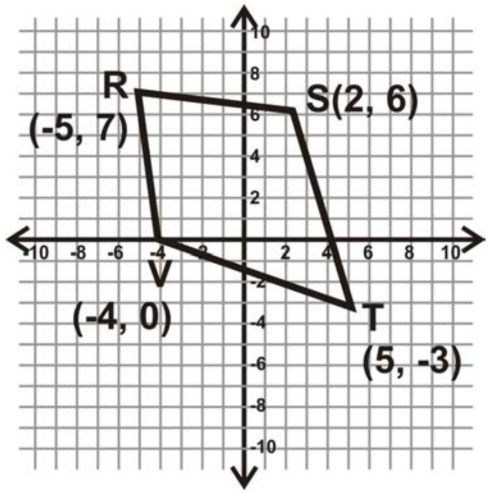
$$\begin{aligned}6^2 + 5^2 &= h^2\\ 36 + 25 &= h^2\\ 61 &= h^2\\ \sqrt{61} &= h\end{aligned} \qquad\qquad \begin{aligned}12^2 + 5^2 &= j^2\\ 144 + 25 &= j^2\\ 169 &= j^2\\ 13 &= j\end{aligned}$$

Check This Out!!!

http://ggbtu.be/b73479

Kites and Trapezoids in the Coordinate Plane

Example 7: Determine what type of quadrilateral $RSTV$ is.



Solution: Find the lengths of all the sides.

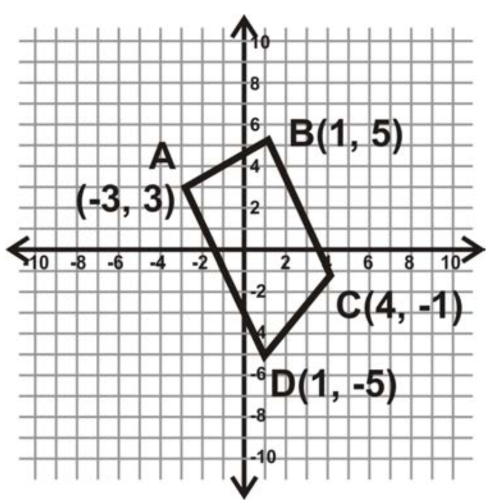
$$\begin{aligned} RS &= \sqrt{(-5-2)^2+(7-6)^2} \\ &= \sqrt{(-7)^2+1^2} \\ &= \sqrt{50} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(2-5)^2+(6-(-3))^2} \\ &= \sqrt{(-3)^2+9^2} \\ &= \sqrt{90} \end{aligned}$$

$$\begin{aligned} RV &= \sqrt{(-5-(-4))^2+(7-0)^2} \\ &= \sqrt{(-1)^2+7^2} \\ &= \sqrt{50} \end{aligned} \qquad \begin{aligned} VT &= \sqrt{(-4-5)^2+(0-(-3))^2} \\ &= \sqrt{(-9)^2+3^2} \\ &= \sqrt{90} \end{aligned}$$

From this we see that the adjacent sides are congruent. Therefore, $RSTV$ is a kite.

Example 8: Determine what type of quadrilateral $ABCD$ is. $A(-3,3), B(1,5), C(4,-1), D(1,-5)$.

Solution: First, graph $ABCD$. This will make it easier to figure out what type of quadrilateral it is. From the graph, we can tell this is not a parallelogram. Find the slopes of $\overline{BC}$ and $\overline{AD}$ to see if they are parallel.



Slope of $\overline{BC} = \frac{5-(-1)}{1-4} = \frac{6}{-3} = -2$

Slope of $\overline{AD} = \frac{3-(-5)}{-3-1} = \frac{8}{-4} = -2$

$\overline{BC} \| \overline{AD}$, so $ABCD$ is a trapezoid. To determine if it is an isosceles trapezoid, find AB and CD.

$$\begin{aligned} AB &= \sqrt{(-3-1)^2+(3-5)^2} \\ &= \sqrt{(-4)^2+(-2)^2} \\ &= \sqrt{20} = 2\sqrt{5} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(4-1)^2+(-1-(-5))^2} \\ &= \sqrt{3^2+4^2} \\ &= \sqrt{25} = 5 \end{aligned}$$

$AB \neq CD$, therefore this is only a trapezoid and not an isosceles trapezoid.

Example 9: Determine what type of quadrilateral $EFGH$ is. $E(5,-1), F(11,-3), G(5,-5), H(-1,-3)$

Solution: To contrast with Example 8, we will not graph this example. Let's find the length of all four sides.

$$\begin{aligned} EF &= \sqrt{(5-11)^2+(-1-(-3))^2} \\ &= \sqrt{(-6)^2+2^2} = \sqrt{40} \\ GH &= \sqrt{(5-(-1))^2+(-5-(-3))^2} \\ &= \sqrt{6^2+(-2)^2} = \sqrt{40} \end{aligned} \qquad \begin{aligned} \sqrt{FG} &= \sqrt{(11-5)^2+(-3-(-5))^2} \\ &= \sqrt{6^2+2^2} = \sqrt{40} \\ HE &= \sqrt{(-1-5)^2+(-3-(-1))^2} \\ &= \sqrt{(-6)^2+(-2)^2} = \sqrt{40} \end{aligned}$$

All four sides are equal. This quadrilateral is either a ***rhombus*** or a ***square***. We can determine if this quadrilateral is a ***square*** be checking the diagonals. If the diagonals are equal in length the quadrilateral is a ***square***. If the diagonals are different lengths the quadrilateral is a ***rhombus***. Let's find the length of the diagonals.

$$\begin{aligned} EG &= \sqrt{(5-5)^2+(-1-(-5))^2} \\ &= \sqrt{0^2+4^2} \\ &= \sqrt{16} = 4 \end{aligned} \qquad \begin{aligned} FH &= \sqrt{(11-(-1))^2+(-3-(-3))^2} \\ &= \sqrt{12^2+0^2} \\ &= \sqrt{144} = 12 \end{aligned}$$

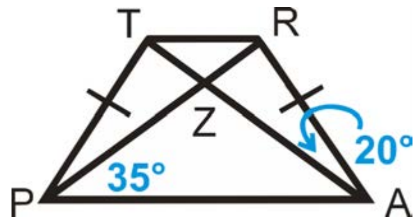
The diagonals are not congruent, so $EFGH$ is a rhombus.

Know What? Revisited If the diagonals (pieces of wood) are 36 inches and 54 inches, x is half of 36, or 18 inches. Then, $2x$ is 36.

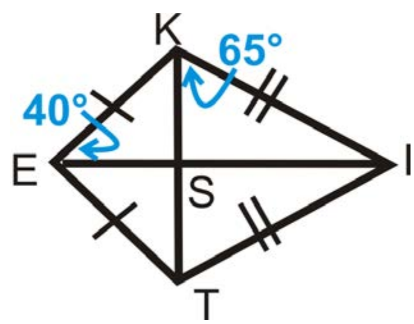
Practice Problems

- Questions 1 and 2 are similar to Examples 1, 2, 5 and 6.
- Questions 3 and 4 use the definitions of trapezoids and kites.
- Questions 5-10 are similar to Example 4.
- Questions 11-16 are similar to Examples 5 and 6.
- Questions 17-22 are similar to Examples 4-6.
- Questions 23 and 24 are similar to Example 3.
- Questions 25-28 are similar to Examples 7-9.
- Questions 29 and 30 are similar to the proof of Theorem 6-21.

1. $TRAP$an isosceles trapezoid. Find:
   a. $m\angle TPA$
   b. $m\angle PTR$
   c. $m\angle ZRA$
   d. $m\angle PZA$



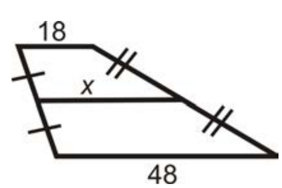
2. $KITE$is a kite. Find:
   a. $m\angle ETS$
   b. $m\angle KIT$
   c. $m\angle IST$
   d. $m\angle SIT$
   e. $m\angle ETI$



3. ***Writing*** Can the parallel sides of a trapezoid be congruent? Why or why not?
4. ***Writing*** Besides a kite and a rhombus, can you find another quadrilateral with perpendicular diagonals? Explain and draw a picture.

For questions 5-10, find the length of the midsegment or missing side.

5.



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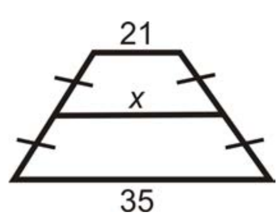

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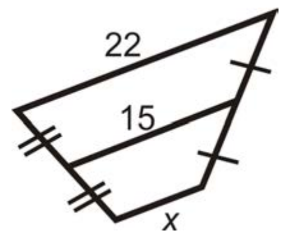

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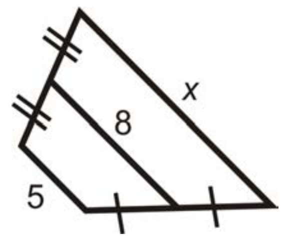

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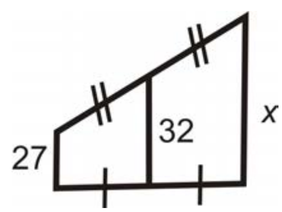

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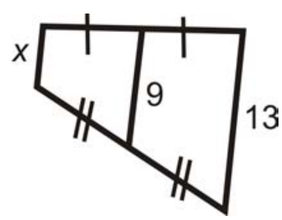
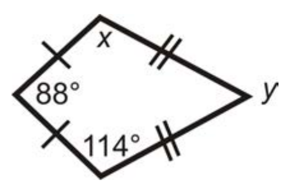


For questions 11-16, find the value of the missing variable(s). All figures are kites.

11.



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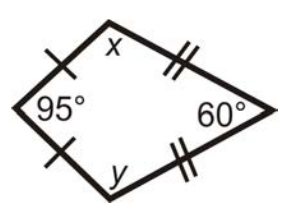

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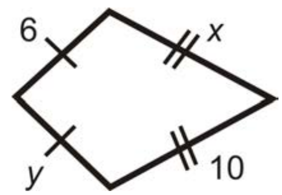

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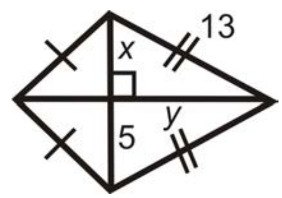

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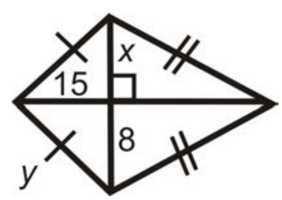

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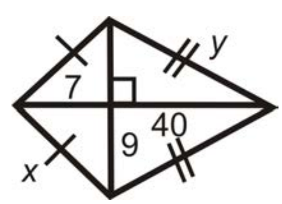
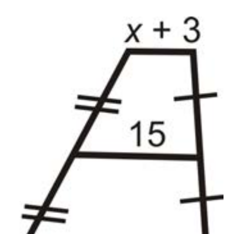


Algebra Connection For questions 17-22, find the value of the missing variable(s).

17



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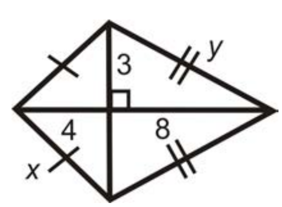

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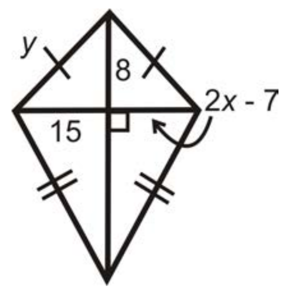

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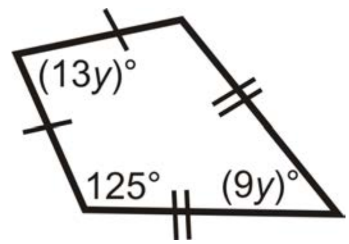

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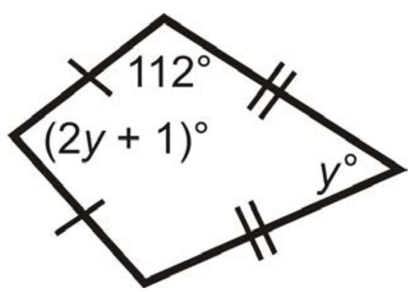

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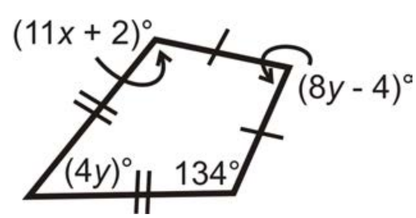


Find the lengths of the diagonals of the trapezoids below to determine if it is isosceles.

23. $A(-3,2), B(1,3), C(3,-1), D(-4,-2)$
24. $A(-3,3), B(2,-2), C(-6,-6), D(-7,1)$

For questions 25-28, determine what type of quadrilateral *ABCD* is. *ABCD* could be any quadrilateral that we have learned in this chapter. If it is none of these, write none.

25. $A(1,-2), B(7,-5), C(4,-8), D(-2,-5)$
26. $A(6,6), B(10,8), C(12,4), D(8,2)$

27. $A(-1,8), B(1,4), C(-5,-4), D(-5,6)$
28. $A(5,-1), B(9,-4), C(6,-10), D(3,-5)$

Fill in the blanks to the proofs below.

29. Given: $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$Prove: $\overline{EI}$ is the angle bisector of $\angle KET$ and $\angle KIT$

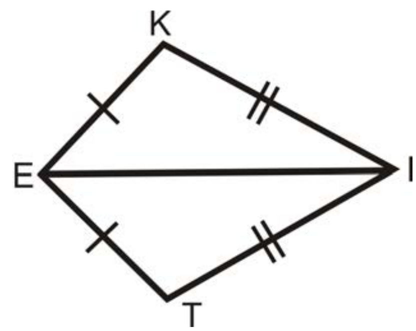


TABLE 7.13:

Statement	Reason
1. $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$	
2. $\overline{EI} \cong \overline{EI}$	
3. $\triangle EKI \cong \triangle ETI$	
4.	CPCTC
5. $\overline{EI}$ is the angle bisector of $\angle KET$ and $\angle KIT$	

30. Given: $\overline{EK} \cong \overline{ET}, \overline{KI} \cong \overline{IT}$Prove: $\overline{KT} \perp \overline{EI}$

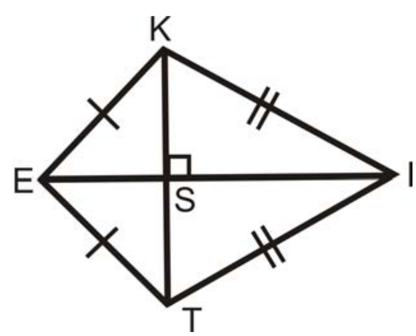


TABLE 7.14:

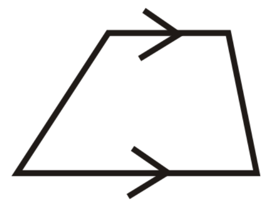
Statement	Reason
1. $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$	
2.	Definition of isosceles triangles
3. $\overline{EI}$ is the angle bisector of $\angle KET$ and $\angle KIT$	
4.	Isosceles Triangle Theorem
5. $\overline{KT} \perp \overline{EI}$	

Review and Reflect

31. Compare and contrast the properties of a trapezoid and an isosceles trapezoid.
32. In a kite, could BOTH pairs of opposite angles be congruent? Why or why not?

Warm-Up Answers

1.



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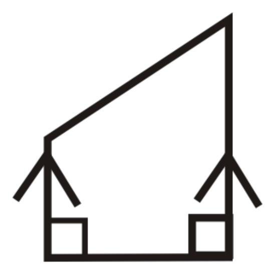

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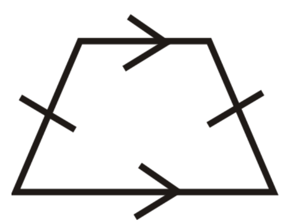

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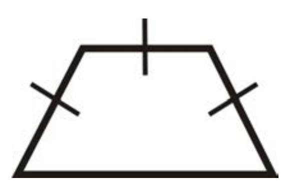


7.6 Chapter 7 Review

Keywords and Theorems

Angles in Polygons

- Polygon Sum Formula
- Equiangular Polygon Formula
- Regular Polygon
- Exterior Angle Sum Theorem

Properties of Parallelograms

- Parallelogram
- Opposite Sides Theorem
- Opposite Angles Theorem
- Consecutive Angles Theorem
- Parallelogram Diagonals Theorem

Proving Quadrilaterals are Parallelograms

- Opposite Sides Theorem Converse
- Opposite Angles Theorem Converse
- Consecutive Angles Theorem Converse
- Parallelogram Diagonals Theorem Converse
- Theorem 6-10

Rectangles, Rhombuses, and Squares

- Rectangle Theorem
- Rhombus Theorem
- Square Theorem
- Theorem 6-14
- Theorem 6-15
- Theorem 6-16

Trapezoids and Kites

- Trapezoid
- Isosceles Trapezoid
- Theorem 6-17
- Theorem 6-17 Converse
- Isosceles Trapezoid Diagonals Theorem
- Midsegment (of a trapezoid)
- Midsegment Theorem
- Kite
- Theorem 6-21
- Theorem 6-22
- Kite Diagonals Theorem

Quadrilateral Flow Chart

Fill in the flow chart according to what you know about the quadrilaterals we have learned in this chapter.

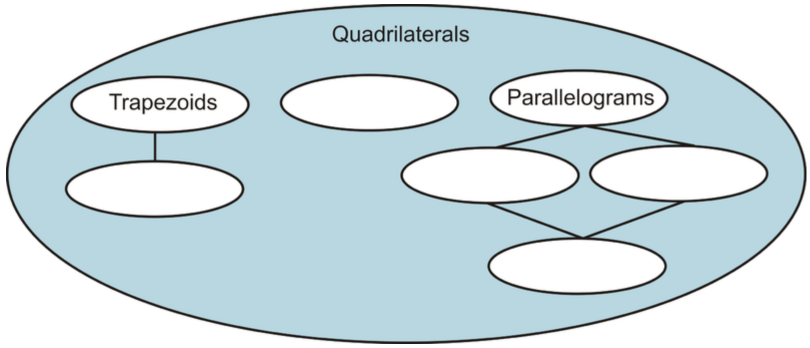


Table Summary

Determine if each quadrilateral has the given properties. If so, write yes or state how many sides (or angles) are congruent, parallel, or perpendicular.

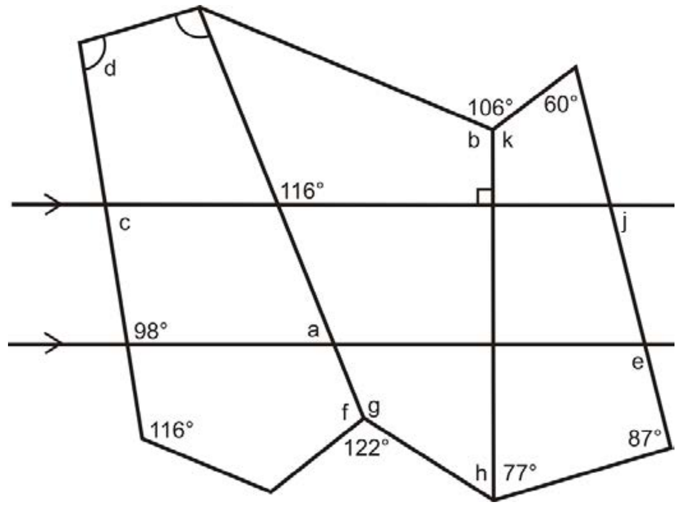
TABLE 7.15:

	Opposite sides $\parallel$	Diagonals bisect each other	Diagonals $\perp$	Opposite sides $\cong$	Opposite angles $\cong$	Consecutive Angles add up to 180°
Trapezoid						
Isosceles Trapezoid						
Kite						
Parallelogram						
Rectangle						
Rhombus						
Square						

1. How many degrees are in a:

   a. triangle
   b. quadrilateral
   c. pentagon
   d. hexagon

2. Find the measure of all the lettered angles below. The missing angle in the pentagon (at the bottom of the drawing), is 138°.



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9691 .

7.7 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

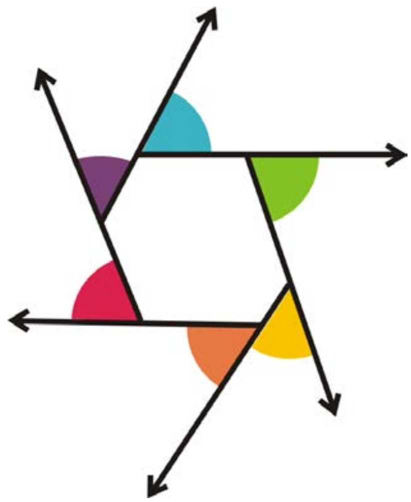
1^{st} **Section: Angles in Polygons**

Polygon Sum Formula

Equiangular Polygon Formula

Regular Polygon

Exterior Angle Sum Theorem



Homework:

2^{nd} **Section: Properties of Parallelograms**

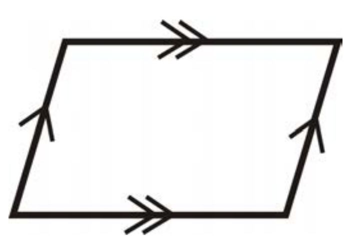
Parallelogram

Opposite Sides Theorem

Opposite Angles Theorem

Consecutive Angles Theorem

Parallelogram Diagonals Theorem



Homework:

3^{rd} **Section: Proving Quadrilaterals are Parallelograms**

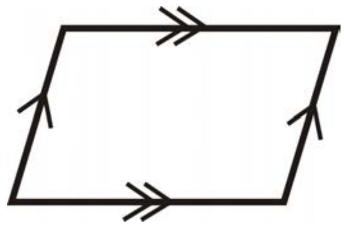
Opposite Sides Theorem Converse

Opposite Angles Theorem Converse

Consecutive Angles Theorem Converse

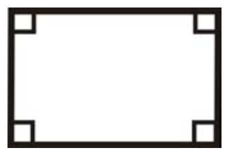
Parallelogram Diagonals Theorem Converse

Theorem 6-10



Homework:

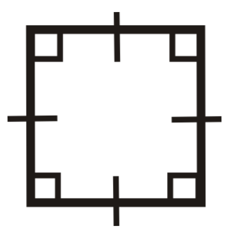
4^{th}**Section: Rectangles, Rhombuses, and Squares**



Rectangle Theorem

Rhombus Theorem

Square Theorem



Theorem 6-14

Theorem 6-15

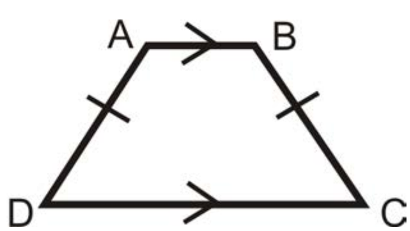
Theorem 6-16

Homework:

5^{th}**Section: Trapezoids and Kites**

Trapezoid

Isosceles Trapezoid



Theorem 6-17

Theorem 6-17 Converse

Isosceles Trapezoid Diagonals Theorem

Midsegment (of a trapezoid)

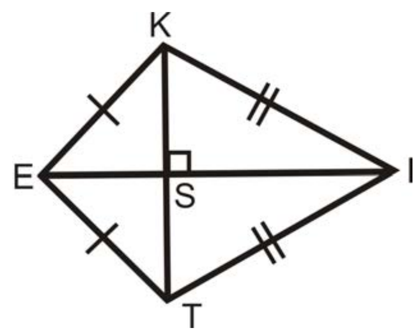
Midsegment Theorem

Kite

Theorem 6-21

Theorem 6-22

Kite Diagonals Theorem



Homework:

7.8 References

1. . https://encrypted-tbn3.gstatic.com/images?q=tbn:ANd9GcRL0V8_I21dq7s9iSmaaMDkbBDfZi52GprdfoO4KIcvkNzjQ6YONQ .
2. . https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcRnhqdjzgWUwGxKOnOTbO4UeK0ofKQjwfsrdQ0g-xfIMhNRbAEC .

CHAPTER 8 Right Triangle Trigonometry

Chapter Outline

Chapter 8 takes a look at right triangles. A right triangle is a triangle with one right angle(90°). In this chapter, we will prove the Pythagorean Theorem and its converse. Then, we will discuss trigonometric ratios and inverse ratios.

8.1 The Pythagorean Theorem

TEKS G(1)B, G(6)D, G(9)B

Learning Objectives

- Review simplifying and reducing radicals
- Prove and use the Pythagorean Theorem
- Use the Pythagorean Theorem to derive the distance formula
- Identify Pythagorean Triples

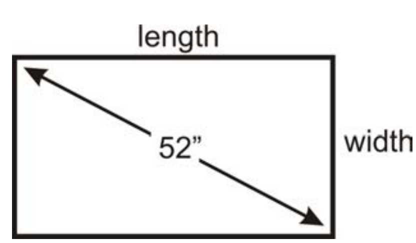
Vocabulary

- Radical
- Radicand
- Simplest Radical Form
- Pythagorean Theorem
- Pythagorean Triples

Warm-Up

1. Draw a right scalene triangle.
2. Draw an isosceles right triangle.
3. List all the factors of 75.
4. Write the prime factorization of 75.

Know What? For a 52" TV, 52" is the length of the diagonal. High Definition Televisions (HDTVs) have sides in a ratio of 16:9. What are the length and width of a 52" HDTV?



Simplifying and Reducing Radicals

Before we can simplify radicals we need to understand the square root function. The symbol for square root is the **radical** sign, or $\sqrt{\ }$. The number under the radical is called the **radicand**.

ADD PICTURE

With radicals there are many ways to write equivalent things. For instance,

$\sqrt{32} = 2\sqrt{8} = 4\sqrt{2} \approx 5.65685.$

In Geometry it is important that you do not round your calculations too many times. Rounding at each step of a problem can loose the integrity of the final answer. Therefore, we want to keep the radicals throughout each step. Which Radical do we use? $\sqrt{32}$? $2\sqrt{8}$? or $4\sqrt{2}$? All answers should be in simplest radical form.

Simplest Radical Form:

- There cannot be any perfect squares in the radicand(under the square root) except for 1.
- The radicand cannot contain fractions
- There cannot be a square root in the denominator of a fraction.

In order to meet the criteria of Simplest Radical Form, we are going to use some properties of radicals.

Properties of Radical

1. $\sqrt{ab} = \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ Any two radicals can be multiplied together.
2. $x\sqrt{a} \cdot y\sqrt{b} = (x \cdot y)\sqrt{a \cdot b}$ When multiplying two radicals you multiply the coefficients and the radicands.
3. $\sqrt{a} \cdot \sqrt{a} = (\sqrt{a})^2 = \sqrt{a^2} = a$ The square and square root cancel each other out.
4. $\sqrt{\frac{a}{b}} =. \frac{\sqrt{a}}{\sqrt{b}}$ Radicals of fractions can be broken into two radicals.
5. $x\sqrt{a} \pm y\sqrt{a} = (x \pm y)\sqrt{a}$ Radicals can only be added/subtracted if the radicands are the same

Perfect Squares

Perfect squares are created by multiplying a whole number by itself.

1, 4, 9, 16 ,25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400

are the first twenty perfect squares.

TABLE 8.1: Perfect Squares

Whole Number	$number^2$	$\sqrt{PerfectSquare}$
1	$1^2 = 1$	$\sqrt{1} = 1$
2	$2^2 = 4$	$\sqrt{4} = 2$
3	$3^2 = 9$	$\sqrt{9} = 3$
4	$4^2 = 16$	$\sqrt{16} = 4$
5	$5^2 = 25$	$\sqrt{25} = 5$
6	$6^2 = 36$	$\sqrt{36} = 6$
7	$7^2 = 49$	$\sqrt{49} = 7$
8	$8^2 = 64$	$\sqrt{64} = 8$
9	$9^2 = 81$	$\sqrt{81} = 9$
10	$10^2 = 100$	$\sqrt{100} = 10$

These are necessary when simplifying radicals because we have to pull them out of the radical.

Example 1: Simplify the radicals.

a) $\sqrt{50}$

b) $\sqrt{27}$

c) $\sqrt{272}$

Solution: For each radical, find the perfect squares that are factors.

a) $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$

b) $\sqrt{27} = \sqrt{9 \cdot 3} = \sqrt{9} \cdot \sqrt{3} = 3\sqrt{3}$

c) $\sqrt{272} = \sqrt{16 \cdot 17} = \sqrt{16} \cdot \sqrt{17} = 4\sqrt{17}$

Watch This!!!

https://www.youtube.com/watch?v=QiTe36zF9ew

Example 2: Simplify the radicals.

a) $4\sqrt{3} \cdot 5\sqrt{7}$

b) $5\sqrt{6} \cdot 4\sqrt{18}$

c) $\sqrt{8} \cdot 12\sqrt{2}$

Watch This!!!

https://www.youtube.com/watch?v=oPA8h7eccT8

Solution:

a) $4\sqrt{3} \cdot 5\sqrt{7} = 4 \cdot 5\sqrt{3 \cdot 7} = 20\sqrt{21}$

b) $5\sqrt{6} \cdot 4\sqrt{18} = 5 \cdot 4\sqrt{6 \cdot 18} = 20\sqrt{108} = 20\sqrt{36 \cdot 3} = 20 \cdot 6\sqrt{3} = 120\sqrt{3}$

c) $\sqrt{8} \cdot 12\sqrt{2} = 12\sqrt{8 \cdot 2} = 12\sqrt{16} = 12 \cdot 4 = 48$

Example 3: Simplify the radicals.

a) $(\sqrt{7})^2$

b) $(5\sqrt{2})^2$

Solution:

a) $(\sqrt{7})^2 = (\sqrt{7}) \cdot (\sqrt{7}) = \sqrt{49} = 7$

b) $\left(5\sqrt{2}\right)^2 = 5^2\left(\sqrt{2}\right)^2 = 25 \cdot 2 = 50 \rightarrow$ the $\sqrt{}$ and the $()^2$ cancel each other out.

To divide radicals, you need to simplify the denominator, which means multiplying the top and bottom of the fraction by the radical in the denominator.

Example 4: Divide and simplify the radicals.

a) $4\sqrt{6} \div \sqrt{3}$

b) $\frac{\sqrt{30}}{\sqrt{8}}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}}$

Solution: Rewrite all division problems like a fraction.

a)$4\sqrt{6} \div \sqrt{3} = \frac{4\sqrt{6}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{18}}{\sqrt{9}} = \frac{4\sqrt{9 \cdot 2}}{3} = \frac{4 \cdot 3\sqrt{2}}{3} = 4\sqrt{2}$

b) $\frac{\sqrt{30}}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{240}}{\sqrt{64}} = \frac{\sqrt{16 \cdot 15}}{8} = \frac{4\sqrt{15}}{8} = \frac{\sqrt{15}}{2}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{8\sqrt{14}}{6 \cdot 7} = \frac{4\sqrt{14}}{3 \cdot 7} = \frac{4\sqrt{14}}{21}$

Notice, we do not really "divide" radicals, but get them out of the denominator of a fraction.

Watch This!!!

Another way to divide radicals is to simplify the fraction before you rationalize the denominator. This creates smaller numbers that may be easier to deal with.

https://www.youtube.com/watch?v=oPA8h7eccT8

Example 5: Simplify the radicals.

a) $3\sqrt{5} + 8\sqrt{5}$

b) $2\sqrt{10} + \sqrt{160}$

c) $2\sqrt{6} - 5\sqrt{3}$

Solution:

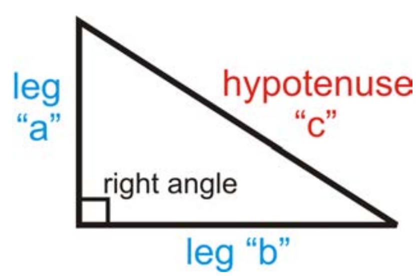
a) $3\sqrt{5} + 8\sqrt{5} = (3+8)\sqrt{5} = 11\sqrt{5}$

b) Simplify $\sqrt{160}$ before adding: $2\sqrt{10} + \sqrt{160} = 2\sqrt{10} + \sqrt{16 \cdot 10} = 2\sqrt{10} + 4\sqrt{10} = 6\sqrt{10}$

c) $2\sqrt{6} - 5\sqrt{3}$ The radicands are not the same. We can not subtract these.

The Pythagorean Theorem

We have used the Pythagorean Theorem already in this text, but have not proved it. Recall that the sides of a right triangle are the **legs** (the sides of the right angle) and the **hypotenuse** (the side opposite the right angle). For the Pythagorean Theorem, the legs are "a" and "b" and the hypotenuse is "c".



Pythagorean Theorem

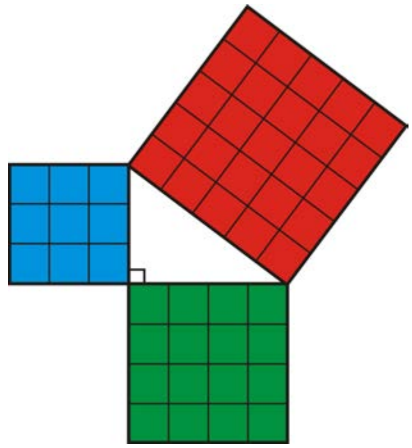
Given a right triangles with legs of lengths a and b and hypotenuse of length c, then $a^2 + b^2 = c^2$

Investigation 8-1: Proof of the Pythagorean Theorem

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in. square and a right triangle with legs of 3 in. and 4 in.

2. Cut out the triangle and square and arrange them like the picture on the right.



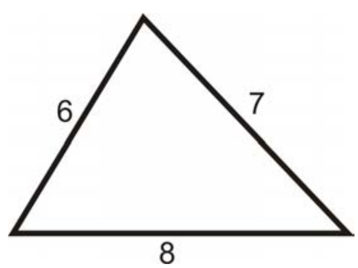
3. This theorem relies on area. Recall that the area of a square is *side*^{2}. In this case, we have three squares with sides 3 in., 4 in., and 5 in. What is the area of each square?
4. Now, we know that $9+16=25$, or $3^2+4^2=5^2$. Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

For two more proofs, go to: http://www.mathsisfun.com/pythagoras.html and scroll down to "And You Can Prove the Theorem Yourself."

Using the Pythagorean Theorem

Here are several examples of the Pythagorean Theorem in action.

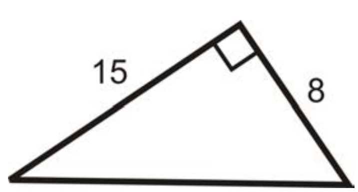
Example 6: Do 6, 7, and 8 make the sides of a right triangle?



Solution: Plug in the three numbers to the Pythagorean Theorem. ***The largest length will always be the hypotenuse.*** If $6^2+7^2=8^2$, then they are the sides of a right triangle.

$$6^2+7^2=36+49=85$$
$$8^2=64 \qquad 85 \neq 64, \text{ so the lengths are not the sides of a right triangle.}$$

Example 7: Find the length of the hypotenuse.

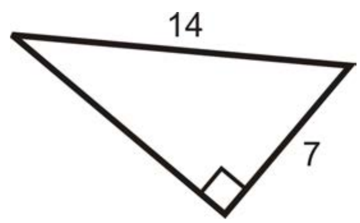


Solution: Use the Pythagorean Theorem. Set $a = 8$ and $b = 15$. Solve for c.

$$\begin{aligned}
8^2 + 15^2 &= c^2 \\
64 + 225 &= c^2 \\
289 &= c^2 && \textit{Take the square root of both sides.} \\
17 &= c
\end{aligned}$$

When you take the square root of an equation, the answer is 17 or -17. ***Length is never negative,*** which makes 17 the answer.

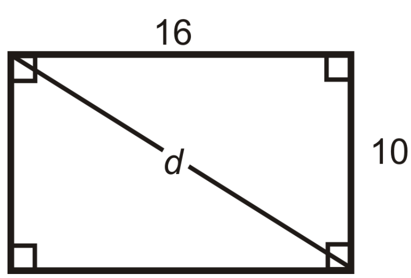
Example 8: Find the missing side of the right triangle below.



Solution: Here, we are given the hypotenuse and a leg. Let's solve for b.

$$\begin{aligned}
7^2 + b^2 &= 14^2 \\
49 + b^2 &= 196 \\
b^2 &= 147 \\
b &= \sqrt{147} = \sqrt{49 \cdot 3} = 7\sqrt{3}
\end{aligned}$$

Example 9: What is the diagonal of a rectangle with sides 10 and 16?



Solution: For any square and rectangle, you can use the Pythagorean Theorem to find the length of a diagonal. Plug in the sides to find d.

$$\begin{aligned}
10^2 + 16^2 &= d^2 \\
100 + 256 &= d^2 \\
356 &= d^2 \\
d &= \sqrt{356} = 2\sqrt{89} \approx 18.87
\end{aligned}$$

Pythagorean Triples

In Example 7, the sides of the triangle were 8, 15, and 17. This combination of numbers is called a ***Pythagorean triple.***

Pythagorean Triple: A set of three whole numbers that makes the Pythagorean Theorem true.

$$3,4,5 \qquad 5,12,13 \qquad 7,24,25 \qquad 8,15,17 \qquad 9,12,15 \qquad 10,24,26$$

Any multiple of a Pythagorean triple is also considered a triple because it would still be three whole numbers. Multiplying 3, 4, 5 by 2 gives 6, 8, 10, which is another triple. To see if a set of numbers makes a triple, plug them into the Pythagorean Theorem.

Example 10: Is 20, 21, 29 a Pythagorean triple?

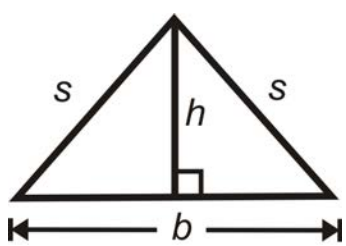
Solution: If $20^2 + 21^2 = 29^2$, then the set is a Pythagorean triple.

$$20^2 + 21^2 = 400 + 441 = 841$$
$$29^2 = 841$$

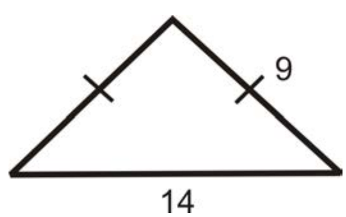
Therefore, 20, 21, and 29 is a Pythagorean triple.

Height of an Isosceles Triangle

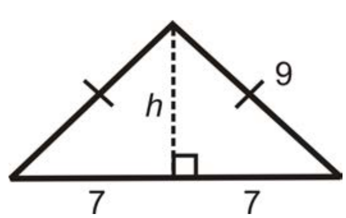
One way to use The Pythagorean Theorem is to find the height of an isosceles triangle.



Example 11: What is the height of the isosceles triangle?



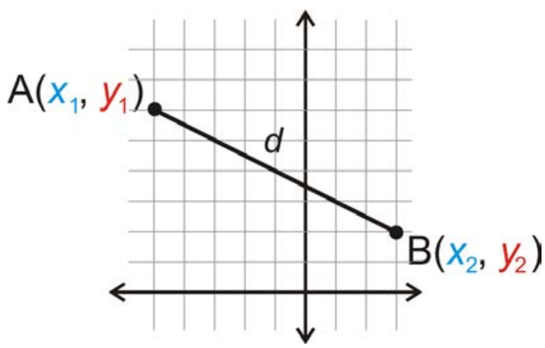
Solution: Draw the altitude from the vertex between the congruent sides, which bisect the base.



$$\begin{aligned}7^2+h^2&=9^2\\49+h^2&=81\\h^2&=32\\h&=\sqrt{32}=\sqrt{16\cdot 2}=4\sqrt{2}\end{aligned}$$

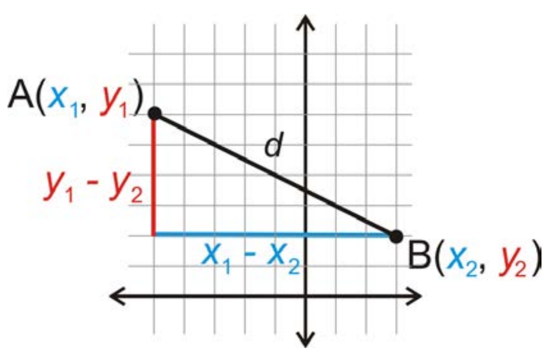
The Distance Formula

Another application of the Pythagorean Theorem is the Distance Formula. We will prove it here.



Let's start with point $A(x_1,y_1)$ and point $B(x_2,y_2)$, to the left. We will call the distance between A and B,d.

Draw the vertical and horizontal lengths to make a right triangle.



Now that we have a right triangle, we can use the Pythagorean Theorem to find the hypotenuse, d.

$$\begin{aligned}d^2&=(x_1-x_2)^2+(y_1-y_2)^2\\d&=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\end{aligned}$$

Distance Formula: The distance $A(x_1,y_1)$ and $B(x_2,y_2)$ is $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Example 12: Find the distance between (1, 5) and (5, 2).

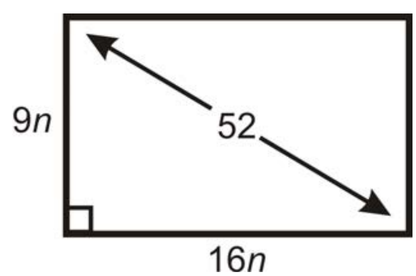
Solution: Make $A(1,5)$ and $B(5,2)$. Plug into the distance formula.

$$\begin{aligned} d &= \sqrt{(1-5)^2+(5-2)^2} \\ &= \sqrt{(-4)^2+(3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5 \end{aligned}$$

Just like the lengths of the sides of a triangle, distances are always positive.

Know What? Revisited To find the length and width of a 52" HDTV, plug in the ratios and 52 into the Pythagorean Theorem. We know that the sides are going to be a multiple of 16 and 9, which we will call n.

$$\begin{aligned} (16n)^2+(9n)^2 &= 52^2 \\ 256n^2+81n^2 &= 2704 \\ 337n^2 &= 2704 \\ n^2 &= 8.024 \\ n &= 2.83 \end{aligned}$$



The dimensions of the TV are $16(2.83'') \times 9(2.83'')$, or $45.3'' \times 25.5''$.

Practice Problems

- Questions 1-9 are similar to Examples 1-5.
- Questions 10-15 are similar to Example 7 and 8.
- Questions 16-19 are similar to Example 9.
- Questions 20-25 are similar to Example 10.
- Questions 26-28 are similar to Example 11.
- Questions 29-31 are similar to Example 12.
- Questions 32 and 33 are similar to the Know What?
- Question 34 and 35 are a challenge and similar to Example 11.

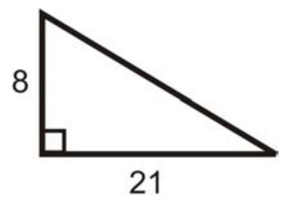
Simplify the radicals.

1. $2\sqrt{5}+\sqrt{20}$
2. $\sqrt{24}$
3. $\left(6\sqrt{3}\right)^2$
4. $8\sqrt{8}\cdot\sqrt{10}$
5. $\left(2\sqrt{30}\right)^2$

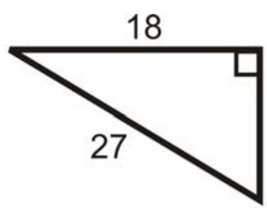
6. $\sqrt{320}$
7. $\frac{4\sqrt{5}}{\sqrt{6}}$
8. $\frac{12}{\sqrt{10}}$
9. $\frac{21\sqrt{5}}{9\sqrt{15}}$

Find the length of the missing side. Simplify all radicals.

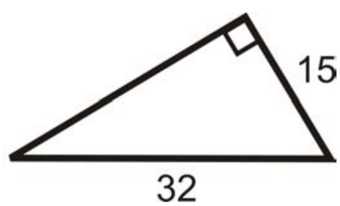
10.



11.



12.



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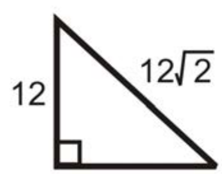

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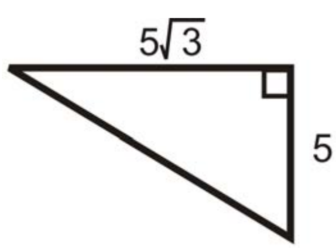

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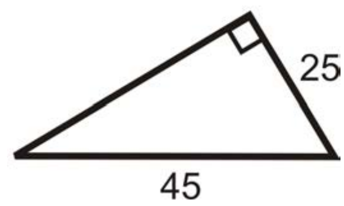


16. If the legs of a right triangle are 10 and 24, then the hypotenuse is ___________.

17. If the sides of a rectangle are 12 and 15, then the diagonal is __________.

18. If the sides of a square are 16, then the diagonal is _________.

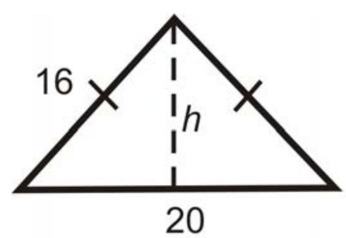
19. If the sides of a square are 9, then the diagonal is _________.

Determine if the following sets of numbers are Pythagorean Triples.

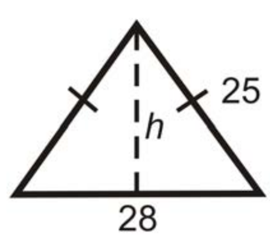
20. 12, 35, 37
21. 9, 17, 18
22. 10, 15, 21
23. 11, 60, 61
24. 15, 20, 25
25. 18, 73, 75

Find the height of each isosceles triangle below. Simplify all radicals.

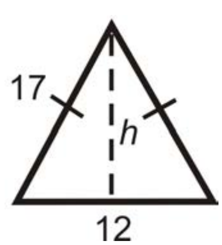
26.



27.

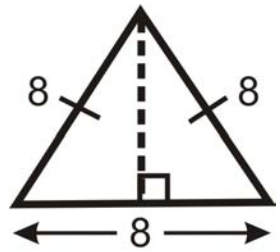


28.



Find the length between each pair of points.

29. (-1, 6) and (7, 2)
30. (10, -3) and (-12, -6)
31. (1, 3) and (-8, 16)
32. What are the length and width of a 42" HDTV? Round your answer to the nearest tenth.
33. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a 42" Standard definition TV? Round your answer to the nearest tenth.
34. ***Challenge*** An equilateral triangle is an isosceles triangle. If all the sides of an equilateral triangle are 8, find the height. Leave your answer in simplest radical form.



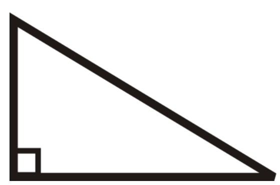
35. If the sides are length s, what would the height be?

Review and Reflect

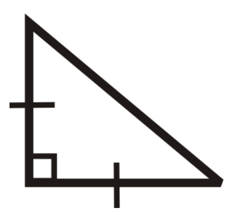
36. True or false? Two right with the same hypotenuse have the same area.
37. Compare and Contrast the distance formula and Pythagorean Theorem.

Warm-Up Answers

1.



2.



3. Factors of 75: 1, 3, 5, 15, 25, 75

4.Prime Factorization of 75: $3 \cdot 5 \cdot 5$

8.2 Converse of the Pythagorean Theorem

TEKS G(1)E, G(4)B, G(6)D, G(9)B

Learning Objectives

- Understand the converse of the Pythagorean Theorem
- Determine if a triangle is acute or obtuse from side measures

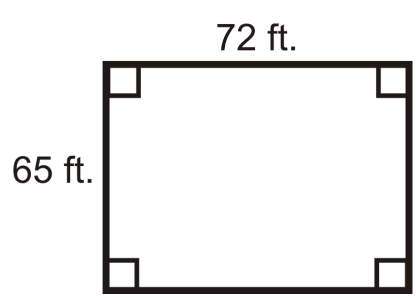
Vocabulary

- Pythagorean Theorem

Warm-Up

1. Determine if the following sets of numbers are Pythagorean triples.
   a. 14, 48, 50
   b. 9, 40, 41
   c. 12, 43, 44
   d. 12, 35, 37
2. Simplify the radicals.
   a. $\left(5\sqrt{12}\right)^2$
   b. $\frac{14}{\sqrt{2}}$
   c. $\frac{18}{\sqrt{3}}$

Know What? A friend of yours is designing a building and wants it to be rectangular. One wall 65 ft. long and the other is 72 ft. long. How can he ensure the walls are going to be perpendicular?

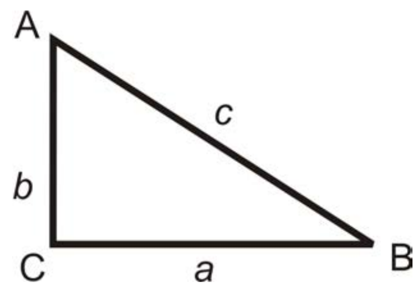


Converse of the Pythagorean Theorem

Pythagorean Theorem Converse

If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangles is a right triangle

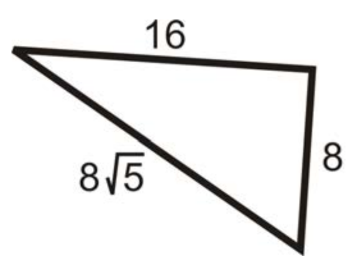
If $a^2+b^2=c^2$, then $\triangle ABC$ is a right triangle.



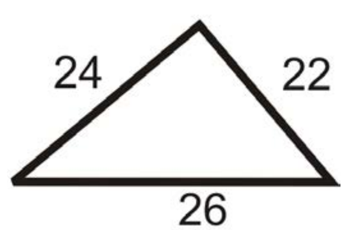
With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any angle measures.

Example 1: Determine if the triangles below are right triangles.

a)



b)



Solution: Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest side represent c.

a) $a^2+b^2=c^2$

$8^2+16^2 \stackrel{?}{=} \left(8\sqrt{5}\right)^2$

$64+256 \stackrel{?}{=} 64 \cdot 5$
$320 = 320$ Yes

b) $a^2+b^2=c^2$
$22^2+24^2 \stackrel{?}{=} 26^2$
$484+576 \stackrel{?}{=} 676$
$1060 \neq 676$ No

Example 2: Do the following lengths make a right triangle?

a) $\sqrt{5}, 3, \sqrt{14}$

b) $6, 2\sqrt{3}, 8$

c) $3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Solution: Even though there is no picture, you can still use the Pythagorean Theorem. Again, the longest length will be c.

a)

b)

c) This is a multiple of $\sqrt{2}$ of a 3, 4, 5 right triangle. Yes, this is a right triangle.

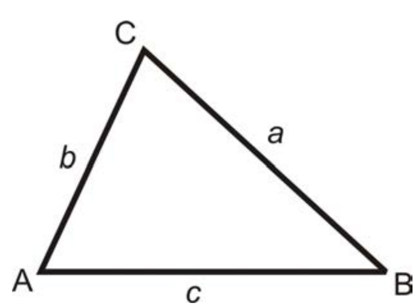
Identifying Acute and Obtuse Triangles

We can extend the converse of the Pythagorean Theorem to determine if a triangle is an obtuse or acute triangle.

Theorem 8-3

If the square of the longest sides is ***less than*** the sum of the squares of the two shorter sides in a right triangle, then the triangle is ***acute***

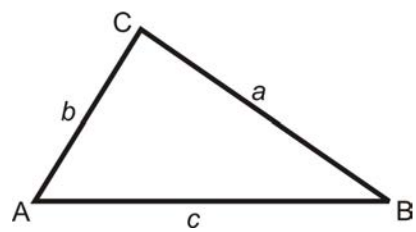
If $c^2 < a^2+b^2$, then the triangle is acute.



Theorem 8-4

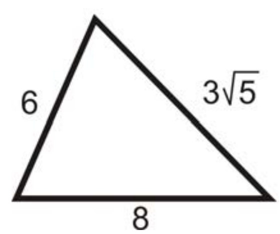
If the square of the longest side is ***greater than*** the sum of the squares of the two shorter sides in a right triangles, then the triangle is obtuse

If $c^2 > a^2 + b^2$, then the triangle is obtuse.

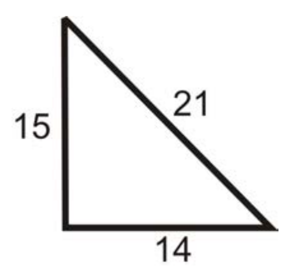


Example 3: Determine if the following triangles are acute, right or obtuse.

a)



b)



Solution: Set the longest side equal to c.

a) $8^2 ? 6^2 + \left(3\sqrt{5}\right)^2$
$64 ? 36 + 45$
$64 < 81$

The triangle is acute.

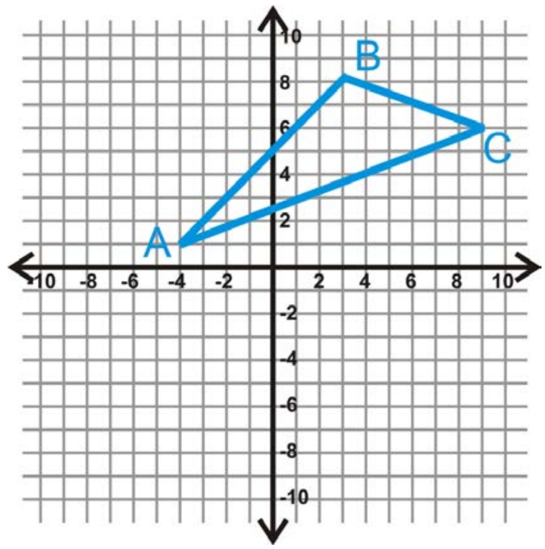
b) $21^2 ? 15^2 + 14^2$
$441 ? 225 + 196$
$441 > 421$

The triangle is obtuse.

Watch This!!!

http://youtu.be/kue5yLHdYII

Example 4: Graph $A(-4,1), B(3,8)$, and $C(9,6)$. Determine if $\triangle ABC$ is acute, obtuse, or right.



Solution: Use the distance formula to find the length of each side.

$$AB = \sqrt{(-4-3)^2+(1-8)^2} = \sqrt{49+49} = \sqrt{98}$$
$$BC = \sqrt{(3-9)^2+(8-6)^2} = \sqrt{36+4} = \sqrt{40}$$
$$AC = \sqrt{(-4-9)^2+(1-6)^2} = \sqrt{169+25} = \sqrt{194}$$

Plug these lengths into the Pythagorean Theorem.

$$(\sqrt{194})^2 \ ? \ (\sqrt{98})^2 + (\sqrt{40})^2$$
$$194 \ ? \ 98 + 40$$
$$194 > 138$$

$\triangle ABC$ is an obtuse triangle.

Know What? Revisited Find the length of the diagonal.

$$65^2 + 72^2 = c^2$$
$$4225 + 5184 = c^2$$
$$9409 = c^2$$
$$97 = c$$

To make the building rectangular, both diagonals must be 97 feet.

Practice Problems

- Questions 1-6 are similar to Examples 1 and 2.

- Questions 7-15 are similar to Example 3.
- Questions 16-20 are similar to Example 4.
- Questions 21-24 use the Pythagorean Theorem.
- Question 25 uses the definition of similar triangles.

Determine if the following lengths make a right triangle.

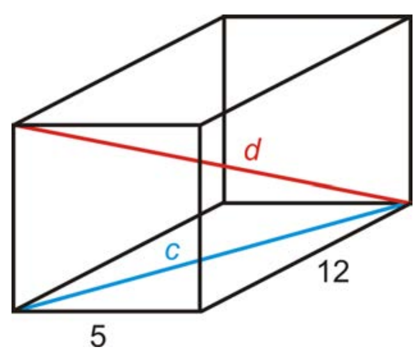
1. 7, 24, 25
2. $\sqrt{5}, 2\sqrt{10}, 3\sqrt{5}$
3. $2\sqrt{3}, \sqrt{6}, 8$
4. 15, 20, 25
5. 20, 25, 30
6. $8\sqrt{3}, 6, 2\sqrt{39}$

Determine if the following triangles are acute, right or obtuse.

7. 7, 8, 9
8. 14, 48, 50
9. 5, 12, 15
10. 13, 84, 85
11. 20, 20, 24
12. 35, 40, 51
13. 39, 80, 89
14. 20, 21, 38
15. 48, 55, 76

Graph each set of points and determine if $\triangle ABC$ is acute, right, or obtuse, using the distance formula.

16. $A(3,-5), B(-5,-8), C(-2,7)$
17. $A(5,3), B(2,-7), C(-1,5)$
18. $A(1,6), B(5,2), C(-2,3)$
19. $A(-6,1), B(-4,-5), C(5,-2)$
20. Show that #18 is a right triangle by using the slopes of the sides of the triangle. The figure to the right is a rectangular prism. All sides (or faces) are either squares (the front and back) or rectangles (the four around the middle). All faces are perpendicular.

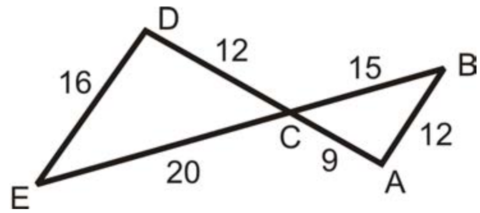


21. Find c.
22. Find d.

Now, the figure is a cube, where all the sides are squares. If all the sides have length 4, find:

23. Find c.

24. Find d.



25. ***Writing*** Explain why $m\angle A = 90^\circ$.

Review and Reflect

26. In order to apply the Pythagorean Theorem you need to first proof that the sides can create a triangle, describe these steps.
27. When applying the Converse of the Pythagorean Theorem, describe the process used and what concerns you need to consider.

Warm-Up Answers

1.
   a. Yes
   b. Yes
   c. No
   d. Yes
2.
   a. $\left(5\sqrt{12}\right)^2 = 5^2 \cdot 12 = 25 \cdot 12 = 300$
   b. $\frac{14}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{14\sqrt{2}}{2} = 7\sqrt{2}$
   c. $\frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$

8.3 Using Similar Right Triangles

TEKS G(1)G, G(5)A, G(7)A, G(8)A, G(8)B

Learning Objectives

- Identify similar triangles inscribed in a larger triangle
- Use proportions in similar right triangles

Vocabulary

- Geometric Mean

Warm-Up

1. Solve the following ratios.
   a. $\frac{3}{x} = \frac{x}{27}$
   b. $\frac{\sqrt{6}}{x} = \frac{x}{9\sqrt{6}}$
   c. $\frac{x}{15} = \frac{12}{x}$
2. If the legs of an isosceles right triangle are 4, find the length of the hypotenuse. Draw a picture and simplify the radical.

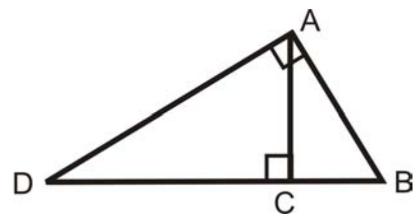
Know What? The bridge to the right is called a truss bridge. It is a steel bridge with a series of right triangles that are connected as support. All the red right triangles are similar. Can you find x, y and z?

Inscribed Similar Triangles

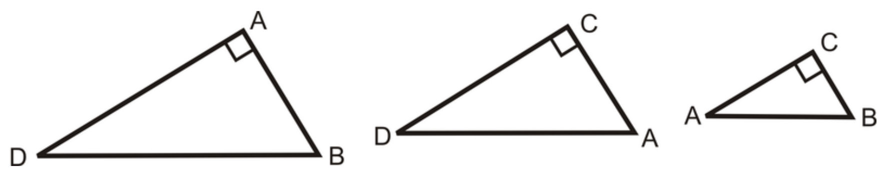
You may recall that if two objects are similar, corresponding angles are congruent and their sides are proportional in length.

Theorem 8-5

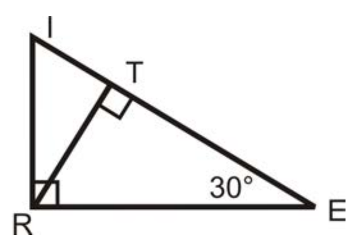
If an altitude is drawn from the right angles of any right triangles, then the two triangles formed are similar to the original triangles and all three triangles are similar to each other



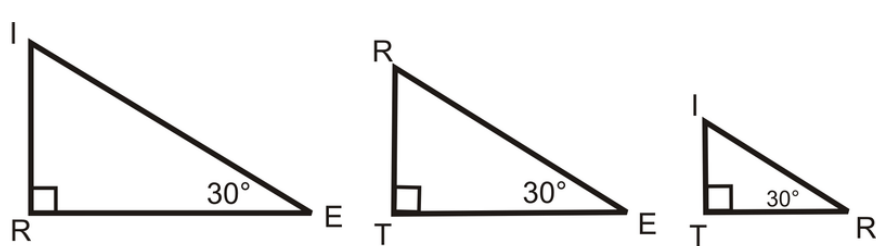
In $\triangle ADB, m\angle A = 90^\circ$ and $\overline{AC} \perp \overline{DB}$, then $\triangle ADB \sim \triangle CDA \sim \triangle CAB$.



Example 1: Write the similarity statement for the triangles below.



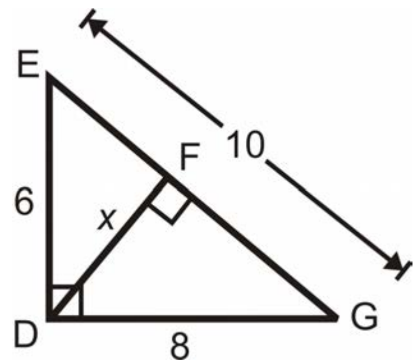
Solution: Separate out the three triangles.



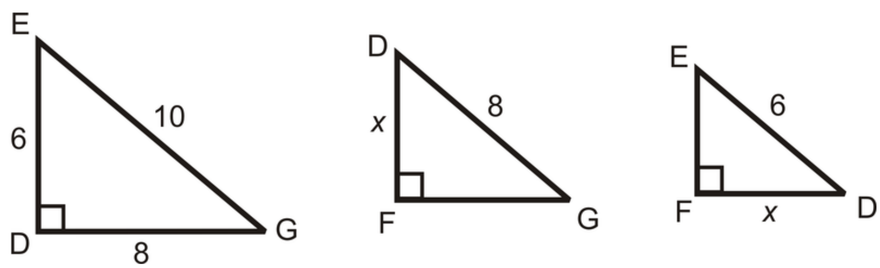
Line up the congruent angles: $\triangle IRE \sim \triangle ITR \sim \triangle RTE$

We can also use the side proportions to find the length of the altitude.

Example 2: Find the value of x.



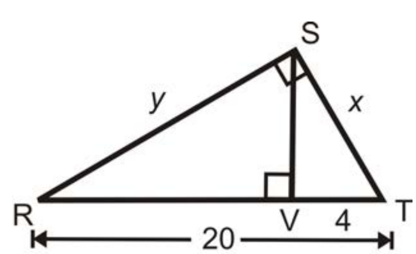
Solution: Separate the triangles to find the corresponding sides.



Set up a proportion.

$$\frac{\text{shorter leg in } \triangle EDG}{\text{shorter leg in } \triangle DFG} = \frac{\text{hypotenuse in } \triangle EDG}{\text{hypotenuse in } \triangle DFG}$$
$$\frac{6}{x} = \frac{10}{8}$$
$$48 = 10x$$
$$4.8 = x$$

Example 3: Find the value of x.



Solution: Set up a proportion.

$$\frac{\text{shorter leg in } \triangle SVT}{\text{shorter leg in } \triangle RST} = \frac{\text{hypotenuse in } \triangle SVT}{\text{hypotenuse in } \triangle RST}$$
$$\frac{4}{x} = \frac{x}{20}$$
$$x^2 = 80$$
$$x = \sqrt{80} = 4\sqrt{5}$$

Example 4: Find the value of y in $\triangle RST$ above.

Solution: Use the Pythagorean Theorem.

$$\begin{aligned}
y^2 + \left(4\sqrt{5}\right)^2 &= 20^2 \\
y^2 + 80 &= 400 \\
y^2 &= 320 \\
y &= \sqrt{320} = 8\sqrt{5}
\end{aligned}$$

The Geometric Mean

Geometric Mean: The geometric mean of two positive numbers a and b is the positive number x, such that $\frac{a}{x} = \frac{x}{b}$ or $x^2 = ab$ and $x = \sqrt{ab}$.

Example 5: Find the geometric mean of 24 and 36.

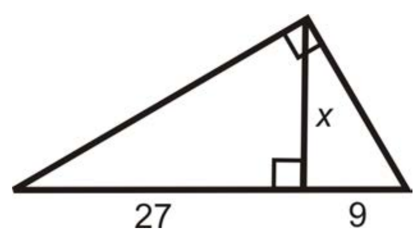
Solution: $x = \sqrt{24 \cdot 36} = \sqrt{12 \cdot 2 \cdot 12 \cdot 3} = 12\sqrt{6}$

Example 6: Find the geometric mean of 18 and 54.

Solution: $x = \sqrt{18 \cdot 54} = \sqrt{18 \cdot 18 \cdot 3} = 18\sqrt{3}$

In both of these examples, we did not multiply the numbers together. This makes it easier to simplify the radical. A practical application of the geometric mean is to find the altitude of a right triangle.

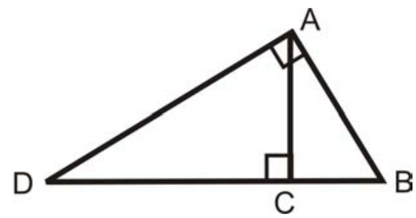
Example 7: Find the value of x.



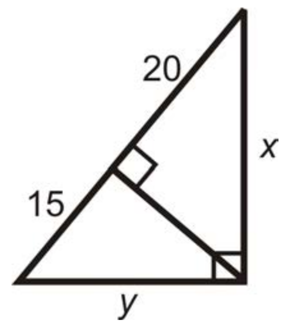
Solution: Set up a proportion.

$$\begin{aligned}
\frac{shortest\ leg\ of\ smallest\ \triangle}{shortest\ leg\ of\ middle\ \triangle} &= \frac{longer\ leg\ of\ smallest\ \triangle}{longer\ leg\ of\ middle\ \triangle} \\
\frac{9}{x} &= \frac{x}{27} \\
x^2 &= 243 \\
x &= \sqrt{243} = 9\sqrt{3}
\end{aligned}$$

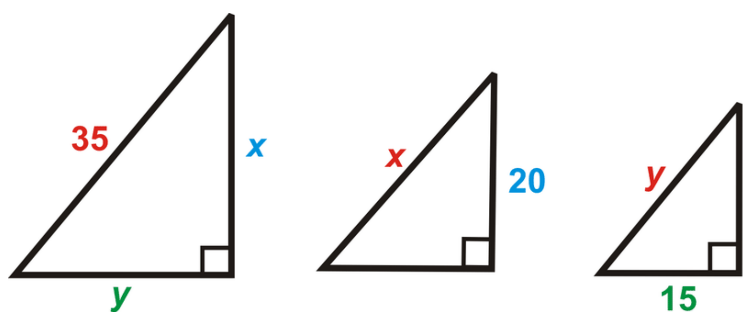
In Example 7, $\frac{9}{x} = \frac{x}{27}$ is in the definition of the geometric mean. So, the altitude is the geometric mean of the two segments that it divides the hypotenuse into. In other words, $\frac{BC}{AC} = \frac{AC}{DC}$. Two other true proportions are $\frac{BC}{AB} = \frac{AB}{DB}$ and $\frac{DC}{AD} = \frac{AD}{DB}$.



Example 8: Find the value of x and y.



Solution: Separate the triangles. Write a proportion for x.



$$\frac{20}{x} = \frac{x}{35}$$
$$x^2 = 20 \cdot 35$$
$$x = \sqrt{20 \cdot 35}$$
$$x = 10\sqrt{7}$$

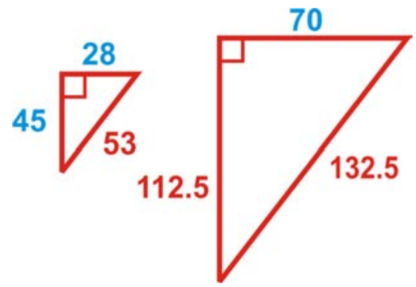
Set up a proportion for y. Or, you can use the Pythagorean Theorem to solve for y.

$$\frac{15}{y} = \frac{y}{35}$$
$$y^2 = 15 \cdot 35$$
$$y = \sqrt{15 \cdot 35}$$
$$y = 5\sqrt{21}$$

$$(10\sqrt{7})^2 + y^2 = 35^2$$
$$700 + y^2 = 1225$$
$$y = \sqrt{525} = 5\sqrt{21}$$

Use the method you feel most comfortable with.

Know What? Revisited To find the hypotenuse of the smallest triangle, do the Pythagorean Theorem.



$$45^2 + 28^2 = x^2$$
$$2809 = x^2$$
$$53 = x$$

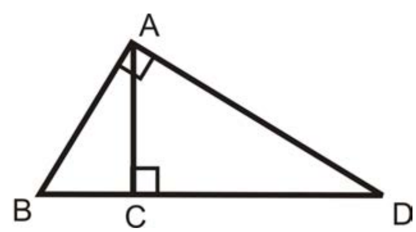
Because the triangles are similar, find the scale factor of $\frac{70}{28} = 2.5$.

$y = 45 \cdot 2.5 = 112.5$ and $z = 53 \cdot 2.5 = 135.5$

Practice Problems

- Questions 1-4 use the ratios of similar right triangles.
- Questions 5-8 are similar to Example 1.
- Questions 9-11 are similar to Examples 2-4
- Questions 12-17 are similar to Examples 5 and 6.
- Questions 18-29 are similar to Examples 2, 3, 4, 7, and 8.
- Question 30 is a proof of theorem 8-5.

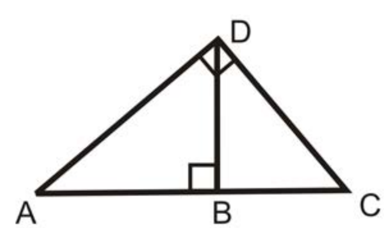
Fill in the blanks.



1. $\triangle BAD \sim \triangle$____ $\sim \triangle$____
2. $\frac{BC}{?} = \frac{?}{CD}$
3. $\frac{BC}{AB} = \frac{AB}{?}$
4. $\frac{?}{AD} = \frac{AD}{BD}$

Write the similarity statement for the right triangles in each diagram.

5.



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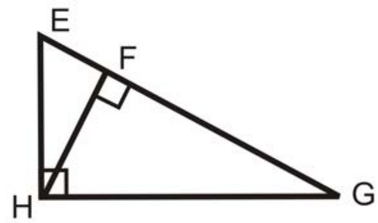

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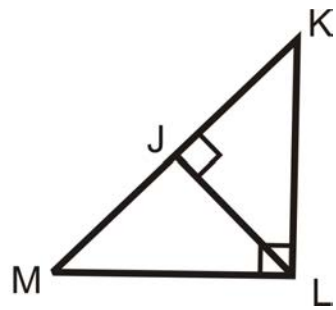
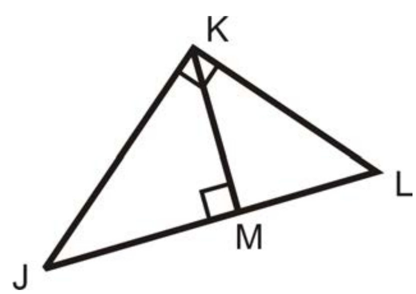


Use the diagram to answer questions 8-11.



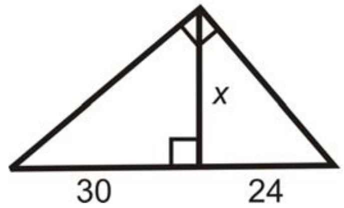
8. Write the similarity statement for the three triangles in the diagram.
9. If $JM = 12$ and $ML = 9$, find KM.
10. Find JK.
11. Find KL.

Find the geometric mean between the following two numbers. Simplify all radicals.

12. 16 and 32
13. 45 and 35
14. 10 and 14
15. 28 and 42
16. 40 and 100
17. 51 and 8

Find the length of the missing variable(s). Simplify all radicals.

18.



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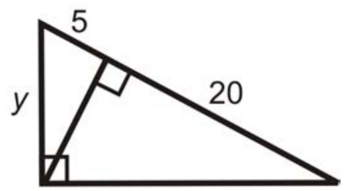

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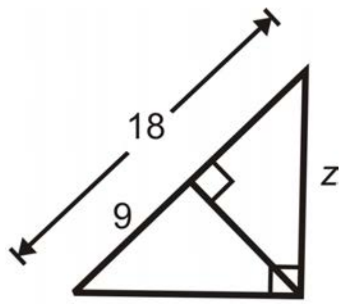

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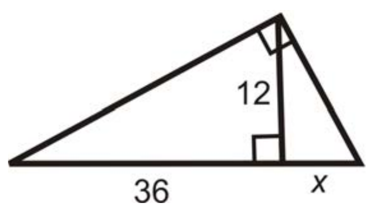

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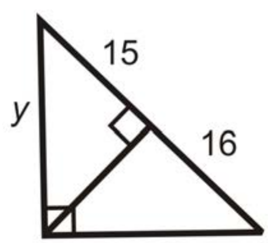

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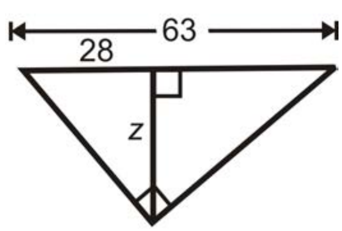

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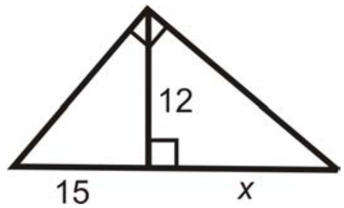

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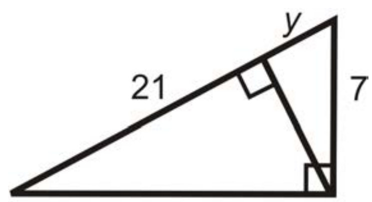

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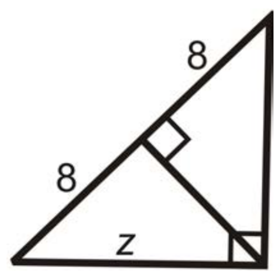

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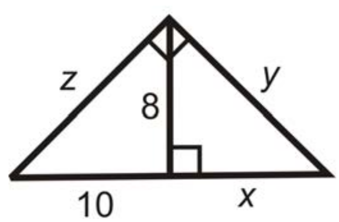

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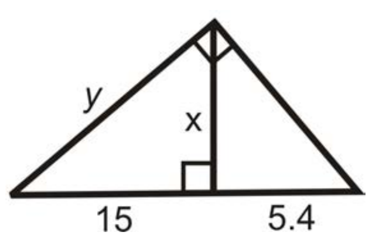

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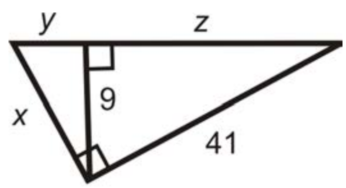
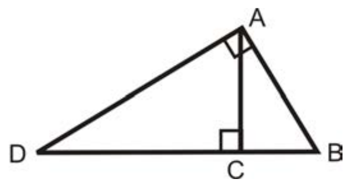


30. Fill in the blanks of the proof for Theorem 8-5.



Given: $\triangle ABD$with $\overline{AC} \perp \overline{DB}$and $\angle DAB$is a right angle. Prove: $\triangle ABD \sim \triangle CBA \sim \triangle CAD$

TABLE 8.2:

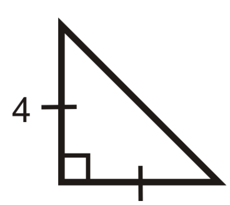
Statement	***Reason***
1.	Given
2. $\angle DCA$ and $\angle ACB$ are right angles	
3. $\angle DAB \cong \angle DCA \cong \angle ACB$	
4.	Reflexive PoC
5.	AA Similarity Postulate
6. $B \cong \angle B$	
7. $\triangle CBA \cong \triangle ABD$	
8. $\triangle CAD \cong \triangle CBA$	

Review and Reflect

31. Find a pair of whole number that have a geometric mean that is also a whole number.
32. Describe the method used above.

Warm-Up Answers

1.
   a. $\frac{3}{x} = \frac{x}{27} \rightarrow x^2 = 81 \rightarrow x = 9$
   b. $\frac{\sqrt{6}}{x} = \frac{x}{9\sqrt{6}} \rightarrow x^2 = 54 \rightarrow x = \sqrt{54} = \sqrt{9 \cdot 6} = 3\sqrt{6}$
   c. $\frac{x}{15} = \frac{12}{x} \rightarrow x^2 = 180 \rightarrow x = \sqrt{180} = \sqrt{4 \cdot 9 \cdot 5} = 2 \cdot 3\sqrt{5} = 6\sqrt{5}$
2. $4^2 + 4^2 = h^2$
   $h = \sqrt{32} = 4\sqrt{2}$



8.4 Special Right Triangles

TEKS G(1)F, G(6)D, G(9)B

Learning Objectives

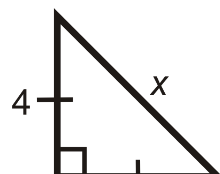
- Learn and use the 45°- 45°- 90° triangle ratio
- Learn and use the 30°- 60°- 90° triangle ratio

Vocabulary

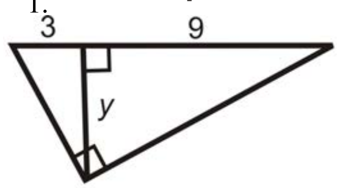
- Special Right Triangles (45°- 45°- 90°, 30°- 60°- 90°)

Warm-Up

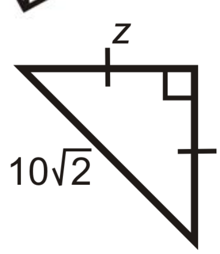
Find the value of the missing variables. Simplify all radicals.



1.



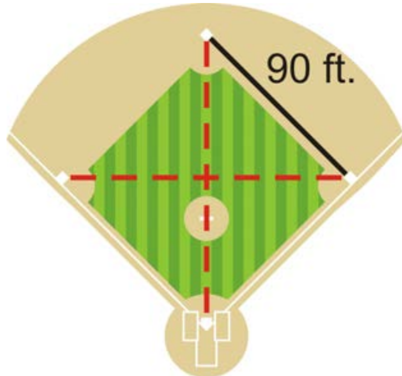
2.



3.

4. Is 9, 12, and 15 a right triangle?
5. Is 3, $3\sqrt{3}$, and 6 a right triangle?

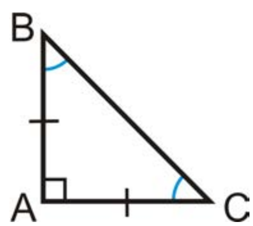
Know What? A baseball diamond is a square with sides that are 90 feet long. Each base is a corner of the square. What is the length between 1^{st} and 3^{rd} base and between 2^{nd} base and home plate? (the red dotted lines in the diagram).



Isosceles Right Triangles

There are two special right triangles. The first is an isosceles right triangle.

Isosceles Right Triangle: A right triangle with congruent legs and acute angles. This triangle is also called a 45°- 45°- 90° triangle (after the angle measures).



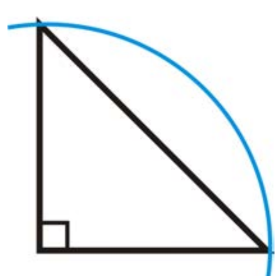
$\triangle ABC$ is a right triangle with:

$$m\angle A = 90^\circ$$
$$\overline{AB} \cong \overline{AC}$$
$$m\angle B = m\angle C = 45^\circ$$

Investigation 8-2: Properties of an Isosceles Right Triangle

Tools Needed: Pencil, paper, compass, ruler, protractor

1. Draw an isosceles right triangle with 2 inch legs and the 90° angle between them.

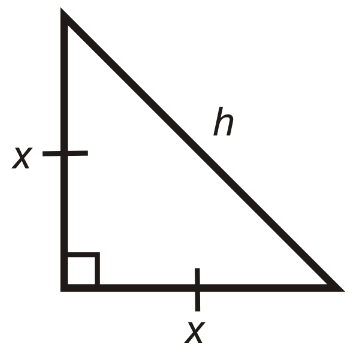


2. Find the measure of the hypotenuse, using the Pythagorean Theorem. Simplify the radical.

$$\begin{aligned}2^2+2^2&=c^2\\ 8&=c^2\\ c&=\sqrt{8}=\sqrt{4\cdot 2}=2\sqrt{2}\end{aligned}$$

What do you notice about the length of the legs and hypotenuse?

3. Now, let's say the legs are of length x and the hypotenuse is h. Use the Pythagorean Theorem to find the hypotenuse. How is it similar to your answer in #2?



$$\begin{aligned}x^2+x^2&=h^2\\ 2x^2&=h^2\\ x\sqrt{2}&=h\end{aligned}$$

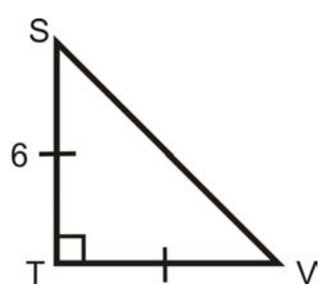
45°- 45°- 90º Theorem

If a right triangle is isosceles, then its sides are $x:x:x\sqrt{2}$

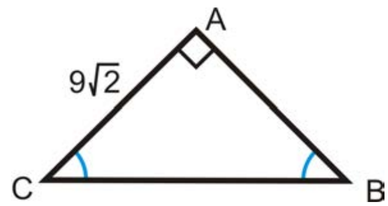
For any isosceles right triangle, the legs are x and ***the hypotenuse is always*** $x\sqrt{2}$. Because the three angles are always $45^\circ, 45^\circ$, and 90°, ***all isosceles right triangles are similar.***

Example 1: Find the length of the missing sides.

a)



b)



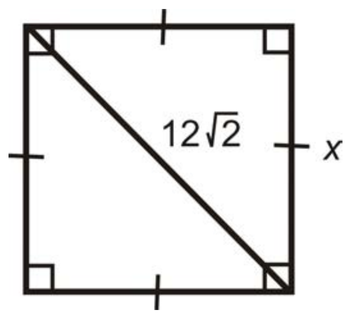
Solution: Use the $x : x : x\sqrt{2}$ ratio.

a) $TV = 6$ because it is equal to ST. So, $SV = 6 \cdot \sqrt{2} = 6\sqrt{2}$.

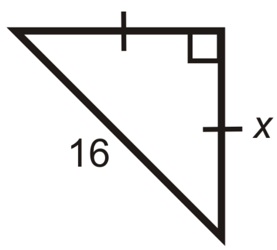
b) $AB = 9\sqrt{2}$ because it is equal to AC. So, $BC = 9\sqrt{2} \cdot \sqrt{2} = 9 \cdot 2 = 18$.

Example 2: Find the length of x.

a)



b)



Solution: Use the $x : x : x\sqrt{2}$ ratio.

a) $12\sqrt{2}$ is the diagonal of the square. Remember that the diagonal of a square bisects each angle, so it splits the square into two 45°- 45°- 90° triangles. $12\sqrt{2}$ would be the hypotenuse, or equal to $x\sqrt{2}$.

$$12\sqrt{2} = x\sqrt{2}$$
$$12 = x$$

b) Here, we are given the hypotenuse. Solve for x in the ratio.

$$x\sqrt{2} = 16$$
$$x = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$$

In part b, we ***rationalized the denominator*** which we learned in the first section.

30°- 60°- 90° Triangles

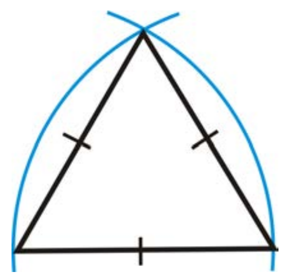
The second special right triangle is called a 30°-60°-90° triangle, after the three angles.

To draw a 30°- 60°- 90° triangle, start with an equilateral triangle.

Investigation 8-3: Properties of a 30°- 60°- 90° Triangle

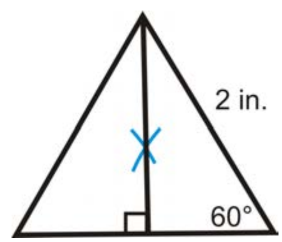
Tools Needed: Pencil, paper, ruler, compass

1. Construct an equilateral triangle with 2 inch sides.



http://www.mathsisfun.com/geometry/construct-equitriangle.html

2. Draw or construct the altitude from the top vertex to form two congruent triangles.



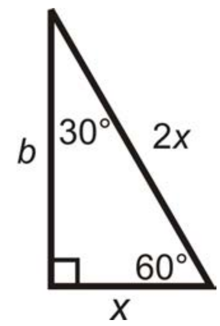
3. Find the measure of the two angles at the top vertex and the length of the shorter leg.

The top angles are each 30° *and the shorter leg is 1 in because the altitude of an equilateral triangle is also the angle and perpendicular bisector.*

4. Find the length of the longer leg, using the Pythagorean Theorem. Simplify the radical.

$$\begin{aligned} 1^2+b^2 &= 2^2 \\ 1+b^2 &= 4 \\ b^2 &= 3 \\ b &= \sqrt{3} \end{aligned}$$

5. Now, let's say the shorter leg is length x and the hypotenuse is $2x$. Use the Pythagorean Theorem to find the longer leg. How is this similar to your answer in #4?



$$
\begin{aligned}
x^2 + b^2 &= (2x)^2 \\
x^2 + b^2 &= 4x^2 \\
b^2 &= 3x^2 \\
b &= x\sqrt{3}
\end{aligned}
$$

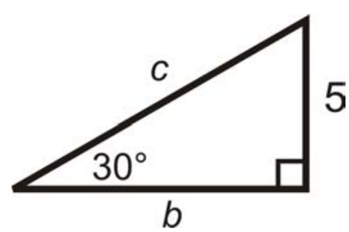
30°- 60°- 90° Theorem

If a triangle has angle measures of 30º, 60º, and 90º, then the sides are $x : x\sqrt{3} : 2x$

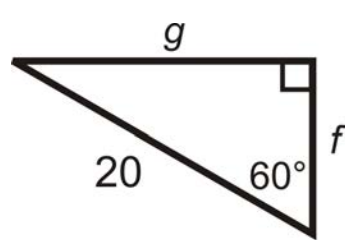
The shortest leg is always x, the longest leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$. If you ever forget these theorems, you can still use the Pythagorean Theorem.

Example 3: Find the length of the missing sides.

a)



b)



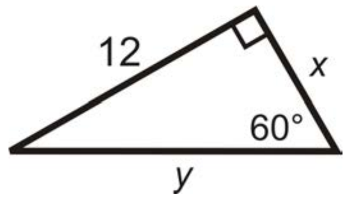
Solution: In part a, we are given the shortest leg and in part b, we are given the hypotenuse.

a) If $x = 5$, then the longer leg, $b = 5\sqrt{3}$, and the hypotenuse, $c = 2(5) = 10$.

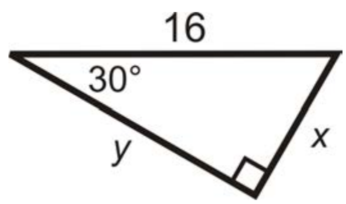
b) Now, $2x = 20$, so the shorter leg, $f = \frac{20}{2} = 10$, and the longer leg, $g = 10\sqrt{3}$.

Example 4: Find the value of x and y.

a)



b)



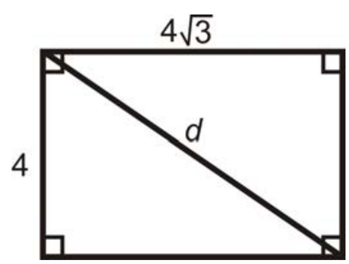
Solution: In part a, we are given the longer leg and in part b, we are given the hypotenuse.

a) $x\sqrt{3} = 12$
$x = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$
The hypotenuse is
$y = 2(4\sqrt{3}) = 8\sqrt{3}$

b) $2x = 16$
$x = 8$
The longer leg is
$y = 8 \cdot \sqrt{3} = 8\sqrt{3}$

Example 5: A rectangle has sides 4 and $4\sqrt{3}$. What is the length of the diagonal?

Solution: If you are not given a picture, draw one.



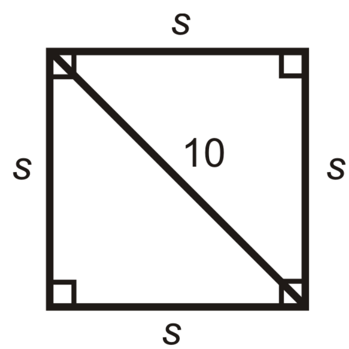
The two lengths are $x, x\sqrt{3}$, so the diagonal would be $2x$, or $2(4) = 8$.

If you did not recognize this is a 30°- 60°- 90° triangle, you can use the Pythagorean Theorem too.

$$4^2 + \left(4\sqrt{3}\right)^2 = d^2$$
$$16 + 48 = d^2$$
$$d = \sqrt{64} = 8$$

Example 6: A square has a diagonal with length 10, what are the sides?

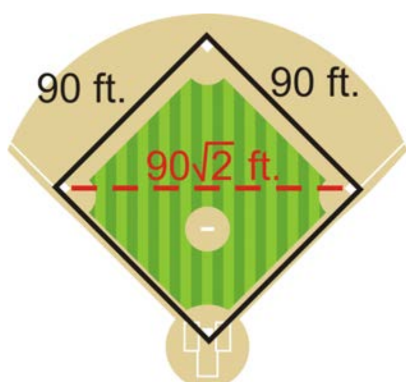
Solution: Draw a picture.



We know half of a square is a 45°- 45°- 90° triangle, so $10 = s\sqrt{2}$.

$$s\sqrt{2} = 10$$
$$s = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Know What? Revisited The distance between 1^{st} and 3^{rd} base is one of the diagonals of the square. So, it would be the same as the hypotenuse of a 45°- 45°- 90° triangle. Using our ratios, the distance is $90\sqrt{2} \approx 127.3\ ft$. The distance between 2^{nd} base and home plate is the same length.



Practice Problems

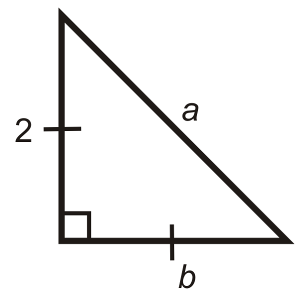
- Questions 1-4 are similar to Example 1-4.
- Questions 5-8 are similar to Examples 5 and 6.
- Questions 9-23 are similar to Examples 1-4.
- Questions 24 and 25 are a challenge.

1. In an isosceles right triangle, if a leg is 4, then the hypotenuse is __________.
2. In a 30°- 60°- 90° triangle, if the shorter leg is 5, then the longer leg is __________ and the hypotenuse is __________.
3. In an isosceles right triangle, if a leg is x, then the hypotenuse is __________.
4. In a 30°- 60°- 90° triangle, if the shorter leg is x, then the longer leg is __________ and the hypotenuse is __________.

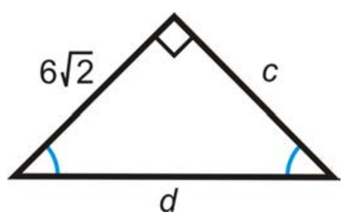
5. A square has sides of length 15. What is the length of the diagonal?
6. A square's diagonal is 22. What is the length of each side?
7. A rectangle has sides of length 6 and $6\sqrt{3}$. What is the length of the diagonal?
8. Two (opposite) sides of a rectangle are 10 and the diagonal is 20. What is the length of the other two sides?

For questions 9-23, find the lengths of the missing sides. Simplify all radicals.

9.



10.



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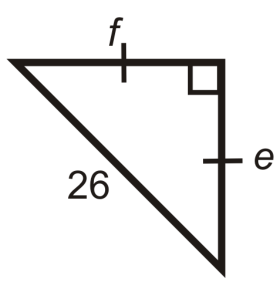

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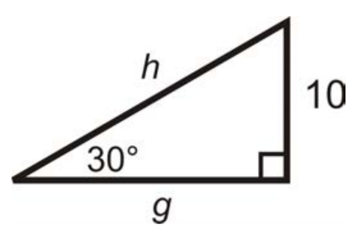

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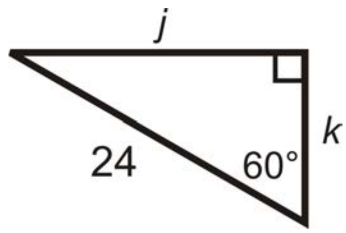

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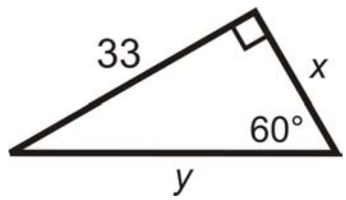

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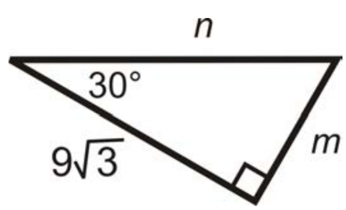

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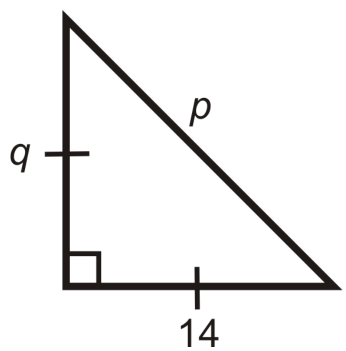

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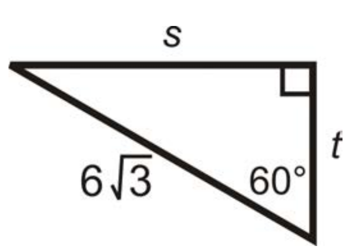

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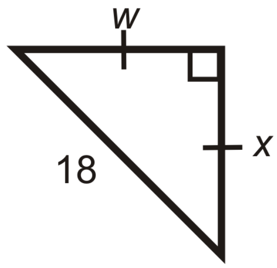

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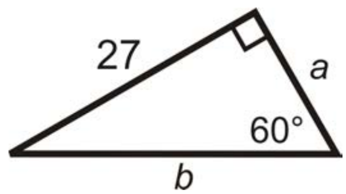

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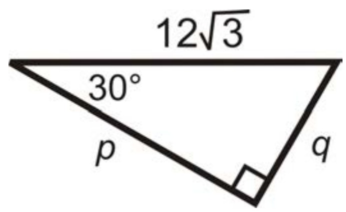

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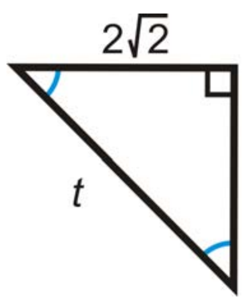

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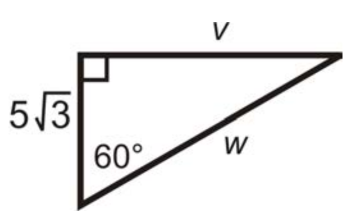
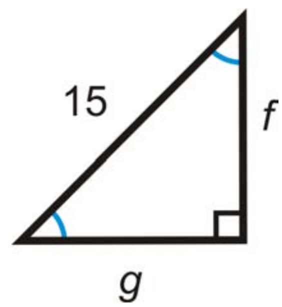
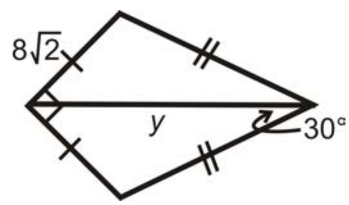


23.

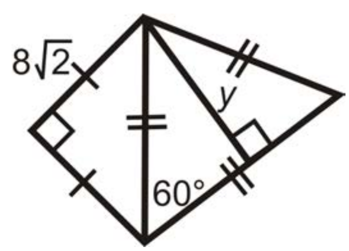


Challenge For 24 and 25, find the value of y. You may need to draw in additional lines. Round all answers to the nearest hundredth.

24.



25.



Review and Reflect

26. In the practice problems above which of them were more difficult? Is there a pattern? Describe your findings?

Warm-Up Answers

1. $4^2+4^2=x^2$
$$32=x^2$$
$$x=4\sqrt{2}$$
2. $3^2+y^2=6^2$
$$y^2=27$$
$$y=3\sqrt{3}$$
3. $x^2+x^2=\left(10\sqrt{2}\right)^2$
$$2x^2=200$$
$$x^2=100$$
$$x=10$$
4. Yes, $9^2+12^2=15^2 \rightarrow 81+144=225$
5. Yes, $3^2+\left(3\sqrt{3}\right)^2=6^2 \rightarrow 9+27=36$

8.5 Sine, Cosine and Tangent

TEKS G(1)A, G(9)A, G(9)B

Learning Objectives

- Use the tangent, sine, and cosine ratios
- Use a scientific calculator to find sine, cosine, and tangent
- Use trigonometric ratios in real-life situations

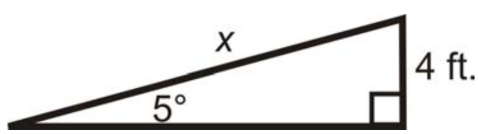
Vocabulary

- Trigonometry
- SIne Ratio
- Cosine Ratio
- Tangent Ratio
- Angle of Depression
- Angle of Elevation

Warm-Up

1. The legs of an isosceles right triangle have length 14. What is the hypotenuse?
2. Do the lengths 8, 16, 20 make a right triangle? If not, is the triangle obtuse or acute?
3. In a 30-60-90 triangle, what do the 30, 60, and 90 refer to?

Know What? A restaurant is building a wheelchair ramp. The angle of elevation for the ramp is 5°. If the vertical distance from the sidewalk to the front door is 4 feet, how long will the ramp be (x)? Round your answers to the nearest hundredth.

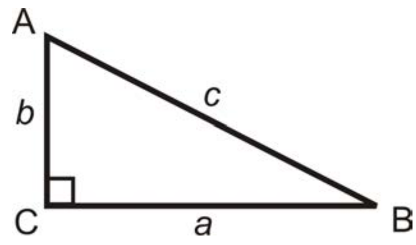


What is Trigonometry?

In this lesson we will define three trigonometric (or trig) ratios. Once we have defined these ratios, we will be able to solve problems like the **Know What?** above.

Trigonometry: The study of the relationships between the sides and angles of right triangles.

The hypotenuse should be identified first. The legs are called ***adjacent*** or ***opposite*** depending on which ***acute*** angle is being used.



c is the *hypotenuse*

a is *adjacent* to $\angle B$ a is *opposite* $\angle A$

b is *adjacent* to $\angle A$ b is *opposite* $\angle B$

Sine, Cosine, and Tangent Ratios

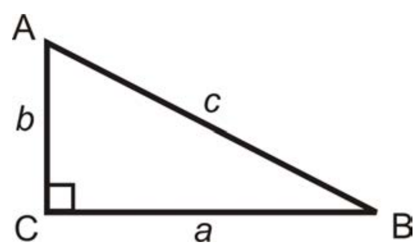
The three basic trig ratios are called, sine, cosine and tangent. For now, we will only take the sine, cosine and tangent of ***acute*** angles. However, you can use these ratios with obtuse angles as well.

For right triangle $\triangle ABC$, we have:

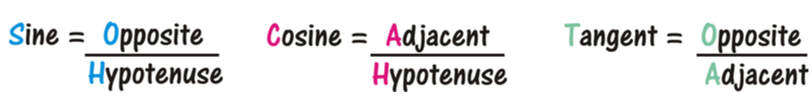
Sine Ratio: $\frac{opposite\ leg}{hypotenuse}$ $\sin A = \frac{a}{c}$ or $\sin B = \frac{b}{c}$

Cosine Ratio: $\frac{adjacent\ leg}{hypotenuse}$ $\cos A = \frac{b}{c}$ or $\cos B = \frac{a}{c}$

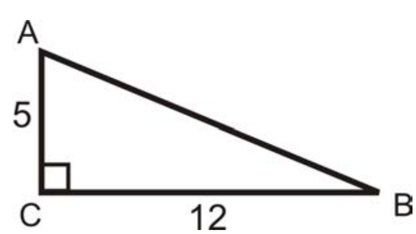
Tangent Ratio: $\frac{opposite\ leg}{adjacent\ leg}$ $\tan A = \frac{a}{b}$ or $\tan B = \frac{b}{a}$



An easy way to remember ratios is to use SOH-CAH-TOA.



Example 1: Find the sine, cosine and tangent ratios of $\angle A$.



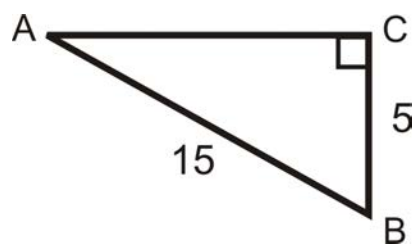
Solution: First, we need to use the Pythagorean Theorem to find the length of the hypotenuse.

$$5^2+12^2=h^2$$
$$13=h$$
$$\sin A=\frac{leg\ opposite\ \angle A}{hypotenuse}=\frac{12}{13} \qquad \cos A=\frac{leg\ adjacent\ to\ \angle A}{hypotenuse}=\frac{5}{13},$$
$$\tan A=\frac{leg\ opposite\ \angle A}{leg\ adjacent\ to\ \angle A}=\frac{12}{5}$$

A few important points:

- Always **reduce ratios** (fractions) when you can.
- Use the Pythagorean Theorem to find the missing side (if there is one).
- If there is a radical in the denominator, **rationalize the denominator.**

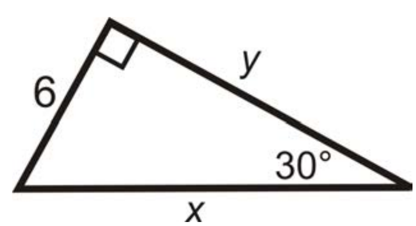
Example 2: Find the sine, cosine, and tangent of $\angle B$.



Solution: Find the length of the missing side.

$$AC^2+5^2=15^2$$
$$AC^2=200$$
$$AC=10\sqrt{2}$$
$$\sin B=\frac{10\sqrt{2}}{15}=\frac{2\sqrt{2}}{3} \qquad \cos B=\frac{5}{15}=\frac{1}{3} \qquad \tan B=\frac{10\sqrt{2}}{5}=2\sqrt{2}$$

Example 3: Find the sine, cosine and tangent of 30°.



Solution: This is a 30°- 60°- 90° triangle. The short leg is 6, $y=6\sqrt{3}$ and $x=12$.

$$\sin 30^\circ=\frac{6}{12}=\frac{1}{2} \qquad \cos 30^\circ=\frac{6\sqrt{3}}{12}=\frac{\sqrt{3}}{2} \qquad \tan 30^\circ=\frac{6}{6\sqrt{3}}=\frac{1}{\sqrt{3}}\cdot\frac{3}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

Sine, Cosine, and Tangent with a Calculator

From Example 3, we can conclude that ***there is a fixed sine, cosine, and tangent value for every angle, from*** $0°$ ***to*** $90°$. Your scientific (or graphing) calculator knows all the trigonometric values for any angle. Your calculator, should have [SIN], [COS], and [TAN] buttons. In order to use these functions, your calculator needs to be in degree mode.

Example 4: Find the trigonometric value, using your calculator. Round to 4 decimal places.

a) $\sin 78°$

b) $\cos 60°$

c) $\tan 15°$

Solution: Depending on your calculator, you enter the degree and then press the trig button or the other way around. Also, make sure the mode of your calculator is in *DEGREES*.

a) $\sin 78° = 0.9781$

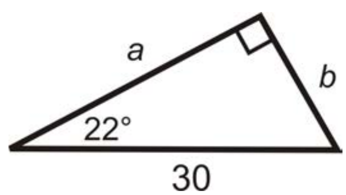
b) $\cos 60° = 0.5$

c) $\tan 15° = 0.2679$

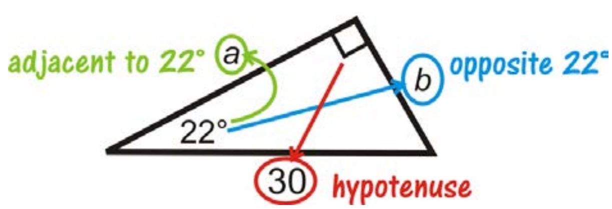
Finding the Sides of a Triangle using Trig Ratios

One application of the trigonometric ratios is to use them to find the missing sides of a right triangle. Notice that when we used Pythagorean Theorem, we needed to know the length of two sides of the right triangle. If we know the length of one side of the right triangle and the measure of an angle we can use Trigonometry to find the lengths of the other two sides.

Example 5: Find the value of each variable. Round your answer to the nearest tenth.

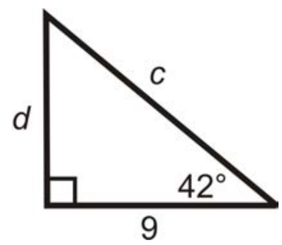


Solution: We are given the hypotenuse. Use *sine* to find b, and *cosine* to find a.



$$\sin 22° = \frac{b}{30} \qquad\qquad \cos 22° = \frac{a}{30}$$
$$30 \cdot \sin 22° = b \qquad\qquad 30 \cdot \cos 22° = a$$
$$b \approx 11.2 \qquad\qquad a \approx 27.8$$

Example 6: Find the value of each variable. Round your answer to the nearest tenth.



Solution: We are given the adjacent leg to 42°. To find c, use *cosine* and *tangent* to find d.

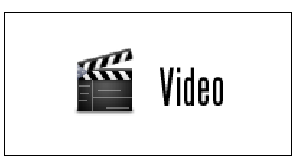
$$\cos 42^\circ = \frac{adjacent}{hypotenuse} = \frac{9}{c} \qquad \tan 42^\circ = \frac{opposite}{adjacent} = \frac{d}{9}$$

$$c \cdot \cos 42^\circ = 9 \qquad 9 \cdot \tan 42^\circ = d$$

$$c = \frac{9}{\cos 42^\circ} \approx 12.1 \qquad d \approx 8.1$$

Anytime you use trigonometric ratios, only use the information that you are given. You don't want to round $\cos 42^\circ$ Save all the rounding until the end. This will give the most accurate answers.

Watch This!!!



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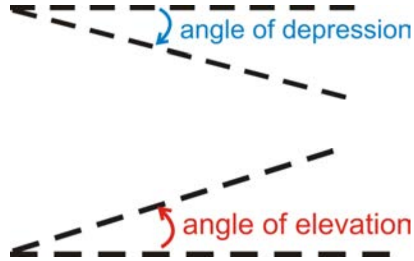
Angles of Depression and Elevation

Another application of the trigonometric ratios is to find lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

Angle of Depression: The angle measured from the horizon or horizontal line, down.



Angle of Elevation: The angle measure from the horizon or horizontal line, up.

Example 7: A math student is standing 25 feet from the base of the Washington Monument. The angle of elevation from her horizontal line of sight is 87.4°. If her "eye height" is 5 ft, how tall is the monument?

Solution: We can find the height of the monument by using the tangent ratio.

$$\tan 87.4^\circ = \frac{h}{25}$$
$$h = 25 \cdot \tan 87.4^\circ = 550.54$$

Adding 5 ft, the total height of the Washington Monument is 555.54 ft.

Know What? Revisited To find the length of the ramp, we need to use sine.

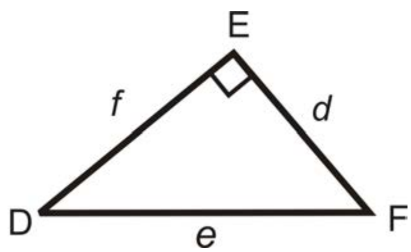
$$\sin 5^\circ = \frac{4}{x}$$
$$y = \frac{2}{\sin 5^\circ} = 22.95$$

Practice Problems

- Questions 1-8 use the definitions of sine, cosine and tangent.
- Questions 9-16 are similar to Example 4.
- Questions 17-22 are similar to Examples 1-3.
- Questions 23-28 are similar to Examples 5 and 6.

- Questions 29 and 30 are similar to Example 7.

Use the diagram to fill in the blanks below.



1. $\tan D = \frac{?}{?}$
2. $\sin F = \frac{?}{?}$
3. $\tan F = \frac{?}{?}$
4. $\cos F = \frac{?}{?}$
5. $\sin D = \frac{?}{?}$
6. $\cos D = \frac{?}{?}$

From questions 1-6, we can conclude the following. Fill in the blanks.

7. $\cos \_\_\_ = \sin F$ and $\sin \_\_\_ = \cos F$.
8. $\tan D$ and $\tan F$ are __________ of each other.

Use your calculator to find the value of each trig function below. Round to four decimal places.

9. $\sin 24^\circ$
10. $\cos 45^\circ$
11. $\tan 88^\circ$
12. $\sin 43^\circ$
13. $\tan 12^\circ$
14. $\cos 79^\circ$
15. $\sin 82^\circ$
16. $\tan 45^\circ$

Find the sine, cosine and tangent of $\angle A$. Reduce all fractions and radicals

17.

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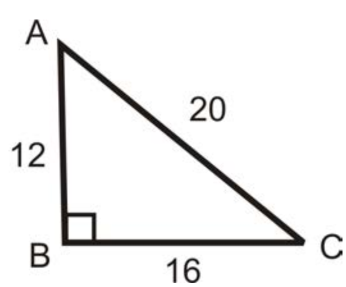

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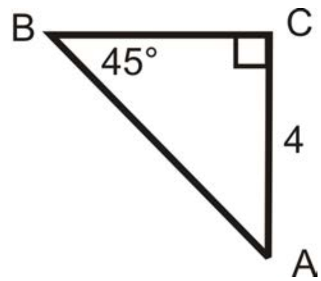

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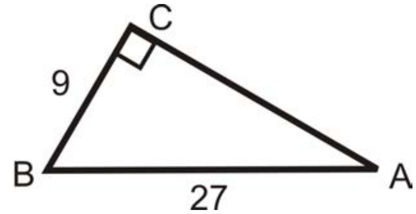

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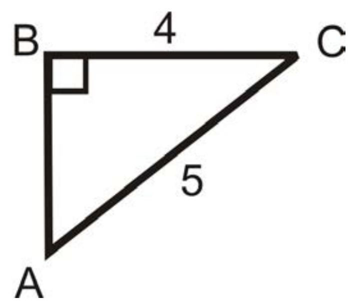

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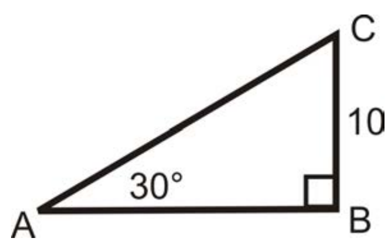

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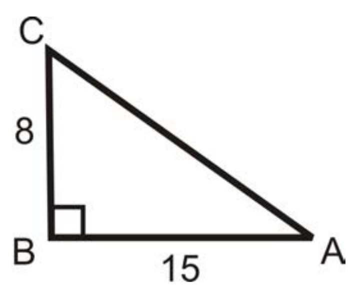
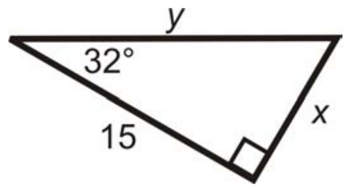


Find the length of the missing sides. Round your answers to the nearest tenth.

23.



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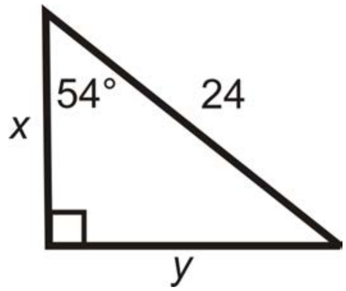

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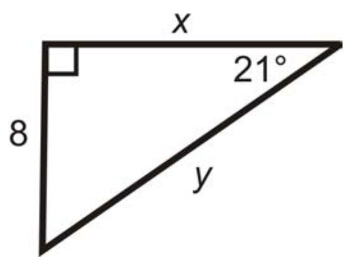

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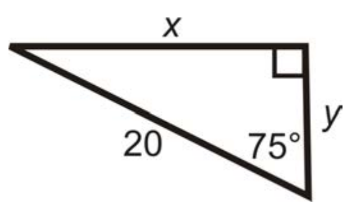

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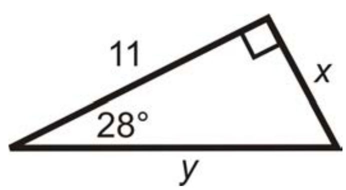

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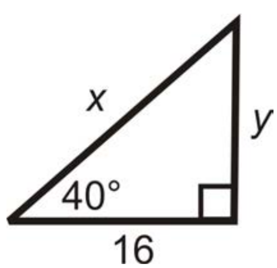
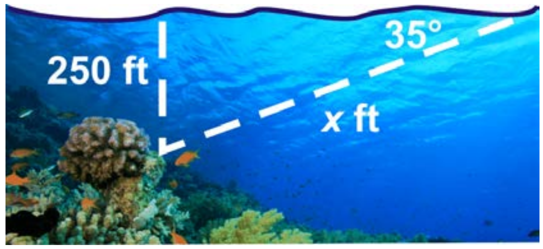


29. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35°and the depth of the ocean, at that point is 250 feet. How far away is she from the reef?



30. The Leaning Tower of Piza currently "leans" at a 4°angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?

Review and Reflect

31. When dealing with right triangles, we can use Pythagorean Theorem or Trigonometry(Sine, Cosine and Tangent), how do you know which method to use.

Warm-Up Answers

1. The hypotenuse is $14\sqrt{2}$.
2. No, $8^2+16^2<20^2$, the triangle is obtuse.
3. $30^\circ, 60^\circ$, and 90° refer to the angle measures in the special right triangle.

8.6 Inverse Trigonometric Ratios

TEKS G(1)F, G(9)A

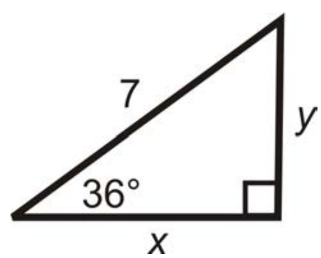
Learning Objectives

- Use the inverse trigonometric ratios to find an angle in a right triangle
- Solve a right triangle

Warm-Up

Find the lengths of the missing sides. Round your answer to the nearest tenth.

1.



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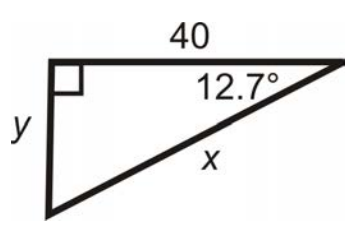


3. Draw an isosceles right triangle with legs of length 3. What is the length of the hypotenuse?

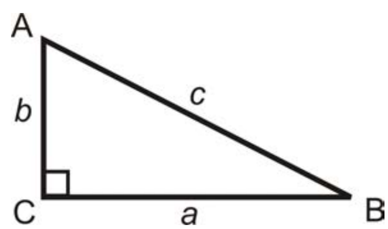
4. Use the triangle from #3 to find sine, cosine, and tangent of 45°.

Know What? The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115 ft. high. What is the angle of elevation, x°, of this escalator?

Inverse Trigonometric Ratios

In mathematics, the word inverse means "undo." For example, addition and subtraction are inverses of each other because one undoes the other. When we apply inverses to the trigonometric ratios, we can find acute angle measures as long as we are given two sides.

Inverse Tangent: Labeled $\tan^{-1}$, the "-1" means inverse.



$$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B \qquad \tan^{-1}\left(\frac{a}{b}\right) = m\angle A$$

Inverse Sine: Labeled $\sin^{-1}$.

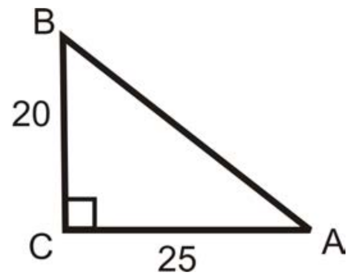
$$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B \qquad \sin^{-1}\left(\frac{a}{c}\right) = m\angle A$$

Inverse Cosine: Labeled $\cos^{-1}$.

$$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B \qquad \cos^{-1}\left(\frac{b}{c}\right) = m\angle A$$

In order to find the measure of the angles, you will need you use your calculator. On most scientific and graphing calculators, the buttons look like [SIN^{-1}], [COS^{-1}], and [TAN^{-1}]. You might also have to hit a shift or 2^{nd} button to access these functions.

Example 1: Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.



Solution: In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$. So, $\tan^{-1} \frac{4}{5} = m\angle A$. Now, use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: $[2^{nd}]$[TAN]$\left(\frac{4}{5}\right)$**[ENTER]** and the screen looks like:

$$m\angle A = 38.7^\circ$$

Example 2: $\angle A$ is an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

a) $\sin A = 0.68$

b) $\cos A = 0.85$

c) $\tan A = 0.34$

Solution:

a) $m\angle A = \sin^{-1} 0.68 = 42.8^\circ$

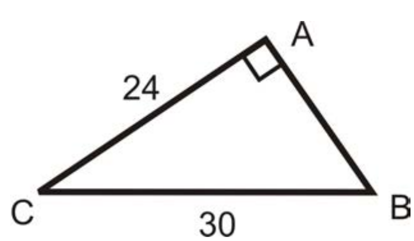
b) $m\angle A = \cos^{-1} 0.85 = 31.8^\circ$

c) $m\angle A = \tan^{-1} 0.34 = 18.8^\circ$

Solving Triangles

To solve a right triangle, you need to find all sides and angles in a right triangle, using sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem.

Example 3: Solve the right triangle.



Solution: To solve this right triangle, we need to find $AB, m\angle C$ and $m\angle B$. Only use the values you are given.

$\underline{AB}$: Use the Pythagorean Theorem.

$$\begin{aligned} 24^2 + AB^2 &= 30^2 \\ 576 + AB^2 &= 900 \\ AB^2 &= 324 \\ AB &= \sqrt{324} = 18 \end{aligned}$$

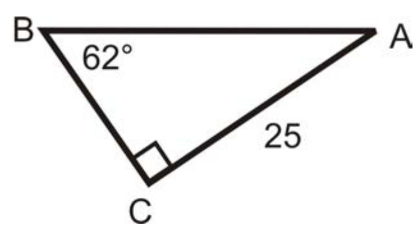
<u>$m\angle B$</u>: Use the inverse sine ratio.

$$\begin{aligned} \sin B &= \frac{24}{30} = \frac{4}{5} \\ \sin^{-1}\left(\frac{4}{5}\right) &= 53.1^\circ = m\angle B \end{aligned}$$

<u>$m\angle C$</u>: Use the inverse cosine ratio.

$$\cos C = \frac{24}{30} = \frac{4}{5} \longrightarrow \cos^{-1}\left(\frac{4}{5}\right) = 36.9^\circ = m\angle C$$

Example 4: Solve the right triangle.



Solution: To solve this right triangle, we need to find AB, BC and $m\angle A$.

<u>AB</u>: Use sine ratio.

$$\begin{aligned} \sin 62^\circ &= \frac{25}{AB} \\ AB &= \frac{25}{\sin 62^\circ} \\ AB &\approx 28.31 \end{aligned}$$

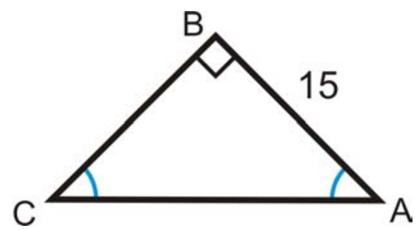
<u>BC</u>: Use tangent ratio.

$$\begin{aligned} \tan 62^\circ &= \frac{25}{BC} \\ BC &= \frac{25}{\tan 62^\circ} \\ BC &\approx 13.30 \end{aligned}$$

<u>$m\angle A$</u>: Use Triangle Sum Theorem

$$62^\circ + 90^\circ + m\angle A = 180^\circ$$
$$m\angle A = 28^\circ$$

Example 5: Solve the right triangle.



Solution: The two acute angles are congruent, making them both 45°. This is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\tan 45^\circ = \frac{15}{BC} \qquad\qquad \sin 45^\circ = \frac{15}{AC}$$
$$BC = \frac{15}{\tan 45^\circ} = 15 \qquad\qquad AC = \frac{15}{\sin 45^\circ} \approx 21.21$$

45-45-90 Triangle Ratios

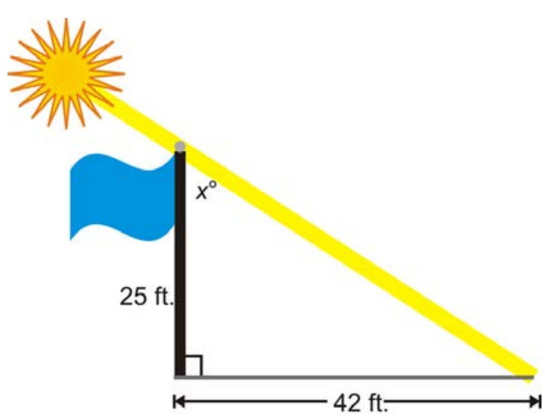
$$BC = AB = 15, AC = 15\sqrt{2} \approx 21.21$$

Watch This!!!

https://s3.amazonaws.com/ck12bg.ck12.org/curriculum/106223/video.mp4

Real-Life Situations

Example 6: A 25 foot tall flagpole casts a 42 feet shadow. What is the angle that the sun hits the flagpole?



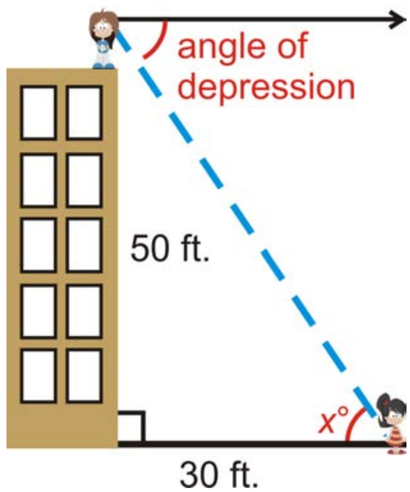
Solution: Draw a picture. The angle that the sun hits the flagpole is x°. We need to use the inverse tangent ratio.

$$\tan x = \frac{42}{25}$$
$$\tan^{-1}\frac{42}{25} \approx 59.2^\circ = x$$

Example 7: Elise is standing on top of a 50 foot building and sees her friend, Molly. If Molly is 30 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise's eye height is 4.5 feet.

Solution: Because of parallel lines, the angle of depression is equal to the angle at Molly, or x°. We can use the inverse tangent ratio.

$$\tan^{-1}\left(\frac{54.5}{30}\right) = 61.2^\circ = x$$



Know What? Revisited To find the escalator's angle of elevation, use the inverse sine.

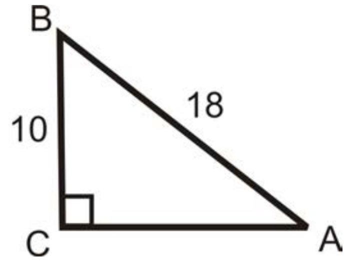
$$\sin^{-1}\left(\frac{115}{230}\right) = 30^\circ \qquad \text{The angle of elevation is } 30^\circ.$$

Practice Problems

- Questions 1-6 are similar to Example 1.
- Questions 7-12 are similar to Example 2.
- Questions13-21 are similar to Examples 3 and 4.
- Questions 22-24 are similar to Examples 6 and 7.
- Questions 25-30 are a review of the trigonometric ratios.

Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

1.



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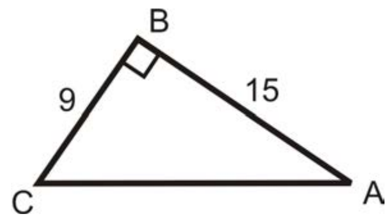

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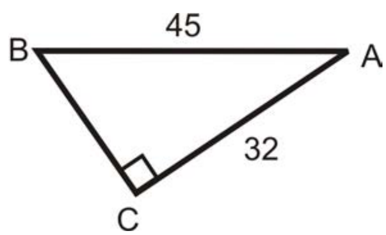

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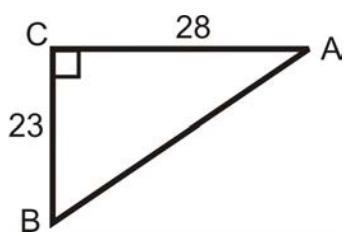

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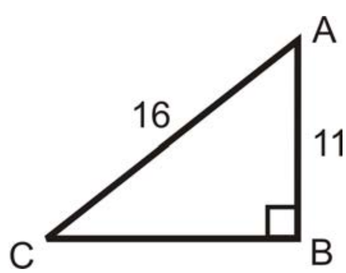

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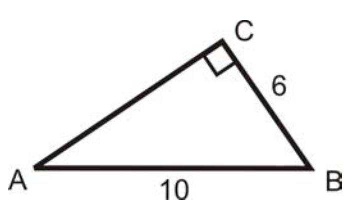
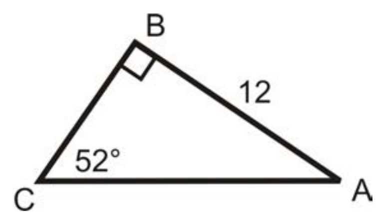


Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$
10. $\cos^{-1} 0.9845$
11. $\tan^{-1} 15.93$
12. $\sin^{-1} 0.7851$

Solving the following right triangles. Find all missing sides and angles. Round any decimal answers to the nearest tenth

13.



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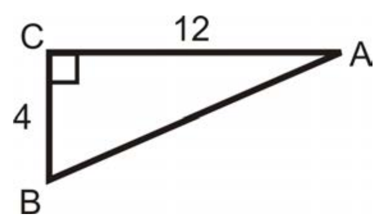

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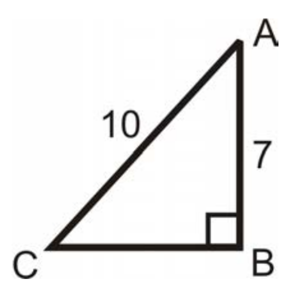

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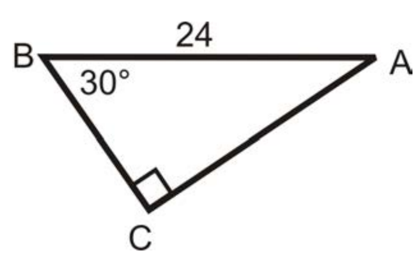

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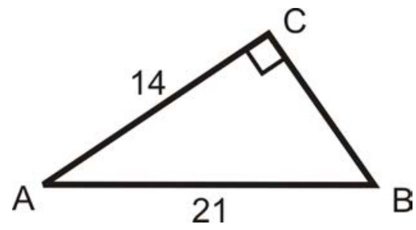

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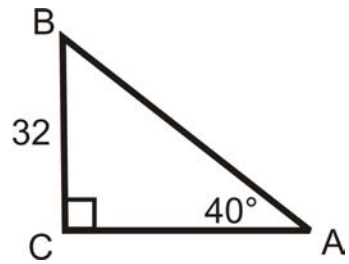

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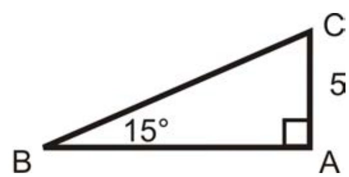

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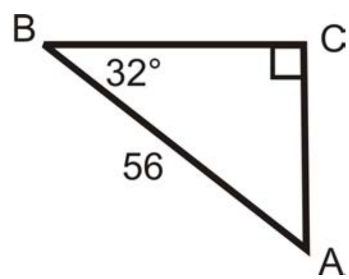

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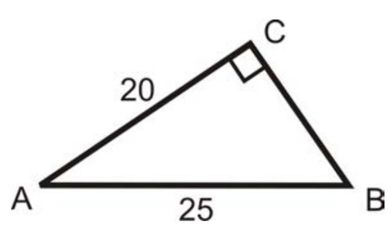


Real-Life Situations Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

22. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
23. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?

24. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?

Examining Patterns Below is a table that shows the sine, cosine, and tangent values for eight different angle measures. Answer the following questions.

TABLE 8.3:

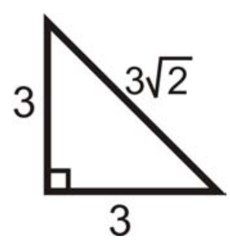
	10°	20°	30°	40°	50°	60°	70°	80°
Sine	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848
Cosine	0.9848	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1736
Tangent	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713

25. What value is equal to $\sin 40^\circ$?
26. What value is equal to $\cos 70^\circ$?
27. Describe what happens to the sine values as the angle measures increase.
28. Describe what happens to the cosine values as the angle measures increase.
29. What two numbers are the sine and cosine values between?
30. Find $\tan 85^\circ, \tan 89^\circ$, and $\tan 89.5^\circ$ using your calculator. Now, describe what happens to the tangent values as the angle measures increase.

Review and Reflect

31. Describe what the inverse trigonometric functions are used for.
32. Why does the $\sin^{-1}(2)$ give you an error?

Warm-Up Answers

1. $\sin 36^\circ = \frac{y}{7}$ $\quad$ $\cos 36^\circ = \frac{x}{7}$
   $y = 4.11$ $\quad$ $x = 5.66$
2. $\cos 12.7^\circ = \frac{40}{x}$ $\quad$ $\tan 12.7^\circ = \frac{y}{40}$
   $x = 41.00$ $\quad$ $y = 9.01$
3. 

4. $\sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
   $\cos 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
   $\tan 45^\circ = \frac{3}{3} = 1$

8.7 Chapter 8 Review

Keywords & Theorems

The Pythagorean Theorem

- Pythagorean Theorem
- Pythagorean Triple
- Distance Formula

The Pythagorean Theorem Converse

- Pythagorean Theorem Converse
- Theorem 8-3
- Theorem 8-4

Similar Right Triangles

- Theorem 8-5
- Geometric Mean

Special Right Triangles

- Isosceles Right (45-45-90) Triangle
- 30-60-90 Triangle
- 45-45-90 Theorem
- 30-60-90 Theorem

Tangent, Sine and Cosine Ratios

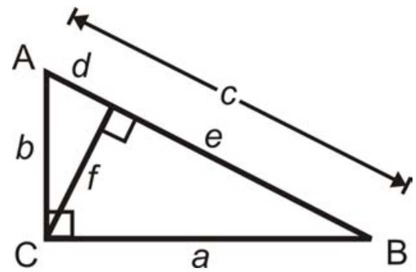
- Trigonometry
- Adjacent (Leg)
- Opposite (Leg)
- Sine Ratio
- Cosine Ratio
- Tangent Ratio
- Angle of Depression
- Angle of Elevation

Solving Right Triangles

- Inverse Tangent
- Inverse Sine
- Inverse Cosine

Review

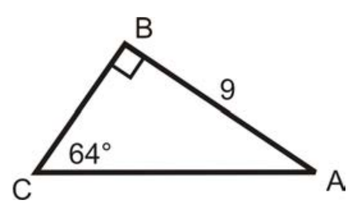
Fill in the blanks using right triangle $\triangle ABC$.



1. $a^2 + \_\_\_\_^2 = c^2$
2. $\sin \_\_\_\_ = \frac{b}{c}$
3. $\tan \_\_\_\_ = \frac{f}{d}$
4. $\cos \_\_\_\_ = \frac{b}{c}$
5. $\tan^{-1}\left(\frac{f}{e}\right) = \_\_\_\_$
6. $\sin^{-1}\left(\frac{f}{b}\right) = \_\_\_\_$
7. $\_\_\_\_^2 + d^2 = b^2$
8. $\frac{?}{b} = \frac{b}{c}$
9. $\frac{e}{?} = \frac{?}{c}$
10. $\frac{d}{f} = \frac{f}{?}$

Solve the following right triangles using the Pythagorean Theorem, the trigonometric ratios, and the inverse trigonometric ratios. When possible, simplify the radical. If not, round all decimal answers to the nearest tenth

11.



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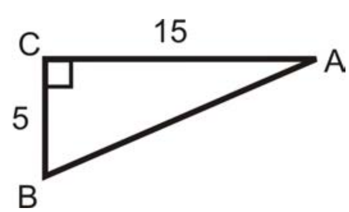

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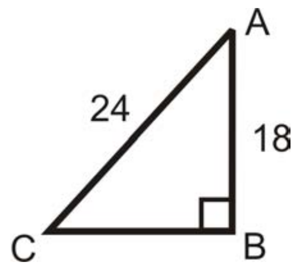

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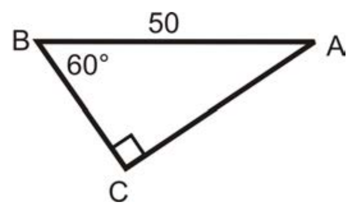

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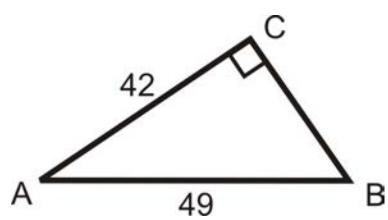

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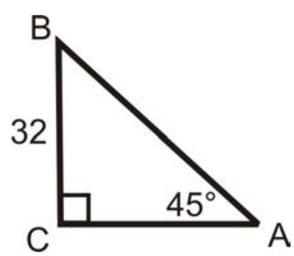

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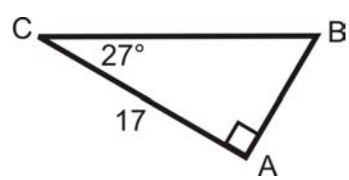

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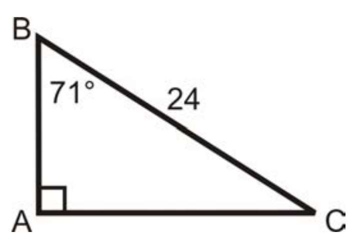

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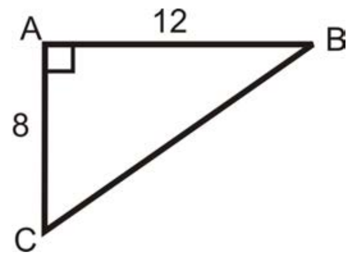
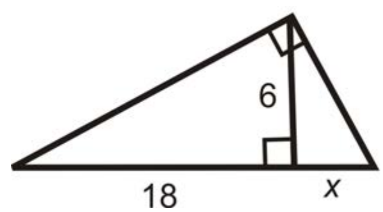


Determine if the following lengths make an acute, right, or obtuse triangle. If they make a right triangle, determine if the lengths are a Pythagorean triple.

20. 11, 12, 13
21. 16, 30, 34
22. 20, 25, 42
23. $10\sqrt{6}, 30, 10\sqrt{15}$
24. 22, 25, 31
25. 47, 27, 35

Find the value of x.

26.



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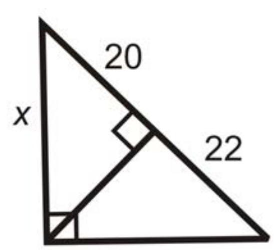

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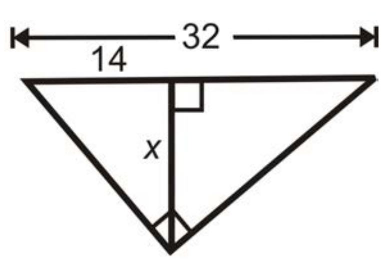


26. The angle of elevation from the base of a mountain to its peak is 76°. If its height is 2500 feet, what is the length to reach the top? Round the answer to the nearest tenth.
27. Taylor is taking an aerial tour of San Francisco in a helicopter. He spots ATT Park (baseball stadium) at a horizontal distance of 850 feet and down (vertical) 475 feet. What is the angle of depression from the helicopter to the park? Round the answer to the nearest tenth.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9693 .

8.8 Study Guide

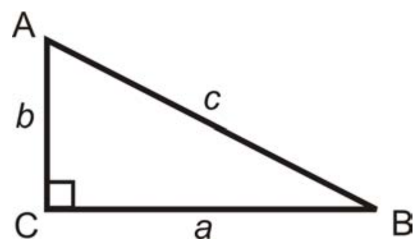
Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st **Section: The Pythagorean Theorem**

Pythagorean Theorem

Pythagorean Triple

Distance Formula



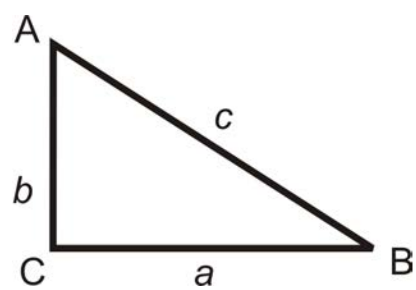
Homework:

2nd **Section: The Pythagorean Theorem Converse**

Pythagorean Theorem Converse

Theorem 8-3

Theorem 8-4

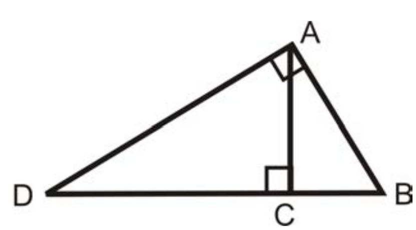


Homework:

3rd **Section: Similar Right Triangles**

Theorem 8-5

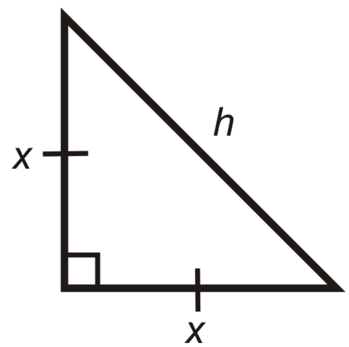
Geometric Mean



Homework:

4^{th}**Section: Special Right Triangles**

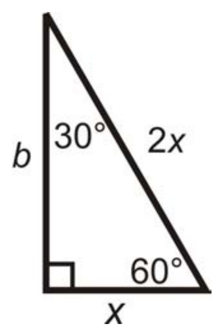
Isosceles Right (45-45-90) Triangle



30-60-90 Triangle

45-45-90 Theorem

30-60-90 Theorem



Homework:

5^{th}**Section: Tangent, Sine and Cosine Ratios**

Trigonometry

Adjacent (Leg)

Opposite (Leg)

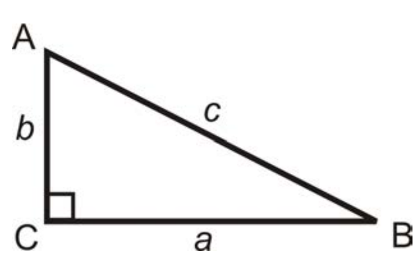
Sine Ratio

Cosine Ratio

Tangent Ratio

Angle of Depression

Angle of Elevation



Homework:

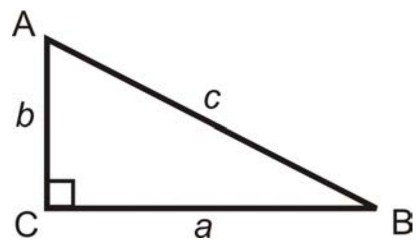
6^{th}**Section: Solving Right Triangles**

Inverse Tangent

Inverse Sine

Inverse Cosine

Solving Right Triangles



Homework:

CHAPTER 9 Perimeter and Area

Chapter Outline

Now that we have explored triangles, quadrilaterals, polygons, and circles, we are going to learn how to find the perimeter and area of each.

9.1 Triangles and Parallelograms

TEKS G(4)B, G(4)C, G(11)A, G(11)B

Learning Objectives

- Understand the basic concepts of area
- Use formulas to find the area of triangles and parallelograms

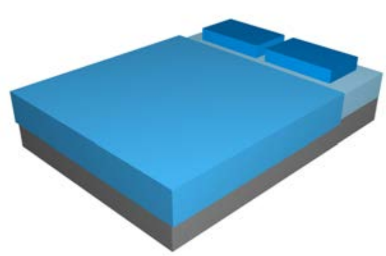
Vocabulary

- Area
- Perimeter

Warm-Up

1. Define perimeter and area, in your own words.
2. Solve the equations below. Simplify any radicals.
   a. $x^2 = 121$
   b. $4x + 6 = 80$
   c. $x^2 - 6x + 8 = 0$
   d. $\frac{1}{2}x - 3 = 5$
   e. $x^2 + 2x - 15 = 0$
   f. $x^2 - x - 12 = 0$

Know What? Ed's parents are getting him a new king bed. Upon further research, Ed discovered there are two types of king beds, and Eastern (or standard) King and a California King. The Eastern King has $76'' \times 80''$ dimensions, while the California King is $72'' \times 84''$ (both dimensions are $width \times length$). Which bed has a larger area to lie on?



Areas and Perimeters of Squares and Rectangles

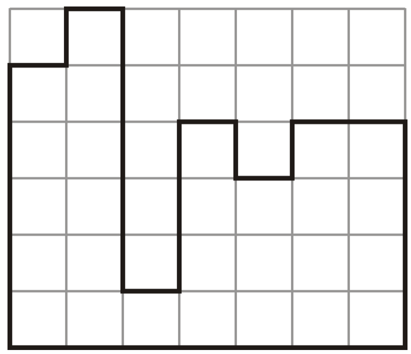
Perimeter: The distance around a shape.

The perimeter of any figure must have a unit of measurement attached to it. If no specific units are given (feet, inches, centimeters, etc), write "units."

Example 1: Find the perimeter of the figure to the left.

Solution: Here, we can use the grid as our units. Count around the figure to find the perimeter.

$$5+1+1+1+5+1+3+1+1+1+1+2+4+7 = 34 \ units$$

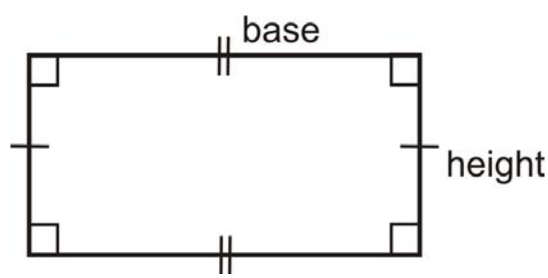


You are probably familiar with the area of squares and rectangles from a previous math class. Recall that you must always establish a unit of measure for area. Area is always measured in square units, square feet $(ft.^2)$, square inches $(in.^2)$. square centimeters $(cm.^2)$, etc. If no specific units are given, write "$units^2$".

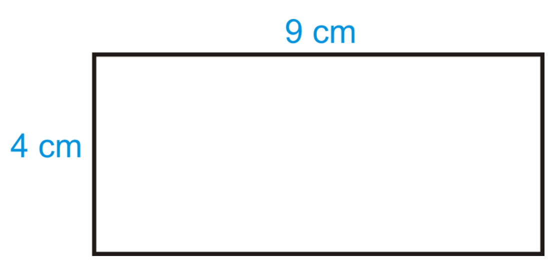
Example 2: Find the area of the figure from Example 1.

Solution: Count the number of squares within the figure. If we start on the left and count each column. $5+6+1+4+3+4+4 = 27 \ units^2$

Area of a Rectangle: $A = bh$, where b is the base (width) and h is the height (length).



Example 3: Find the area and perimeter of a rectangle with sides 4 cm by 9 cm.

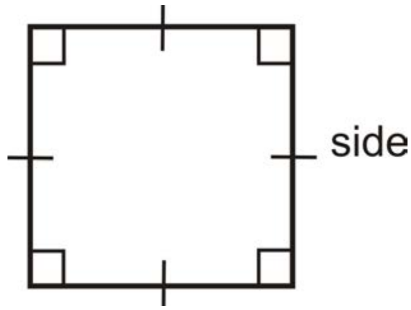


Solution: The perimeter is $4+9+4+9 = 26 \ cm$. The area is $A = 9 \cdot 4 = 36 \ cm^2$.

Perimeter of a Rectangle: $P = 2b+2h$.

If a rectangle is a square, with sides of length s, the formulas are as follows:

Perimeter of a Square: $P_{square} = 2s + 2s = 4s$



Area of a Square: $A_{sqaure} = s \cdot s = s^2$

Example 4: The area of a square is 75 in^2. Find the perimeter.

Solution: To find the perimeter, we need to find the length of the sides.

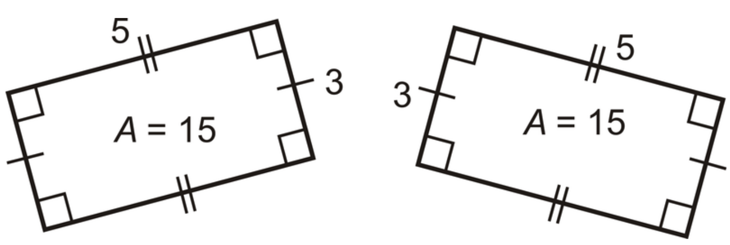
$$A = s^2 = 75\ in^2$$
$$s = \sqrt{75} = 5\sqrt{3}\ in$$

From this, $P = 4\left(5\sqrt{3}\right) = 20\sqrt{3}\ in$.

Area Postulates

Congruent Area Postulate

If two figures are congruent, then they have the same area



Example 5: Draw two different rectangles with an area of 36 cm^2.

Solution: Think of all the different factors of 36. These can all be dimensions of the different rectangles.

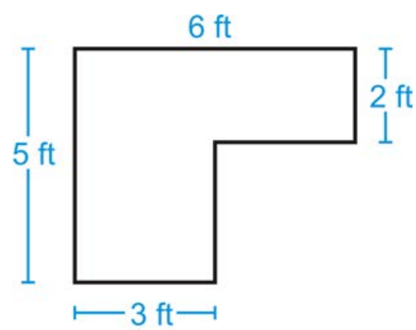
Other possibilities could be $6\times 6, 2\times 18$, and 1×36.

Example 5 shows two rectangles with the same area and are not congruent. This tells us that the converse of the Congruent Areas Postulate is not true.

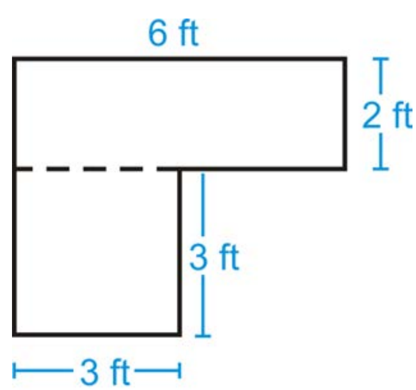
Area Addition Postulate

If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts

Example 6: Find the area of the figure below. You may assume all sides are perpendicular.



Solution: Split the shape into two rectangles and find the area of each.

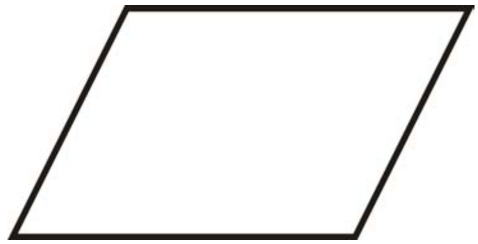


$$A_{top\ rectangle} = 6\cdot 2 = 12\ ft^2$$
$$A_{bottom\ square} = 3\cdot 3 = 9\ ft^2$$

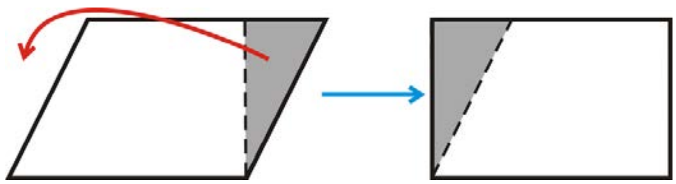
The total area is $12+9=21\ ft^2$.

Area of a Parallelogram

Recall that a parallelogram is a quadrilateral whose opposite sides are parallel.



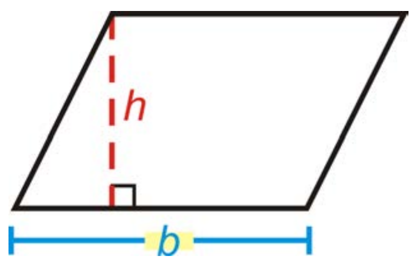
To find the area of a parallelogram, make it into a rectangle.



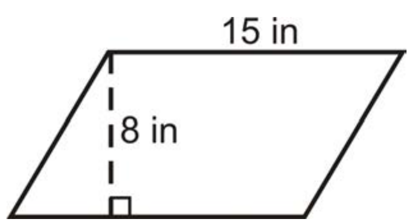
From this, we see that the area of a parallelogram is the same as the area of a rectangle.

Area of a Parallelogram: The area of a parallelogram is $A = bh$.

The height of a parallelogram is always perpendicular to the base. This means that the sides are ***not*** the height.



Example 7: Find the area of the parallelogram.



Solution: $A = 15 \cdot 8 = 120\ in^2$

Example 8: If the area of a parallelogram is 56 $units^2$ and the base is 4 units, what is the height?

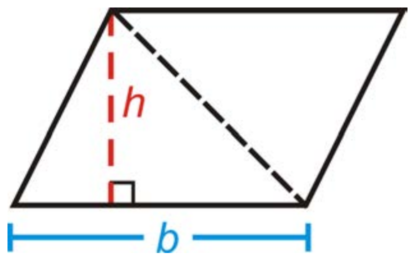
Solution: Solve for the height in $A = bh$.

$$\begin{aligned} 56 &= 4h \\ 14 &= h \end{aligned}$$

Watch This!!!

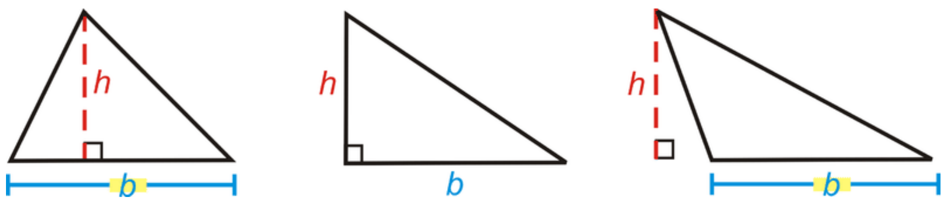
https://learnzillion.com/lessons/1058-find-the-area-of-a-parallelogram-by-decomposing

Area of a Triangle

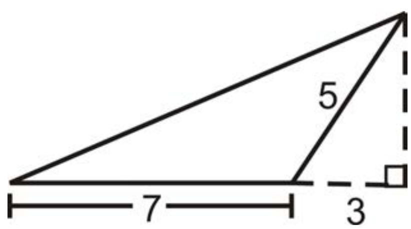


If we take parallelogram and cut it in half, along a diagonal, we would have two congruent triangles. The formula for the area of a triangle is half the area of a parallelogram.

Area of a Triangle: $A = \frac{1}{2}\, bh$ or $A = \frac{bh}{2}$.



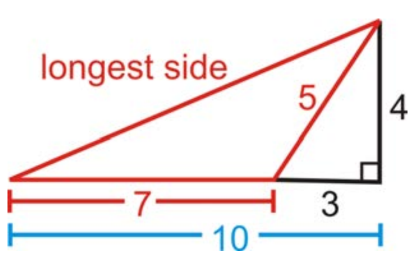
Example 9: Find the area of the triangle.



Solution: To find the area, we need to find the height of the triangle. We are given the two sides of the small right triangle, where the hypotenuse is also the short side of the obtuse triangle.

$$\begin{aligned}
3^2 + h^2 &= 5^2 \\
9 + h^2 &= 25 \\
h^2 &= 16 \\
h &= 4 \\
A &= \frac{1}{2}(4)(7) = 14\ units^2
\end{aligned}$$

Example 10: Find the perimeter of the triangle in Example 9.

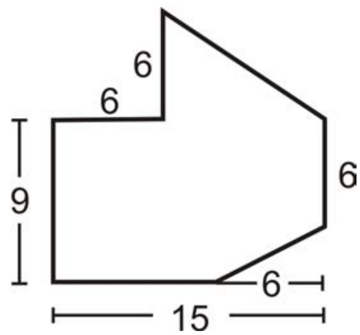


Solution: To find the perimeter, we need to find the longest side of the obtuse triangle. If we used the black lines in the picture, we would see that the longest side is also the hypotenuse of the right triangle with legs 4 and 10.

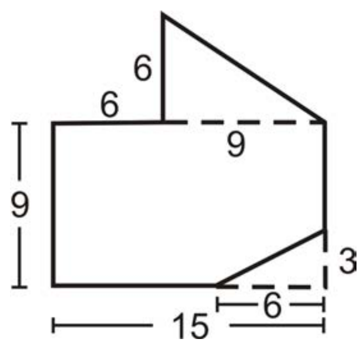
$$4^2 + 10^2 = c^2$$
$$16 + 100 = c^2$$
$$c = \sqrt{116} \approx 10.77$$

The perimeter is $7 + 5 + 10.77 = 22.77\ units$

Example 11: Find the area of the figure below.



Solution: Divide the figure into a triangle and a rectangle with a small rectangle cut out of the lower right-hand corner.



$$A = A_{top\ triangle} + A_{rectangle} - A_{small\ triangle}$$
$$A = \left(\frac{1}{2} \cdot 6 \cdot 9\right) + (9 \cdot 15) - \left(\frac{1}{2} \cdot 3 \cdot 6\right)$$
$$A = 27 + 135 - 9$$
$$A = 153\ units^2$$

Know What? Revisited The area of an Eastern King is 6080 in^2 and the California King is 6048 in^2.

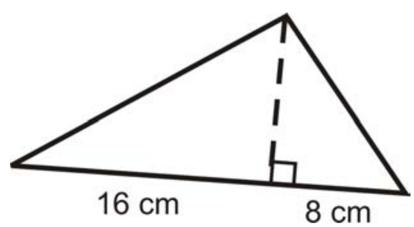
Practice Problems

- Questions 1-12 are similar to Examples 3-5, 7-9.

- Questions 13-18 are similar to Examples 9 and 10.
- Questions 19-24 are similar to Examples 7 and 9.
- Questions 25-30 are similar to Examples 6 and 11.
- Questions 31-36 use the formula for the area of a triangle.

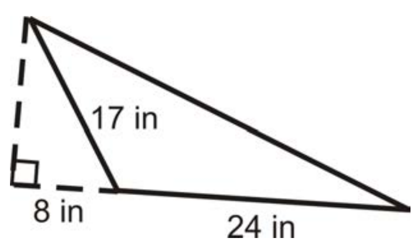
1. Find the area and perimeter of a square with sides of length 12 in.
2. Find the area and perimeter of a rectangle with height of 9 cm and base of 16 cm.
3. Find the area of a parallelogram with height of 20 m and base of 18 m.
4. Find the area and perimeter of a rectangle if the height is 8 and the base is 14.
5. Find the area and perimeter of a square if the sides are 18 ft.
6. If the area of a square is 81 ft^2, find the perimeter.
7. If the perimeter of a square is 24 in, find the area.
8. Find the area of a triangle with base of length 28 cm and height of 15 cm.
9. What is the height of a triangle with area 144 m^2 and a base of 24 m?
10. The perimeter of a rectangle is 32. Find two different dimensions that the rectangle could be.
11. Draw two different rectangles that haven an area of 90 mm^2.
12. Write the converse of the Congruent Areas Postulate. Determine if it is a true statement. If not, write a counterexample. If it is true, explain why.

Use the triangle to answer the following questions.



13. Find the height of the triangle by using the geometric mean.
14. Find the perimeter.
15. Find the area.

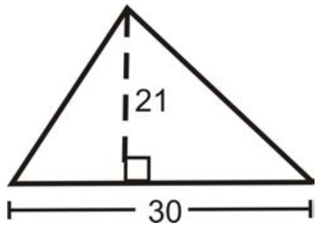
Use the triangle to answer the following questions.



16. Find the height of the triangle.
17. Find the perimeter.
18. Find the area.

Find the area of the following shapes.

19.



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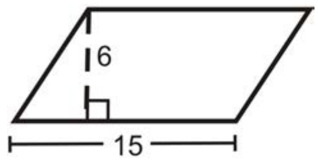

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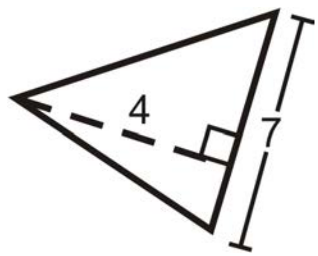

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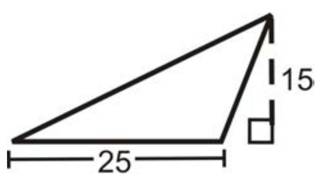

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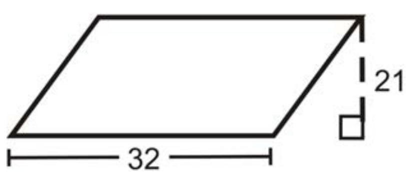
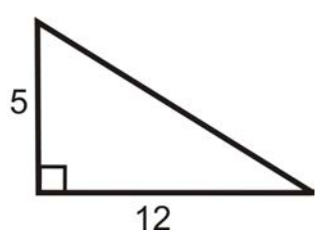
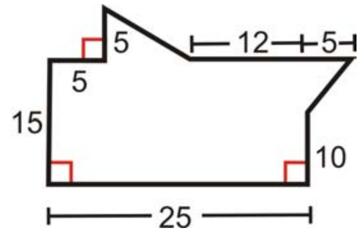


24.



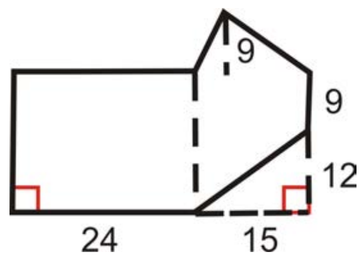
25. .25.

a. Divide the shape into two triangles and one rectangle.
b. Find the area of the two triangles and rectangle.
c. Find the area of the entire shape.

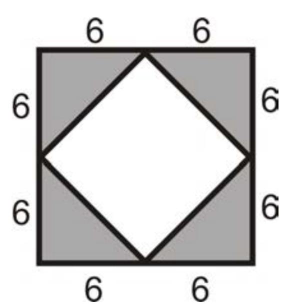


26.26.

A. a. Divide the shape into two rectangles and one triangle.
b. Find the area of the two rectangles and triangle.
c. Find the area of the entire shape (you will need to subtract the area of the small triangle in the lower right-hand corner).

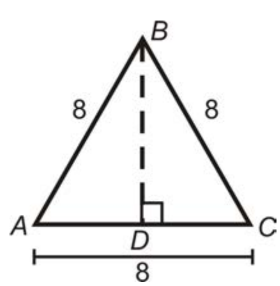


Use the picture below for questions 27-30. Both figures are squares.



27. Find the area of the outer square.
28. Find the area of one grey triangle.
29. Find the area of all four grey triangles.
30. Find the area of the inner square.

In questions 31-36 we are going to derive a formula for the area of an equilateral triangle.



31. What kind of triangle is $\triangle ABD$? Find AD and BD.
32. Find the area of $\triangle ABC$.
33. If each side is x, what is AD and BD?
34. If each side is x, find the area of $\triangle ABC$.
35. Using your formula from #34, find the area of an equilateral triangle with 12 inch sides.
36. Using your formula from #34, find the area of an equilateral triangle with 5 inch sides.

Review and Reflect

37. In your own words, explain why the area formula of a triangle is half that of a parallelogram.
38. Is the area of a parallelogram always larger than its perimeter? If you chose no provide a counterexample.
39. Given a parallelogram with no right angles and a rectangle that have the same area, will the rectangle *always*, *sometime* or *never* have the larger perimeter?

Warm-Up Answers

1. *Possible* Answers

Perimeter: The distance around a shape.

Area: The space inside a shape.

2. (a) $x = \pm 11$

(b) $x = 18.5$

(c) $x = 4, 2$

(d) $x = 16$

(e) $x = 3, -5$

(f) $x = 4, -3$

9.2 Trapezoids, Rhombi, and Kites

TEKS G(2)B, G(11)B

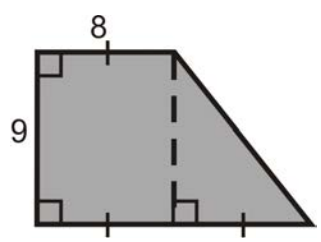
Learning Objectives

- Derive and use the area formulas for trapezoids, rhombi, and kites

Warm-Up

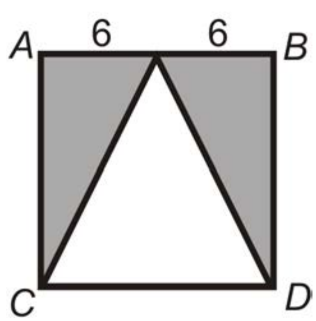
Find the area of the *shaded* regions in the figures below.

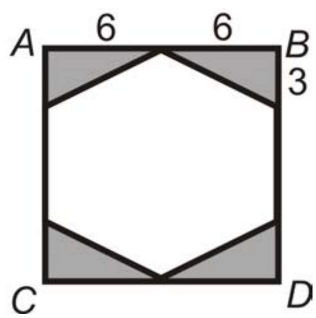
1.



2. *ABCD* is a square

.



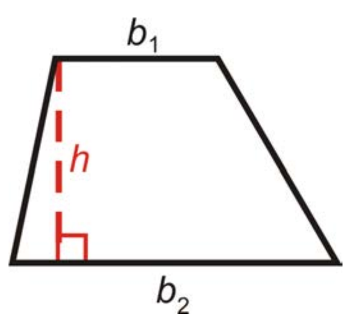


Know What? The Brazilian flag is to the right. The flag has dimensions of 20×14 (units vary depending on the size, so we will not use any here). The vertices of the yellow rhombus in the middle are 1.7 units from the midpoint of each side.

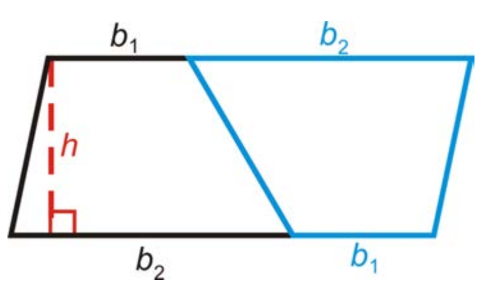
Find the area of the rhombus (including the circle). *Do not round your answer.*

Area of a Trapezoid

Recall that a trapezoid is a quadrilateral with one pair of parallel sides. The lengths of the parallel sides are the bases and the perpendicular distance between the parallel sides is the height of the trapezoid.



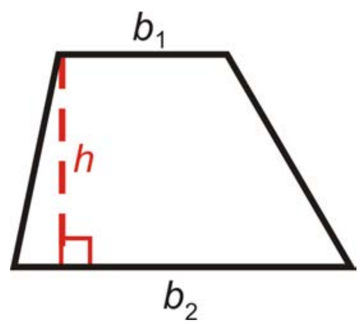
To find the area of the trapezoid, make a copy of the trapezoid and then rotate the copy 180°. Now, this is a parallelogram with height h and base $b_1 + b_2$. The area of this shape is $A = h(b_1 + b_2)$.



Because the area of this parallelogram is two congruent trapezoids, the area of one trapezoid would be $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$

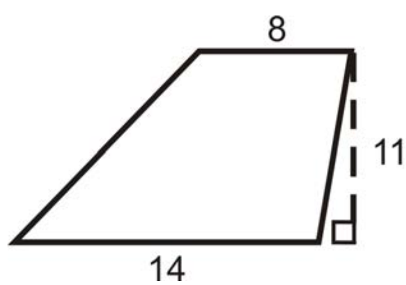
h is ***always*** perpendicular to the bases.



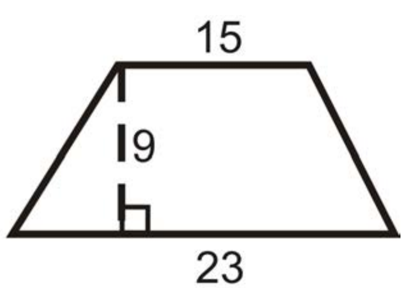
You could also say the area of a trapezoid is the average of the bases times the height.

Example 1: Find the area of the trapezoids below.

a)



b)

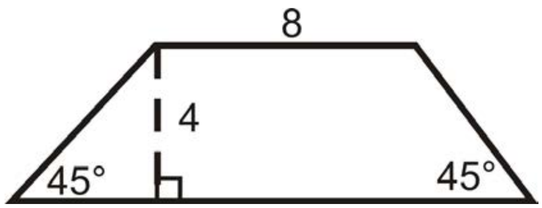


Solution:

a) $A = \frac{1}{2}(11)(14+8)$
$A = \frac{1}{2}(11)(22)$
$A = 121 \ units^2$

b) $A = \frac{1}{2}(9)(15+23)$
$A = \frac{1}{2}(9)(38)$
$A = 171 \ units^2$

Example 2: Find the perimeter and area of the trapezoid.



Solution: Even though we are not told the length of the second base, we can find it using special right triangles. Both triangles at the ends of this trapezoid are isosceles right triangles, so the hypotenuses are $4\sqrt{2}$ and the other legs are of length 4.

$$P = 8 + 4\sqrt{2} + 16 + 4\sqrt{2} \qquad A = \frac{1}{2}(4)(8+16)$$
$$P = 24 + 8\sqrt{2} \approx 35.3 \text{ } units \qquad A = 48 \text{ } units^2$$

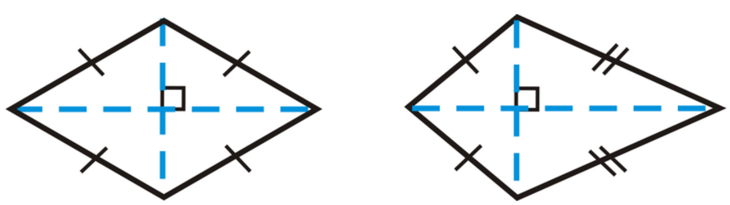
Watch This!!!

https://youtu.be/yTnYRpcZA9c

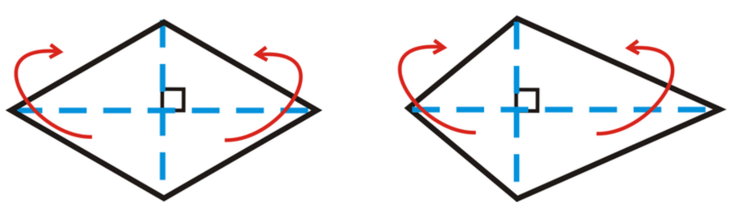
Area of a Rhombus and Kite

Recall that a rhombus is an equilateral quadrilateral and a kite has adjacent congruent sides.

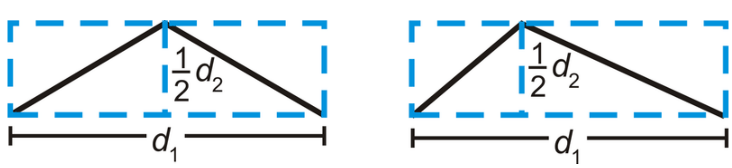
Both of these quadrilaterals have perpendicular diagonals, which is how we are going to find their areas.



Notice that the diagonals divide each quadrilateral into 4 triangles. If we move the two triangles on the bottom of each quadrilateral so that they match up with the triangles above the horizontal diagonal, we would have two rectangles.

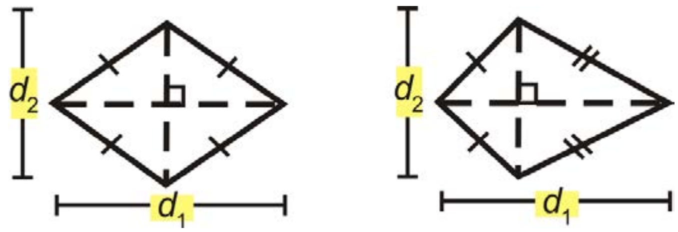


So, the height of these rectangles is half of one of the diagonals and the base is the length of the other diagonal.



Area of a Rhombus: $A = \frac{1}{2}d_1d_2$

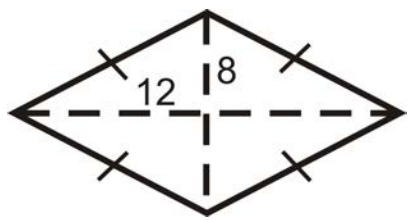
The area is half the product of the diagonals.



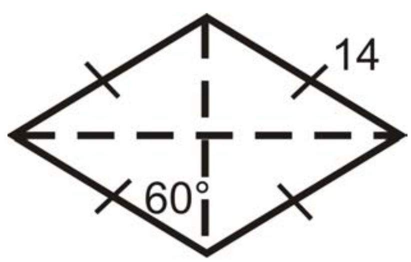
Area of a Kite: $A = \frac{1}{2}d_1d_2$

Example 3: Find the perimeter and area of the rhombi below.

a)



b)



Solution: In a rhombus, all four triangles created by the diagonals are congruent.

a) To find the perimeter, you must find the length of each side, which would be the hypotenuse of one of the four triangles. Use the Pythagorean Theorem.

$$\begin{aligned} 12^2 + 8^2 &= side^2 \\ 144 + 64 &= side^2 \\ side &= \sqrt{208} = 4\sqrt{13} \\ P &= 4\left(4\sqrt{13}\right) = 16\sqrt{13} \end{aligned}$$

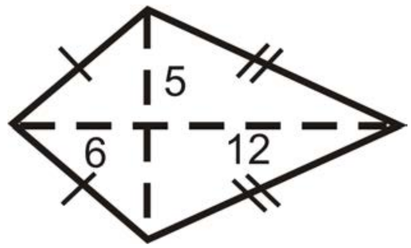
$$\begin{aligned} A &= \frac{1}{2} \cdot 16 \cdot 24 \\ A &= 192 \end{aligned}$$

b) Here, each triangle is a 30-60-90 triangle with a hypotenuse of 14. From the special right triangle ratios the short leg is 7 and the long leg is $7\sqrt{3}$.

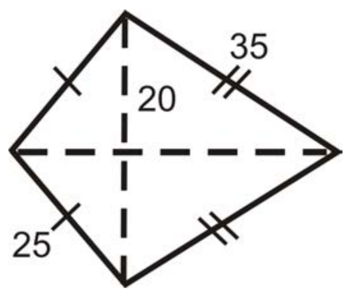
$$P = 4 \cdot 14 = 56 \qquad A = \frac{1}{2} \cdot 14 \cdot 14\sqrt{3} = 98\sqrt{3}$$

Example 4: Find the perimeter and area of the kites below.

a)



b)



Solution: In a kite, there are two pairs of congruent triangles. Use the Pythagorean Theorem in both problems to find the length of sides or diagonals.

a)

Shorter sides of kite

$$\begin{aligned} 6^2+5^2 &= s_1^2 \\ 36+25 &= s_1^2 \\ s_1 &= \sqrt{61} \end{aligned}$$

Longer sides of kite

$$\begin{aligned} 12^2+5^2 &= s_2^2 \\ 144+25 &= s_2^2 \\ s_2 &= \sqrt{169}=13 \end{aligned}$$

$$P=2\left(\sqrt{61}\right)+2(13)=2\sqrt{61}+26\approx 41.6 \qquad A=\frac{1}{2}(10)(18)=90$$

b)

Smaller diagonal portion

$$\begin{aligned} 20^2+d_s^2 &= 25^2 \\ d_s^2 &= 225 \\ d_s &= 15 \end{aligned}$$

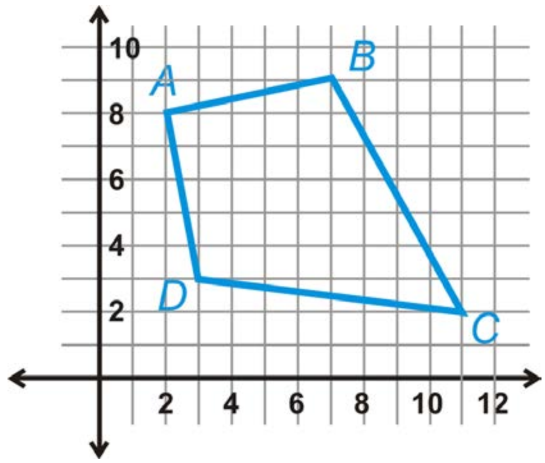
Larger diagonal portion

$$\begin{aligned} 20^2+d_l^2 &= 35^2 \\ d_l^2 &= 825 \\ d_l &= 5\sqrt{33} \end{aligned}$$

$$A=\frac{1}{2}\left(15+5\sqrt{33}\right)(40)\approx 874.5 \qquad P=2(25)+2(35)=120$$

Example 5: The vertices of a quadrilateral are $A(2,8), B(7,9), C(11,2)$, and $D(3,3)$. Show $ABCD$ is a kite and find its area.

Solution: After plotting the points, it looks like a kite. $AB=AD$ and $BC=DC$. The diagonals are perpendicular if the slopes are opposite signs and flipped.



$$m_{AC} = \frac{2-8}{11-2} = -\frac{6}{9} = -\frac{2}{3}$$
$$m_{BD} = \frac{9-3}{7-3} = \frac{6}{4} = \frac{3}{2}$$

The diagonals are perpendicular, so $ABCD$ is a kite. To find the area, we need to find the length of the diagonals.

$$\begin{aligned} d_1 &= \sqrt{(2-11)^2+(8-2)^2} \\ &= \sqrt{(-9)^2+6^2} \\ &= \sqrt{81+36} = \sqrt{117} = 3\sqrt{13} \end{aligned} \qquad \begin{aligned} d_2 &= \sqrt{(7-3)^2+(9-3)^2} \\ &= \sqrt{4^2+6^2} \\ &= \sqrt{16+36} = \sqrt{52} = 2\sqrt{13} \end{aligned}$$

Plug these lengths into the area formula for a kite. $A = \frac{1}{2}\left(3\sqrt{13}\right)\left(2\sqrt{13}\right) = 39\ units^2$

Know What? Revisited To find the area of the rhombus, we need to find the length of the diagonals. One diagonal is $20-1.7-1.7=16.6$ and the other is $14-1.7-1.7=10.6$. The area is $A=\frac{1}{2}(16.6)(10.6)=87.98\ units^2$.

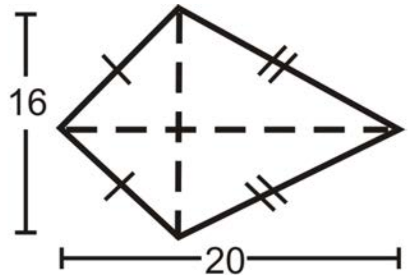
Practice Problems

- Question 1 uses the formula of the area of a kite and rhombus.
- Questions 2-16 are similar to Examples 1-4.
- Questions 17-23 are similar to Example 5.
- Questions 24-27 use the area formula for a kite and rhombus and factors.
- Questions 28-30 are similar to Example 4.

1. Do you think all rhombi and kites with the same diagonal lengths have the same area? *Explain* your answer.

Find the area of the following shapes. *Round your answers to the nearest hundredth.*

2.



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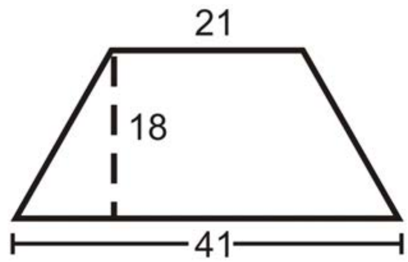

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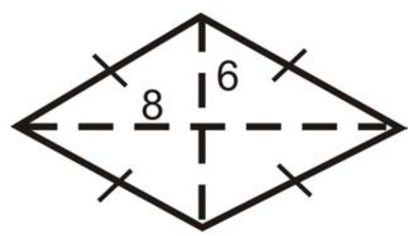
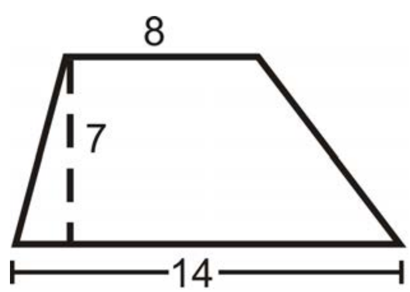


5.



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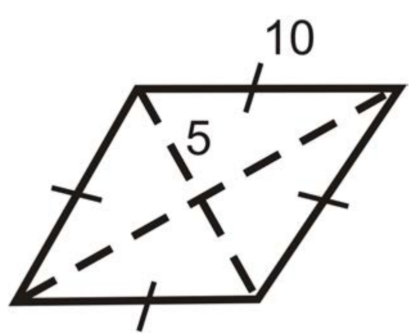

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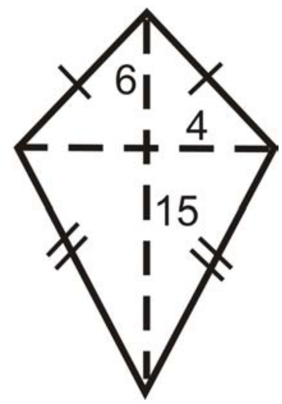

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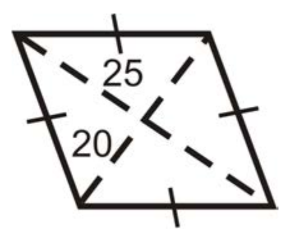

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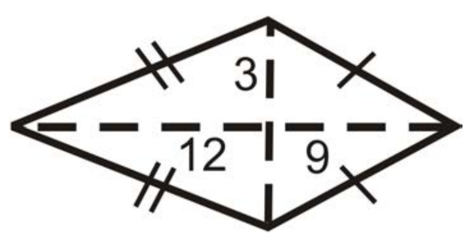

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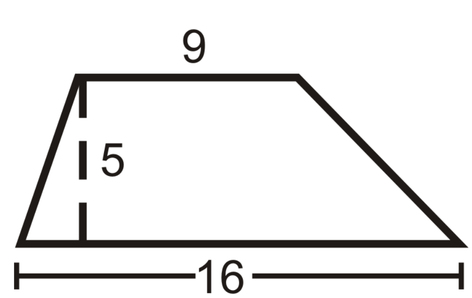
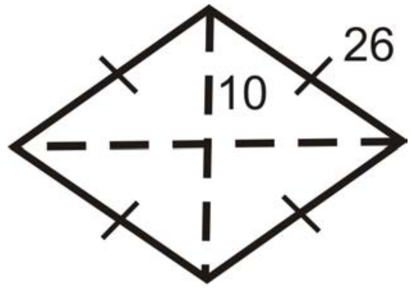


Find the area and perimeter of the following shapes. *Round your answers to the nearest hundredth.*

11.



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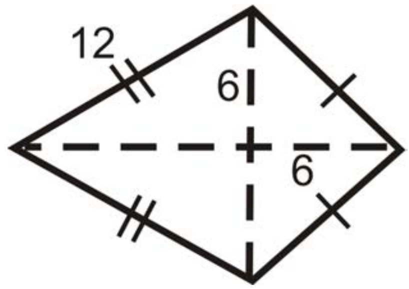

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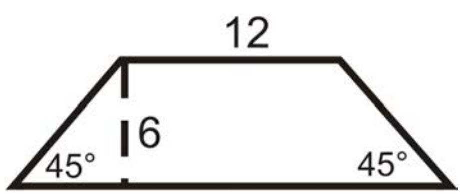

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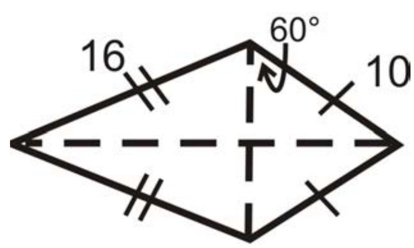

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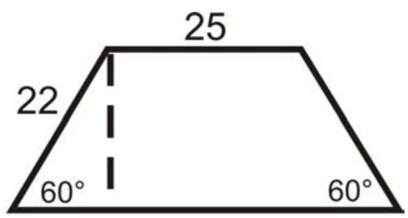
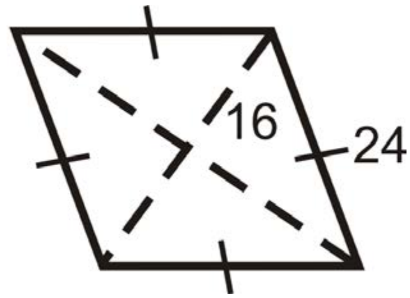


16.



Quadrilateral $ABCD$ has vertices $A(-2,0), B(0,2), C(4,2)$, and $D(0,-2)$. Leave your answers in simplest radical form.

17. Find the slopes of $\overline{AB}$ and $\overline{DC}$. What type of quadrilateral is this? *Plotting the points will help you find the answer.*
18. Find the slope of $\overline{AD}$. Is it perpendicular to $\overline{AB}$ and $\overline{DC}$?
19. Find AB, AD, and DC.
20. Use #19 to find the area of the shape.

Quadrilateral $EFGH$ has vertices $E(2,-1), F(6,-4), G(2,-7)$, and $H(-2,-4)$.

21. Find the slopes of all the sides and diagonals. What type of quadrilateral is this? *Plotting the points will help you find the answer.*
22. Find HF and EG.
23. Use #22 to find the area of the shape.

For Questions 24 and 25, the area of a rhombus is 32 $units^2$.

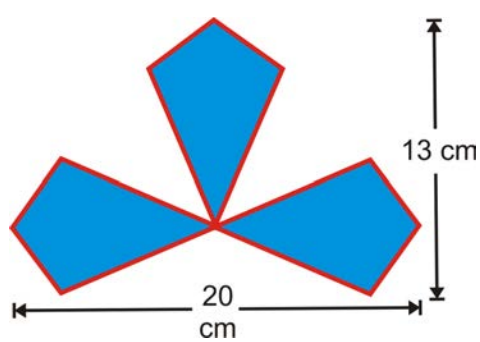
24. What would the product of the diagonals have to be for the area to be 32 $units^2$?
25. List two possibilities for the length of the diagonals, based on your answer from #24.

For Questions 26 and 27, the area of a kite is 54 $units^2$.

26. What would the product of the diagonals have to be for the area to be 54 $units^2$?
27. List two possibilities for the length of the diagonals, based on your answer from #26.

Sherry designed the logo for a new company, made up of 3 congruent kites.

28. What are the lengths of the diagonals for one kite?
29. Find the area of one kite.
30. Find the area of the entire logo.



Review and Reflect

31. Can the area of a rhombus formula be used for any other quadrilaterals?
32. If you knew all of the sides length would that be enough information to find the area of a trapezoid? Rhombus? Kite?

Warm-Up Answers

1. $A = 9(8) + \left[\frac{1}{2}(9)(8)\right] = 72 + 36 = 108\ units^2$
2. $A = \frac{1}{2}(6)(12)2 = 72\ units^2$
3. $A = 4\left[\frac{1}{2}(6)(3)\right] = 36\ units^2$

9.3 Area of Regular Polygons

Here you'll learn how to calculate the area and perimeter of a regular polygon.

TEKS G(11)A

Learning Objectives

- Derive and use the area of regular polygon formula

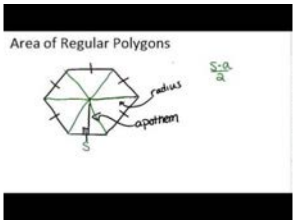
Vocabulary

- regular polygon
- center
- radius
- apothem

Warm-Up

What if you were asked to find the distance across The Pentagon in Arlington, VA? The Pentagon, which also houses the Department of Defense, is composed of two regular pentagons with the same center. The entire area of the building is 29 acres (40,000 square feet in an acre), with an additional 5 acre courtyard in the center. The length of each outer wall is 921 feet. What is the total distance across the pentagon? Round your answer to the nearest hundredth.

Watch This



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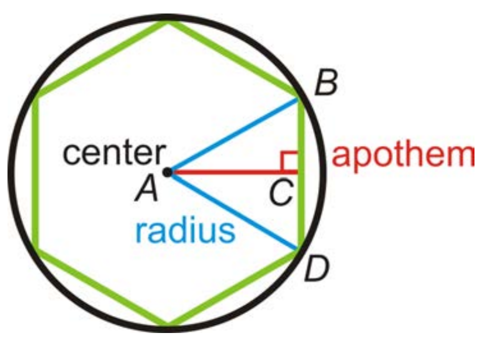
CK-12 Foundation: Chapter10AreaofRegularPolygonsA

Learn more about the area of regular polygons by watching the video at this link.

A **regular polygon** is a polygon with congruent sides and angles. Recall that the perimeter of a square is 4 times the length of a side because each side is congruent. We can extend this concept to any regular polygon.

Perimeter of a Regular Polygon: If the length of a side is s and there are n sides in a regular polygon, then the perimeter is $P = ns$.

In order to find the area of a regular polygon, we need to define some new terminology. First, all regular polygons can be inscribed in a circle. So, regular polygons have a **center** and **radius**, which are the center and radius of the circumscribed circle. Also like a circle, a regular polygon will have a central angle formed. In a regular polygon, however, the central angle is the angle formed by two radii drawn to consecutive vertices of the polygon. In the picture below, the central angle is $\angle BAD$. Also, notice that $\triangle BAD$ is an isosceles triangle. Every regular polygon with n sides is formed by n isosceles triangles. The height of these isosceles triangles is called the **apothem**.

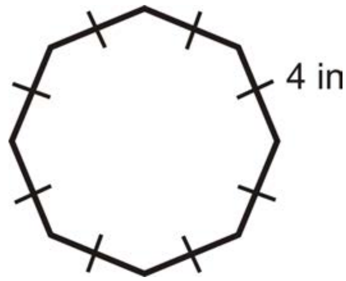


The area of each triangle is $A_\triangle = \frac{1}{2}bh = \frac{1}{2}sa$, where s is the length of a side and a is the apothem. If there are n sides in the regular polygon, then it is made up of n congruent triangles.

Area of a Regular Polygon: If there are n sides with length s in a regular polygon and a is the apothem, then $A = \frac{1}{2}asn$ or $A = \frac{1}{2}aP$, where P is the perimeter.

Example 1: What is the perimeter of a regular octagon with 4 inch sides?

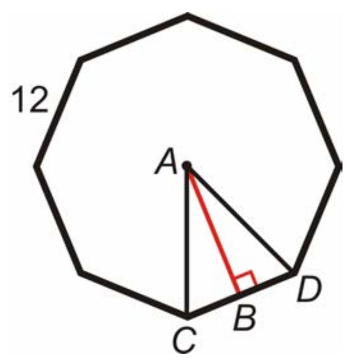
Solution: If each side is 4 inches and there are 8 sides, that means the perimeter is 8(4 in) = 32 inches.



Example 2: The perimeter of a regular heptagon is 35 cm. What is the length of each side?

Solution: If $P = ns$, then $35\ cm = 7s$. Therefore, $s = 5\ cm$.

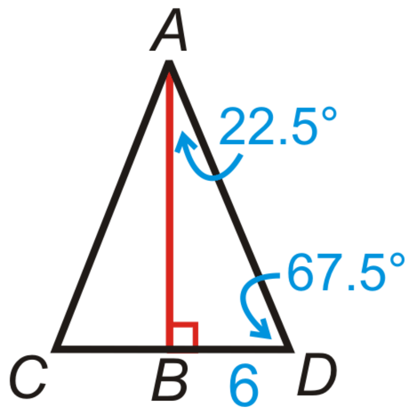
Example 3: Find the length of the apothem in the regular octagon. Round your answer to the nearest hundredth.



Solution: To find the length of the apothem, AB, you will need to use the trig ratios. First, find $m\angle CAD$. There are 360° around a point, so $m\angle CAD = \frac{360^\circ}{8} = 45^\circ$. Now, we can use this to find the other two angles in $\triangle CAD$. $m\angle ACB$ and $m\angle ADC$ are equal because $\triangle CAD$ is a right triangle.

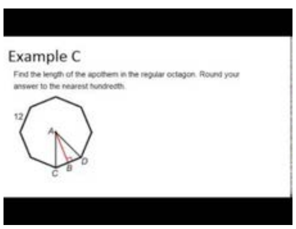
$$\begin{aligned} m\angle CAD + m\angle ACB + m\angle ADC &= 180^\circ \\ 45^\circ + 2m\angle ACB &= 180^\circ \\ 2m\angle ACB &= 135^\circ \\ m\angle ACB &= 67.5^\circ \end{aligned}$$

To find AB, we must use the tangent ratio. You can use either acute angle.



$$\tan 67.5^\circ = \frac{AB}{6}$$
$$AB = 6 \cdot \tan 67.5^\circ \approx 14.49$$

Watch this video for help with the Examples above.



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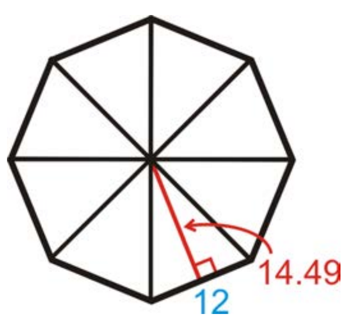
CK-12 Foundation: Chapter10AreaofRegularPolygonsB

Example 4: Find the area of the regular octagon in Example 3.

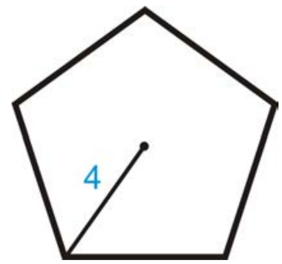


Solution:

Recall the formula for finding area of a regular polygon is $A = \frac{1}{2}aP$. Therefor, with a side length of 12 units and an apothem of 14.49 units the area of this regular polygon is

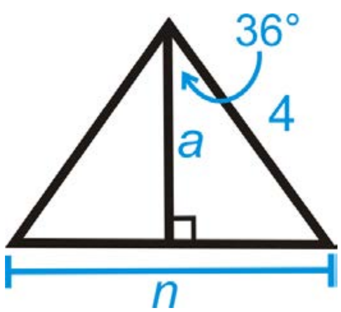
$$A_{octagon} = \left(\frac{1}{2} \cdot 96 \cdot 14.49\right) = 695.52 \ units^2$$

Example 5: Find the area of the regular polygon with radius 4.



Solution:

In this problem we need to find the apothem and the length of the side before we can find the area of the entire polygon. Each central angle for a regular pentagon is $\frac{360^\circ}{5} = 72^\circ$. So, half of that, to make a right triangle with the apothem, is 36°. We need to use sine and cosine. Notice the length of the side of the polygon is n, when using trig to find the parts needed we will use $.5n$ (half of the side length) for each ratio. That way, once we find n, we can multiply by the number of sides to find the perimeter.



$$\begin{aligned} \sin 36^\circ &= \frac{.5n}{4} \\ 4\sin 36^\circ &= \frac{1}{2}n \\ 8\sin 36^\circ &= n \\ n &\approx 4.7 \end{aligned} \qquad\qquad \begin{aligned} \cos 36^\circ &= \frac{a}{4} \\ 4\cos 36^\circ &= a \\ a &\approx 3.24 \end{aligned}$$

Using these two pieces of information, we can now find the area. $A = \frac{1}{2}(3.24)(23.5) \approx 38.07\ units^2$

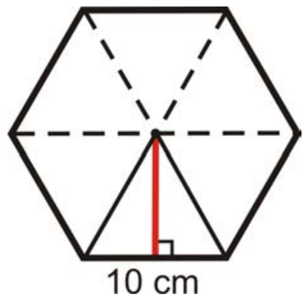
Example 6: The area of a regular hexagon is $54\sqrt{3}$ and the perimeter is 36. Find the length of the sides and the apothem.

Solution: Remeber a hexagon is a 6 sided figure. Plug in what you know into both the area and perimeter formulas to solve for the length of the side and the apothem.

$$\begin{aligned} P &= sn \\ 36 &= 6s \\ s &= 6 \end{aligned} \qquad\qquad \begin{aligned} A &= \frac{1}{2}aP \\ 54\sqrt{3} &= \frac{1}{2}a(36) \\ 54\sqrt{3} &= 18a \\ 3\sqrt{3} &= a \end{aligned}$$

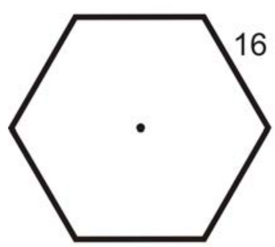
Practice Problems

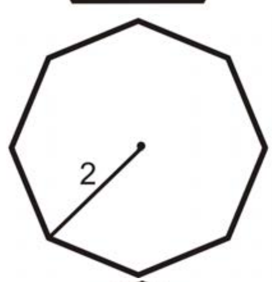
Use the regular hexagon below to answer the following questions. Each side is 10 cm long.

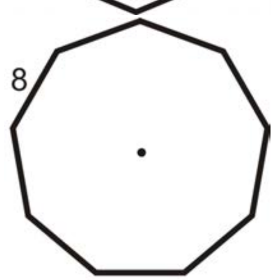


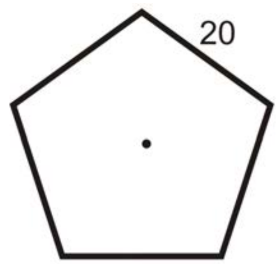
1. Each dashed line segment is $a(n)$ ________________.
2. The red line segment is $a(n)$ __________________.
3. There are _____ congruent triangles in a regular hexagon.
4. In a regular hexagon, all the triangles are _________________.
5. Find the radius of this hexagon.
6. Find the apothem.
7. Find the perimeter.
8. Find the area.

Find the area and perimeter of each of the following regular polygons. Round your answer to the nearest hundredth.

9. 

10. 

11. 

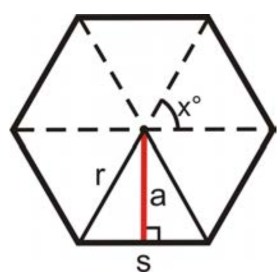


12.

13. If the perimeter of a regular decagon is 65, what is the length of each side?
14. A regular polygon has a perimeter of 132 and the sides are 11 units long. How many sides does the polygon have?
15. The area of a regular pentagon is 440.44 in^2 and the perimeter is 80 in. Find the length of the apothem of the pentagon.
16. The area of a regular octagon is 695.3 cm^2 and the sides are 12 cm. What is the length of the apothem?

A regular 20-gon and a regular 40-gon are inscribed in a circle with a radius of 15 units.

17. ***Challenge*** Derive a formula for the area of a regular hexagon with sides of length s. Your only variable will be s. HINT: Use 30-60-90 triangle ratios.
18. ***Challenge*** in the following steps you will derive an alternate formula for finding the area of a regular polygon with n sides.



We are going to start by thinking of a polygon with n sides as n congruent isosceles triangles. We will find the sum of the areas of these triangles using trigonometry. First, the area of a triangle is $\frac{1}{2}bh$. In the diagram to the right, this area formula would be $\frac{1}{2}sa$, where s is the length of a side and a is the length of the apothem. In the diagram, $x$represents the measure of the vertex angle of each isosceles triangle.

1. The apothem, a, divides the triangle into two congruent right triangles. The top angle in each is $\frac{x^\circ}{2}$. Find $\sin\left(\frac{x^\circ}{2}\right)$ and $\cos\left(\frac{x^\circ}{2}\right)$.
2. Solve your sin equation to find an expression for s in terms of r and x.
3. Solve your cos equation to find an expression for a in terms of r and x.
4. Substitute these expressions into the equation for the area of one of the triangles, $\frac{1}{2}sa$.
5. Since there will be n triangles in an n-gon, you need to multiply your expression from part d by n to get the total area.
6. How would you tell someone to find the value of x for a regular n-gon?

Use the formula you derived in problem 18 to find the area of the regular polygons described in problems 19-22. Round your answers)to the nearest hundredth.

19. Decagon with radius 12 cm.
20. 20-gon with radius 5 in.
21. 15-gon with radius length 8 cm.
22. 45-gon with radius length 7 in.

Review and Reflect

23. Which regular polygons would allow the use of special right triangles. (30°-60°-90° or 45°-45°-90°) Explain your reasoning.

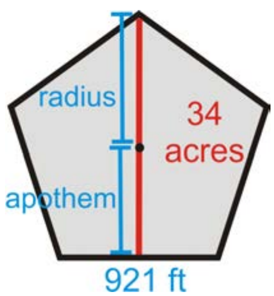
24. Describe the process of finding the area of a stop sign.

Warm-up Answer

From the picture below, we can see that the total distance across the Pentagon is the length of the apothem plus the length of the radius. If the total area of the Pentagon is 34 acres, that is 2,720,000 square feet. Therefore, the area equation is $2720000 = \frac{1}{2}a(921)(5)$ and the apothem is 590.66 ft. To find the radius, we can either use the Pythagorean Theorem, with the apothem and half the length of a side or the sine ratio. Recall from Example 5, that each central angle in a pentagon is 72°, so we would use half of that for the right triangle.

$$\sin 36^\circ = \frac{460.5}{r} \rightarrow r = \frac{460.5}{\sin 36^\circ} \approx 783.45\ ft.$$

Therefore, the total distance across is $590.66 + 783.45 = 1374.11\ ft.$



9.4 Areas of Similar Polygons

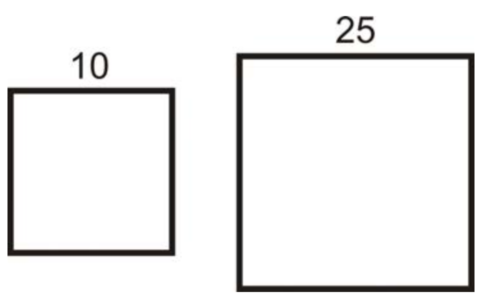
TEKS G(10)B

Learning Objectives

- Understand the relationship between the scale factor of similar polygons and their areas
- Apply scale factors to solve problems about areas of similar polygons

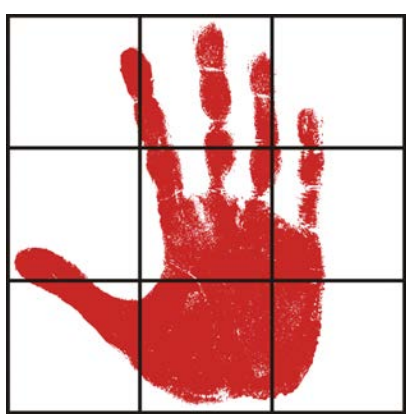
Warm-Up

1. Are two squares similar? Are two rectangles?



2. Find the scale factor of the sides of the similar shapes. Both figures are squares.
3. Find the area of each square.
4. Find the ratio of the smaller square's area to the larger square's area. Reduce it.

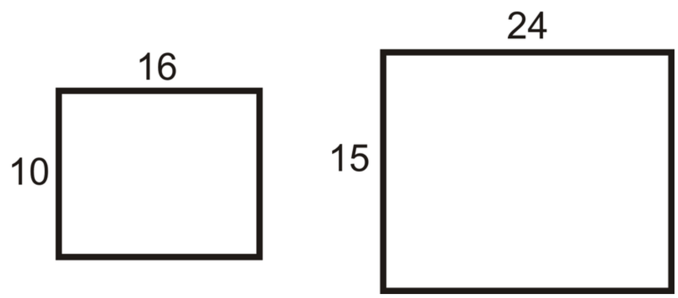
Know What? One use of scale factors and areas is scale drawings. This technique takes a small object, like the handprint to the right, divides it up into smaller squares and then blows up the individual squares. In this Know What? you are going to make a scale drawing of your own hand. Trace your hand on a piece of paper. Then, divide your hand into 9 squares, like the one to the right, 2 $in \times 2$ in. Take a larger piece of paper and blow up each square to be 6 $in \times 6$ in (you will need at least an 18 in square piece of paper). Once you have your 6 $in \times 6$ in squares drawn, use the proportions and area to draw in your enlarged handprint.



Areas of Similar Polygons

In Chapter 7, we learned about similar polygons. Polygons are similar when the corresponding angles are equal and the corresponding sides are in the same proportion.

Example 1: The two rectangles below are similar. Find the scale factor and the ratio of the perimeters.



Solution: The scale factor is $\frac{16}{24} = \frac{2}{3}$.

$$P_{small} = 2(10) + 2(16) = 52 \ units$$
$$P_{large} = 2(15) + 2(24) = 78 \ units$$

The ratio of the perimeters is $\frac{52}{78} = \frac{2}{3}$.

The ratio of the perimeters is the same as the scale factor. In fact, the ratio of any part of two similar shapes (diagonals, medians, midsegments, altitudes, etc.) is the same as the scale factor.

Example 2: Find the area of each rectangle from Example 1. Then, find the ratio of the areas.

Solution:

$$A_{small} = 10 \cdot 16 = 160 \ units^2$$
$$A_{large} = 15 \cdot 24 = 360 \ units^2$$

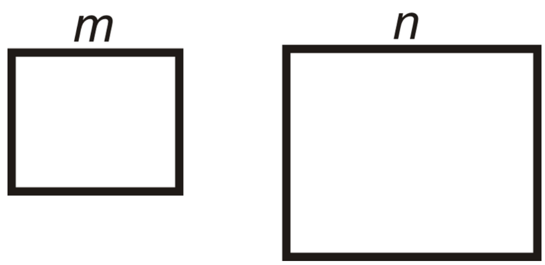
The ratio of the areas would be $\frac{160}{360} = \frac{4}{9}$.

The ratio of the sides, or scale factor was $\frac{2}{3}$ and the ratio of the areas is $\frac{4}{9}$. Notice that the ratio of the areas is the ***square*** of the scale factor.

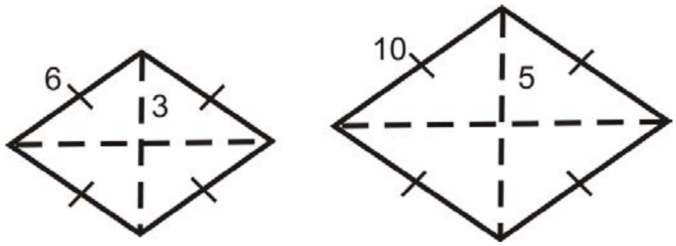
Area of Similar Polygons Theorem

If the scale factor of the sides of two similar polygons is $\frac{m}{n}$, then the ratios of areas would be $\left(\frac{m}{n}\right)^2$

If the scale factor is $\frac{m}{n}$, then the ratio of the areas is $\left(\frac{m}{n}\right)^2$.



Example 3: Find the ratio of the areas of the rhombi below. The rhombi are similar.



Solution: Find the ratio of the sides and square it.

$$\left(\frac{3}{5}\right)^2 = \frac{9}{25}$$

Watch This!!!

http://youtu.be/g6nCK4dlqpo

Example 4: Two trapezoids are similar. If the scale factor is $\frac{3}{4}$ and the area of the smaller trapezoid is 81 cm^2, what is the area of the larger trapezoid?

Solution: First, the ratio of the areas would be $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. Now, we need the area of the larger trapezoid. To find this, set up a proportion using the area ratio.

$$\frac{9}{16} = \frac{81}{A} \rightarrow 9A = 1296$$
$$A = 144\ cm^2$$

Example 5: Two triangles are similar. The ratio of the areas is $\frac{25}{64}$. What is the scale factor?

Solution: The scale factor is $\sqrt{\frac{25}{64}} = \frac{5}{8}$.

Example 6: Using the ratios from Example 5, find the length of the base of the smaller triangle if the length of the base of the larger triangle is 24 units.

Solution: Set up a proportion using the scale factor.

$$\frac{5}{8} = \frac{b}{24} \rightarrow 8b = 120$$
$$b = 15\ units$$

Know What? Revisited You should end up with an 18 *in* × 18 *in* drawing of your handprint.

Practice Problems

- Questions 1-4 are similar to Example 3.
- Questions 5-8 are similar to Example 5.
- Questions 9-18 are similar to Examples 1-3, and 5.
- Questions 19-22 are similar to Examples 4 and 6.
- Questions 23-26 are similar to Examples 5 and 6.

Determine the ratio of the areas, given the ratio of the sides of a polygon.

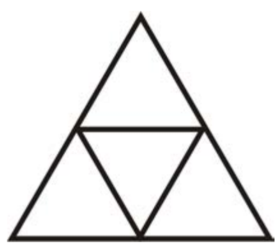
1. $\frac{3}{5}$
2. $\frac{1}{4}$
3. $\frac{7}{2}$
4. $\frac{6}{11}$

Determine the ratio of the sides of a polygon, given the ratio of the areas.

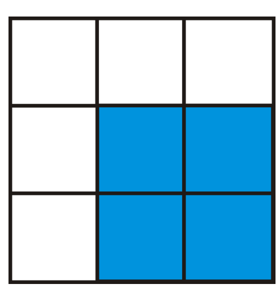
5. $\frac{1}{36}$
6. $\frac{4}{81}$
7. $\frac{49}{9}$
8. $\frac{25}{144}$

This is an equilateral triangle made up of 4 congruent equilateral triangles.

9. What is the ratio of the areas of the large triangle to one of the small triangles?



10. What is the scale factor of large to small triangle?
11. If the area of the large triangle is 20 $units^2$, what is the area of a small triangle?
12. If the length of the altitude of a small triangle is $2\sqrt{3}$, find the perimeter of the large triangle.



13. Find the perimeter of the large square and the blue square.
14. Find the scale factor of the blue square and large square.
15. Find the ratio of their perimeters.
16. Find the area of the blue and large squares.
17. Find the ratio of their areas.
18. Find the length of the diagonals of the blue and large squares. Put them into a ratio. Which ratio is this the same as?
19. Two rectangles are similar with a scale factor of $\frac{4}{7}$. If the area of the larger rectangle is 294 in^2, find the area of the smaller rectangle.
20. Two triangles are similar with a scale factor of $\frac{1}{3}$. If the area of the smaller triangle is 22 ft^2, find the area of the larger triangle.
21. The ratio of the areas of two similar squares is $\frac{16}{81}$. If the length of a side of the smaller square is 24 units, find the length of a side in the larger square.
22. The ratio of the areas of two right triangles is $\frac{4}{9}$. If the length of the hypotenuse of the larger triangle is 48 units, find the length of the smaller triangle's hypotenuse.

Questions 23-26 build off of each other. You may assume the problems are connected.

23. Two similar rhombi have areas of 72 $units^2$ and 162 $units^2$. Find the ratio of the areas.
24. Find the scale factor.
25. The diagonals in these rhombi are congruent. Find the length of the diagonals and the sides.
26. What type of rhombi are these quadrilaterals?

Review and Reflect

27. If you want to double the area of a given rectangle by what scale factor should you multiple the sides by? Triple?
28. In your own words, explain the process involved in finding the area of an enlarged polygon if you are given the scale factor and the area of the original polygon?
29. What is the difference between two similar figures and two congruent figures?

Warm-Up Answers

1. Two squares are always similar. Two rectangles can be similar as long as the sides are in the same proportion.
2. $\frac{10}{25} = \frac{2}{5}$
3. $A_{small} = 100, A_{large} = 625$

9.5 Circumference and Arc Length

TEKS G(5)A, G(11)B, G(12)B

Learning Objectives

- Find the circumference of a circle
- Define the length of an arc and find arc length

Vocabulary

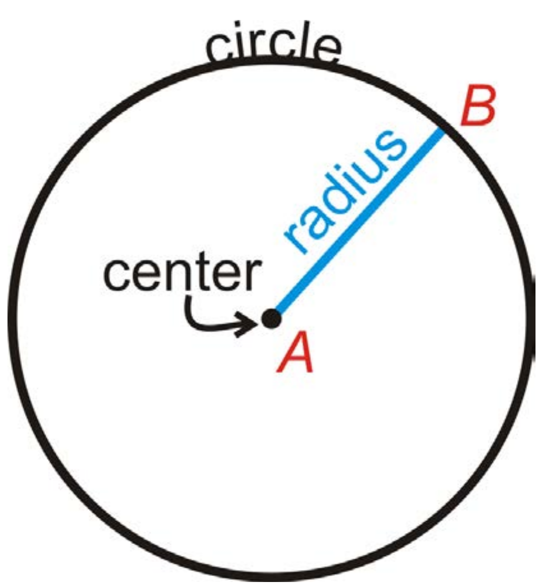
- Radius
- Diameter
- Circumference
- pi
- Arc Length
- Radian

Defining Terms

Circle: The set of all points that are the same distance away from a specific point, called the ***center***.

The center of the circle is point A. We call this circle, "circle A," and it is labeled $\odot A$.

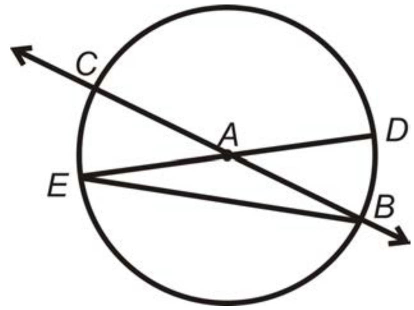
Radii (the plural of radius) are line segments. There are infinitely many radii in any circle and they are all equal.



Radius: The distance from the center to the circle.

Warm-Up

1. Find a central angle in that intercepts $\widehat{CE}$



2. Find an inscribed angle that intercepts $\widehat{CE}$.
3. How many degrees are in a circle? Find $m\widehat{ECD}$.
4. If $m\widehat{CE} = 26^\circ$, find $m\widehat{CD}$ and $m\angle CBE$.

Know What? A typical large pizza has a diameter of 14 inches and is cut into 8 pieces. Think of the crust as the circumference of the pizza. Find the *length* of the crust for the entire pizza. Then, find the length of the crust for one piece of pizza if the entire pizza is cut into 8 pieces.

Circumference of a Circle

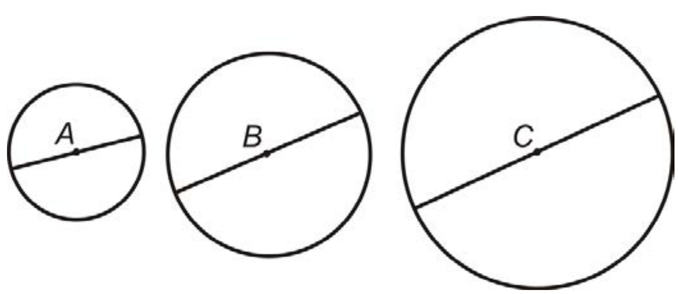
Circumference: The distance around a circle.

The circumference can also be called the perimeter of a circle. However, we use the term circumference for circles because they are round. In order to find the circumference of a circle, we need to explore π (pi).

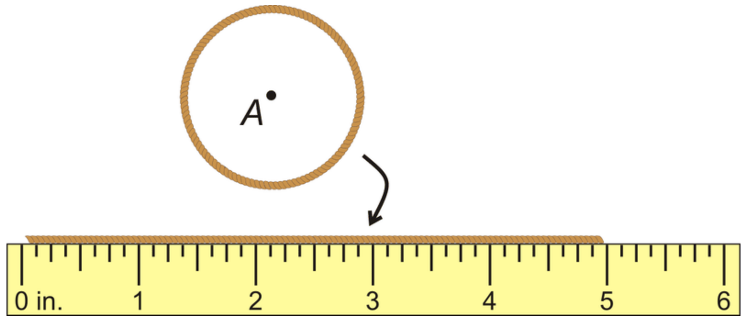
Investigation 10-1: Finding π (pi)

Tools Needed: paper, pencil, compass, ruler, string, and scissors

1. Draw three circles with radii of 2 in, 3 in, and 4 in. Label the centers of each A, B, and C.
2. Draw in the diameters and determine their lengths.



3. Take the string and outline each circle with it. Cut the string so that it perfectly outlines the circle. Then, lay it out straight and measure it in inches. Round your answer to the nearest $\frac{1}{8}$-inch. Repeat this for the other two circles.



4. Find $\frac{circumference}{diameter}$ for each circle. Record your answers to the nearest thousandth.

You should see that $\frac{circumference}{diameter}$ approaches 3.14159... We call this number π, the Greek letter "pi." When finding the circumference and area of circles, we must use π.

Watch This!!!

https://youtu.be/TlY-Sh9Rzas

π, or **"pi":** The ratio of the circumference of a circle to its diameter. It is approximately equal to 3.14159265358979323846...

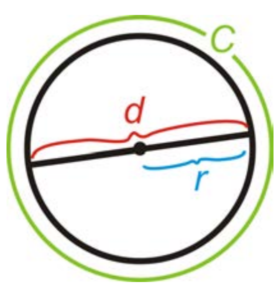
To see more digits of π, go to http://www.eveandersson.com/pi/digits/ .

From Investigation 10-1, we found that $\frac{circumference}{diameter} = \pi$. Let's solve for the circumference, C.

$$\frac{C}{d} = \pi$$
$$C = \pi d$$

We can also say $C = 2\pi r$ because $d = 2r$.

Circumference Formula: $C = \pi d$ or $C = 2\pi r$



$d = 2r$

Example 1: Find the circumference of a circle with a radius of 7 cm.

Solution: Plug the radius into the formula.

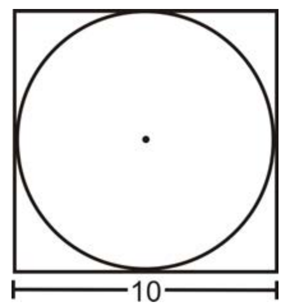
$$C = 2\pi(7) = 14\pi \approx 44\ cm$$

Example 2: The circumference of a circle is 64π. Find the diameter.

Solution: Again, you can plug in what you know into the circumference formula and solve for d.

$$64\pi = \pi d$$
$$64 = d$$

Example 3: A circle is inscribed in a square with 10 in. sides. What is the circumference of the circle? Leave your answer in terms of π.



Solution: From the picture, we can see that the diameter of the circle is equal to the length of a side. $C = 10\pi$ *in*.

Example 4: Find the perimeter of the square. Is it more or less than the circumference of the circle? Why?

Solution: The perimeter is $P = 4(10) = 40$ *in*. In order to compare the perimeter with the circumference we should change the circumference into a decimal.

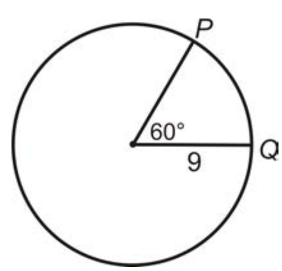
$C = 10\pi \approx 31.42$ *in*. This is less than the perimeter of the square, which makes sense because the circle is inside the square.

Arc Length

In Chapter 9, we measured arcs in degrees. This was called the "arc measure" or "degree measure." Arcs can also be measured in length, as a portion of the circumference.

Arc Length: The length of an arc or a portion of a circle's circumference.

The arc length is directly related to the degree arc measure.

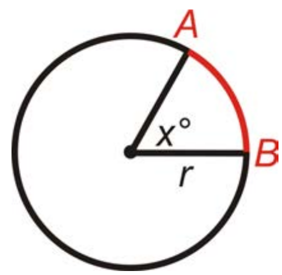


Example 5: Find the length of $\widehat{PQ}$. Leave your answer in terms of π.

Solution: In the picture, the central angle that corresponds with $\widehat{PQ}$ is 60°. This means that $m\widehat{PQ} = 60^\circ$. Think of the arc length as a portion of the circumference. There are 360° in a circle, so 60° would be $\frac{1}{6}$ of that $\left(\frac{60^\circ}{360^\circ} = \frac{1}{6}\right)$. Therefore, the length of $\widehat{PQ}$ is $\frac{1}{6}$ of the circumference. *length of* $\widehat{PQ} = \frac{1}{6} \cdot 2\pi(9) = 3\pi$

Arc Length Formula: The length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$.

Another way to write this could be $\frac{x^\circ}{360^\circ} \cdot 2\pi r$, where x is the central angle.

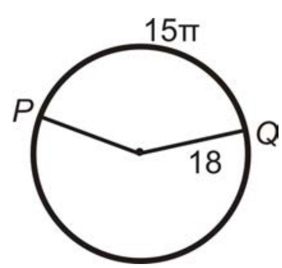


Example 6: The arc length of $\widehat{AB} = 6\pi$ and is $\frac{1}{4}$ the circumference. Find the radius of the circle.

Solution: If 6π is $\frac{1}{4}$ the circumference, then the total circumference is $4(6\pi) = 24\pi$. To find the radius, plug this into the circumference formula and solve for r.

$$\begin{aligned} 24\pi &= 2\pi r \\ 12 &= r \end{aligned}$$

Example 7: Find the measure of the central angle or $\widehat{PQ}$.



Solution: Let's plug in what we know to the Arc Length Formula.

$$\begin{aligned} 15\pi &= \frac{m\widehat{PQ}}{360^\circ} \cdot 2\pi(18) \\ 15 &= \frac{m\widehat{PQ}}{10^\circ} \\ 150^\circ &= m\widehat{PQ} \end{aligned}$$

Example 8: The tires on a compact car are 18 inches in diameter. How far does the car travel after the tires turn once? How far does the car travel after 2500 rotations of the tires?

Solution: One turn of the tire is the circumference. This would be $C = 18\pi \approx 56.55$ *in*. 2500 rotations would be $2500 \cdot 56.55$ *in* $= 141371.67$ *in*, 11781 ft, or 2.23 miles.

Know What? Revisited The entire length of the crust, or the circumference of the pizza is $14\pi \approx 44$ *in*. In $\frac{1}{8}$ of the pizza, one piece would have $\frac{44}{8} \approx 5.5$ inches of crust.

Practice Problems

- Questions 1-10 are similar to Examples 1 and 2.
- Questions 11-14 are similar to Examples 3 and 4.
- Questions 15-20 are similar to Example 5.
- Questions 21-23 are similar to Example 6.
- Questions 24-26 are similar to Example 7.
- Questions 27-30 are similar to Example 8.

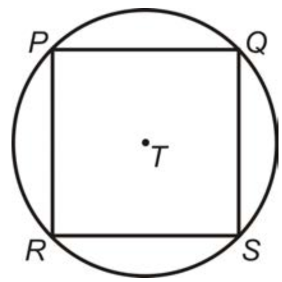
Fill in the following table. Leave all answers in terms of π.

TABLE 9.1:

	diameter	*radius*	*circumference*
1.	15		
2.		4	
3.	6		
4.			84π
5.		9	
6.			25π
7.			2π
8.	36		

9. Find the radius of circle with circumference 88 in.
10. Find the circumference of a circle with $d = \frac{20}{\pi}$ *cm*.

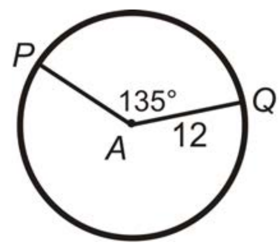
Square $PQSR$ is inscribed in $\odot T$. $RS = 8\sqrt{2}$.



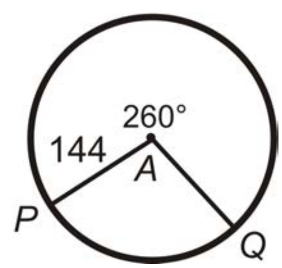
11. Find the length of the diameter of $\odot T$.
12. How does the diameter relate to $PQSR$?
13. Find the perimeter of $PQSR$.
14. Find the circumference of $\odot T$.

Find the arc length of $\widehat{PQ}$ in $\odot A$. Leave your answers in terms of π.

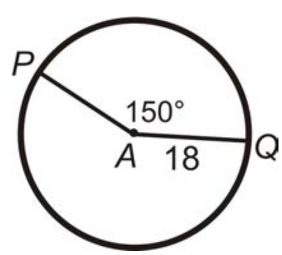
15.



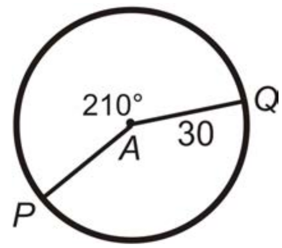
16.



17.



18.



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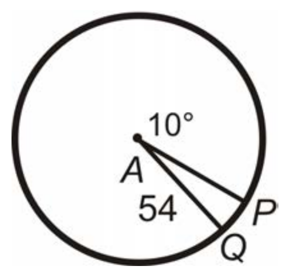
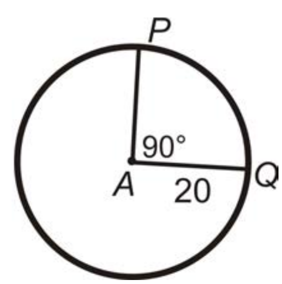
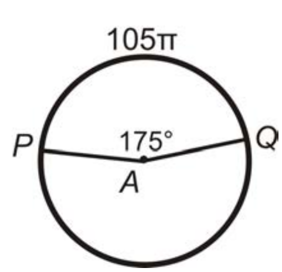


20.

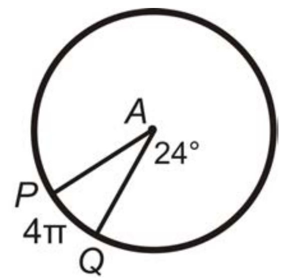


Find *PA* (the radius) in $\odot A$. Leave your answer in terms of π.

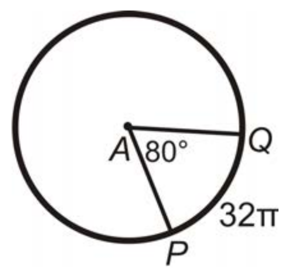
21.



22.

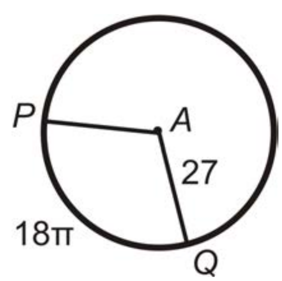


23.

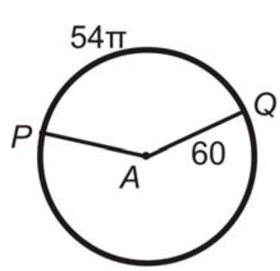


Find the central angle or $m\widehat{PQ}$ in $\odot A$. Round any decimal answers to the nearest tenth.

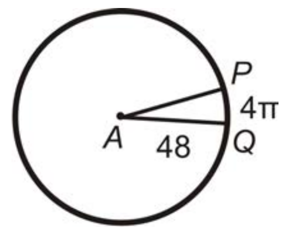
24.



25.



26.



For questions 27-30, a truck has tires with a 26 in diameter.

27. How far does the truck travel every time a tire turns exactly once? What is this the same as?
28. How many times will the tire turn after the truck travels 1 mile? (1 mile = 5280 feet)
29. The truck has traveled 4072 tire rotations. How many miles is this?
30. The average recommendation for the life of a tire is 30,000 miles. How many rotations is this?

Review and Reflect

31. Given a clock with a radius of 12 in, how far will the tip of the minute hand travel over a course of 1 hour and 35 minutes?
32. Describe the meaning of pi.

Warm-Up Answers

1. $\angle CAE$
2. $\angle CBE$
3. $360^\circ, 180^\circ$
4. $m\widehat{CD} = 180^\circ - 26^\circ = 154^\circ, m\angle CBE = 13^\circ$

9.6 Areas of Circles and Sectors

TEKS G(5)A, G(11)B, G(12)B, G(12)C

Learning Objectives

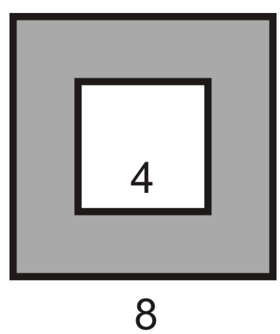
- Find the area of the circles, sectors, and segments

Vocabulary

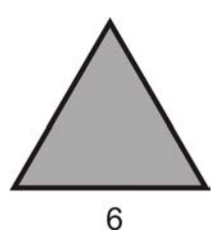
- Cicumference
- Arc Length
- Sector

Warm-Up

1. Find the area of both squares.
2. Find the area of the shaded region.



3. The triangle to the right is an equilateral triangle.
   a. Find the height of the triangle.
   b. Find the area of the triangle.



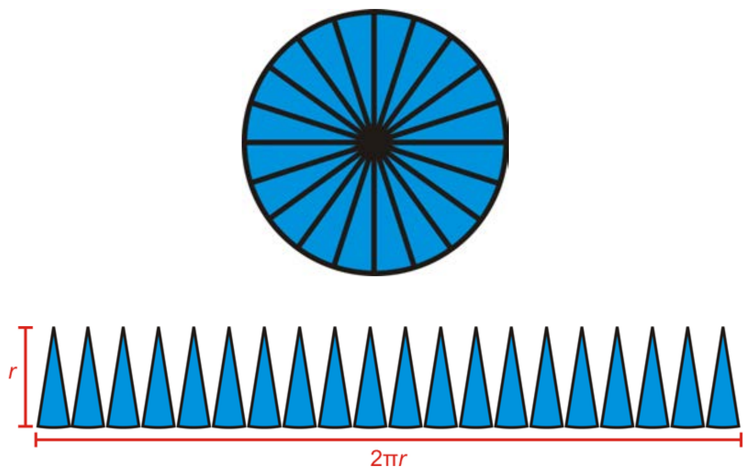
Know What? Back to the pizza. In the previous section, we found the length of the crust for a 14 in pizza. However, crust typically takes up some area on a pizza. Round your answers to the nearest hundredth.

a) Find the area of the crust of a deep-dish 16 in pizza. A typical deep-dish pizza has 1 in of crust around the toppings.

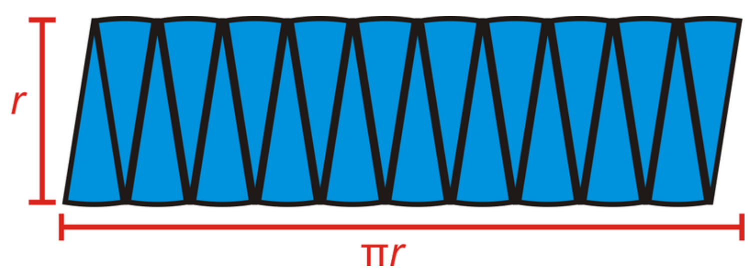
b) A thin crust pizza has $\frac{1}{2}$-in of crust around the edge of the pizza. Find the area of a thin crust 16 in pizza.

Area of a Circle

Take a circle and divide it into several wedges. Then, unfold the wedges so they are in a line, with the points at the top.



The height of the wedges is the radius and the length is the circumference of the circle. Now, take half of these wedges and flip them upside-down and place them so they all fit together.



Now our circle looks like a parallelogram. The area of this parallelogram is $A = bh = \pi r \cdot r = \pi r^2$.

To see an animation of this derivation, see http://www.rkm.com.au/ANIMATIONS/animation-Circle-Area-Derivation.html , by Russell Knightley.

Area of a Circle: If r is the radius of a circle, then $A = \pi r^2$.

Example 1: Find the area of a circle with a diameter of 12 cm.

Solution: If $d = 12\ cm$, then $r = 6\ cm$. The area is $A = \pi\left(6^2\right) = 36\pi\ cm^2$.

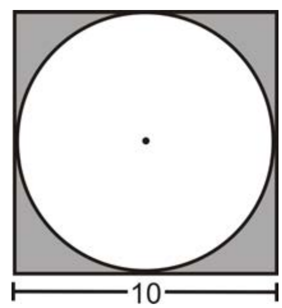
Example 2: If the area of a circle is 20π, what is the radius?

Solution: Plug in the area and solve for the radius.

$$\begin{aligned} 20\pi &= \pi r^2 \\ 20 &= r^2 \\ r &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Just like the circumference, we will leave our answers in terms of π, unless otherwise specified.

Example 3: A circle is inscribed in a square. Each side of the square is 10 cm long. What is the area of the circle?



Solution: The diameter of the circle is the same as the length of a side of the square. Therefore, the radius is 5 cm.

$$A = \pi 5^2 = 25\pi\ cm^2$$

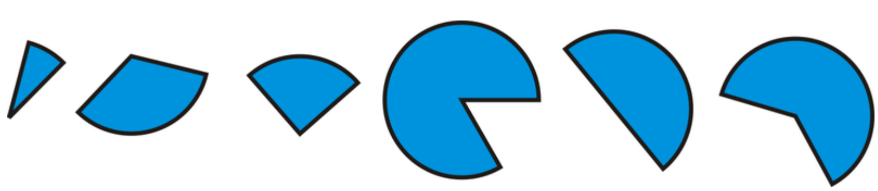
Example 4: Find the area of the shaded region.

Solution: The area of the shaded region would be the area of the square minus the area of the circle.

$$A = 10^2 - 25\pi = 100 - 25\pi \approx 21.46\ cm^2$$

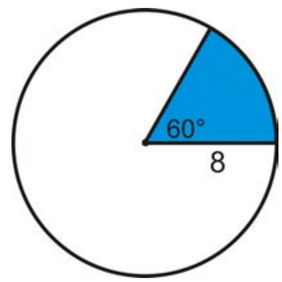
Area of a Sector

Sector of a Circle: The area bounded by two radii and the arc between the endpoints of the radii.



Area of a Sector: If r is the radius and $\widehat{AB}$ is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Example 5: Find the area of the blue sector. Leave your answer in terms of π.

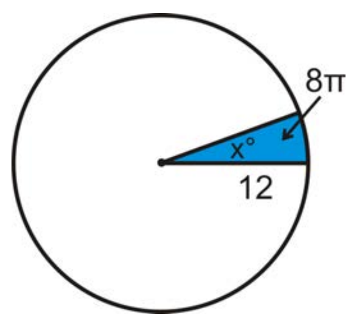


Solution: In the picture, the central angle that corresponds with the sector is 60°. 60° would be $\frac{1}{6}$ of 360°, so this sector is $\frac{1}{6}$ of the total area. $area\ of\ blue\ sector = \frac{1}{6} \cdot \pi 8^2 = \frac{32}{3}\pi$

Another way to write the sector formula is $A = \frac{central\ angle}{360^\circ} \cdot \pi r^2$.

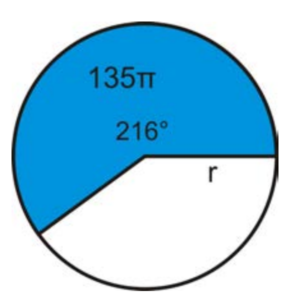
Example 6: The area of a sector is 8π and the radius of the circle is 12. What is the central angle?

Solution: Plug in what you know to the sector area formula and then solve for the central angle, we will call it x.



$$\begin{aligned} 8\pi &= \frac{x}{360^\circ} \cdot \pi 12^2 \\ 8\pi &= \frac{x}{360^\circ} \cdot 144\pi \\ 8 &= \frac{2x}{5^\circ} \\ x &= 8 \cdot \frac{5^\circ}{2} = 20^\circ \end{aligned}$$

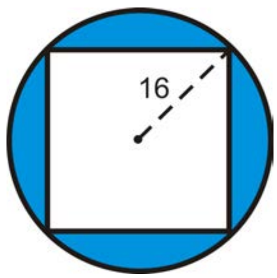
Example 7: The area of a sector is 135π and the arc measure is 216°. What is the radius of the circle?



Solution: Plug in what you know to the sector area formula and solve for r.

$$135\pi = \frac{216^\circ}{360^\circ} \cdot \pi r^2$$
$$135 = \frac{3}{5} \cdot r^2$$
$$\frac{5}{3} \cdot 135 = r^2$$
$$225 = r^2 \rightarrow r = \sqrt{225} = 15$$

Example 8: Find the area of the shaded region. The quadrilateral is a square.



Solution: The radius of the circle is 16, which is also half of the diagonal of the square. So, the diagonal is 32 and the sides would be $\frac{32}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 16\sqrt{2}$ because each half of a square is a 45-45-90 triangle.

$$A_{circle} = 16^2\pi = 256\pi$$
$$A_{square} = \left(16\sqrt{2}\right)^2 = 256 \cdot 2 = 512$$

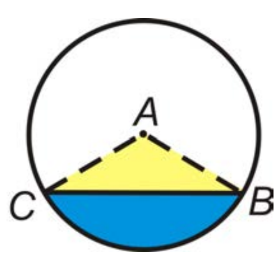
The area of the shaded region is $256\pi - 512 \approx 292.25$

Segments of a Circle

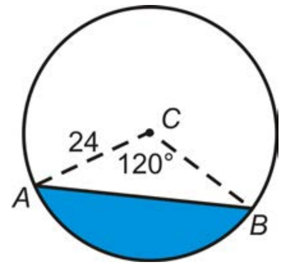
The last part of a circle that we can find the area of is called a segment, not to be confused with a line segment.

Segment of a Circle: The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

$$A_{segment} = A_{sector} - A_{\triangle ABC}$$



Example 9: Find the area of the blue segment below.



Solution: The area of the segment is the area of the sector minus the area of the isosceles triangle made by the radii. If we split the isosceles triangle in half, each half is a 30-60-90 triangle, where the radius is the hypotenuse. The height of $\triangle ABC$ is 12 and the base would be $2\left(12\sqrt{3}\right) = 24\sqrt{3}$.

$$A_{sector} = \frac{120}{360}\pi \cdot 24^2 \qquad A_{\triangle} = \frac{1}{2}\left(24\sqrt{3}\right)(12)$$
$$= 192\pi \qquad = 144\sqrt{3}$$

The area of the segment is $A = 192\pi - 144\sqrt{3} \approx 353.8$.

Know What? Revisited The area of the crust for a deep-dish pizza is $8^2\pi - 7^2\pi = 15\pi$. The area of the crust of the thin crust pizza is $8^2\pi - 7.5^2\pi = \frac{31}{4}\pi$.

Practice Problems

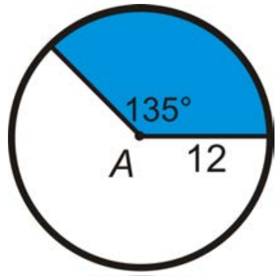
- Questions 1-10 are similar to Examples 1 and 2.
- Questions 11-16 are similar to Example 5.
- Questions 17-19 are similar to Example 7.
- Questions 20-22 are similar to Example 6.
- Questions 23-25 are similar to Examples 3, 4, and 8.
- Questions 26-31 are similar to Example 9.

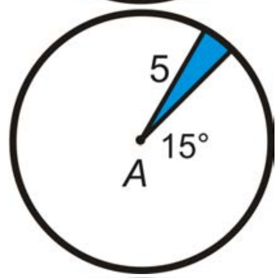
Fill in the following table. Leave all answers in terms of π.

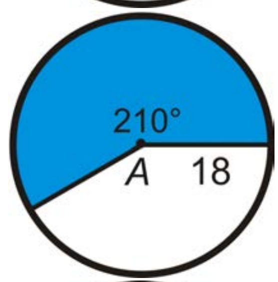
TABLE 9.2:

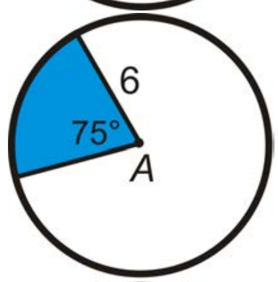
	radius	*Area*	*circumference*
1.	2		
2.		16π	
3.			10π
4.			24π
5.	9		
6.		90π	
7.			35π
8.	$\frac{7}{\pi}$		
9.			60
10.		36	

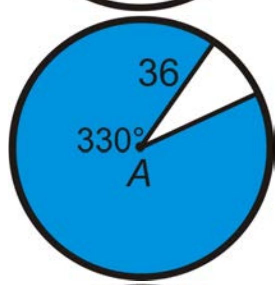
Find the area of the blue sector or segment in $\odot A$. Leave your answers in terms of π. Round any decimal answers to the nearest hundredth.

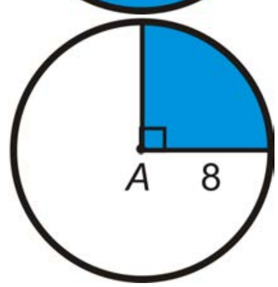
11. 

12. 

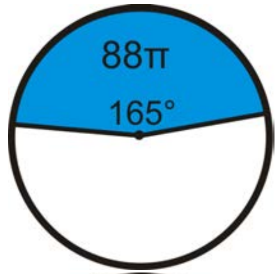
13. 

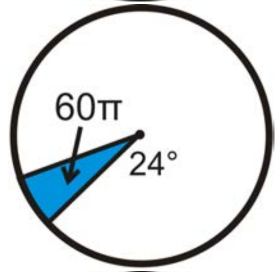
14. 

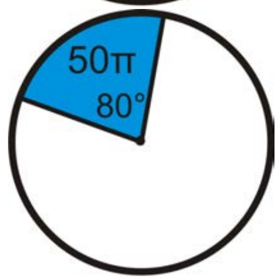
15. 

16. 

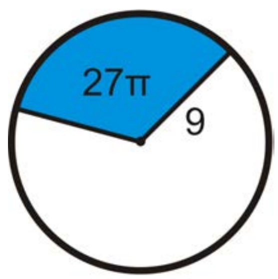
Find the radius of the circle. Leave your answer in terms of π.

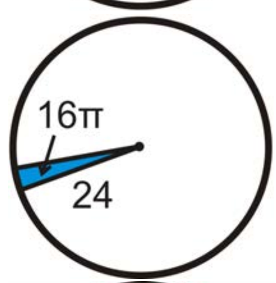
17. 

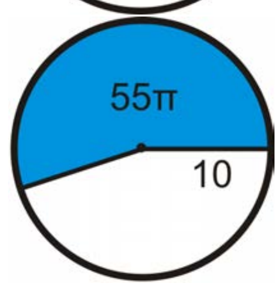
18. 

19. 

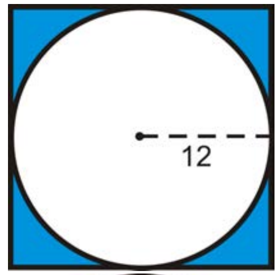
Find the central angle of each blue sector. Round any decimal answers to the nearest tenth.

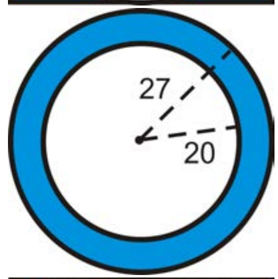
20. 

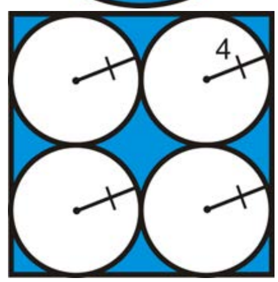
21. 

22. 

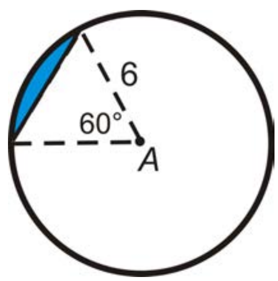
Find the area of the shaded region. Round your answer to the nearest hundredth.

23. 

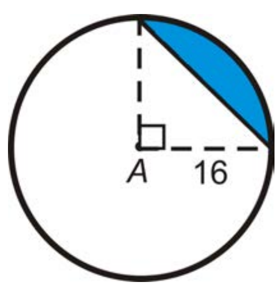
24. 

25. 

26. Find the area of the sector in $\odot A$. Leave your answer in terms of π.



27. Find the area of the equilateral triangle.
28. Find the area of the segment. Round your answer to the nearest hundredth.
29. Find the area of the sector in $\odot A$. Leave your answer in terms of π.
30. Find the area of the right triangle.



31. Find the area of the segment. Round your answer to the nearest hundredth.

Review and Reflect

32. If you were given a circle and had to draw a sector of the circle that is 65% of the circle, what would the central angle be?

33. If the radius of a circle is tripled, will the area of a given sector triple?

Warm-Up Answers

1. $8^2 - 4^2 = 64 - 16 = 48$
2. $6(10) - \frac{1}{2}(7)(3) = 60 - 10.5 = 49.5$
3. $\frac{1}{2}(6)\left(3\sqrt{3}\right) = 9\sqrt{3}$
4. $\frac{1}{2}(s)\left(\frac{1}{2}s\sqrt{3}\right) = \frac{1}{4}s^2\sqrt{3}$

9.7 Area of Composite Shapes

Here you'll learn how to find the area of a composite shape.

TEKS G(1)A, G(11)B

Learning Objectives

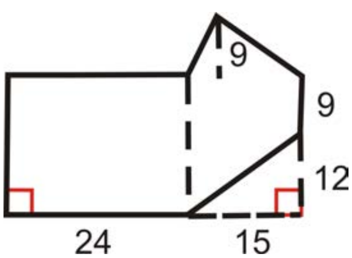
- determine the area of composite two dimensional figures

Vocabulary

- perimeter
- area
- composite shapes

Warm-Up

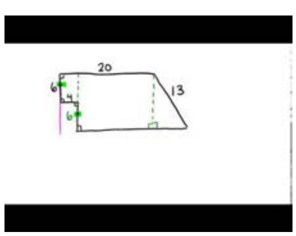
What if you wanted to find the area of a shape that was made up of other shapes? How could you use your knowledge of the area of rectangles, parallelograms, and triangles to help you? After completing this Concept, you'll be able to answer questions like these.



1. Find the area of the rectangles and triangle.

2. Find the area of the whole shape.

Watch This



MEDIA

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URL: https://www.ck12.org/flx/render/embeddedobject/137564

CK-12 Foundation: Chapter10AreaofCompositeShapesA

Perimeter is the distance around a shape. The perimeter of any figure must have a unit of measurement attached to it. If no specific units are given (feet, inches, centimeters, etc), write "units." **Area** is the amount of space inside a figure.

Congruent Area Postulate

If two figures are congruent, then they have the same area.

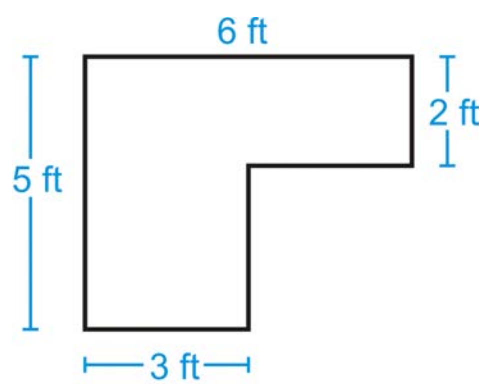
This postulate needs no proof because congruent figures have the same amount of space inside them. Keep in mind that two figures with the same area are not necessarily congruent.

A **composite shape** is a shape made up of other shapes. To find the area of such a shape, simply find the area of each part and add them up.

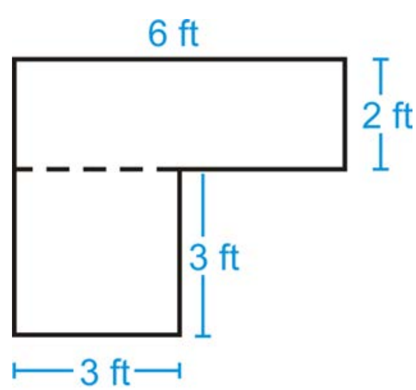
Area Addition Postulate

If a figure is composed of two or more parts that do not overlap each other, then the are of the figure is the sum of the areas of the parts.

Example 1: Find the area of the figure below. You may assume all sides are perpendicular.



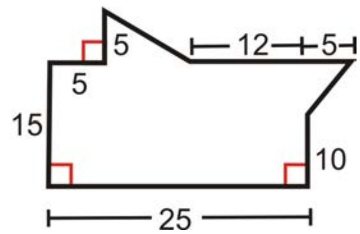
Solution: Split the shape into two rectangles and find the area of each.



$$A_{top\ rectangle} = 6 \cdot 2 = 12\ ft^2$$
$$A_{bottom\ square} = 3 \cdot 3 = 9\ ft^2$$

The total area is $12 + 9 = 21\ ft^2$.

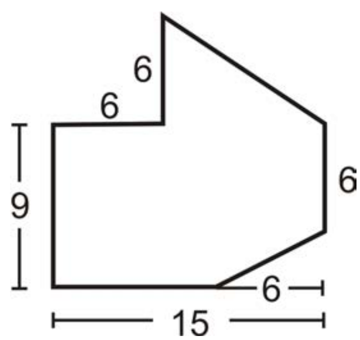
Example 2: Divide the shape below into two triangles and one rectangle. Find the area of the two triangles and rectangle. Then find the area of the entire shape.



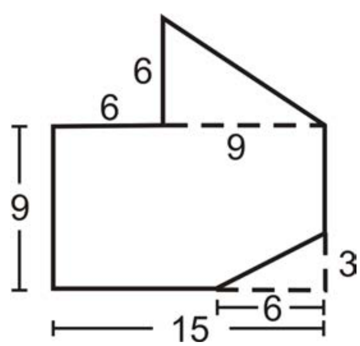
Solution:

- One triangle on the top and one on the right. Rectangle is the rest.
- Area of triangle on top is $\frac{8(5)}{2} = 20\ units^2$. Area of triangle on right is $\frac{5(5)}{2} = 12.5\ units^2$. Area of rectangle is $375\ units^2$.
- Total area is $407.5\ units^2$.

Example 3: Find the area of the figure below.

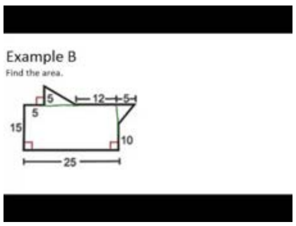


Solution: Divide the figure into a triangle and a rectangle with a small triangle cut out of the lower right-hand corner.



$$A = A_{top\ triangle} + A_{rectangle} - A_{small\ triangle}$$
$$A = \left(\frac{1}{2} \cdot 6 \cdot 9\right) + (9 \cdot 15)\Big) - (\frac{1}{2} \cdot 3 \cdot 6\Big)$$
$$A = 27 + 135 - 9$$
$$A = 153\ units^2$$

Watch this video for help with the Examples above.



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CK-12 Foundation: Chapter10AreaofCompositeShapesB

Interactive Practice

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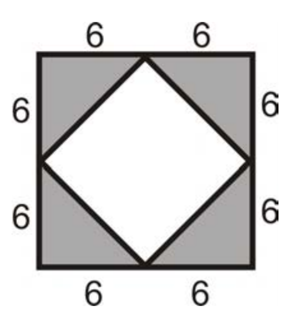
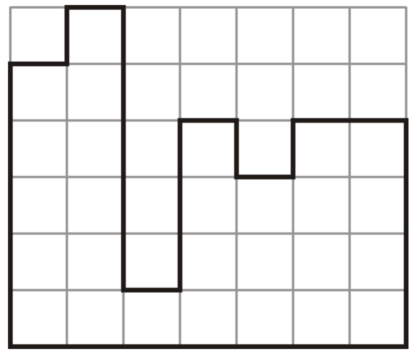
Practice Problems

Use the picture below for questions 1-2. Both figures are squares.

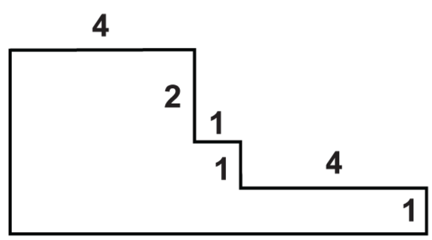


1. Find the area of the unshaded region.
2. Find the area of the shaded region.

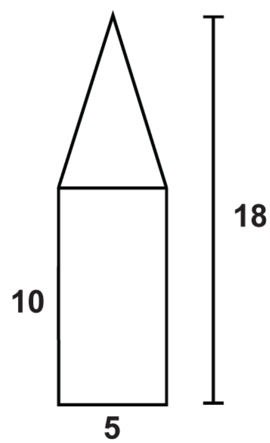
Find the area of the figure below. You may assume all sides are perpendicular.

3. 

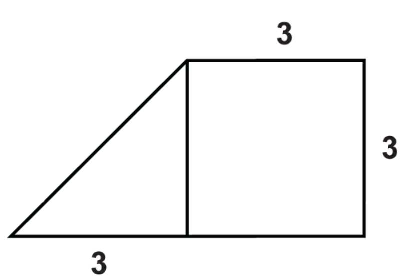
Find the areas of the composite figures.



4.



5.



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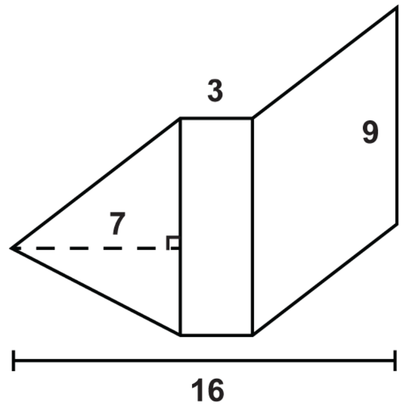

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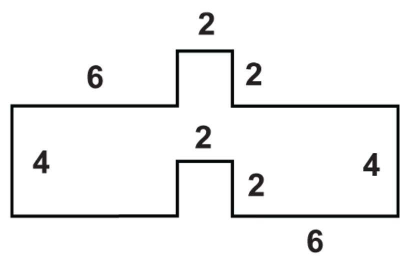

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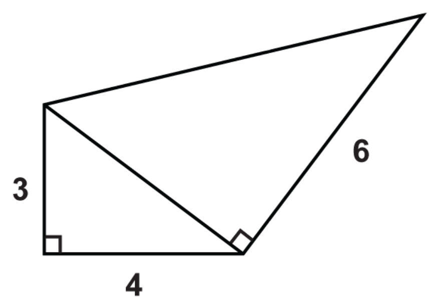
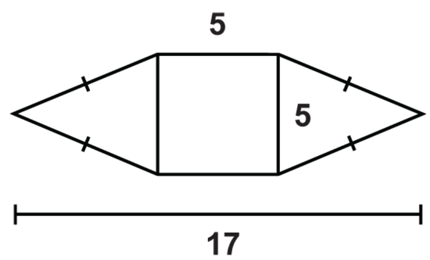


9.

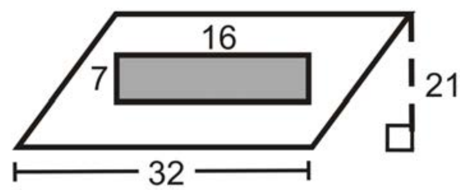
Use the figure to answer the questions.



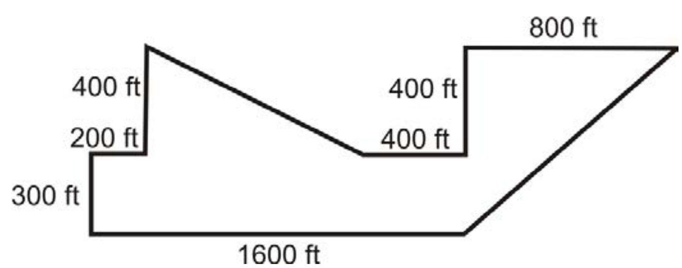
10. What is the area of the square?
11. What is the area of the triangle on the left?
12. What is the area of the composite figure?

Find the area of the following figures.

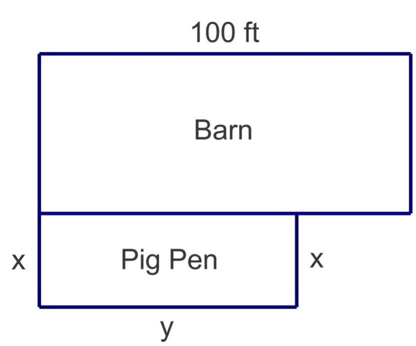
13. Find the area of the unshaded region.



14. Lin bought a tract of land for a new apartment complex. The drawing below shows the measurements of the sides of the tract. Approximately how many acres of land did Lin buy? You may assume any angles that look like right angles are 90°. (1 acre $\approx$ 40,000 square feet)



15. Linus has 100 ft of fencing to use in order to enclose a 1200 square foot rectangular pig pen. The pig pen is adjacent to the barn so he only needs to form three sides of the rectangular area as shown below. What dimensions should the pen be?



Review and Reflect

16. Give at least 2 examples of when you would need to find area of composite figures in the real world.
17. Explain your process for finding total area in number 14.

Warm-Up Answers

1. Rectangle #1: Area $= 24(9+12) = 504\ units^2$. Rectangle #2: Area $= 15(9+12) = 315\ units^2$. Triangle: Area $= \frac{15(9)}{2} = 67.5\ units^2$.

2. You need to subtract the area of the triangle from the bottom right corner. Total Area $= 504+315+67.5-\frac{15(12)}{2} = 796.5\ units^2$

9.8 Chapter 9 Review

Keywords, Theorems and Formulas

Triangles and Parallelograms

- Perimeter
- Area of a Rectangle: $A = bh$
- Perimeter of a Rectangle $P = 2b + 2h$
- Perimeter of a Square: $P = 4s$
- Area of a Square: $A = s^2$
- Congruent Areas Postulate
- Area Addition Postulate
- Area of a Parallelogram: $A = bh$
- Area of a Triangle: $A = \frac{1}{2}\, bh$ or $A = \frac{bh}{2}$

Trapezoids, Rhombi, and Kites

- Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$
- Area of a Rhombus: $A = \frac{1}{2}d_1 d_2$
- Area of a Kite: $A = \frac{1}{2}d_1 d_2$

Area of Similar Polygons

- Area of Similar Polygons Theorem

Circumference and Arc Length

- π
- Circumference: $C = \pi d$ or $C = 2\pi r$
- Arc Length
- Arc Length Formula: length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$

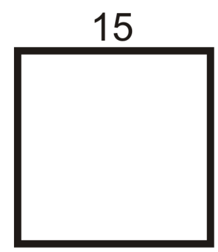
Area of Circles and Sectors

- Area of a Circle: $A = \pi r^2$
- Sector
- Area of a Sector: $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$
- Segment of a Circle

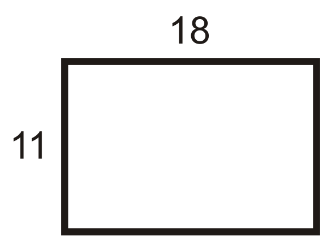
Review Questions

Find the area and perimeter of the following figures. Round your answers to the nearest hundredth.

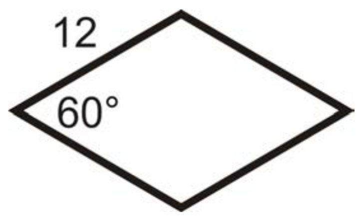
1. square



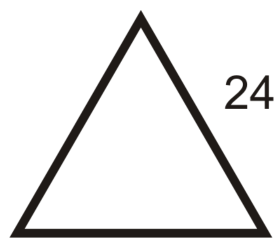
2. rectangle



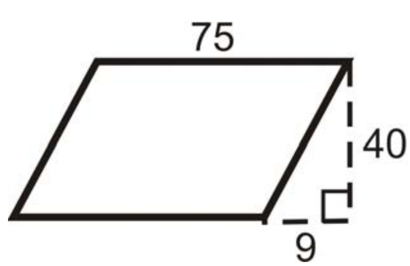
3. rhombus



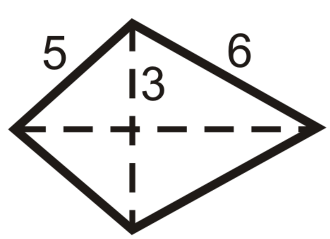
4. equilateral triangle



5. parallelogram

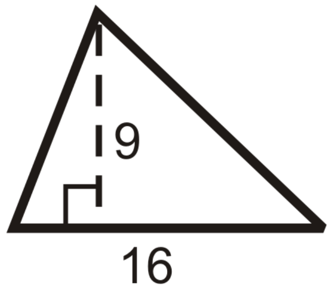


6. kite

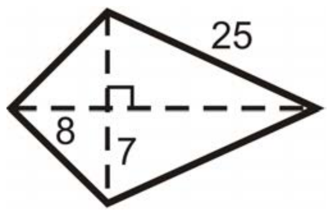


Find the area of the following figures. Leave your answers in simplest radical form.

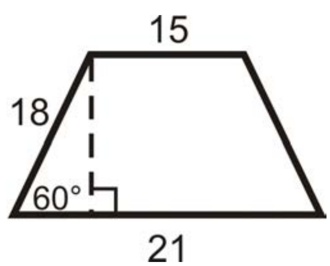
7. triangle



8. kite

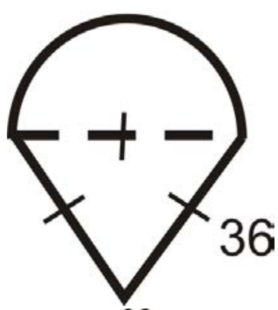


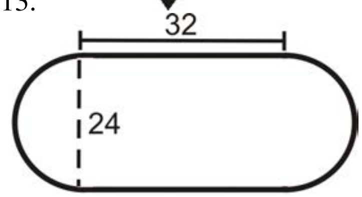
9. isosceles trapezoid



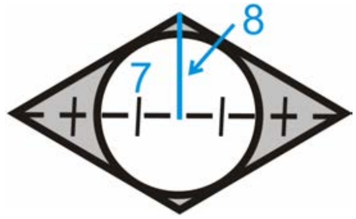
10. Find the area and circumference of a circle with radius 17.
11. Find the area and circumference of a circle with diameter 30.
12. Two similar rectangles have a scale factor $\frac{4}{3}$. If the area of the larger rectangle is 96 $units^2$, find the area of the smaller rectangle.

Find the area of the following figures. Round your answers to the nearest hundredth.

13. 

14. 

15. find the shaded area



(figure is a rhombus)

Texas Instruments Resources

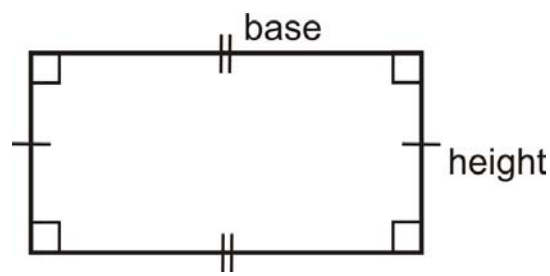
In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9695 .

9.9 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st **Section: Triangles and Parallelograms**

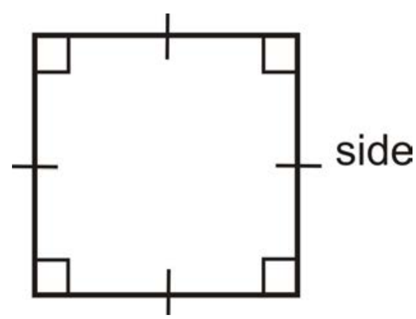
Perimeter



Area of a Rectangle: $A = bh$

Perimeter of a Rectangle $P = 2b + 2h$

Perimeter of a Square: $P = 4s$

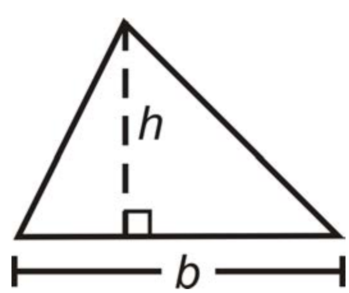


Area of a Square: $A = s^2$

Congruent Areas Postulate

Area Addition Postulate

Area of a Parallelogram: $A = bh$

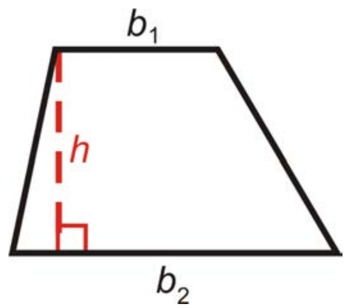


Area of a Triangle: $A = \frac{1}{2}\ bh$ or $A = \frac{bh}{2}$

Homework:

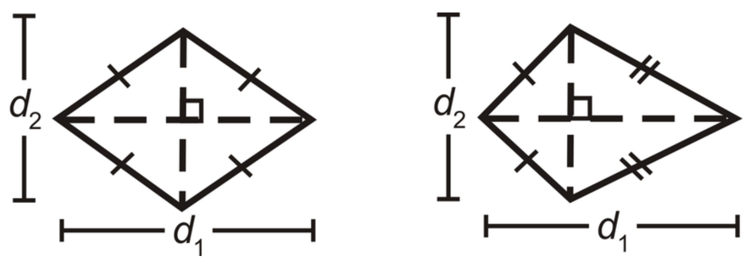
2^{nd}**Section: Trapezoids, Rhombi, and Kites**

Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$



Area of a Rhombus: $A = \frac{1}{2}d_1d_2$

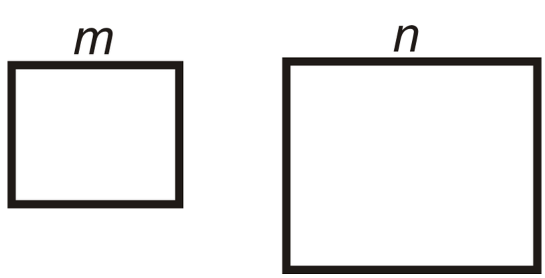
Area of a Kite: $A = \frac{1}{2}d_1d_2$



Homework:

3^{rd}**Section: Area of Similar Polygons**

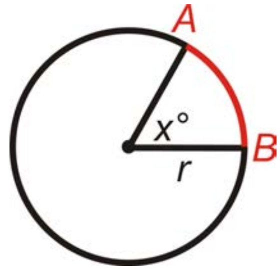
Area of Similar Polygons Theorem



Homework:

4^{th}**Section: Circumference and Arc Length**

π



Circumference: $C = \pi d$ or $C = 2\pi r$

Arc Length

Arc Length Formula: length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$

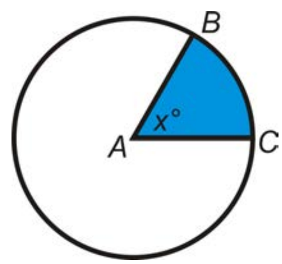
Homework:

5^{th} Section: Area of Circles and Sectors

Area of a Circle: $A = \pi r^2$

Sector

Area of a Sector: $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$



Segment of a Circle

Homework:

CHAPTER 10 Circles

Chapter Outline

10.1 Parts of Circles & Tangent Lines

TEKS G(5)A, G(12)A

Learning Objectives

- Define the parts of a circle
- Discover the properties of tangent lines

Vocabulary

- Center
- Radius
- Chord
- Diameter
- Secant
- Tangent
- Point of Tangency
- Tangent Circles
- Concentric Circles
- Congruent Circles

Warm-Up

1. Find the equation of the line with $m = \frac{2}{5}$ and $y-$intercept of 4.
2. Find the equation of the line with $m = -2$ and passes through (4, -5).
3. Find the equation of the line that passes though (6, 2) and (-3, -1).
4. Find the equation of the line *perpendicular* to the line in #2 and passes through (-8, 11).

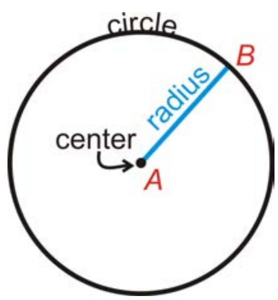
Know What? The clock to the right is an ancient astronomical clock in Prague. It has a large background circle that tells the local time and the "ancient time" and the smaller circle rotates to show the current astrological sign. The yellow point is the center of the larger clock. How does the orange line relate to the small and large circle? How does the hand with the moon on it relate to both circles?

Defining Terms

Circle: The set of all points that are the same distance away from a specific point, called the ***center***.

The center of the circle is point A. We call this circle, "circle A," and it is labeled $\odot A$.

Radii (the plural of radius) are line segments. There are infinitely many radii in any circle and they are all equal.

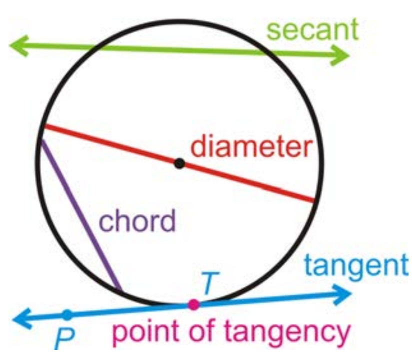


Radius: The distance from the center to the circle.

Chord: A line segment whose endpoints are on a circle.

Diameter: A chord that passes through the center of the circle.

Secant: A line that intersects a circle in two points.



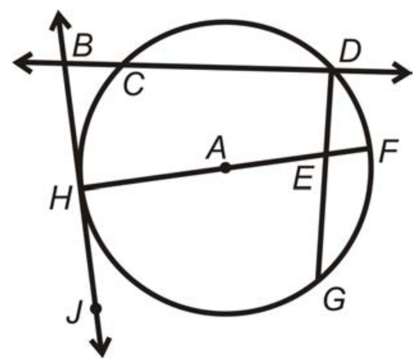
The tangent ray $\overrightarrow{TP}$ and tangent segment $\overline{TP}$ are also called tangents.

The length of a diameter is two times the length of a radius.

Tangent: A line that intersects a circle in exactly one point.

Point of Tangency: The point where the tangent line touches the circle.

Example 1: Find the parts of $\odot A$ that best fit each description.



a) A radius

b) A chord

c) A tangent line

d) A point of tangency

e) A diameter

f) A secant

Solution:

a) $\overline{HA}$ or $\overline{AF}$

b) $\overline{CD}$, $\overline{HF}$, or $\overline{DG}$

c) $\overleftrightarrow{BJ}$

d) *Point H*

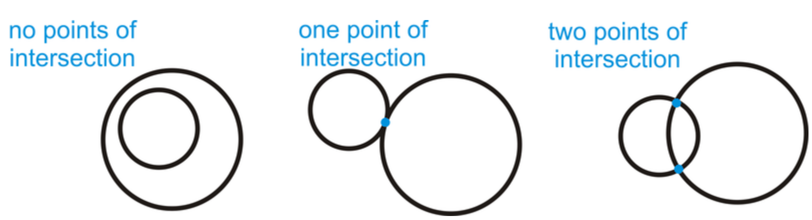
e) $\overline{HF}$

f) $\overleftrightarrow{BD}$

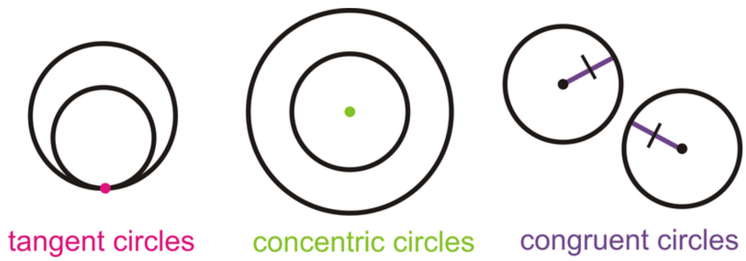
Coplanar Circles

Example 2: Draw an example of how two circles can intersect with no, one and two points of intersection. You will make three separate drawings.

Solution:



Tangent Circles: When two circles intersect at one point.

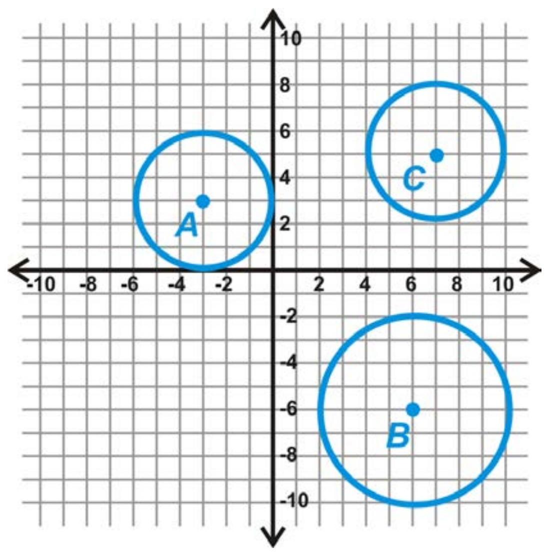


Concentric Circles: When two circles have the same center, but different radii.

Congruent Circles: Two circles with the same radius, but different centers.

If two circles have different radii, they are similar. ***All circles are similar.***

Example 3: Determine if any of the following circles are congruent.



Solution: From each center, count the units to the circle. It is easiest to count vertically or horizontally. Doing this, we have:

$$\text{Radius of } \odot A = 3 \text{ } units$$
$$\text{Radius of } \odot B = 4 \text{ } units$$
$$\text{Radius of } \odot C = 3 \text{ } units$$

From these measurements, we see that $\odot A \cong \odot C$.

Notice the ***circles are congruent***. The ***lengths of the radii*** are ***equal***.

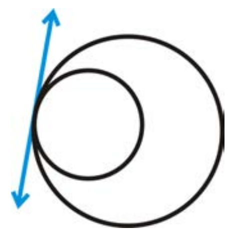
Internally & Externally Tangent

If two circles are tangent to each other, then they are *internally* or *externally* tangent.

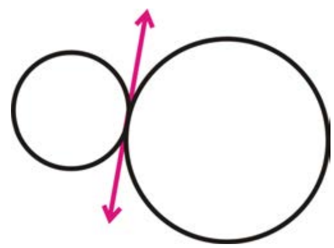
Internally Tangent Circles: When two circles are tangent and one is inside the other.

Externally Tangent Circles: When two circles are tangent and next to each other.

Internally Tangent



Externally Tangent

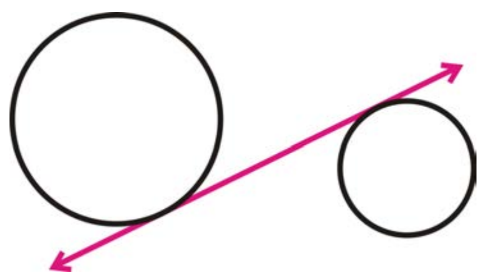


If circles are not tangent, they can still share a tangent line, called a ***common*** tangent.

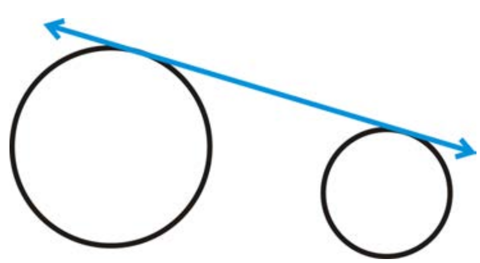
Common Internal Tangent: A line that is tangent to two circles and passes between the circles.

Common External Tangent: A line that is tangent to two circles and stays on the top or bottom of both circles.

Common Internal Tangent



Common External Tangent



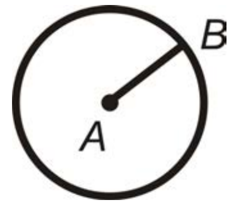
Tangents and Radii

Let's investigate a tangent line and the radius drawn to the point of tangency.

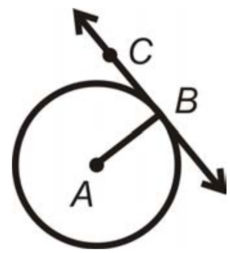
Investigation 9-1: Tangent Line and Radius Property

Tools Needed: compass, ruler, pencil, paper, protractor

1. Using your compass, draw a circle. Locate the center and draw a radius. Label the radius $\overline{AB}$, with A as the center.



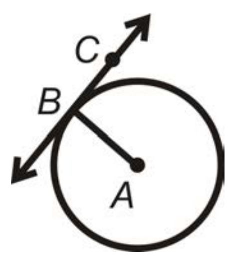
2. Draw a tangent line, $\overleftrightarrow{BC}$, where B is the point of tangency. To draw a tangent line, take your ruler and line it up with point B. B must be the only point on the circle that the line passes through.



3. Find $m\angle ABC$.

Tangent to a Circle Theorem

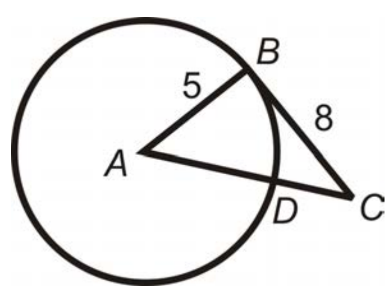
A line is tangent to a circle if and only if the line is perpendicular to the radius drawn to the point of tangency



$\overleftrightarrow{BC}$ is tangent at point B if and only if $\overleftrightarrow{BC} \perp \overline{AB}$.

This theorem uses the words "if and only if," making it a biconditional statement, which means the converse of this theorem is also true.

Example 4: In $\odot A$, $\overline{CB}$ is tangent at point B. Find AC. Reduce any radicals.



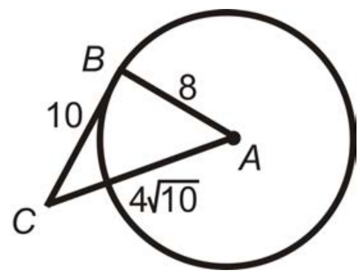
Solution: $\overline{CB}$ is tangent, so $\overline{AB} \perp \overline{CB}$ and $\triangle ABC$ a right triangle. Use the Pythagorean Theorem to find AC.

$$\begin{aligned} 5^2 + 8^2 &= AC^2 \\ 25 + 64 &= AC^2 \\ 89 &= AC^2 \\ AC &= \sqrt{89} \end{aligned}$$

Example 5: Find DC, in $\odot A$. Round your answer to the nearest hundredth.

Solution: $DC = AC - AD$
$DC = \sqrt{89} - 5 \approx 4.43$

Example 6: Determine if the triangle below is a right triangle.

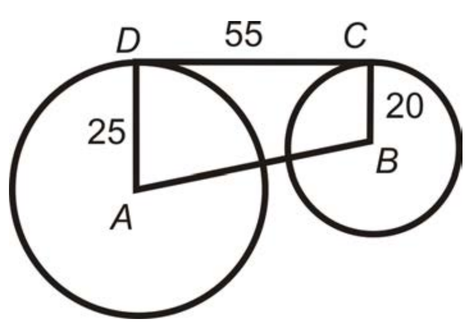


Solution: Again, use the Pythagorean Theorem. $4\sqrt{10}$ is the longest side, so it will be c.

$$\begin{aligned} 8^2 + 10^2 &\ ?\ \left(4\sqrt{10}\right)^2 \\ 64 + 100 &\neq 160 \end{aligned}$$

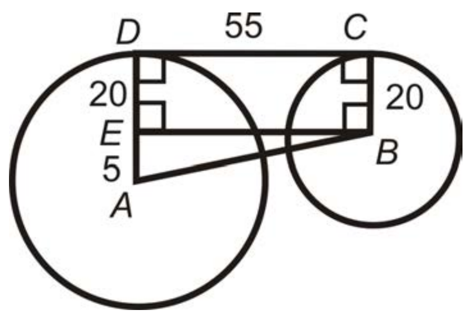
$\triangle ABC$ is not a right triangle. From this, we also find that $\overline{CB}$ is not tangent to $\odot A$.

Example 7: Find AB in $\odot A$ and $\odot B$. Reduce the radical.



Solution: $\overline{AD} \perp \overline{DC}$ and $\overline{DC} \perp \overline{CB}$. Draw in $\overline{BE}$, so $EDCB$ is a rectangle. Use the Pythagorean Theorem to find AB.

$$
\begin{aligned}
5^2 + 55^2 &= AB^2 \\
25 + 3025 &= AB^2 \\
3050 &= AB^2 \\
AB &= \sqrt{3050} = 5\sqrt{122}
\end{aligned}
$$

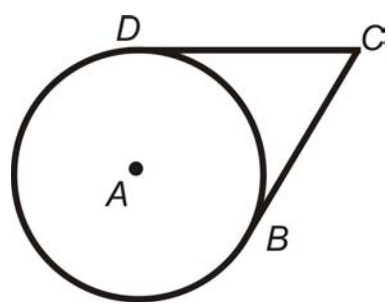


Tangent Segments

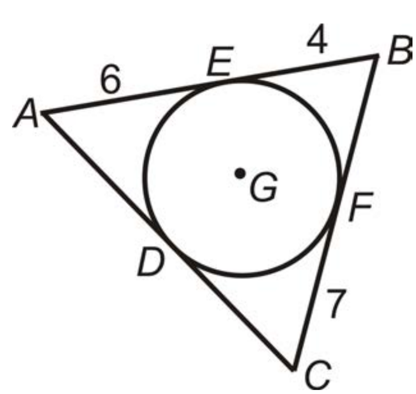
Theorem 10-2

If two tangent segments are drawn from the same external point, then they are equal

$\overline{BC}$ and $\overline{DC}$ have C as an endpoint and are tangent; $\overline{BC} \cong \overline{DC}$.



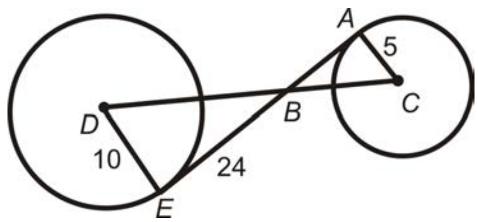
Example 8: Find the perimeter of $\triangle ABC$.



Solution: $AE = AD$, $EB = BF$, and $CF = CD$. Therefore, the perimeter of $\triangle ABC = 6+6+4+4+7+7 = 34$.

$\odot G$ is ***inscribed*** in $\triangle ABC$. A circle is inscribed in a polygon, if every side of the polygon is tangent to the circle.

Example 9: If D and C are the centers and AE is tangent to both circles, find DC.



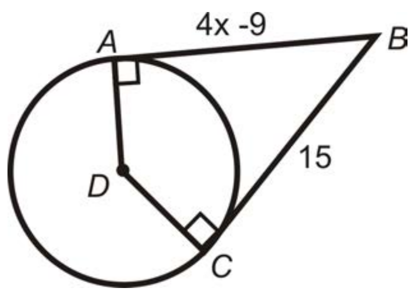
Solution: $\overline{AE} \perp \overline{DE}$ and $\overline{AE} \perp \overline{AC}$ and $\triangle ABC \sim \triangle DBE$.

To find DB, use the Pythagorean Theorem.

$$10^2 + 24^2 = DB^2$$
$$100 + 576 = 676$$
$$DB = \sqrt{676} = 26$$

To find BC, use similar triangles. $\frac{5}{10} = \frac{BC}{26} \longrightarrow BC = 13$. $DC = AB + BC = 26 + 13 = 39$

Example 10: ***Algebra Connection*** Find the value of x.



Solution: $\overline{AB} \cong \overline{CB}$ by Theorem 10-2. Set $AB = CB$ and solve for x.

$$4x - 9 = 15$$
$$4x = 24$$
$$x = 6$$

Know What? Revisited The orange line is a diameter of the smaller circle. Since this line passes through the center of the larger circle (yellow point), it is part of one of its diameters. The "moon" hand is a diameter of the larger circle, but a secant of the smaller circle.

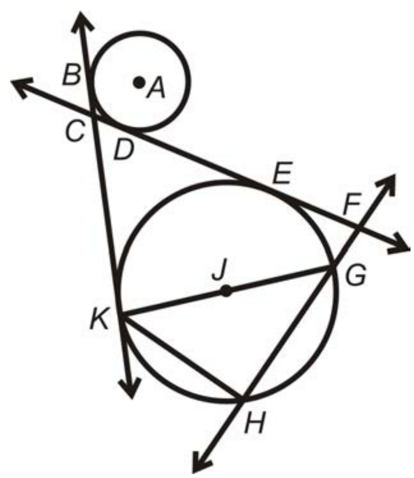
Watch This!!!

https://vimeo.com/45607475

Practice Problems

- Questions 1-9 are similar to Example 1.
- Questions 10-12 are similar to Example 2.
- Questions 13-17 are similar to Example 3.
- Questions 18-20 are similar to Example 6.
- Questions 21-26 are similar to Example 4, 5, 7, and 10.
- Questions 27-31 are similar to Example 9.
- Questions 32-37 are similar to Example 8.
- Question 38 and 39 use the proof of Theorem 10-2.
- Question 40 uses Theorem 10-2.

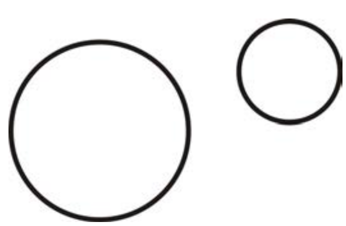
Determine which term best describes each of the following parts of $\odot P$.



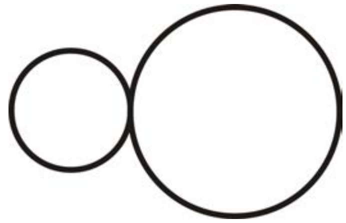
1. $\overline{KG}$
2. $\overleftrightarrow{FH}$
3. $\overline{KH}$
4. E
5. $\overleftrightarrow{BK}$
6. $\overleftrightarrow{CF}$
7. A
8. $\overline{JG}$
9. What is the longest chord in any circle?

Copy each pair of circles. Draw in all common tangents.

10.



11.



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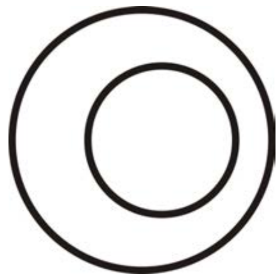
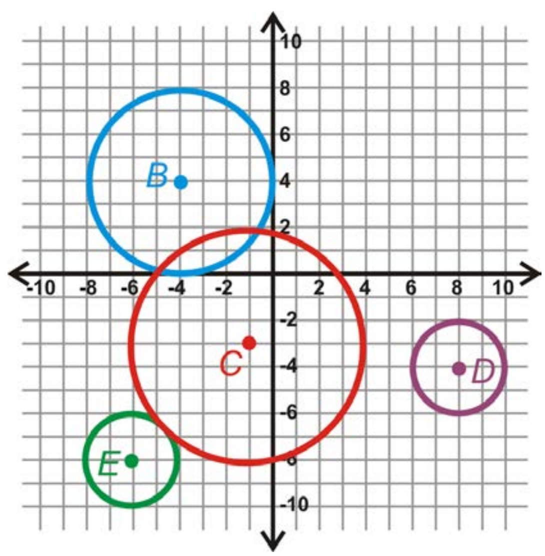


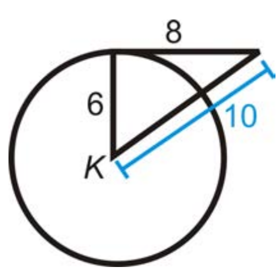
Coordinate Geometry Use the graph below to answer the following questions.

13. Find the radius of each circle.
14. Are any circles congruent? How do you know?
15. Find all the common tangents for $\odot B$ and $\odot C$.
16. $\odot C$ and $\odot E$ are externally tangent. What is CE?
17. Find the equation of $\overline{CE}$.



Determine whether the given segment is tangent to $\odot K$.

18.



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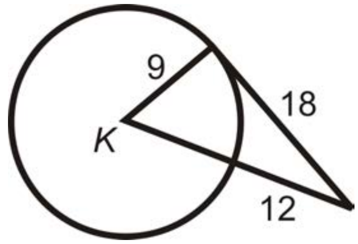

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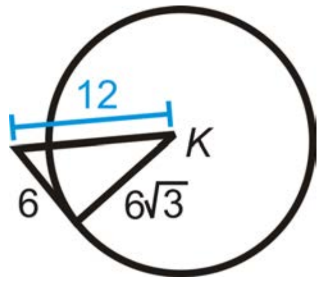
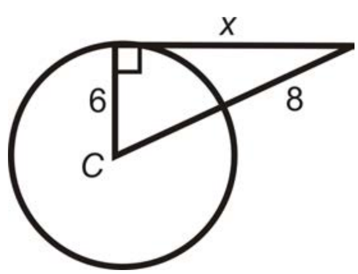


Algebra Connection Find the value of the indicated length(s) in $\odot C$. A and B are points of tangency. Simplify all radicals.

21.



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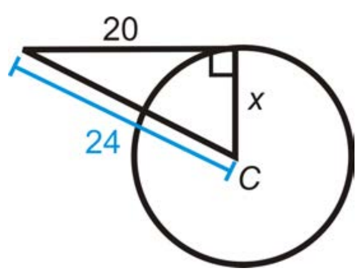

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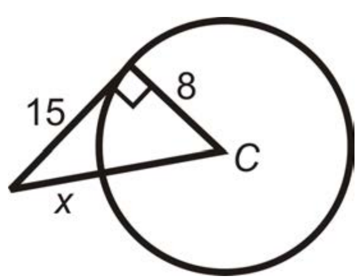

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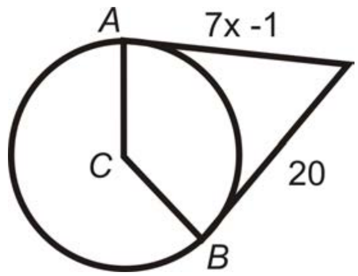

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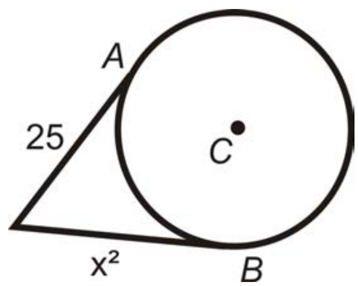
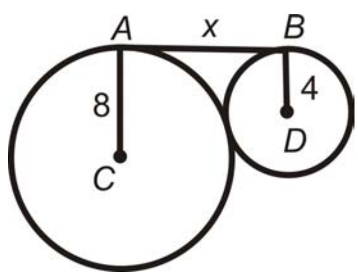
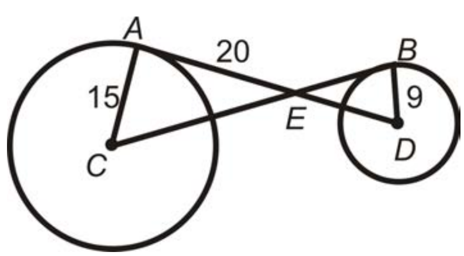


26.

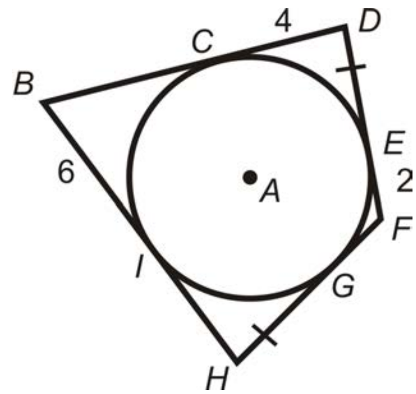


A and B are points of tangency for $\odot C$ and $\odot D$.



27. Is $\triangle AEC \sim \triangle BED$? Why?
28. Find CE.
29. Find BE.
30. Find ED.
31. Find BC and AD.

$\odot A$ is inscribed in $BDFH$.



32. Find the perimeter of $BDFH$.
33. What type of quadrilateral is $BDFH$? How do you know?
34. Draw a circle inscribed in a square. If the radius of the circle is 5, what is the perimeter of the square?
35. Can a circle be inscribed in a rectangle? If so, draw it. If not, explain.
36. Draw a triangle with two sides tangent to a circle, but the third side is not.
37. Can a circle be inscribed in an obtuse triangle? If so, draw it. If not, explain.
38. Fill in the blanks in the proof of Theorem 9-2. Given: $\overline{AB}$ and $\overline{CB}$ with points of tangency at A and C. $\overline{AD}$ and $\overline{DC}$ are radii. Prove: $\overline{AB} \cong \overline{CB}$

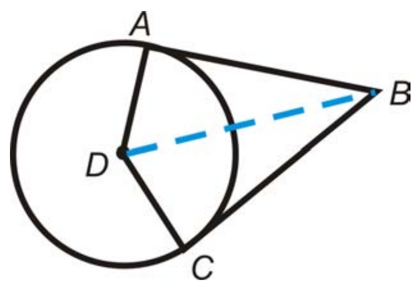


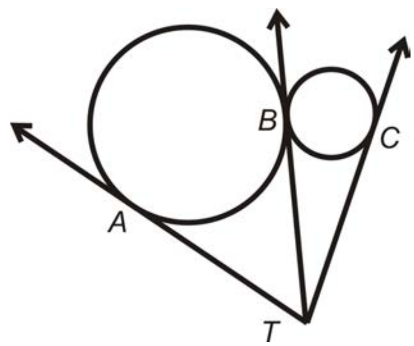
TABLE 10.1:

Statement	***Reason***
1.	
2. $\overline{AD} \cong \overline{DC}$	
3. $\overline{DA} \perp \overline{AB}$ and $\overline{DC} \perp \overline{CB}$	
4.	Definition of perpendicular lines
5.	Connecting two existing points
6. $\triangle ADB$ and $\triangle DCB$ are right triangles	
7. $\overline{DB} \cong \overline{DB}$	
8. $\triangle ABD \cong \triangle CBD$	
9. $\overline{AB} \cong \overline{CB}$	

39. Fill in the blanks, using the proof from #38.
    a. $ABCD$ is a _____________ (type of quadrilateral).
    b. The line that connects the ___________ and the external point B __________ $\angle ABC$.

Review and Reflect

40. Points A, B, and C are points of tangency for the three tangent circles. *Explain* why $\overline{AT} \cong \overline{BT} \cong \overline{CT}$.



Warm-Up Answers

1. $y = \frac{2}{5}x + 4$
2. $y = -2x + 3$
3. $m = \frac{2-(-1)}{6-(-3)} = \frac{3}{9} = \frac{1}{3}$
   $y = \frac{1}{3}x + b \rightarrow \text{plug in}(6,2)$
   $2 = \frac{1}{3}(6) + b$
   $2 = 2 + b \rightarrow b = 0$
   $y = \frac{1}{3}x$
4. $m_{\perp} = -3$
   $11 = -3(-8) + b$
   $11 = 24 + b \rightarrow b = -13$
   $y = -3x - 13$

10.2 Properties of Arcs

TEKS G(5)A

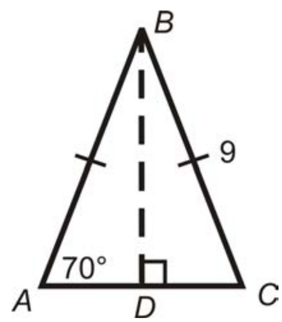
Learning Objectives

- Define and measure central angles, minor arcs, and major arcs

Vocabulary

- Central Angle
- Arc
- Minor Arc
- Major Arc
- Semicircle
- Congruent Arcs

Warm-Up



1. What kind of triangle is $\triangle ABC$?

2. How does $\overline{BD}$ relate to $\triangle ABC$?

3. Find $m\angle ABC$ and $m\angle ABD$.

Round to the nearest tenth. *Use the trig ratios*.

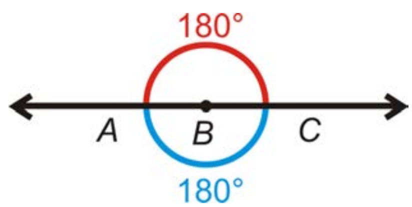
4. Find AD.

5. Find AC.

Know What? The Ferris wheel to the right has equally spaced seats, such that the central angle is 20°. How many seats are on this ride? Why do you think it is important to have equally spaced seats on a Ferris wheel?

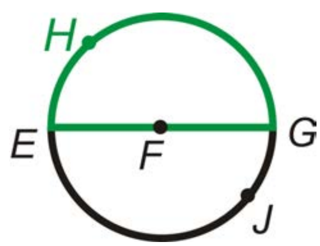
Central Angles & Arcs

Recall that a straight angle is 180°. If take two straight angles and put one on top of the other, we would have a circle. This means that a circle has 360°, $180° + 180°$. This also means that a semicircle, or half circle, is 180°.



Arc: A section of the circle.

Semicircle: An arc that measures 180°.



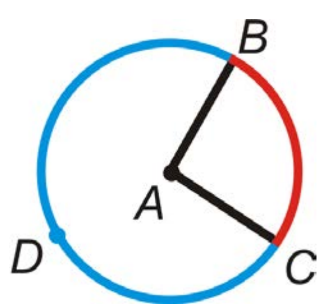
To label an arc, place a curve above the endpoints. You may want to use 3 points to clarify.

$\widehat{EHG}$ and $\widehat{EJG}$ are semicircles $\qquad m\widehat{EHG} = 180°$

Central Angle: The angle formed by two radii and its vertex at the center of the circle.

Minor Arc: An arc that is less than 180°

Major Arc: An arc that is greater than 180°. ***Always*** use 3 letters to label a major arc.



The central angle is $\angle BAC$.

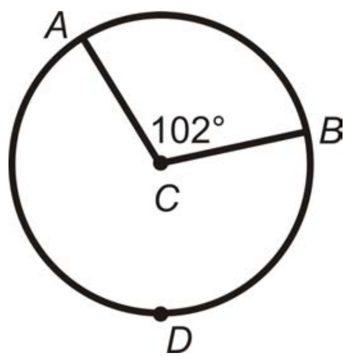
The minor arc is $\widehat{BC}$.

The major arc is $\widehat{BDC}$.

Every central angle divides a circle into two arcs.

An arc can be measured in degrees or in a linear measure (cm, ft, etc.). In this chapter we will use degree measure. ***The measure of the minor arc is the same as the measure of the central angle*** that corresponds to it. The measure of the major arc is 360° minus the measure of the minor arc.

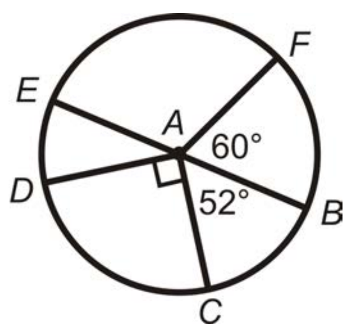
Example 1: Find $m\widehat{AB}$ and $m\widehat{ADB}$ in $\odot C$.



Solution: $m\widehat{AB} = m\widehat{ACB}$. So, $m\widehat{AB} = 102^\circ$.

$$m\widehat{ADB} = 360^\circ - m\widehat{AB} = 360^\circ - 102^\circ = 258^\circ$$

Example 2: Find the measures of the arcs in $\odot A$. $\overline{EB}$ is a diameter.

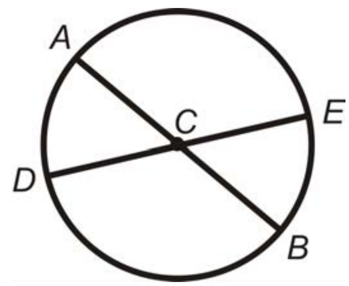


Solution: Because $\overline{EB}$ is a diameter, $m\angle EAB = 180^\circ$. Each arc is the same as its corresponding central angle.

$$\begin{aligned}
m\widehat{BF} &= m\angle FAB = 60^\circ \\
m\widehat{EF} &= m\angle EAF = 120^\circ \rightarrow 180^\circ - 60^\circ \\
m\widehat{ED} &= m\angle EAD = 38^\circ \ \rightarrow 180^\circ - 90^\circ - 52^\circ \\
m\widehat{DC} &= m\angle DAC = 90^\circ \\
m\widehat{BC} &= m\angle BAC = 52^\circ
\end{aligned}$$

Congruent Arcs: Two arcs are congruent if their central angles are congruent.

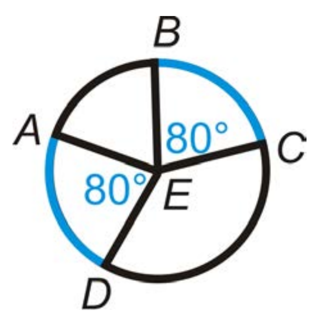
Example 3: List the congruent arcs in $\odot C$ below. $\overline{AB}$ and $\overline{DE}$ are diameters.



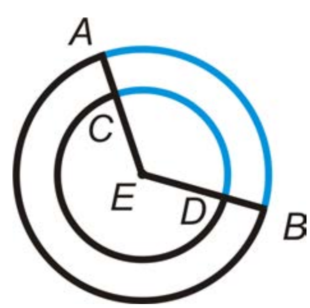
Solution: $\angle ACD = \angle ECB$ because they are vertical angles. $\angle DCB = \angle ACE$ because they are also vertical angles. $\widehat{AD} \cong \widehat{EB}$ and $\widehat{AE} \cong \widehat{DB}$

Example 4: Are the blue arcs congruent? Explain why or why not.

a)



b)



Solution:

a) $\widehat{AD} \cong \widehat{BC}$ because they have the same central angle measure and in the same circle.

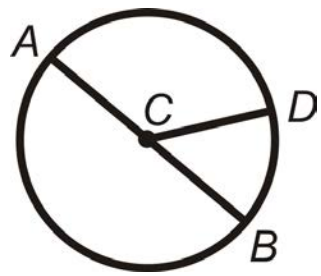
b) The two arcs have the same measure, but are not congruent because the circles have different radii.

Arc Addition Postulate

Just like the Angle Addition Postulate and the Segment Addition Postulate, there is an Arc Addition Postulate.

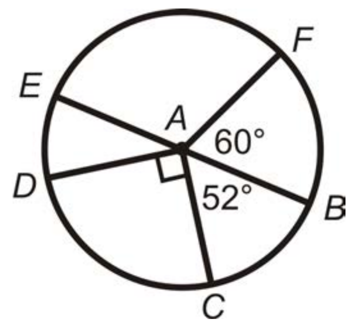
Arc Addition Postulate

The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs



$$m\widehat{AD} + m\widehat{DB} = m\widehat{ADB}$$

Example 5: Find the measure of the arcs in $\odot A$. $\overline{EB}$ is a diameter.



a) $m\widehat{FED}$

b) $m\widehat{CDF}$

c) $m\widehat{DFC}$

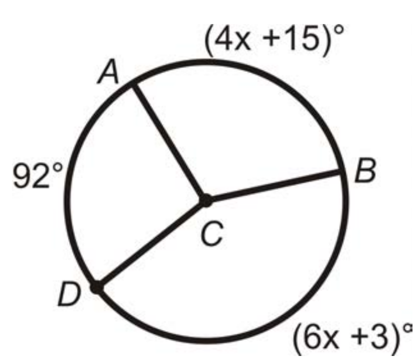
Solution: Use the Arc Addition Postulate.

a) $m\widehat{FED} = m\widehat{FE} + m\widehat{ED} = 120^\circ + 38^\circ = 158^\circ$

b) $m\widehat{CDF} = m\widehat{CD} + m\widehat{DE} + m\widehat{EF} = 90^\circ + 38^\circ + 120^\circ = 248^\circ$

c) $m\widehat{DFC} = 38^\circ + 120^\circ + 60^\circ + 52^\circ = 270^\circ$

Example 6: ***Algebra Connection*** Find the value of x for $\odot C$ below.



Solution:

$$\begin{aligned} m\overset{\frown}{AB}+m\overset{\frown}{AD}+m\overset{\frown}{DB} &= 360^\circ \\ (4x+15)^\circ+92^\circ+(6x+3)^\circ &= 360^\circ \\ 10x+110^\circ &= 360^\circ \\ 10x &= 250^\circ \\ x &= 25^\circ \end{aligned}$$

Know What? Revisited Because the seats are 20° apart, there will be $\frac{360^\circ}{20^\circ}=18$ seats. It is important to have the seats evenly spaced for the balance of the Ferris wheel.

MEDIA

Click image to the left or use the URL below.

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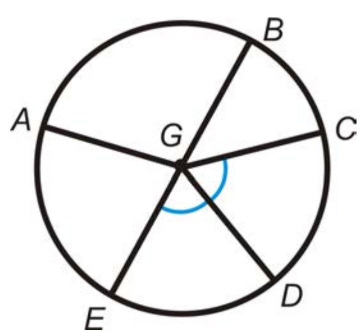
Practice Problems

- Questions 1-6 use the definition of minor arc, major arc, and semicircle.
- Question 7 is similar to Example 3.
- Questions 8 and 9 are similar to Example 5.
- Questions 10-15 are similar to Example 1.
- Questions 16-18 are similar to Example 4.
- Questions 19-26 are similar to Example 2 and 5.
- Questions 27-29 are similar to Example 6.
- Question 30 is a challenge.

Determine if the arcs below are a minor arc, major arc, or semicircle of $\odot G$. $\overline{EB}$ is a diameter.

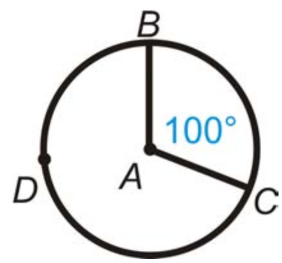


1. $\overset{\frown}{AB}$
2. $\overset{\frown}{ABD}$
3. $\overset{\frown}{BCE}$
4. $\overset{\frown}{CAE}$
5. $\overset{\frown}{ABC}$
6. $\overset{\frown}{EAB}$
7. Are there any congruent arcs? If so, list them.
8. If $m\overset{\frown}{BC}=48^\circ$, find $m\overset{\frown}{CD}$.

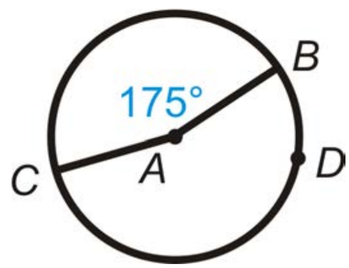
9. Using #8, find $m\widehat{CAE}$.

Find the measure of the minor arc and the major arc in each circle below.

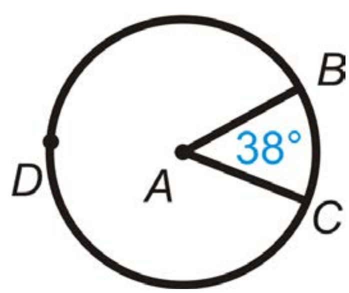
10.



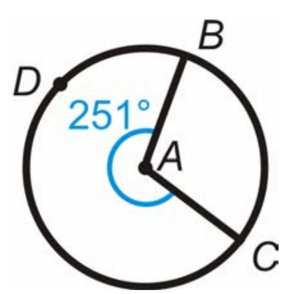
11.



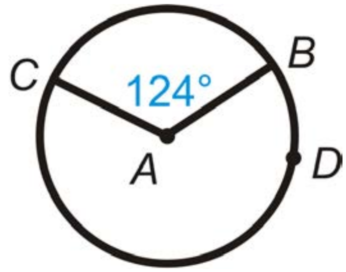
12.



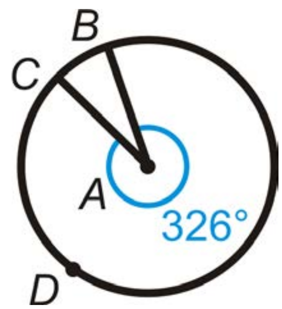
13.



14.

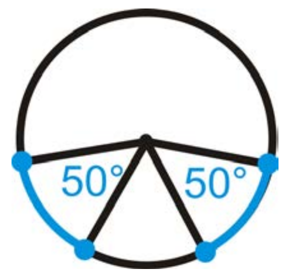


15.

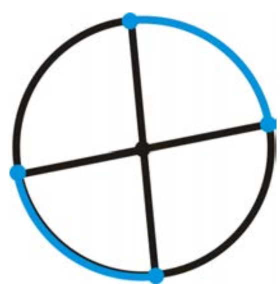


Determine if the blue arcs are congruent. If so, state why.

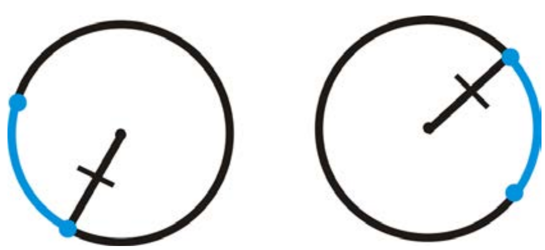
16.



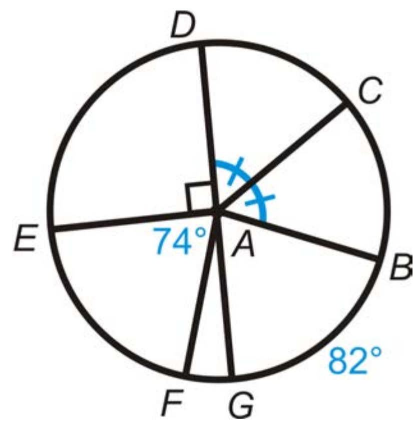
17.



18.



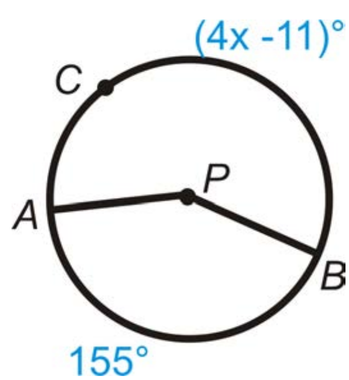
Find the measure of the indicated arcs or central angles in $\odot A$. $\overline{DG}$ is a diameter.



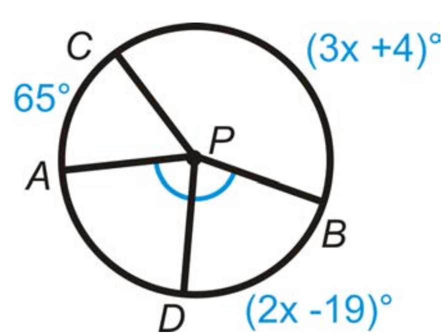
19. $\widehat{DE}$
20. $\widehat{DC}$
21. $\widehat{GAB}$
22. $\widehat{FG}$
23. $\widehat{EDB}$
24. $\widehat{EAB}$
25. $\widehat{DCF}$
26. $\widehat{DBE}$

Algebra Connection Find the measure of x in $\odot P$.

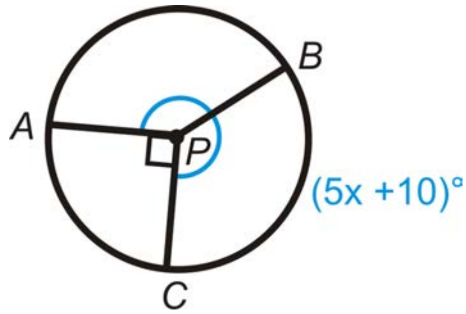
27.



28.

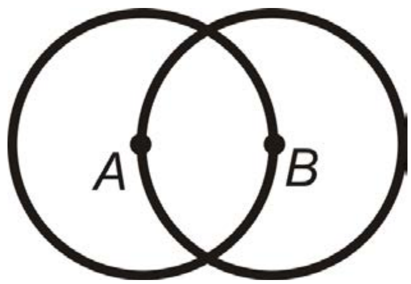


29.



Review and Reflect

30. What can you conclude about $\odot A$ *and* $\odot B$?



Warm-Up Answers

1. isosceles
2. $\overline{BD}$ is the angle bisector of $\angle ABC$ and the perpendicular bisector of $\overline{AC}$.
3. $m\angle ABC = 40^\circ, m\angle ABD = 25^\circ$
4. $\cos 70^\circ = \frac{AD}{9} \rightarrow AD = 9 \cdot \cos 70^\circ = 3.1$
5. $AC = 2 \cdot AD = 2 \cdot 3.1 = 6.2$

10.3 Radian Measure

Here you'll learn what radian measure is, and how to find the radian values for common angles on the unit circle.

TEKS G(5)A, G(12)D

Learning Objectives

- Derive and use radian measures of an angle

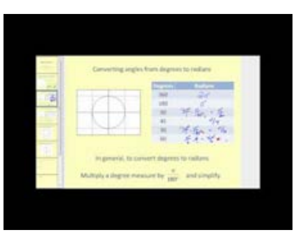
Vocabulary

- degree
- radian

Warm-Up

While working on an experiment in your school science lab, your teacher asks you to turn up a detector by rotating the knob $\frac{\pi}{2}$ radians. You are immediately puzzled since you don't know what a radian measure is. How far will you need to turn the knob?

Watch This



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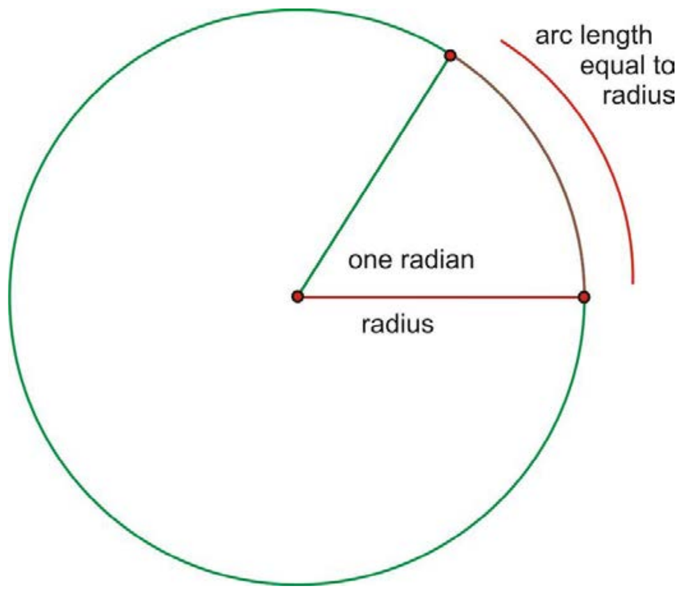
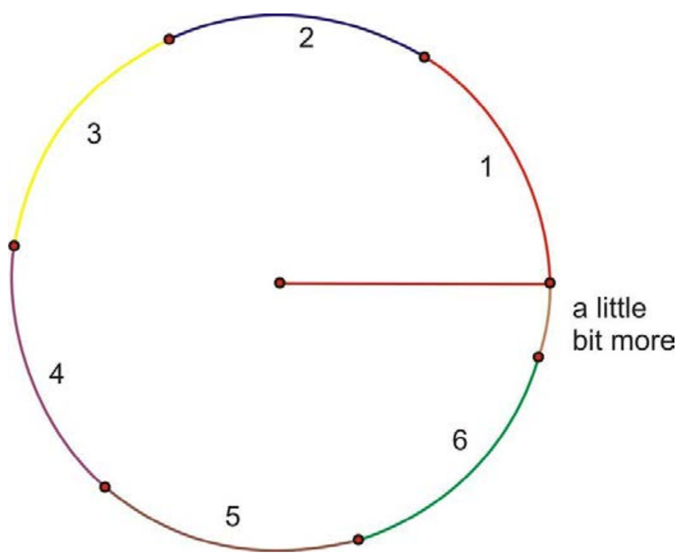
James Sousa: Radian Measure

Until now, we have used degrees to measure angles. But, what exactly is a degree? A **degree** is $\frac{1}{360^{th}}$ of a complete rotation around a circle. **Radians** are alternate units used to measure angles in trigonometry. Just as it sounds, a radian is based on the *radius* of a circle. One **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle. Because a radian is based on an actual part of the circle rather than an arbitrary division, it is a much more natural unit of angle measure for upper level mathematics.



What if we were to rotate all the way around the circle? Continuing to add radius lengths, we find that it takes a little more than 6 of them to complete the rotation.

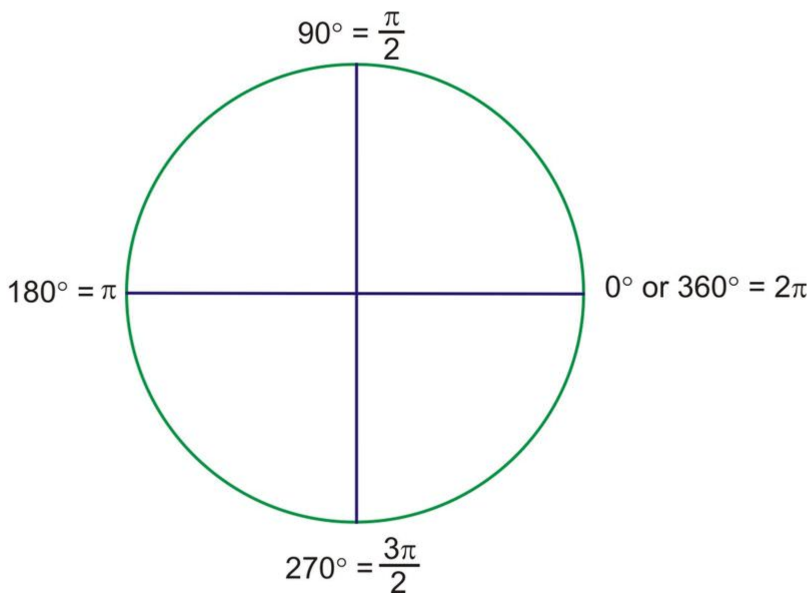


Recall from geometry that the arc length of a complete rotation is the circumference, where the formula is equal to 2π times the length of the radius. 2π is approximately 6.28, so the circumference is a little more than 6 radius lengths. Or, in terms of radian measure, a complete rotation (360 degrees) is 2π radians.

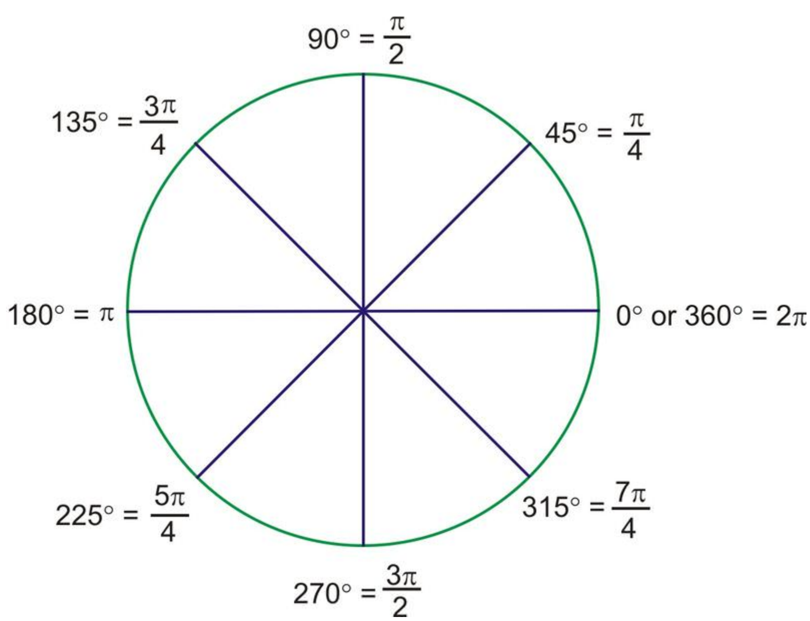
$$360 \text{ degrees} = 2\pi \text{ radians}$$

With this as our starting point, we can find the radian measure of other angles. Half of a rotation, or 180 degrees, must therefore be π radians, and 90 degrees must be $\frac{1}{2}\pi$, written $\frac{\pi}{2}$.

Extending the radian measure past the first quadrant, the quadrantal angles have been determined, except 270°. Because 270° is halfway between $180^\circ(\pi)$ and $360^\circ(2\pi)$, it must be 1.5π, usually written $\frac{3\pi}{2}$.



For the 45° angles, the radians are all multiples of $\frac{\pi}{4}$. For example, 135° is $3 \cdot 45^\circ$. Therefore, the radian measure should be $3 \cdot \frac{\pi}{4}$, or $\frac{3\pi}{4}$. Here are the rest of the multiples of 45°, in radians:



Notice that the additional angles in the drawing all have reference angles of 45 degrees and their radian measures are all multiples of $\frac{\pi}{4}$. All of the even multiples are the quadrantal angles and are reduced, just like any other fraction.

Example 1: Find the radian measure of these angles.

TABLE 10.2:

Angle in Degrees	**Angle in Radians**
90	$\frac{\pi}{2}$
45	
30	

Solution: Because 45 is half of 90, half of $\frac{1}{2}\pi$ is $\frac{1}{4}\pi$. 30 is one-third of a right angle, so multiplying gives:

$$\frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6}$$

and because 60 is twice as large as 30:

$$2 \times \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

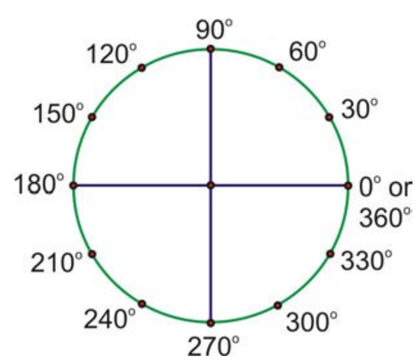
Here is the completed table:

TABLE 10.3:

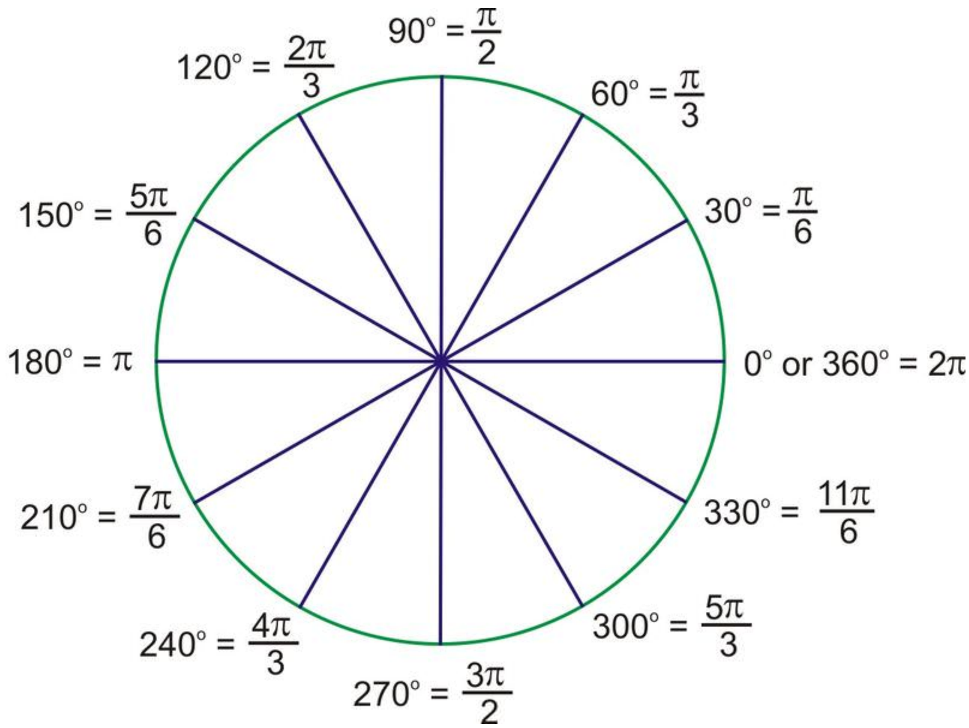
Angle in Degrees	**Angle in Radians**
90	$\frac{\pi}{2}$
45	$\frac{\pi}{4}$
30	$\frac{\pi}{6}$

There is a formula to convert between radians and degrees that you may already have discovered while doing this example. However, many angles that are commonly used can be found easily from the values in this table. For example, most students find it easy to remember 30 and 60. 30 is π over ***6*** and 60 is π over ***3***. Knowing these angles, you can find any of the special angles that have reference angles of 30 and 60 because they will all have the same denominators. The same is true of multiples of $\frac{\pi}{4}$ (45 degrees) and $\frac{\pi}{2}$ (90 degrees).

Example 2: Complete the following radian measures by counting in multiples of $\frac{\pi}{3}$ and $\frac{\pi}{6}$:



Solution:



Notice that all of the angles with 60-degree reference angles are multiples of $\frac{\pi}{3}$, and all of those with 30-degree reference angles are multiples of $\frac{\pi}{6}$. Counting in these terms based on this pattern, rather than converting back to degrees, will help you better understand radians.

Example 3: Find the radian measure of these angles.

TABLE 10.4:

Angle in Degrees	Angle in Radians
120	$\frac{2\pi}{3}$
180	
240	
270	
300	

Solution: Because 30 is one-third of a right angle, multiplying gives:

$$\frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6}$$

adding this to the known value for ninety degrees of $\frac{\pi}{2}$:

$$\frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

Here is the completed table:

TABLE 10.5:

Angle in Degrees	Angle in Radians
120	$\frac{2\pi}{3}$
180	π
240	$\frac{4\pi}{3}$
300	$\frac{5\pi}{3}$

Example 4: Convert the following degrees to radians.

a. Give the radian measure of 60°

b. Give the radian measure of 75°

c. Give the radian measure of 180°

Solutions:

a. 30 is one-third of a right angle. This means that since $90^\circ = \frac{\pi}{2}$, then $30^\circ = \frac{\pi}{6}$. Therefore, multiplying gives:

$\frac{\pi}{6} \times 2 = \frac{\pi}{3}$

b. 15 is one-sixth of a right triangle. This means that since $90^\circ = \frac{\pi}{2}$, then $15^\circ = \frac{\pi}{12}$. Therefore, multiplying gives:

$\frac{\pi}{12} \times 5 = \frac{5\pi}{12}$

c. Since $90^\circ = \frac{\pi}{2}$, then $180^\circ = \frac{2\pi}{2} = \pi$

Check this out! http://www.mathsisfun.com/geometry/radians.html

Practice Problems

- Questions 1-6 are similar to Example 4.
- Questions 7-12 are similar to Example 1-3.

Practice Problems

Find the radian measure of each angle.

1. 90°
2. 120°
3. 300°
4. 330°
5. -45°
6. 135°

Find the degree measure of each angle.

7. $\frac{3\pi}{2}$
8. $\frac{5\pi}{4}$
9. $\frac{7\pi}{6}$
10. $\frac{\pi}{6}$
11. $\frac{5\pi}{3}$
12. π

Review and Reflect

13. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.
14. Explain in your own words why it makes sense that there are 2π radians in a circle.

Warm-Up Answers

Since $45^\circ = \frac{\pi}{4} rad$, then $2 \times \frac{\pi}{4} = \frac{\pi}{2} = 2 \times 45^\circ$. Therefore, a turn of $\frac{\pi}{2}$ is equal to 90°, which is $\frac{1}{4}$ of a complete rotation of the knob.

10.4 Properties of Chords

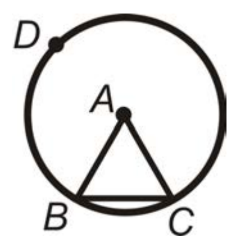
TEKS G(12)A

Learning Objectives

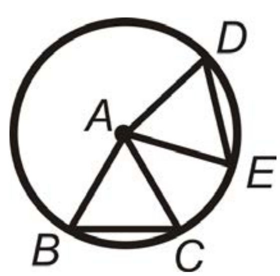
- Find the lengths of chords in a circle
- Discover properties of chords and arcs

Warm-Up

1. Draw a chord in a circle.
2. Draw a diameter in the circle from #1. Is a diameter a chord?
3. $\triangle ABC$ is an equilateral triangle in $\odot A$. Find $m\widehat{BC}$ and $m\widehat{BDC}$.



4. $\triangle ABC$ and $\triangle ADE$ are equilateral triangles in $\odot A$. List a pair of congruent arcs and chords.



Know What? To the right is the Gran Teatro Falla, in Cadiz, Andalucía, Spain. Notice the five windows, $A - E$. $\odot A \cong \odot E$ and $\odot B \cong \odot C \cong \odot D$. Each window is topped with a 240° arc. The gold chord in each circle connects the rectangular portion of the window to the circle. Which chords are congruent?

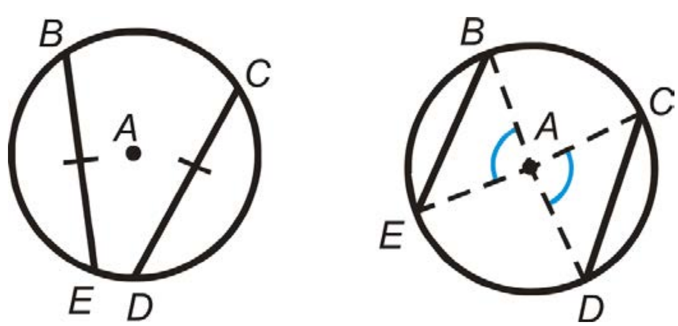
Recall from the first section, a chord is a line segment whose endpoints are on a circle. A diameter is the longest chord in a circle.

Congruent Chords & Congruent Arcs

From #4 in the Review Queue above, we noticed that $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$.

Theorem 10-3

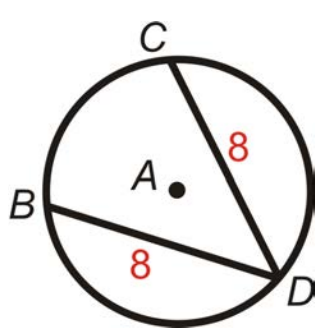
In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent



In both of these pictures, $\overline{BE} \cong \overline{CD}$ and $\widehat{BE} \cong \widehat{CD}$.

In the second circle, $\triangle BAE \cong \triangle CAD$ by SAS.

Example 1: Use $\odot A$ to answer the following.



a) If $m\widehat{BD} = 125^\circ$, find $m\widehat{CD}$.

b) If $m\widehat{BC} = 80^\circ$, find $m\widehat{CD}$.

Solution:

a) $BD = CD$, which means the arcs are equal too. $m\widehat{CD} = 125^\circ$.

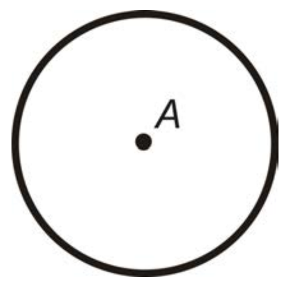
b) $m\widehat{CD} \cong m\widehat{BD}$ because $BD = CD$.

$$\begin{aligned} m\widehat{BC} + m\widehat{CD} + m\widehat{BD} &= 360^\circ \\ 80^\circ + 2m\widehat{CD} &= 360^\circ \\ 2m\widehat{CD} &= 280^\circ \\ m\widehat{CD} &= 140^\circ \end{aligned}$$

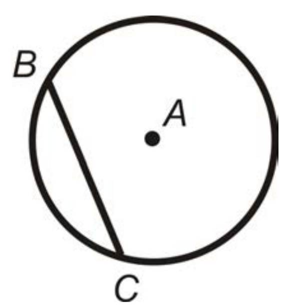
Investigation 10-2: Perpendicular Bisector of a Chord

Tools Needed: paper, pencil, compass, ruler

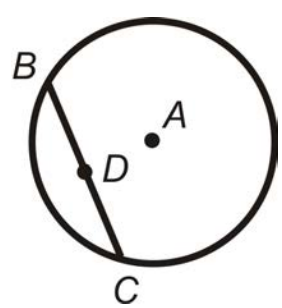
1. Draw a circle. Label the center A.



2. Draw a chord. Label it $\overline{BC}$.



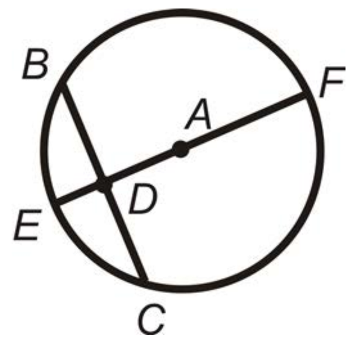
3. Find the midpoint of $\overline{BC}$ using a ruler. Label it D.



4. Connect A and D to form a diameter. How does $\overline{AD}$ relate to $\overline{BC}$?

Theorem 10-4

The perpendicular bisector of a chord is also a diameter



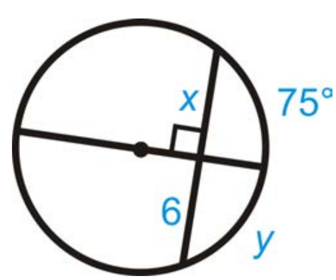
If $\overline{AD} \perp \overline{BC}$ and $\overline{BD} \cong \overline{DC}$ then $\overline{EF}$ is a diameter.

If $\overline{EF} \perp \overline{BC}$, then $\overline{BD} \cong \overline{DC}$ and $\widehat{BE} \cong \widehat{EC}$.

Theorem 10-5

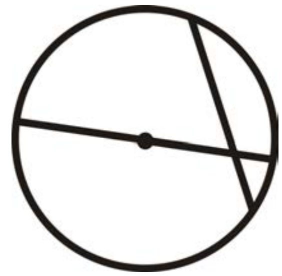
If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc

Example 2: Find the value of x and y.



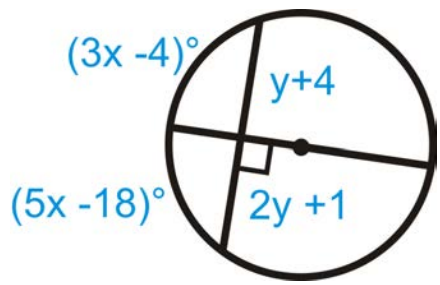
Solution: The diameter perpendicular to the chord. From Theorem 10-5, $x = 6$ and $y = 75^\circ$.

Example 3: Is the converse of Theorem 10-4 true?



Solution: The converse of Theorem 10-4 would be: *A diameter is also the perpendicular bisector of a chord.* This is not true, a diameter cannot always be a perpendicular bisector to every chord. See the picture.

Example 4: ***Algebra Connection*** Find the value of x and y.



Solution: The diameter is perpendicular to the chord, which means it bisects the chord and the arc. Set up an equation for x and y.

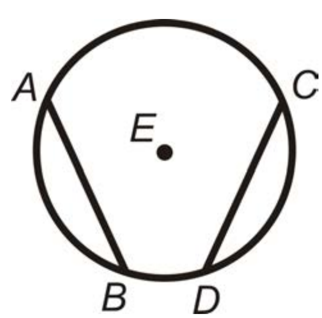
$$\begin{aligned} (3x-4)^\circ &= (5x-18)^\circ & y+4 &= 2y+1 \\ 14^\circ &= 2x & 3 &= y \\ 7^\circ &= x \end{aligned}$$

Equidistant Congruent Chords

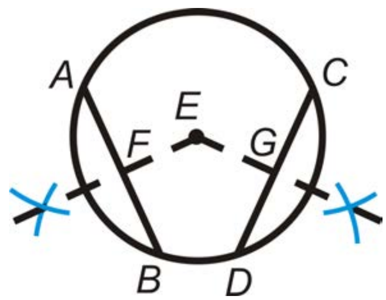
Investigation 10-3: Properties of Congruent Chords

Tools Needed: pencil, paper, compass, ruler

1. Draw a circle with a radius of 2 inches and two chords that are both 3 inches. Label like the picture to the right. *This diagram is drawn* to scale.



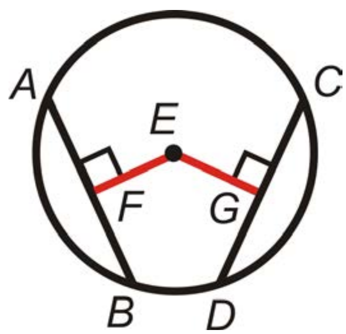
2. From the center, draw the perpendicular segment to $\overline{AB}$ and $\overline{CD}$. You can use Investigation 3-2



3. Erase the arc marks and lines beyond the points of intersection, leaving $\overline{FE}$ and $\overline{EG}$. Find the measure of these segments. What do you notice?

Theorem 10-6

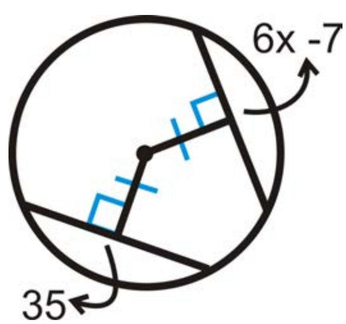
In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center



The shortest distance from any point to a line is the perpendicular line between them.

If $FE = EG$ and $\overline{EF} \perp \overline{EG}$, then $\overline{AB}$ and $\overline{CD}$ are equidistant to the center and $\overline{AB} \cong \overline{CD}$.

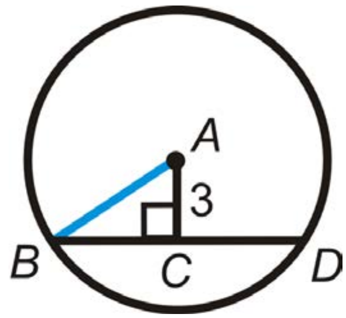
Example 5: ***Algebra Connection*** Find the value of x.



Solution: Because the distance from the center to the chords is congruent and perpendicular to the chords, the chords are equal.

$$\begin{aligned} 6x - 7 &= 35 \\ 6x &= 42 \\ x &= 7 \end{aligned}$$

Example 6: $BD = 12$ and $AC = 3$ in $\odot A$. Find the radius.



Solution: First find the radius. $\overline{AB}$ is a radius, so we can use the right triangle $\triangle ABC$, so $\overline{AB}$ is the hypotenuse. From Theorem 12-5, $BC = 6$.

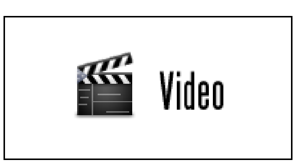
$$\begin{aligned}
3^2 + 6^2 &= AB^2 \\
9 + 36 &= AB^2 \\
AB &= \sqrt{45} = 3\sqrt{5}
\end{aligned}$$

Example 7: Find $m\widehat{BD}$ from Example 6.

Solution: First, find the corresponding central angle, $\angle BAD$. We can find $m\angle BAC$ using the tangent ratio. Then, multiply $m\angle BAC$ by 2 for $m\angle BAD$ and $m\widehat{BD}$.

$$\begin{aligned}
\tan^{-1}\left(\frac{6}{3}\right) &= m\angle BAC \\
m\angle BAC &\approx 63.43^\circ \\
m\angle BAD &\approx 2 \cdot 63.43^\circ \approx 126.86^\circ \approx m\widehat{BD}
\end{aligned}$$

Know What? Revisited In the picture, the chords from $\odot A$ and $\odot E$ are congruent and the chords from $\odot B$, $\odot C$, and $\odot D$ are also congruent. We know this from Theorem 10-3.



MEDIA

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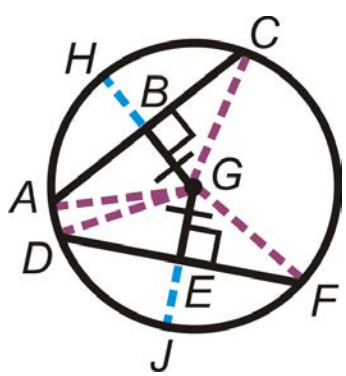
Practice Problems

- Questions 1-3 use the theorems from this section and similar to Example 3.
- Questions 4-10 use the definitions and theorems from this section.
- Questions 11-16 are similar to Example 1 and 2.
- Questions 17-25 are similar to Examples 2, 4, 5, and 6.
- Questions 26 and 27 are similar to Example 7.
- Questions 28-30 use the theorems from this section.

1. Two chords in a circle are perpendicular and congruent. Does one of them have to be a diameter? Why or why not?
2. Write the converse of Theorem 10-5. Is it true? If not, draw a counterexample.
3. Write the original and converse statements from Theorem 10-3 and Theorem 10-6.

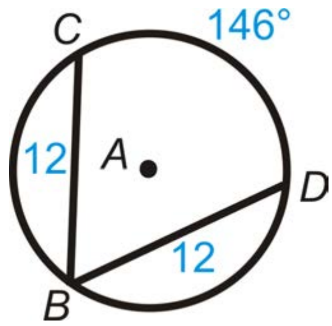
Fill in the blanks.



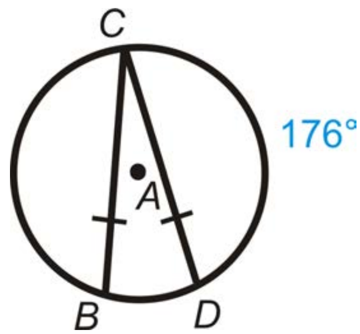
4. ____ $\cong \overline{DF}$
5. $\widehat{AC} \cong$ ____
6. $\widehat{DJ} \cong$ ____
7. ____ $\cong \overline{EJ}$
8. $\angle AGH \cong$ ____
9. $\angle DGF \cong$ ____
10. List all the congruent radii in $\odot G$.

Find the value of the indicated arc in $\odot A$.

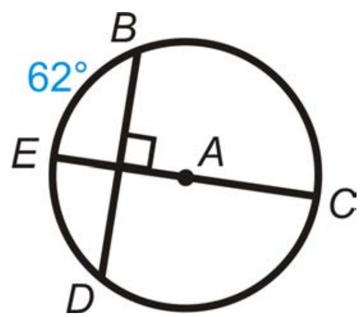
11.$m\widehat{BC}$



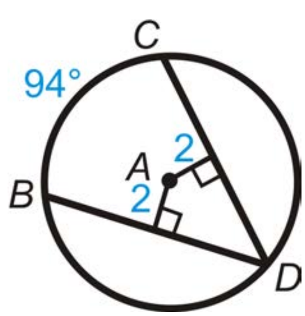
$m\widehat{BD}$12.



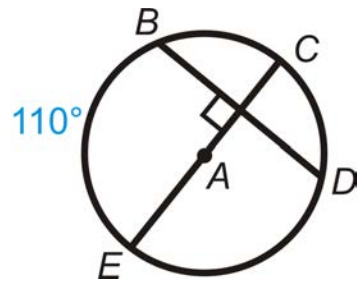
13. $m\widehat{BC}$



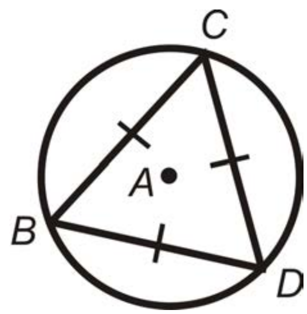
$m\widehat{BD}$14.



15. $m\widehat{BD}$

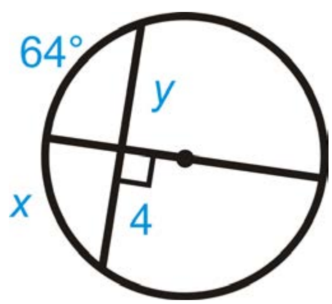


$m\widehat{BD}$16.

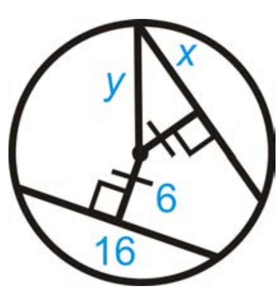


Algebra Connection Find the value of *x* and/or *y*.

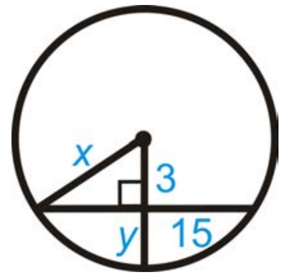
17.



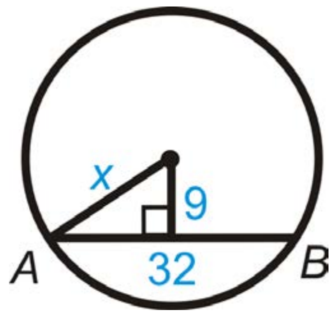
18.



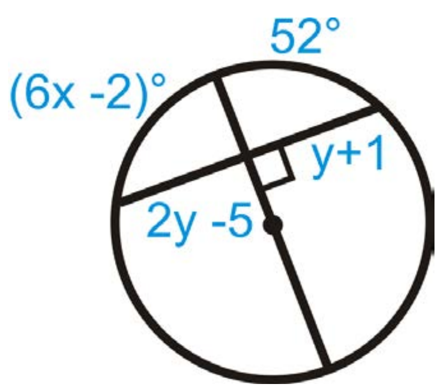
19.



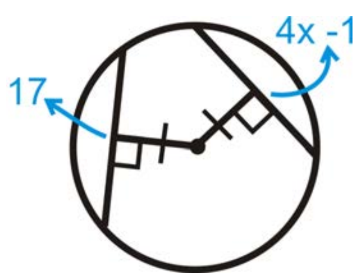
20. $AB = 32$



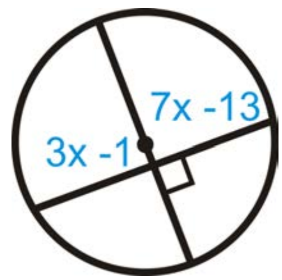
21.



22.



23.



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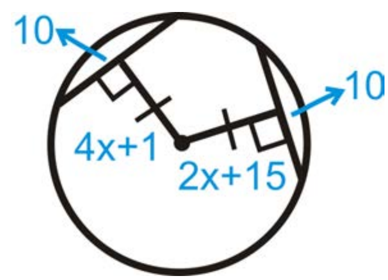
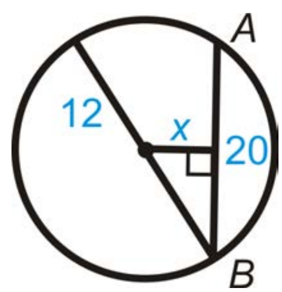


$AB = 2025$.

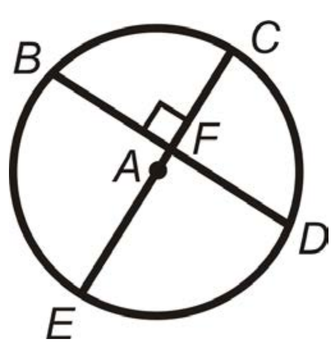


26. Find in Question 20. Round your answer to the nearest tenth of a degree.$m\widehat{AB}$

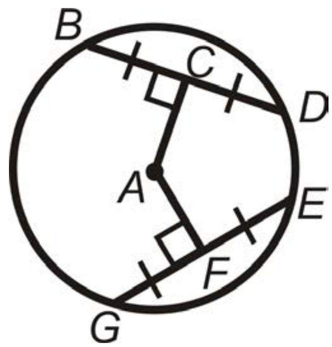
27. Find $m\widehat{AB}$ in Question 25. Round your answer to the nearest tenth of a degree.

In problems 28-30, what can you conclude about the picture? State a theorem that justifies your answer. You may assume that A is the center of the circle.

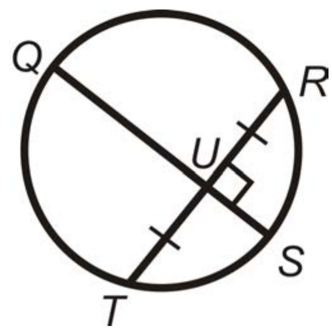
28.



29.



30.

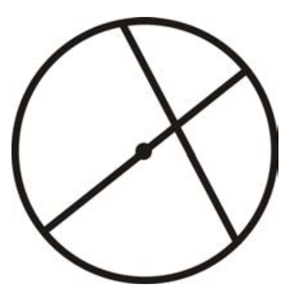


Review and Reflect

31. Theorems 10-3 and 10-6 begin with the phrase "In the same circle or congruent circles", explain why there has to be both stipulations.

Warm-Up Answers

1 & 2. Answers will vary



3. $m\widehat{BC} = 60^\circ, m\widehat{BDC} = 300^\circ$

4. $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$

10.5 Inscribed Angles

TEKS G(12)A

Learning Objectives

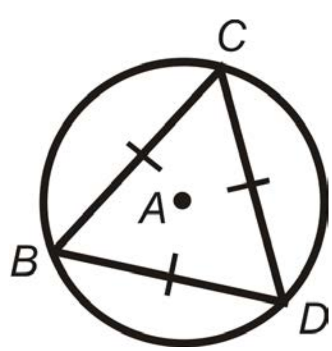
Find the measure of inscribed angles and the arcs they intercept

Vocabulary

- Inscribed Angle
- Intercepted Arc
- Inscribed Polygon

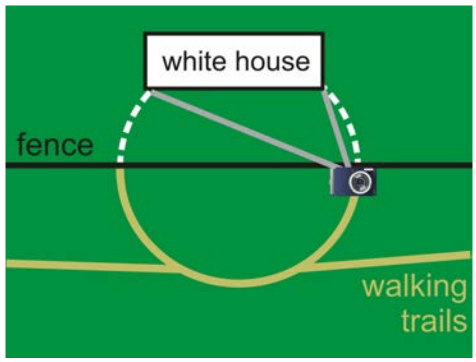
Warm-Up

We are going to use #14 from the homework in the previous section.



1. What is the measure of each angle in the triangle? How do you know?
2. What do you know about the three arcs?
3. What is the measure of each arc?

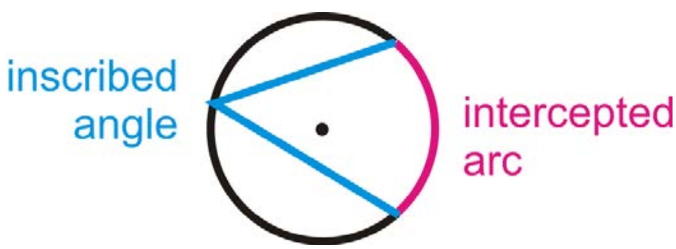
Know What? The closest you can get to the White House are the walking trails on the far right. You want to get as close as you can (on the trail) to the fence to take a picture (you were not allowed to walk on the grass). Where else can you take a picture from to get the same frame of the White House? *Your line of sight in the camera is marked in the picture as the grey lines.*



Inscribed Angles

In addition to central angles, we will now learn about inscribed angles in circles.

Inscribed Angle: An angle with its vertex on the circle and sides are chords.



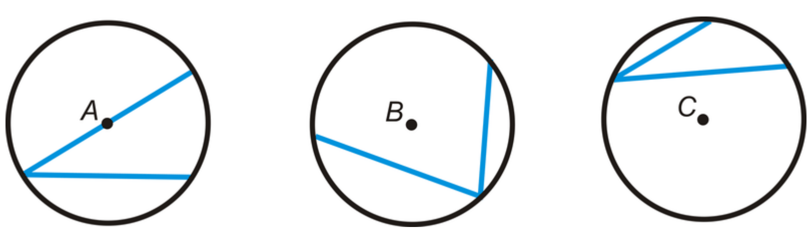
Intercepted Arc: The arc that is inside the inscribed angle and endpoints are on the angle.

The vertex of an inscribed angle can be anywhere on the circle as long as its sides intersect the circle to form an intercepted arc.

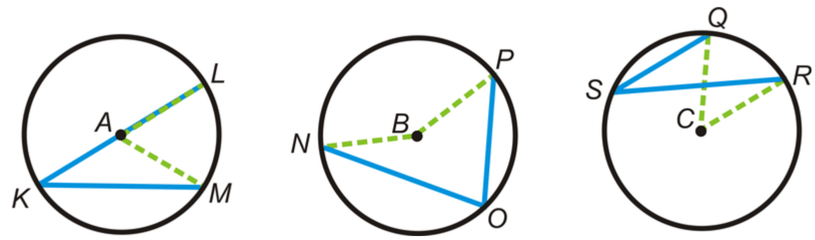
Investigation 10-4: Measuring an Inscribed Angle

Tools Needed: pencil, paper, compass, ruler, protractor

1. Draw three circles with three different inscribed angles. Try to make all the angles different sizes.



2. Using your ruler, draw in the corresponding central angle for each angle and label each set of endpoints.

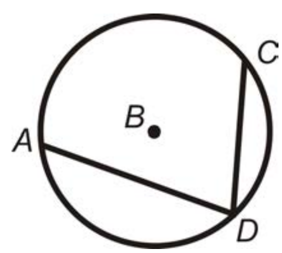


3. Using your protractor measure the six angles and determine if there is a relationship between the central angle, the inscribed angle, and the intercepted arc.

$m\angle LAM =$ _____	$m\angle NBP =$ _____	$m\angle QCR =$ _____
$m\widehat{LM} =$ _____	$m\widehat{NP} =$ _____	$m\widehat{QR} =$ _____
$m\angle LKM =$ _____	$m\angle NOP =$ _____	$m\angle QSR =$ _____

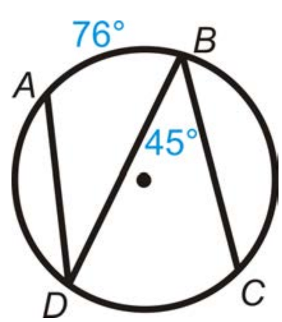
Inscribed Angle Theorem

The measure of an inscribed angle is half the measure of its intercepted arc



$$m\angle ADC = \frac{1}{2}m\widehat{AC}$$
$$m\widehat{AC} = 2m\angle ADC$$

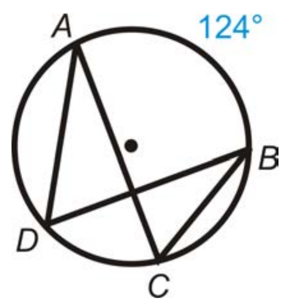
Example 1: Find $m\widehat{DC}$ and $m\angle ADB$.



Solution: From the Inscribed Angle Theorem:

$$m\widehat{DC} = 2 \cdot 45^\circ = 90^\circ$$
$$m\angle ADB = \frac{1}{2} \cdot 76^\circ = 38^\circ$$

Example 2: Find $m\angle ADB$ and $m\angle ACB$.



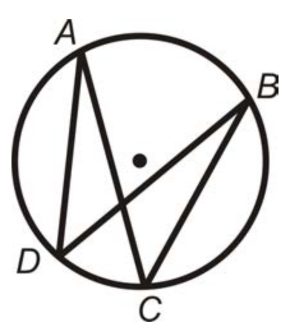
Solution: The intercepted arc for both angles is $\widehat{AB}$. Therefore,

$$m\angle ADB = \frac{1}{2} \cdot 124^\circ = 62^\circ$$
$$m\angle ACB = \frac{1}{2} \cdot 124^\circ = 62^\circ$$

This example leads us to our next theorem.

Theorem 10-8

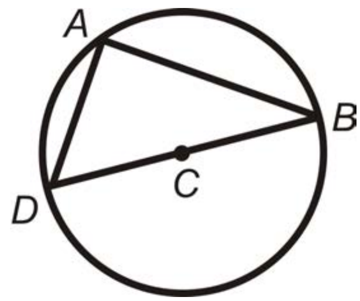
Inscribed angles that intercept the same arc are congruent



$\angle ADB$ and $\angle ACB$ intercept $\widehat{AB}$, so $m\angle ADB = m\angle ACB$.

$\angle DAC$ and $\angle DBC$ intercept $\widehat{DC}$, so $m\angle DAC = m\angle DBC$.

Example 3: Find $m\angle DAB$ in $\odot C$.

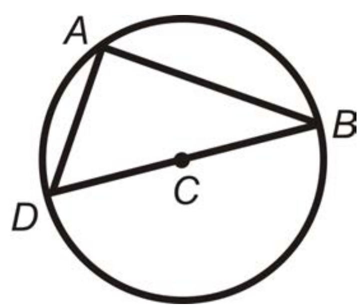


Solution: C is the center, so $\overline{DB}$ is a diameter. $\angle DAB$ endpoints are on the diameter, so the central angle is 180°.

$$m\angle DAB = \frac{1}{2} \cdot 180^\circ = 90^\circ.$$

Theorem 10-9

An angle intercepts a semicircle if and only if it is a right angle.

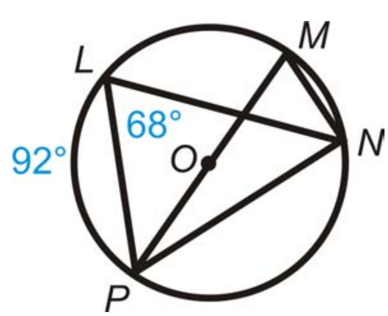


$\angle DAB$ intercepts a semicircle, so $m\angle DAB = 90^\circ$.

$\angle DAB$ is a right angle, so $\widehat{DB}$ is a semicircle.

Anytime a right angle is inscribed in a circle, the endpoints of the angle are the endpoints of a diameter and the diameter is the hypotenuse.

Example 4: Find $m\angle PMN$, $m\widehat{PN}$, $m\angle MNP$, and $m\angle LNP$.



Solution:

$$\begin{aligned} m\angle PMN = m\angle PLN = 68^\circ & \quad \text{by Theorem } 10-8. \\ m\widehat{PN} = 2 \cdot 68^\circ = 136^\circ & \quad \text{from the Inscribed Angle Theorem.} \\ m\angle MNP = 90^\circ & \quad \text{by Theorem } 10-9. \\ m\angle LNP = \frac{1}{2} \cdot 92^\circ = 46^\circ & \quad \text{from the Inscribed Angle Theorem.} \end{aligned}$$

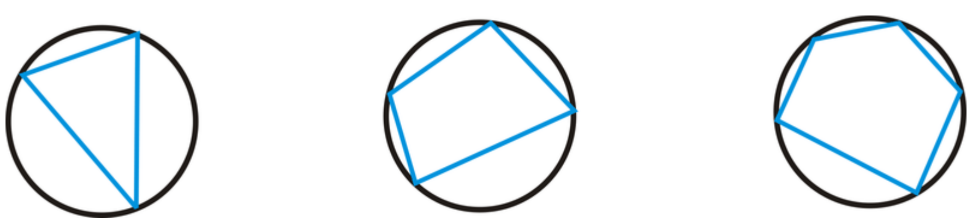
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/160664

Inscribed Quadrilaterals

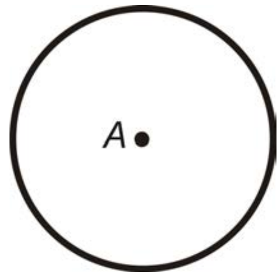
Inscribed Polygon: A polygon where every vertex is on a circle.



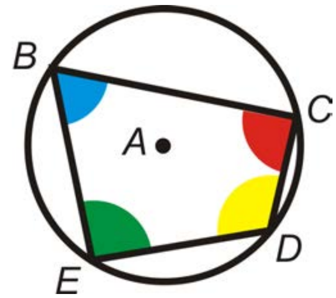
Investigation 10-5: Inscribing Quadrilaterals

Tools Needed: pencil, paper, compass, ruler, colored pencils, scissors

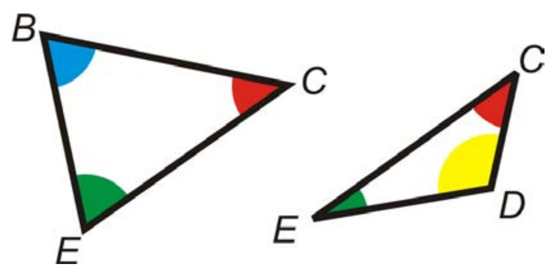
1. Draw a circle. Mark the center point A.



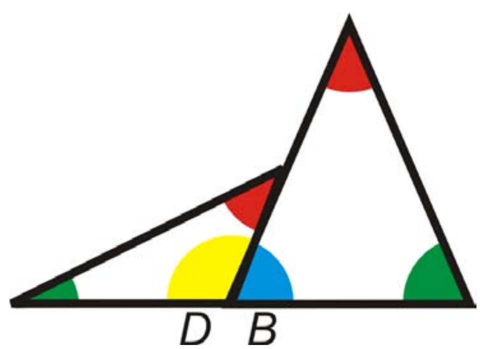
2. Place four points on the circle. Connect them to form a quadrilateral. Color in the 4 angles.



3. Cut out the quadrilateral. Then cut the diagonal $\overline{CE}$, making two triangles.



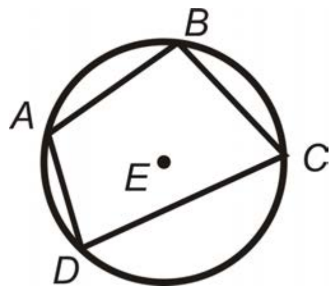
4. Line up $\angle B$ and $\angle D$ so that they are next to each other. What do you notice?



By cutting the quadrilateral in half, we are able to show that $\angle B$ and $\angle D$ form a linear pair when they are placed next to each other, making $\angle B$ and $\angle D$ supplementary.

Theorem 10-10

A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary

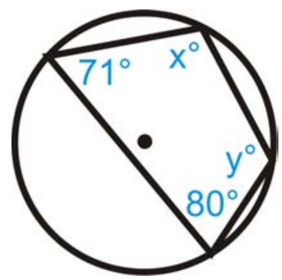


If $ABCD$ is inscribed in $\odot E$, then $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$.

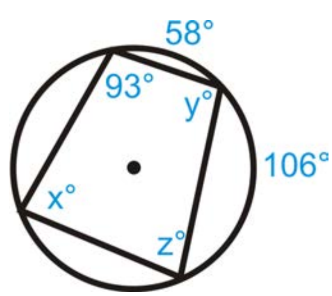
If $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$, then $ABCD$ is inscribed in $\odot E$.

Example 5: Find the value of the missing variables.

a)



b)

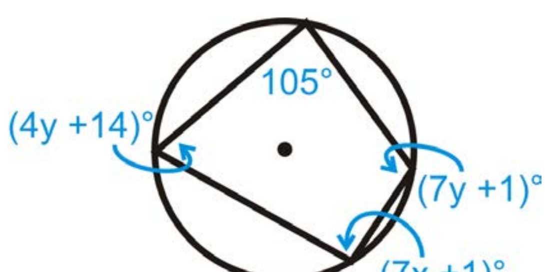


Solution:

a) $x + 80^\circ = 180^\circ$ $\quad$ $y + 71^\circ = 180^\circ$
$x = 100^\circ$ $\quad$ $y = 109^\circ$

b) $z + 93^\circ = 180^\circ$ $\quad$ $x = \frac{1}{2}(58^\circ + 106^\circ)$ $\quad$ $y + 82^\circ = 180^\circ$
$z = 87^\circ$ $\quad$ $x = 82^\circ$ $\quad$ $y = 98^\circ$

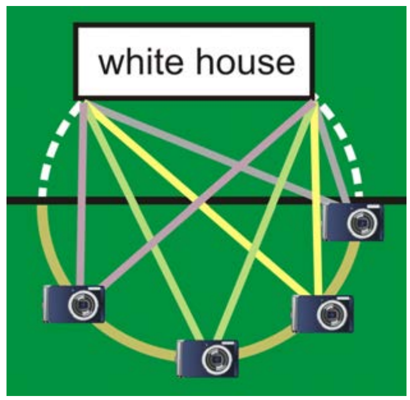
Example 6: ***Algebra Connection*** Find x and y in the picture below.



Solution:

$$\begin{aligned}(7x+1)^\circ + 105^\circ &= 180^\circ \\ 7x + 106^\circ &= 180^\circ \\ 7x &= 74^\circ \\ x &= 10.6^\circ\end{aligned} \qquad \begin{aligned}(4y+14)^\circ + (7y+1)^\circ &= 180^\circ \\ 11y + 15^\circ &= 180^\circ \\ 11y &= 165^\circ \\ y &= 15^\circ\end{aligned}$$

Know What? Revisited You can take the picture from anywhere on the semicircular walking path, the frame will be the same.



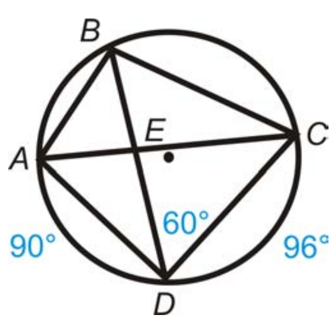
Practice Problems

- Questions 1-8 use the vocabulary and theorems learned in this section.
- Questions 9-27 are similar to Examples 1-5.
- Questions 28-33 are similar to Example 6.
- Question 34 is a proof of the Inscribed Angle Theorem.

Fill in the blanks.

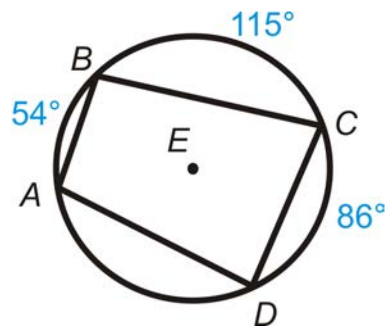
1. A(n) _______________ polygon has all its vertices on a circle.
2. An inscribed angle is ____________ the measure of the intercepted arc.
3. A central angle is ________________ the measure of the intercepted arc.
4. An angle inscribed in a ________________ is 90°.
5. Two inscribed angles that intercept the same arc are _______________.
6. The _____________ angles of an inscribed quadrilateral are ________________.
7. The sides of an inscribed angle are ___________________.
8. Draw inscribed angle $\angle JKL$ in $\odot M$. Then draw central angle $\angle JML$. How do the two angles relate?

Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



9. $m\angle DBC$
10. $m\widehat{BC}$
11. $m\widehat{AB}$
12. $m\angle ACD$
13. $m\angle ADC$
14. $m\angle ACB$

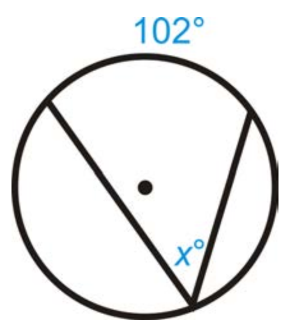
Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



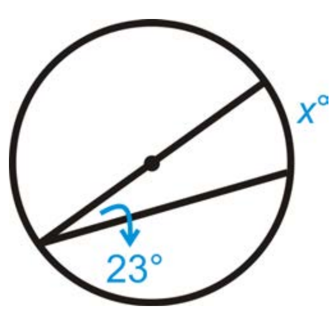
15. $m\angle A$
16. $m\angle B$
17. $m\angle C$
18. $m\angle D$

Find the value of x and/or y in $\odot A$.

19.



20.



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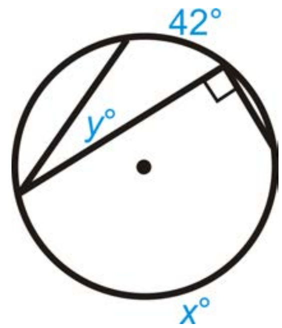

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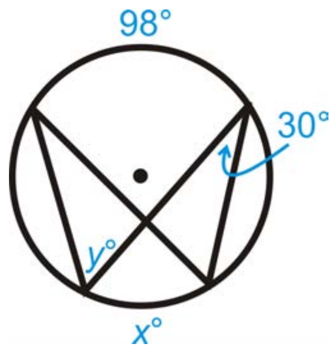

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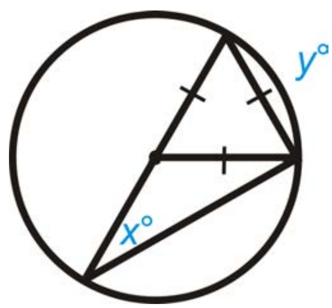

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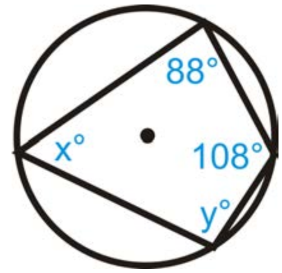

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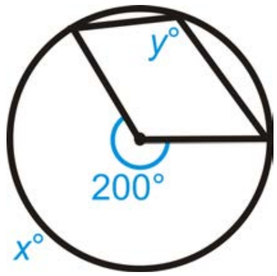

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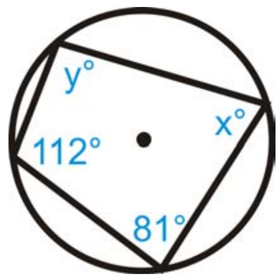
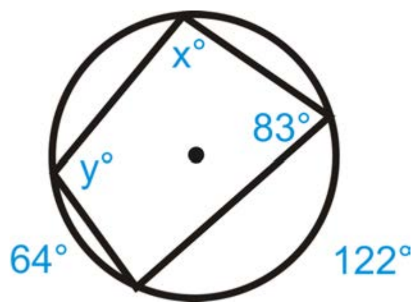
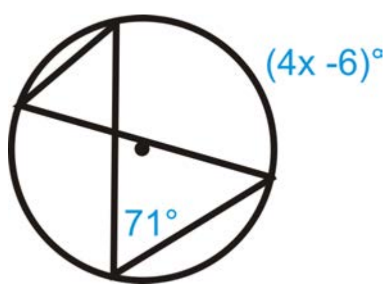


27.



Algebra Connection Solve for *x*.

28.



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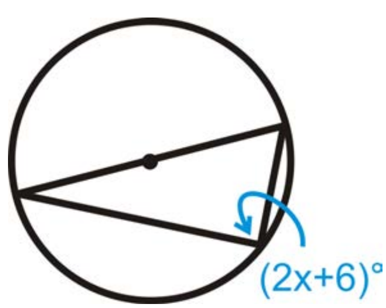

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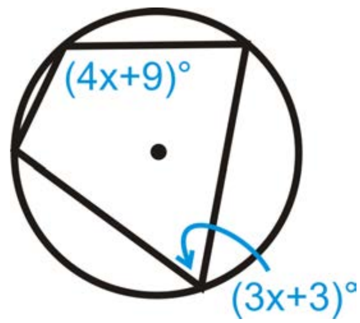
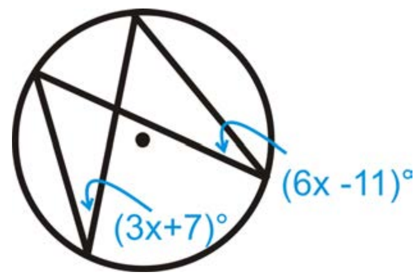


31.



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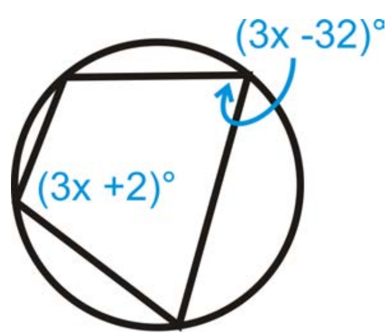

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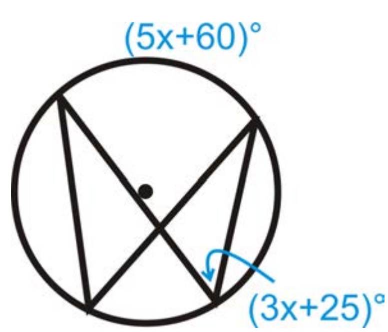
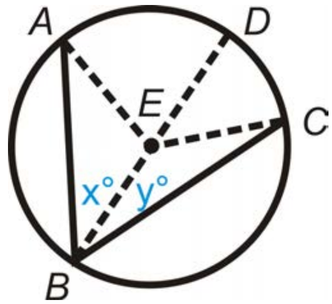


Review and Reflect

34. Fill in the blanks of the Inscribed Angle Theorem proof below.



Given: Inscribed $\angle ABC$and diameter $\overline{BD}$Prove: $m\angle ABC = \frac{1}{2}m\widehat{AC}$

TABLE 10.6:

Statement	***Reason***
1. Inscribed $\angle ABC$ and diameter $\overline{BD}$ $m\angle ABE = x^\circ$ and $m\angle CBE = y^\circ$	
2. $x^\circ + y^\circ = m\angle ABC$	
3.	All radii are congruent
4.	Definition of an isosceles triangle
5. $m\angle EAB = x^\circ$ and $m\angle ECB = y^\circ$	
6. $m\angle AED = 2x^\circ$ and $m\angle CED = 2y^\circ$	
7. $m\widehat{AD} = 2x^\circ$ and $m\widehat{DC} = 2y^\circ$	
8.	Arc Addition Postulate
9. $m\widehat{AC} = 2x^\circ + 2y^\circ$	
10.	Distributive PoE
11. $m\widehat{AC} = 2m\angle ABC$	
12. $m\angle ABC = \frac{1}{2}m\widehat{AC}$	

Warm-Up Answers

1. 60°, it is an equilateral triangle.
2. They are congruent because the chords are congruent.
3. $\frac{360^\circ}{3} = 120^\circ$

10.6 Angles of Chords, Secants, and Tangents

TEKS G(5)A, G(12)A

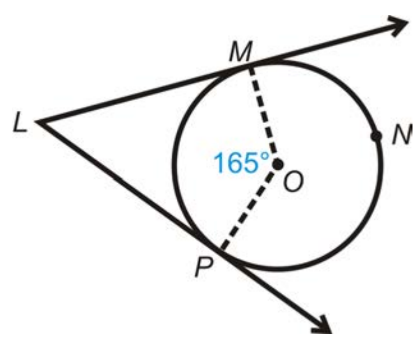
Learning Objectives

- Find the measures of angles formed by chords, secants, and tangents

Vocabulary

- Angle on a circle
- Angle inside of a circle
- Angle outside of circle

Warm-Up



1. What is $m\angle OML$ and $m\angle OPL$? How do you know?
2. Find $m\angle MLP$.
3. Find $m\widehat{MNP}$.

Know What? The sun's rays hit the Earth such that the tangent rays determine when daytime and night time are. If the arc that is exposed to sunlight is 178°, what is the angle at which the sun's rays hit the earth (x°)?

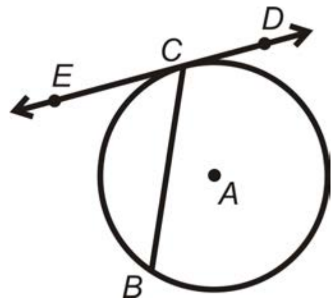
Angle on a Circle

When an angle is ***on*** a circle, the vertex is on the edge of the circle. One type of angle *on* a circle is the inscribed angle, from the previous section. Another type of angle *on* a circle is one formed by a tangent and a chord.

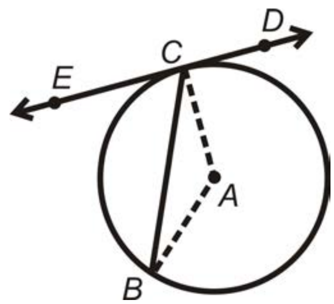
Investigation 10-6: The Measure of an Angle formed by a Tangent and a Chord

Tools Needed: pencil, paper, ruler, compass, protractor

1. Draw $\odot A$ with chord $\overline{BC}$ and tangent line $\overleftrightarrow{ED}$ with point of tangency C.



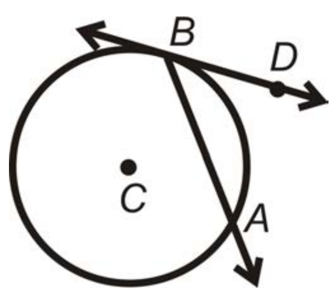
2. Draw in central angle $\angle CAB$. Find $m\angle CAB$ and $m\angle BCE$.



3. Find $m\widehat{BC}$. How does the measure of this arc relate to $m\angle BCE$?

Theorem 10-11

The measure of an angle formed by a chord and tangent that intersect on the circle is half the measure of the intercepted arc

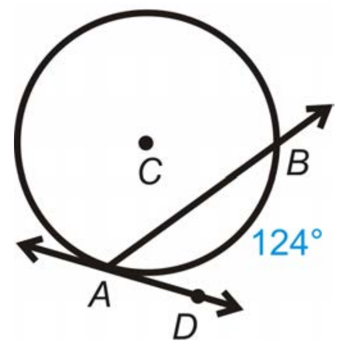


$$m\angle DBA = \frac{1}{2} m\widehat{AB}$$

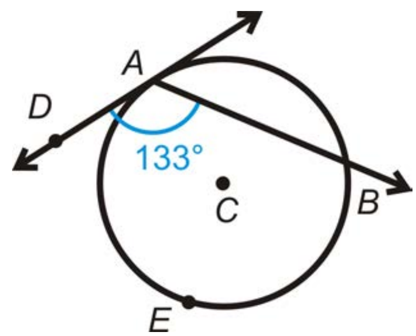
We now know that there are ***two*** types of angles that are ***half*** the measure of the intercepted arc; an ***inscribed angle*** and ***an angle formed by a chord and a tangent***.

Example 1: Find:

a) $m\angle BAD$



b) $m\widehat{AEB}$

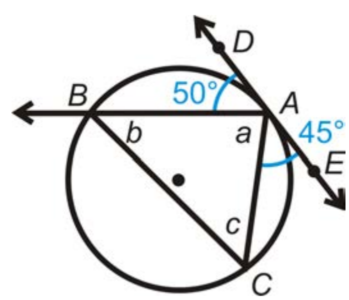


Solution: Use Theorem 10-11.

a) $m\angle BAD = \frac{1}{2} m\widehat{AB} = \frac{1}{2} \cdot 124^\circ = 62^\circ$

b) $m\widehat{AEB} = 2 \cdot m\angle DAB = 2 \cdot 133^\circ = 266^\circ$

Example 2: Find a, b, and c.



Solution:

$$\begin{aligned} 50^\circ + 45^\circ + m\angle a &= 180^\circ \qquad \text{straight angle} \\ m\angle a &= 85^\circ \end{aligned}$$

$$\begin{aligned} m\angle b &= \frac{1}{2} \cdot m\widehat{AC} \\ m\widehat{AC} &= 2 \cdot m\angle EAC = 2 \cdot 45^\circ = 90^\circ \\ m\angle b &= \frac{1}{2} \cdot 90^\circ = 45^\circ \end{aligned}$$

$$85^\circ + 45^\circ + m\angle c = 180^\circ \qquad \text{Triangle Sum Theorem}$$
$$m\angle c = 50^\circ$$

From this example, we see that Theorem 10-8 is true for angles formed by a tangent and chord with the vertex on the circle. ***If two angles, with their vertices on the circle, intercept the same arc then the angles are congruent.***

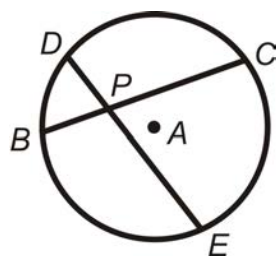
Angles inside a Circle

An angle is *inside* a circle when the vertex anywhere inside the circle, but not on the center.

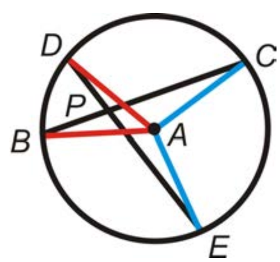
Investigation 10-7: Find the Measure of an Angle *inside* a Circle

Tools Needed: pencil, paper, compass, ruler, protractor, colored pencils (optional)

1. Draw $\odot A$ with chord $\overline{BC}$ and $\overline{DE}$. Label the point of intersection P.



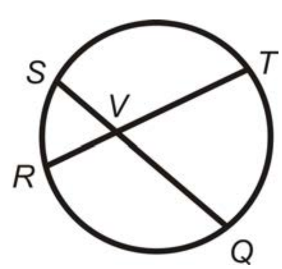
2. Draw central angles $\angle DAB$ and $\angle CAE$. Use colored pencils, if desired.



3. Find $m\angle DPB$, $m\angle DAB$, and $m\angle CAE$. Find $m\widehat{DB}$ and $m\widehat{CE}$.
4. Find $\frac{m\widehat{DB}+m\widehat{CE}}{2}$.
5. What do you notice?

Theorem 10-12

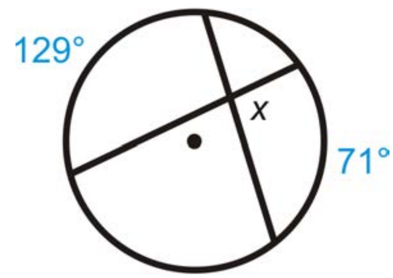
The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs



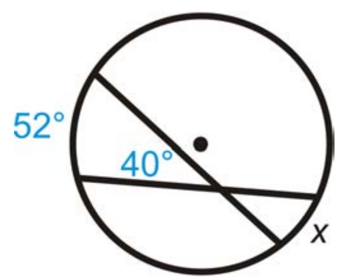
$$m\angle SVR = \frac{1}{2}\left(m\widehat{SR} + m\widehat{TQ}\right) = \frac{m\widehat{SR} + m\widehat{TQ}}{2} = m\angle TVQ$$
$$m\angle SVT = \frac{1}{2}\left(m\widehat{ST} + m\widehat{RQ}\right) = \frac{m\widehat{ST} + m\widehat{RQ}}{2} = m\angle RVQ$$

Example 3: Find x.

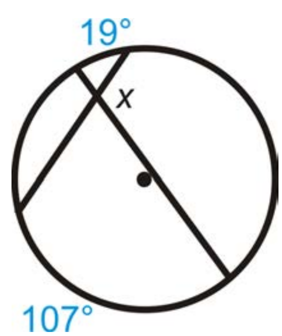
a)



b)



c)



Solution: Use Theorem 10-12 to write an equation.

a) $x = \frac{129^\circ + 71^\circ}{2} = \frac{200^\circ}{2} = 100^\circ$

b) $40^\circ = \frac{52^\circ + x}{2}$
$80^\circ = 52^\circ + x$
$28^\circ = x$

c) x is supplementary to the angle that the average of the given intercepted arcs, y.

$$y = \frac{19^\circ + 107^\circ}{2} = \frac{126^\circ}{2} = 63^\circ \qquad x + 63^\circ = 180^\circ;\ x = 117^\circ$$

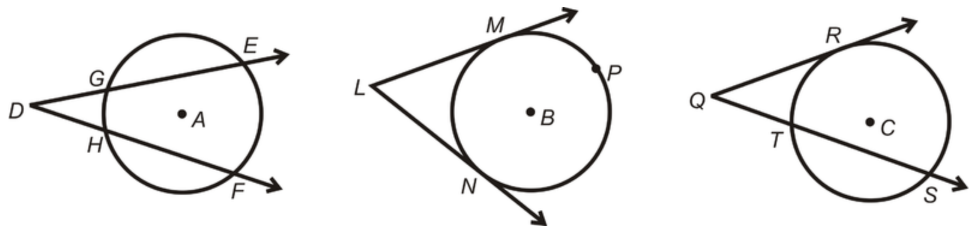
Angles outside a Circle

An angle is *outside* a circle if the vertex of the angle is outside the circle and the sides are tangents or secants. The possibilities are: an angle formed by two tangents, an angle formed by a tangent and a secant, and an angle formed by two secants.

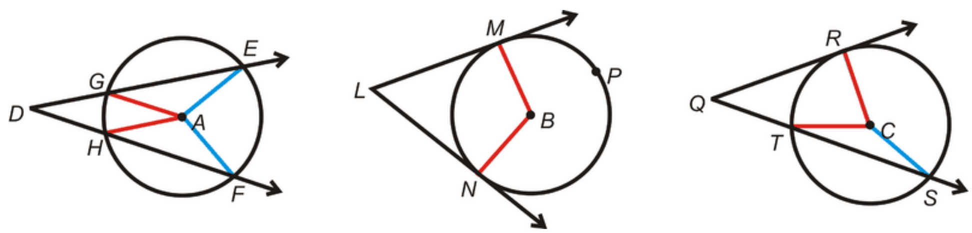
Investigation 10-8: Find the Measure of an Angle outside a Circle

Tools Needed: pencil, paper, ruler, compass, protractor, colored pencils (optional)

1. Draw three circles and label the centers A, B, and C. In $\odot A$ draw two secant rays with the same endpoint. In $\odot B$, draw two tangent rays with the same endpoint. In $\odot C$, draw a tangent ray and a secant ray with the same endpoint. Label the points like the pictures below.



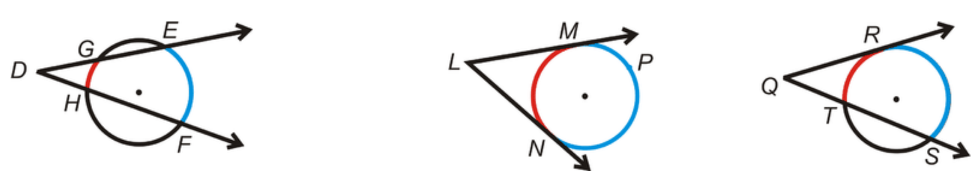
2. Draw in all the central angles. Using a protractor, measure the central angles and find the measures of each intercepted arc.



3. Find $m\angle EDF$, $m\angle MLN$, and $m\angle RQS$.
4. Find $\frac{m\widehat{EF}-m\widehat{GH}}{2}$, $\frac{m\widehat{MPN}-m\widehat{MN}}{2}$, and $\frac{m\widehat{RS}-m\widehat{RT}}{2}$. What do you notice?

Theorem 10-13

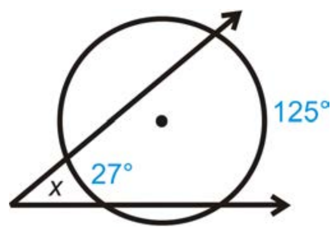
The measure of an angle formed by two secants, two tangents, or a secant and a tangent from a point outside the circle is half the difference of the measure of the intercepted



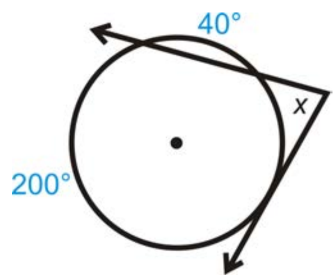
$$m\angle D = \frac{m\widehat{EF} - m\widehat{GH}}{2}$$
$$m\angle L = \frac{m\widehat{MPN} - m\widehat{MN}}{2}$$
$$m\angle Q = \frac{m\widehat{RS} - m\widehat{RT}}{2}$$

Example 4: Find the measure of x.

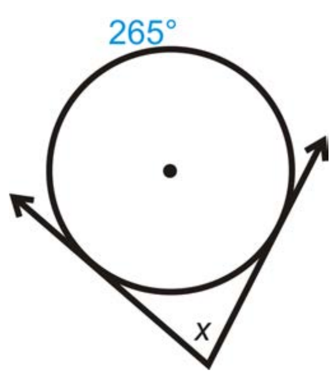
a)



b)



c)



Solution: For all of the above problems we can use Theorem 10-13.

a) $x = \frac{125^\circ - 27^\circ}{2} = \frac{98^\circ}{2} = 49^\circ$

b) 40° is not the intercepted arc. The intercepted arc is 120°, $(360^\circ - 200^\circ - 40^\circ)$. $x = \frac{200^\circ - 120^\circ}{2} = \frac{80^\circ}{2} = 40^\circ$

c) Find the other intercepted arc, $360^\circ - 265^\circ = 95^\circ x = \frac{265^\circ - 95^\circ}{2} = \frac{170^\circ}{2} = 85^\circ$

Know What? Revisited From Theorem 10-13, we know $x = \frac{182^\circ - 178^\circ}{2} = \frac{4^\circ}{2} = 2^\circ$.

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Practice Problems

- Questions 1-3 use the definitions of tangent and secant lines.
- Questions 4-7 use the definition and theorems learned in this section.
- Questions 8-25 are similar to Examples 1-4.
- Questions 26 and 27 are similar to Example 4, but also a challenge.
- Questions 28 and 29 are fill-in-the-blank proofs of Theorems 10-12 and 10-13.

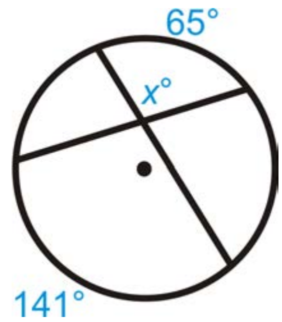
1. Draw two secants that intersect:
   a. inside a circle.
   b. on a circle.
   c. outside a circle.
2. Can two tangent lines intersect inside a circle? Why or why not?
3. Draw a tangent and a secant that intersect:
   a. on a circle.
   b. outside a circle.

Fill in the blanks.

4. If the vertex of an angle is on the ________________ of a circle, then its measure is ________________ to the intercepted arc.
5. If the vertex of an angle is ________________ a circle, then its measure is the average of the ______________-______ arcs.
6. If the vertex of an angle is _________ a circle, then its measure is _______________ the intercepted arc.
7. If the vertex of an angle is _____________ a circle, then its measure is ____________ the difference of the intercepted arcs.

For questions 8-25, find the value of the missing variable(s).

8.



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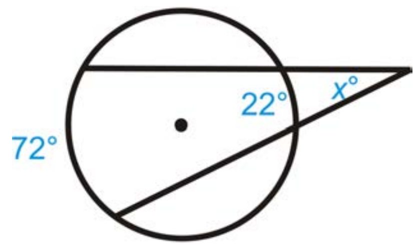

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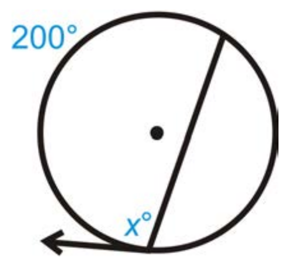

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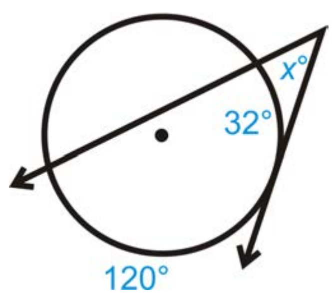
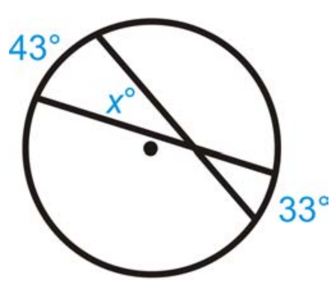
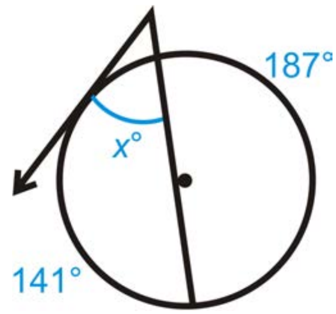


12.



13.



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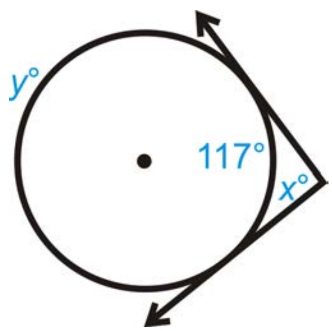

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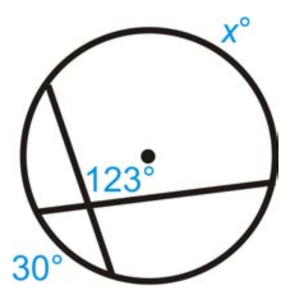

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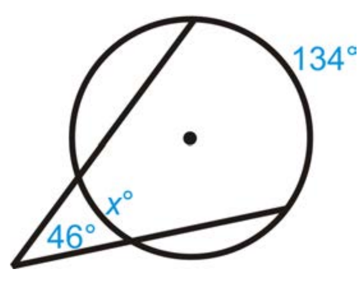

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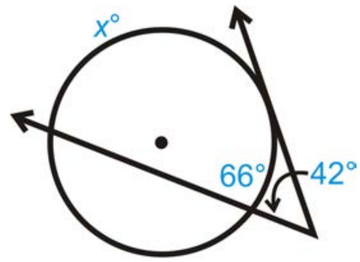

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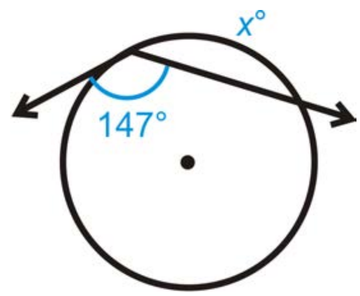

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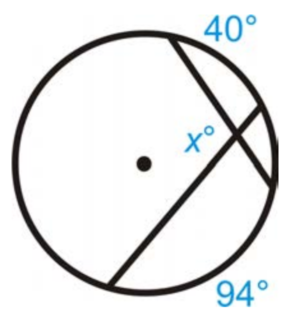

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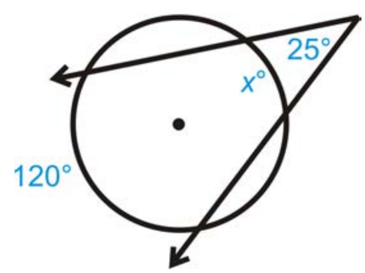

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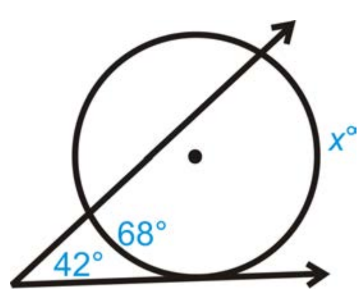

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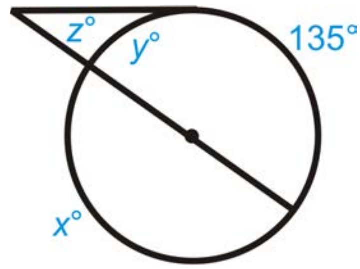
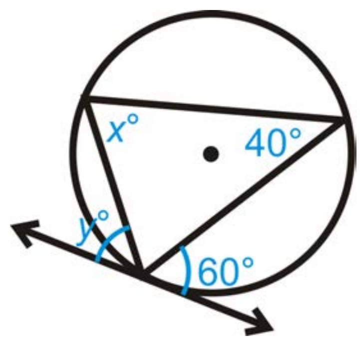


23. $y \neq 60^\circ$



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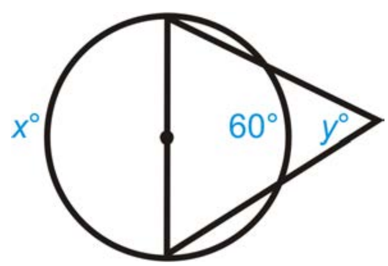

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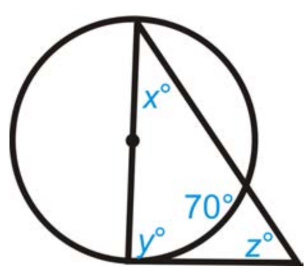
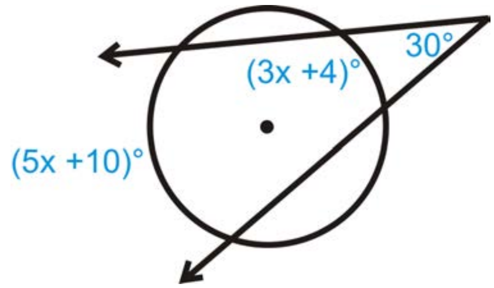
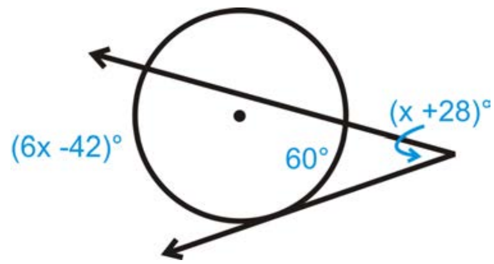


Challenge Solve for *x*.

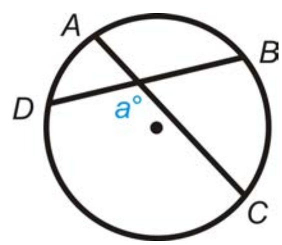
26.



27.

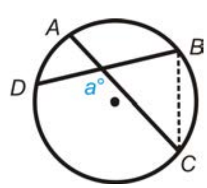


28. Fill in the blanks of the proof for Theorem 10-12.

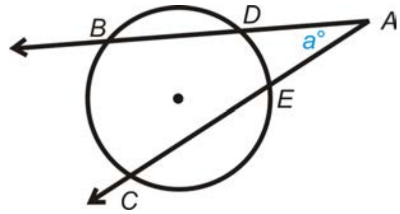


Given: Intersecting chords $\overline{AC}$and $\overline{BD}$. Prove: $m\angle a = \frac{1}{2}\left(m\widehat{DC} + m\widehat{AB}\right)$

TABLE 10.7:

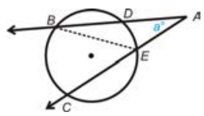
Statement	***Reason***
1. Intersecting chords $\overline{AC}$ and $\overline{BD}$.	
2. Draw $\overline{BC}$ 	Construction
3. $m\angle DBC = \frac{1}{2}m\widehat{DC}$ $\mathrm{m}\angle ACB = \frac{1}{2}m\widehat{AB}$	
4. $m\angle a = m\angle DBC + m\angle ACB$	
5. $m\angle a = \frac{1}{2}m\widehat{DC} + \frac{1}{2}m\widehat{AB}$	

29. Fill in the blanks of the proof for Theorem 10-13.



Given: Secant rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$Prove: $m\angle a = \frac{1}{2}\left(m\widehat{BC} - m\widehat{DE}\right)$

TABLE 10.8:

Statement	*Reason*
1. Intersecting secants $\overrightarrow{AB}$ and $\overrightarrow{AC}$.	
2. Draw $\overline{BE}$. 	Construction
3. $m\angle BEC = \frac{1}{2}m\widehat{BC}$	
m$\angle DBE = \frac{1}{2}m\widehat{DE}$	
5. $m\angle a + m\angle DBE = m\angle BEC$	
6.	Subtraction PoE
7.	Substitution
8. $m\angle a = \frac{1}{2}\left(m\widehat{BC} - m\widehat{DE}\right)$	

Review and Reflect

30. Compare and contrast the formulas of the angles formed by two lines in relation to how the lines interact with the circle.

Warm-Up Answers

1. $m\angle OML = m\angle OPL = 90^\circ$ because a tangent line and a radius drawn to the point of tangency are perpendicular.
2. $165^\circ + m\angle OML + m\angle OPL + m\angle MLP = 360^\circ$
$$165^\circ + 90^\circ + 90^\circ + m\angle MLP = 360^\circ$$
$$m\angle MLP = 15^\circ$$
3. $m\widehat{MNP} = 360^\circ - 165^\circ = 195^\circ$

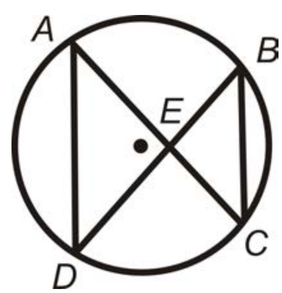
10.7 Segments of Chords, Secants, and Tangents

TEKS G(5)A, G (12)A

Learning Objectives

Find the lengths of segments within circles

Warm-Up



1. What do you know about $m\angle DAC$ and $m\angle DBC$? Why?
2. What do you know about $m\angle AED$ and $m\angle BEC$? Why?
3. Is $\triangle AED \sim \triangle BEC$? How do you know?
4. If $AE = 8,\ ED = 7$, and $BE = 6$, find EC.

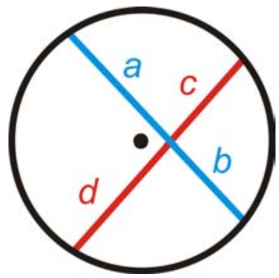
Know What? At a particular time during its orbit, the moon is 238,857 miles from Beijing, China. On the same line, Yukon is 12,451 miles from Beijing. Drawing another line from the moon to Cape Horn we see that Jakarta, Indonesia is collinear. If the distance from the moon to Jakarta is 240,128 miles, what is the distance from Cape Horn to Jakarta?

Segments from Chords

In the Review Queue above, we have two chords that intersect inside a circle. The two triangles are similar, making the sides in each triangle proportional.

Theorem 10-14

If two chords intersect inside a circle so that one is divided into segments of length a and b and the other segments of length c and d, then $ab = cd$

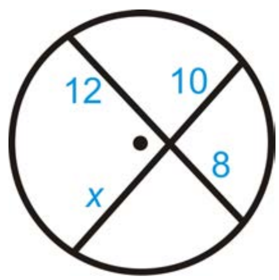


The product of the segments of one chord is equal to the product of segments of the second chord.

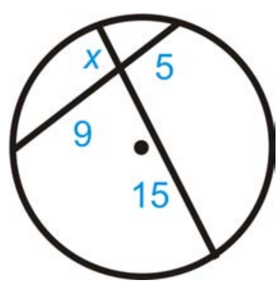
$ab = cd$

Example 1: Find x in each diagram below.

a)



b)



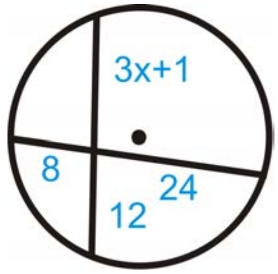
Solution: Use the ratio from Theorem 10-14.

a) $12 \cdot 8 = 10 \cdot x$
$96 = 10x$
$9.6 = x$

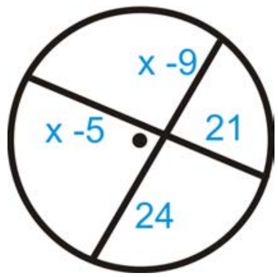
b) $x \cdot 15 = 5 \cdot 9$
$15x = 45$
$x = 3$

Example 2: ***Algebra Connection*** Solve for x.

a)



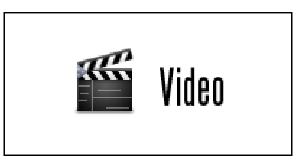
b)



Solution: Use Theorem 10-13.

a) $8 \cdot 24 = (3x+1) \cdot 12$
$192 = 36x + 12$
$180 = 36x$
$5 = x$

b) $(x-5)21 = (x-9)24$
$21x - 105 = 24x - 216$
$111 = 3x$
$37 = x$



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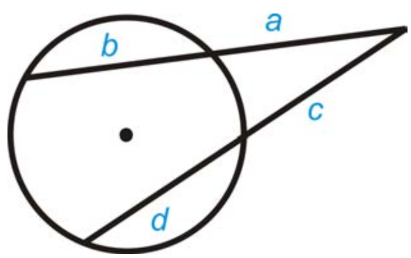
Segments from Secants

In addition to forming an angle outside of a circle, the circle can divide the secants into segments that are proportional with each other.

Theorem 10-15

If two secants are drawn from a common point outside a circle and the segments are labeled as below, then $a(a+b) = c(c+d)$

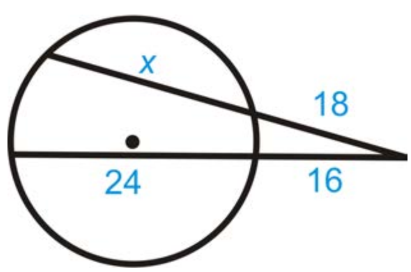


The product of the outer segment and the whole of one secant equals the product of the outer segment and the whole of the other secant.

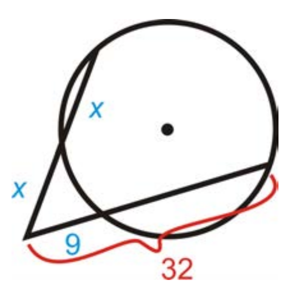
$$a(a+b) = c(c+d)$$

Example 3: Find the value of the missing variable.

a)



b)



Solution: Use Theorem 10-15 to set up an equation.

a) $18 \cdot (18+x) = 16 \cdot (16+24)$
$324+18x = 256+384$
$18x = 316$
$x = 17\frac{5}{9}$

b) $x \cdot (x+x) = 9 \cdot 32$
$2x^2 = 288$
$x^2 = 144$
$x = 12,\ x \neq -12$ (length is not negative)

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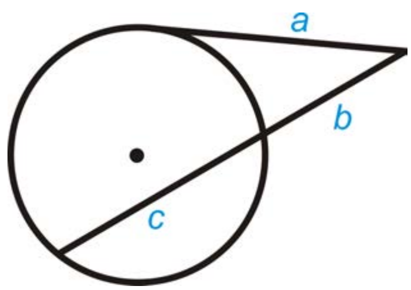
Segments from Secants and Tangents

If a tangent and secant meet at a common point outside a circle, the segments created have a similar relationship to that of two secant rays in Example 3.

Theorem 10-16

If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture below) then $a^2 = b(b+c)$

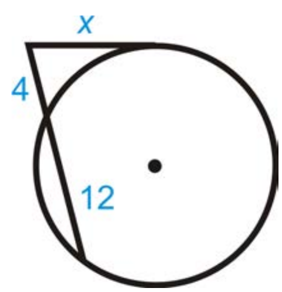


The product of the outside segment of the secant and the whole is equal to the square of the tangent.

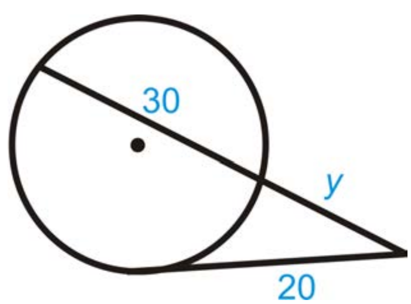
$$a^2 = b(b+c)$$

Example 4: Find the value of the missing segment.

a)



b)



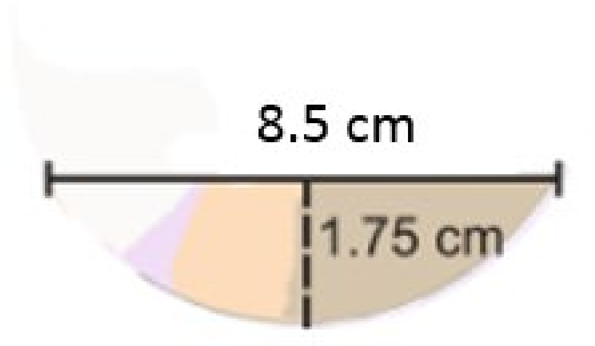
Solution: Use Theorem 10-16.

a) $x^2 = 4(4+12)$
$x^2 = 4 \cdot 16 = 64$
$x = 8$

b) $20^2 = y(y+30)$
$400 = y^2 + 30y$
$0 = y^2 + 30y - 400$
$0 = (y+40)(y-10)$
$y = \cancel{-40}, 10$

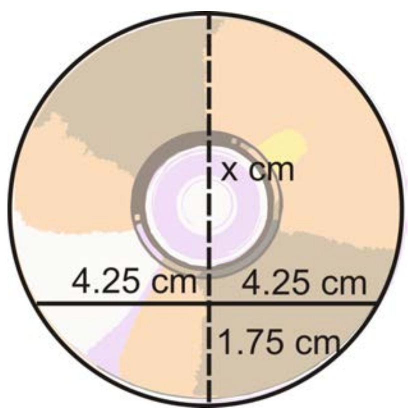
When you have to factor a quadratic equation to find an answer, ***always eliminate the negative answer*** because length is never negative.

Example 5: Ishmael found a broken piece of a CD in his car. He places a ruler across two points on the rim, and the length of the chord is 8.5 cm. The distance from the midpoint of this chord to the nearest point on the rim is 1.75 cm. Find the diameter of the CD.



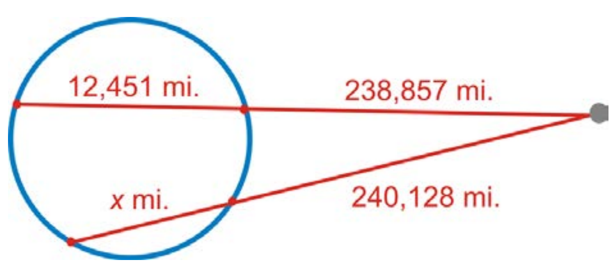
Solution: Think of this as two chords intersecting each other. If we were to extend the 1.75 cm segment, it would be a diameter. So, if we find x, in the diagram to the left, and add it to 1.75 cm, we would find the diameter.

$$\begin{aligned}
4.25 \cdot 4.25 &= 1.75 \cdot x \\
18.0625 &= 1.75x \\
x &\approx 10.3\ cm, \text{ making the diameter } 12\ cm, \text{ which is the} \\
&\quad \text{actual diameter of a CD.}
\end{aligned}$$



Know What? Revisited The given information is to the left. Let's set up an equation using Theorem 10-15.

$$\begin{aligned}
238857 \cdot 251308 &= 240128(240128 + x) \\
60026674956 &= 57661456380 + 240128x \\
2365218572 &= 240128x \\
x &\approx 9849.8\ miles
\end{aligned}$$



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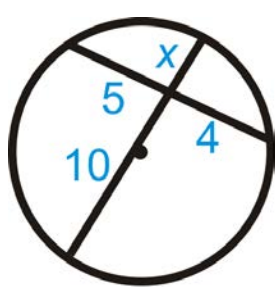
Practice Problems

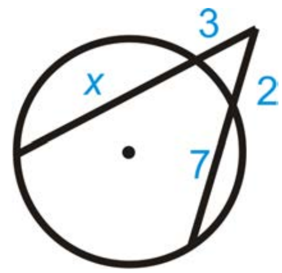
- Questions 1-25 are similar to Examples 1, 3, and 4.
- Questions 26-28 are similar to Example 2.
- Questions 29 is similar to Example 5.
- Questions 30 and 31 are proofs of Theorem 10-14 and 10-15.

Fill in the blanks for each problem below. Then, solve for the missing segment.

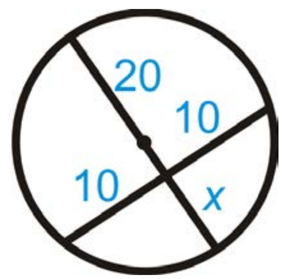
1. $\_\_\_ \cdot 4 = \_\_\_ \cdot x$



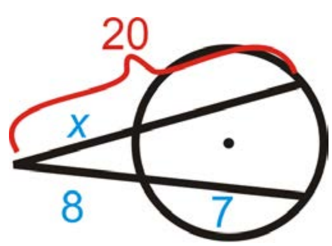
2. $3(\_\_\_ + \_\_\_) = 2(2+7)$



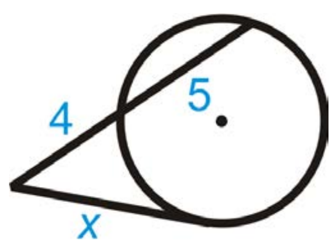
3. $20x = \_\_\_$



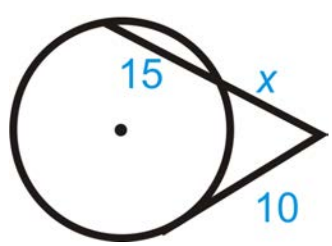
4. $x \cdot \_\_\_ = 8(\_\_\_ + \_\_\_)$



5. $\_\_\_ = \_\_\_(4+5)$

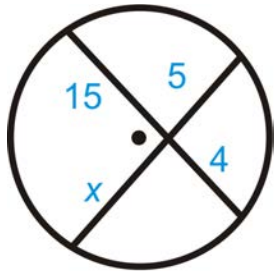


6. $10^2 = x(\_\_\_ + \_\_\_)$



Find x in each diagram below. Simplify any radicals.

7.



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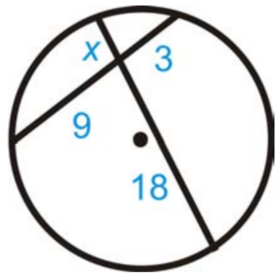

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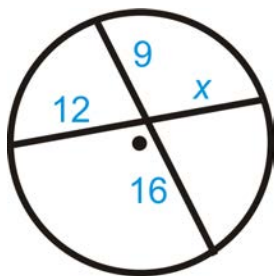

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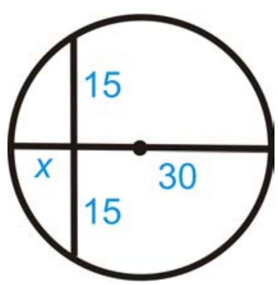

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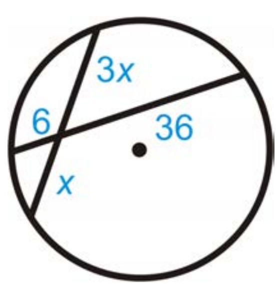

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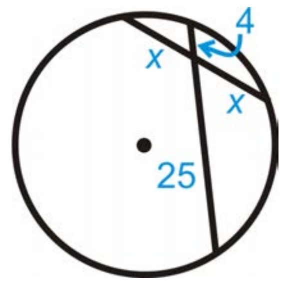

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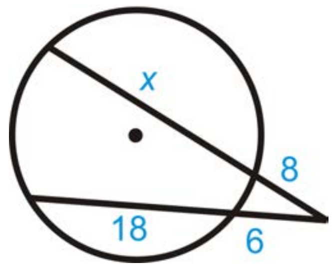

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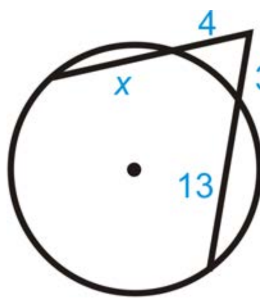

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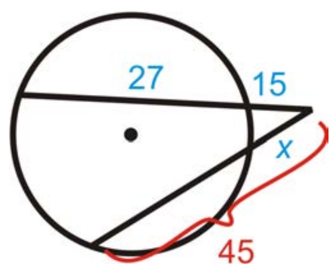

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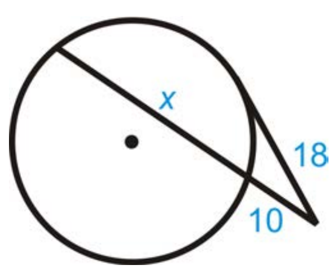

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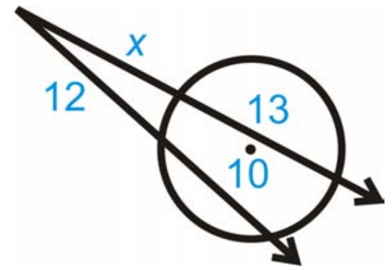

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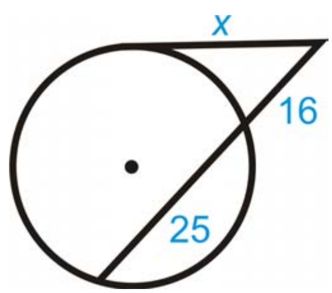

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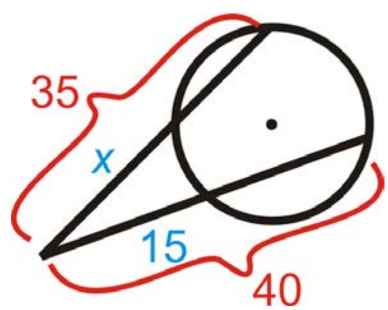

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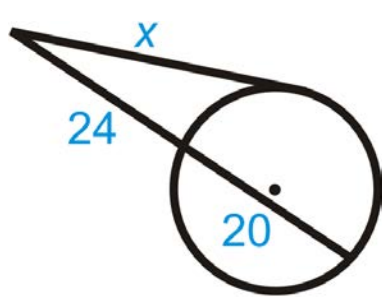

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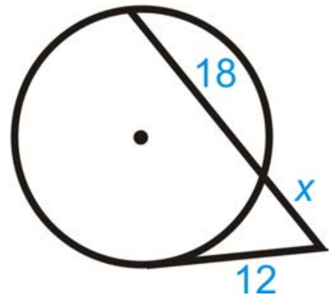

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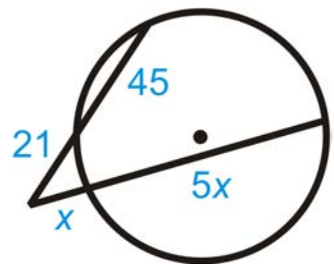

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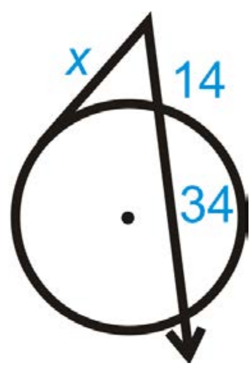

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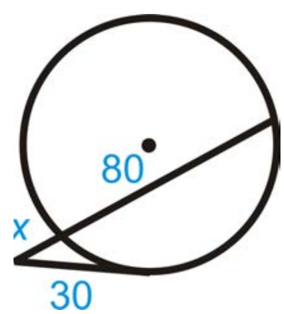
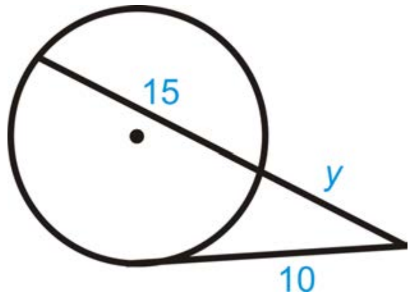


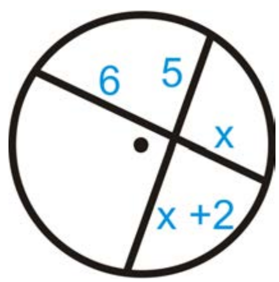
25. ***Error Analysis*** Describe and correct the error in finding y.



$$10 \cdot 10 = y \cdot 15y$$
$$100 = 15y^2$$
$$\frac{20}{3} = y^2$$
$$\frac{2\sqrt{15}}{3} = y \quad \longleftarrow \; y \text{ is not correct}$$

Algebra Connection Find the value of x.

26.



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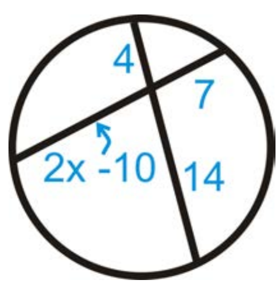

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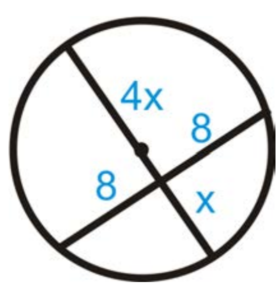
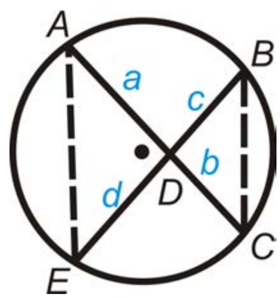


29. Suzie found a piece of a broken plate. She places a ruler across two points on the rim, and the length of the chord is 6 inches. The distance from the midpoint of this chord to the nearest point on the rim is 1 inch. Find the diameter of the plate.
30. Fill in the blanks of the proof of Theorem 10-14.

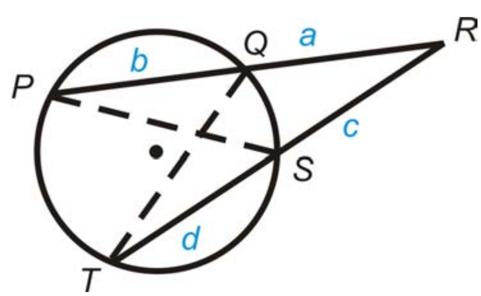


Given: Intersecting chords $\overline{AC}$ and $\overline{BE}$. Prove: $ab = cd$

TABLE 10.9:

Statement	***Reason***
1. Intersecting chords $\overline{AC}$ and $\overline{BE}$ with segments a, b, c, and d.	
2.	Theorem 10-8
3. $\triangle ADE \sim \triangle BDC$	
4.	Corresponding parts of similar triangles are proportional
5. $ab = cd$	

31. Fill in the blanks of the proof of Theorem 10-15.



Given: Secants $\overline{PR}$ and $\overline{RT}$Prove: $a(a+b) = c(c+d)$

TABLE 10.10:

Statement	***Reason***
1. Secants $\overline{PR}$ and $\overline{RT}$ with segments a, b, c, and d.	given
2. $\angle R \cong \angle R$	Reflexive PoC
3. $\angle QPS \cong \angle STQ$	Theorem 10-8
4. $\triangle RPS \sim \triangle RTQ$	AA Similarity Postulate
5. $\frac{a}{c+d} = \frac{c}{a+b}$	Corresponding parts of similar triangles are proportional
6. $a(a+b) = c(c+d)$	Cross multiplication

Review and Reflect

32. Describe the three ways that segments can intersect with respect to a circle(inside, outside and on) and the differences between the equations.

Warm-Up Answers

1. $m\angle DAC = m\angle DBC$ by Theorem 10-8, they are inscribed angles and intercept the same arc.
2. $m\angle AED = m\angle BEC$ by the Vertical Angles Theorem.
3. Yes, by AA Similarity Postulate.
4. $$\begin{aligned} \frac{8}{6} &= \frac{7}{EC} \\ 8 \cdot EC &= 42 \\ EC &= \frac{21}{4} = 5.25 \end{aligned}$$

10.8 Writing and Graphing the Equations of Circles

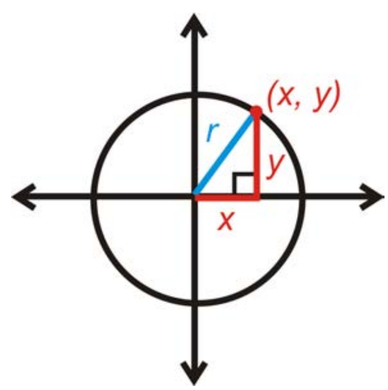
TEKS G(12)E

Learning Objectives

- Graph a circle
- Find the equation of a circle in the x-y plane
- Find the radius and center, given the equation of a circle and vise versa
- Find the equation of a circle, given the center and a point on the circle

Graphing a Circle in the Coordinate Plane

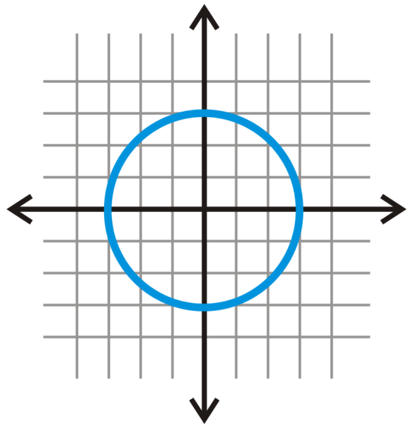
Recall that the definition of a circle is the set of all points that are the same distance from the center. This definition can be used to find an equation of a circle in the coordinate plane.



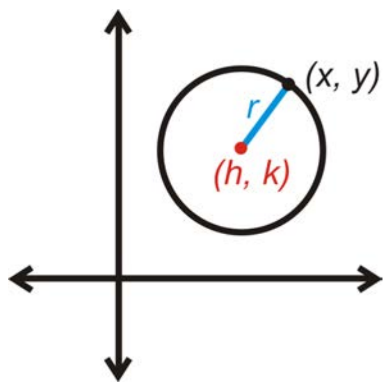
Let's start with the circle centered at (0, 0). If (x,y) is a point on the circle, then the distance from the center to this point would be the radius, r. x is the horizontal distance y is the vertical distance. This forms a right triangle. From the Pythagorean Theorem, the equation of a circle, *centered at the origin* is $x^2 + y^2 = r^2$.

Example 1: Graph $x^2 + y^2 = 9$.

Solution: The center is (0, 0). It's radius is the square root of 9, or 3. Plot the center, and then go out 3 units in every direction and connect them to form a circle.



The center does not always have to be on (0, 0). If it is not, then we label the center (h,k) and would use the distance formula to find the length of the radius.



$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

If you square both sides of this equation, then we would have the standard equation of a circle.

Standard Equation of a Circle: The standard equation of a circle with center (h,k) and radius r is $r^2 = (x-h)^2 + (y-k)^2$.

Example 2: Find the center and radius of the following circles.

a) $(x-3)^2 + (y-1)^2 = 25$

b) $(x+2)^2 + (y-5)^2 = 49$

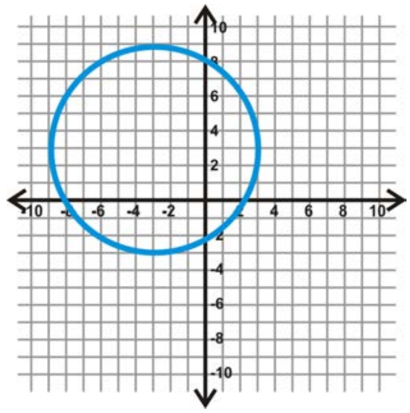
Solution:

a) Rewrite the equation as $(x-3)^2 + (y-1)^2 = 5^2$. The center is (3, 1) and $r = 5$.

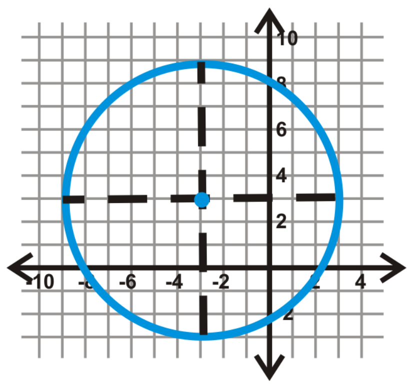
b) Rewrite the equation as $(x-(-2))^2 + (y-5)^2 = 7^2$. The center is (-2, 5) and $r = 7$.

When finding the center of a circle always take the ***opposite sign*** of what the value is in the equation.

Example 3: Find the equation of the circle below.



Solution: First locate the center. Draw in the horizontal and vertical diameters to see where they intersect.



From this, we see that the center is (-3, 3). If we count the units from the center to the circle on either of these diameters, we find $r = 6$. Plugging this into the equation of a circle, we get: $(x-(-3))^2 + (y-3)^2 = 6^2$ or $(x+3)^2 + (y-3)^2 = 36$.

Watch This!!!

https://s3.amazonaws.com/ck12bg.ck12.org/curriculum/108008/video.mp4

Finding the Equation of a Circle

Example 4: Determine if the following points are on $(x+1)^2 + (y-5)^2 = 50$.

a) (8, -3)

b) (-2, -2)

Solution: Plug in the points for x and y in $(x+1)^2 + (y-5)^2 = 50$.

a) $(8+1)^2 + (-3-5)^2 = 50$
$9^2 + (-8)^2 = 50$
$81 + 64 \neq 50$

(8, -3) is *not* on the circle

b) $(-2+1)^2 + (-2-5)^2 = 50$
$(-1)^2 + (-7)^2 = 50$
$1 + 49 = 50$

(-2, -2) is on the circle

Example 5: Find the equation of the circle with center (4, -1) and passes through (-1, 2).

Solution: First plug in the center to the standard equation.

$$(x-4)^2+(y-(-1))^2=r^2$$
$$(x-4)^2+(y+1)^2=r^2$$

Now, plug in (-1, 2) for x and y and solve for r.

$$(-1-4)^2+(2+1)^2=r^2$$
$$(-5)^2+(3)^2=r^2$$
$$25+9=r^2$$
$$34=r^2$$

Substituting in 34 for r^2, the equation is $(x-4)^2+(y+1)^2=34$.

Practice Problems

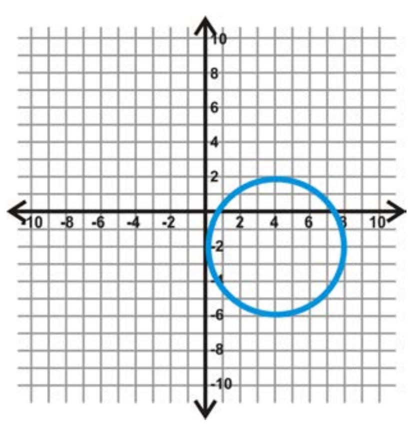
- Questions 1-4 are similar to Examples 1 and 2.
- Questions 5-8 are similar to Example 3.
- Questions 9-11 are similar to Example 4.
- Questions 12-15 are similar to Example 5.

Find the center and radius of each circle. Then, graph each circle.

1. $(x+5)^2+(y-3)^2=16$
2. $x^2+(y+8)^2=4$
3. $(x-7)^2+(y-10)^2=20$
4. $(x+2)^2+y^2=8$

Find the equation of the circles below.

5.



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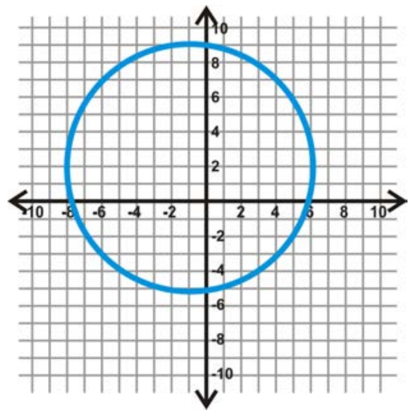

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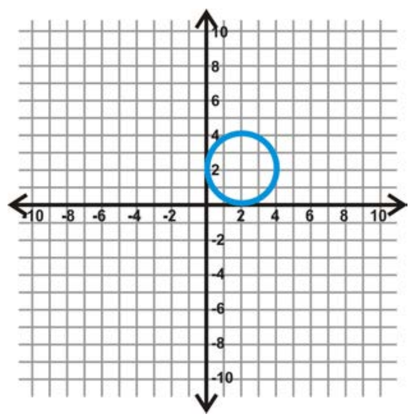

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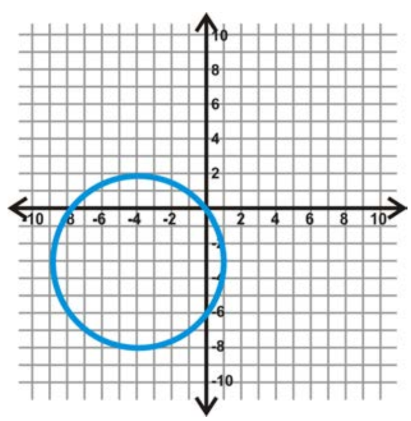


9. Is (-7, 3) on $(x+1)^2+(y-6)^2=45$?

10. Is (9, -1) on $(x-2)^2+(y-2)^2=60$?

11. Is (-4, -3) on $(x+3)^2+(y-3)^2=37$?

12. Is (5, -3) on $(x+1)^2+(y-6)^2=45$?

Find the equation of the circle with the given center and point on the circle.

13. center: (2, 3), point: (-4, -1)
14. center: (10, 0), point: (5, 2)
15. center: (-3, 8), point: (7, -2)
16. center: (6, -6), point: (-9, 4)

10.9 Chapter 10 Review

Keywords & Theorems

Parts of Circles Tangent Lines

- Circle
- Center
- Radius
- Chord
- Diameter
- Secant
- Tangent
- Point of Tangency
- Congruent Circles
- Concentric Circles
- Externally Tangent Circles
- Internally Tangent Circles
- Common Internal Tangent
- Common External Tangent
- Tangent to a Circle Theorem
- Theorem 9-2

Properties of Arcs

- Central Angle
- Arc
- Semicircle
- Minor Arc
- Major Arc
- Congruent Arcs
- Arc Addition Postulate

Properties of Chords

- Theorem 9-3
- Theorem 9-4
- Theorem 9-5
- Theorem 9-6

Inscribed Angles

- Inscribed Angle
- Intercepted Arc
- Inscribed Angle Theorem
- Theorem 9-8

- Theorem 9-9
- Inscribed Polygon
- Theorem 9-10

Angles from Chords, Secants and Tangents

- Theorem 9-11
- Theorem 9-12
- Theorem 9-13

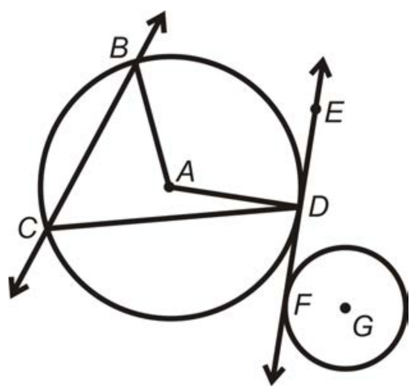
Segments from Secants and Tangents

- Theorem 9-14
- Theorem 9-15
- Theorem 9-16

Extension: Equations of Circles

- Standard Equation of a Circle

Vocabulary



Match the description with the correct label.

1. minor arc - A. $\overline{CD}$
2. chord - B. $\overline{AD}$
3. tangent line - C. $\overleftrightarrow{CB}$
4. central angle - D. $\overleftrightarrow{EF}$
5. secant - E. A
6. radius - F. D
7. inscribed angle - G. $\angle BAD$
8. center - H. $\angle BCD$
9. major arc - I. $\widehat{BD}$
10. point of tangency - J. $\widehat{BCD}$

Texas Instruments Resources

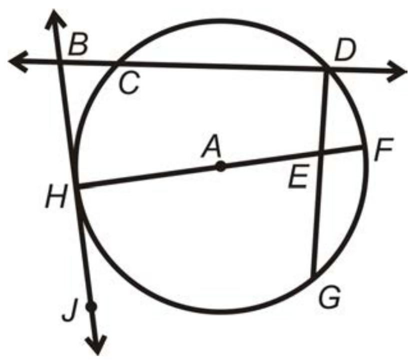
In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9694 .

10.10 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st **Section: Parts of Circles Tangent Lines**

Circle



Radius

Diameter

Tangent

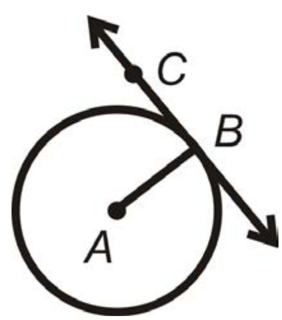
Congruent Circles

Concentric Circles

Externally Tangent Circles

Internally Tangent Circles

Common Internal Tangent



Common External Tangent

Tangent to a Circle Theorem

Theorem 9-2

Center

Chord

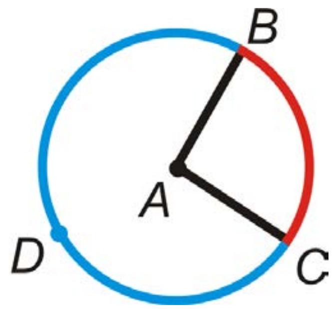
Secant

Point of Tangency

Homework:

2^{nd} **Section: Properties of Arcs**

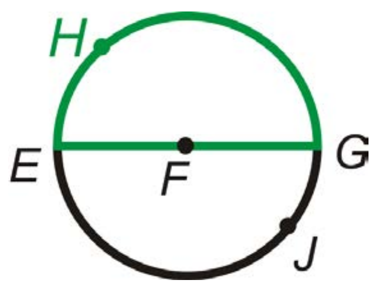
Central Angle



Arc

Semicircle

Minor Arc



Major Arc

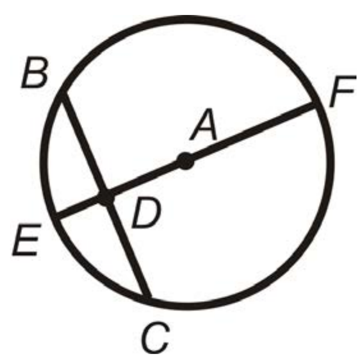
Congruent Arcs

Arc Addition Postulate

Homework:

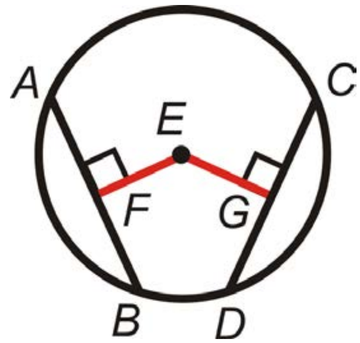
3^{rd} **Section: Properties of Chords**

Theorem 9-3



Theorem 9-4

Theorem 9-5



Theorem 9-6

Homework:

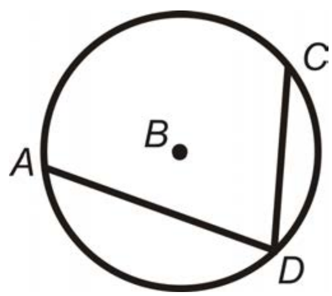
4^{th} **Section: Inscribed Angles**

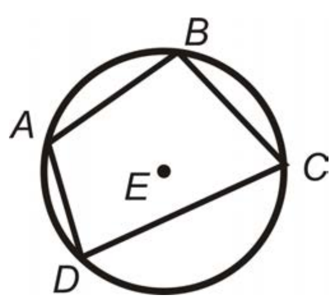
Inscribed Angle

Intercepted Arc

Inscribed Angle Theorem

Theorem 9-8





Theorem 9-9

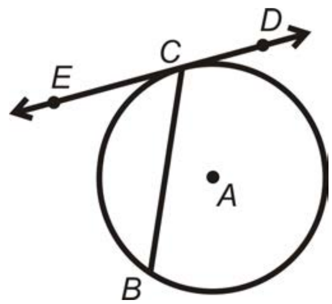
Inscribed Polygon

Theorem 9-10

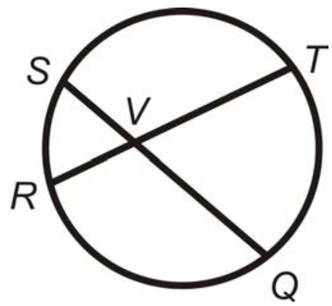
Homework:

5^{th} **Section: Angles from Chords, Secants and Tangents**

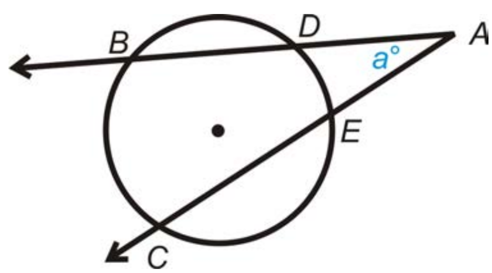
Theorem 9-11



Theorem 9-12



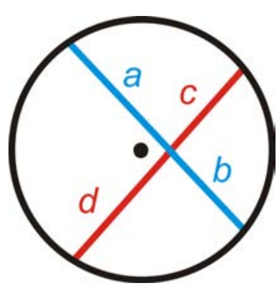
Theorem 9-13



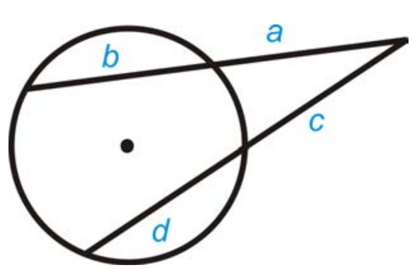
Homework:

6^{th} **Section: Segments from Secants and Tangents**

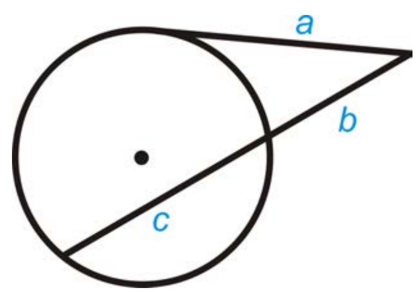
Theorem 9-14



Theorem 9-15



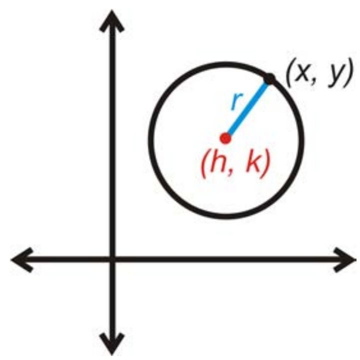
Theorem 9-16



Homework:

Extension: Equations of Circles

Standard Equation of a Circle



Homework:

CHAPTER 11 Rigid Transformations

Chapter Outline

The final chapter of Geometry transforms a figure by moving, flipping, or rotating it. First, we will look at symmetry, followed by the different transformations.

11.1 Exploring Symmetry

TEKS G(3)D

Learning Objectives

Understand line and rotational symmetry

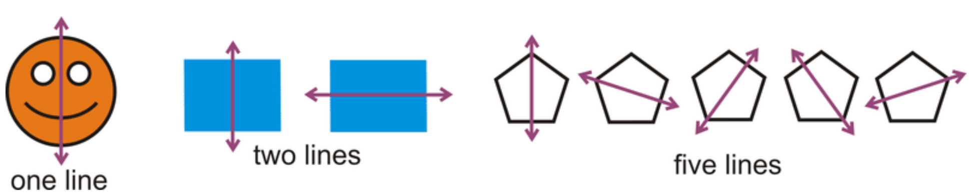
Warm-Up

1. Define symmetry in your own words.
2. Plot the points $A(1,3), B(3,1), C(5,3)$, and $D(3,5)$.
3. Find the slope of each side of the quadrilateral in #2.
4. Find the slope of the diagonals of the quadrilateral. What kind of shape is this?

Know What? Symmetry exists all over nature. One example is a starfish. Determine if the starfish has line symmetry or rotational symmetry.

Lines of Symmetry

Line of Symmetry: A line that passes through a figure such that it splits the figure into two congruent halves.



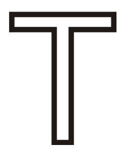
Example 1: Find all lines of symmetry for the shapes below.

a)

b)

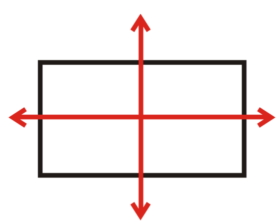
c)

d)

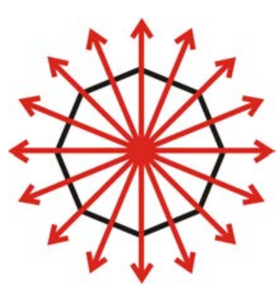


Solution: For each figure, draw lines through the figure so that the lines perfect cut the figure in half. Figure a) has two lines of symmetry, b) has eight, c) has no lines of symmetry, and d) has one.

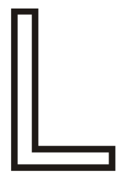
a)



b)



c)



d)

Figures a), b), and d) all have ***line symmetry***.

Line Symmetry: When a figure has one or more lines of symmetry.

These figures have line symmetry:

These figures do not have line symmetry:

Example 2: Do the figures below have line symmetry?

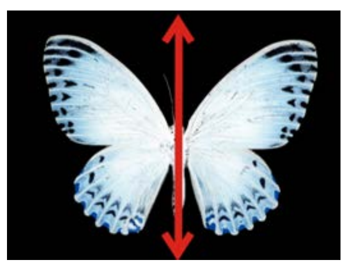
a)

b)

Solution: Yes, both of these figures have line symmetry.

a)

b)

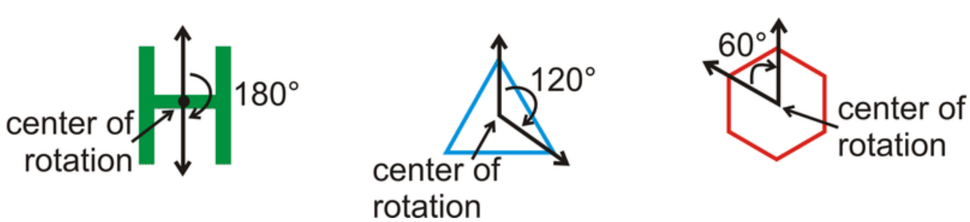


Check This Out!!!

http://www.mathsisfun.com/geometry/symmetry-reflection.html

Rotational Symmetry

Rotational Symmetry: When a figure can be rotated (less that 180°) and it looks like it did before the rotation.



Center of Rotation: The point a figure is rotated around such that the rotational symmetry holds.

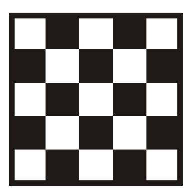
For the H, we can rotate it twice, the triangle can be rotated 3 times and still look the same and the hexagon can be rotated 6 times.

Example 3: Determine if each figure below has rotational symmetry. Find the angle and how many times it can be rotated.

a)

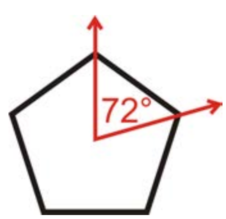
b)

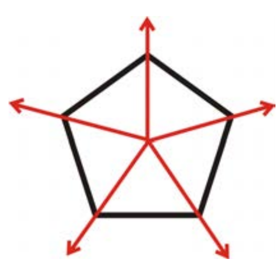
c)



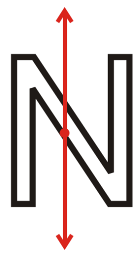
Solution:

a) The pentagon can be rotated 5 times. Because there are 5 lines of rotational symmetry, the angle would be $\frac{360^\circ}{5} 72^\circ$.

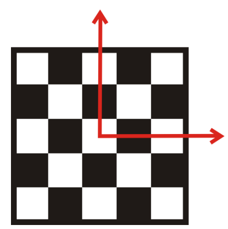


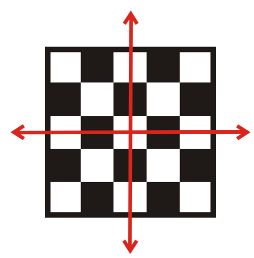


b) The N can be rotated twice. This means the angle of rotation is 180°.



c) The checkerboard can be rotated 4 times. There are 4 lines of rotational symmetry, so the angle of rotation is $\frac{360^\circ}{4} = 90^\circ$.



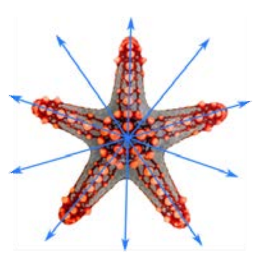


Check This Out!!!

Click the link below to draw pictures of your own that have rotational symmetry.

http://www.mathsisfun.com/geometry/symmetry-artist.html

Know What? Revisited The starfish has 5 lines of symmetry and 5 lines of rotational symmetry. The angle of rotation is 72°. The center of rotation is the center of the starfish.



Practice Problems

- Questions 1-15 use the definitions of figures and symmetry.
- Questions 16-18 ask you to draw figures based on symmetry.
- Questions 19-38 are similar to Examples 1 and 3.
- Questions 39-41 are similar to Example 2.

Fill in the blanks.

1. If a figure has 3 lines of rotational symmetry, it can be rotated _______ times.
2. If a figure can be rotated 6 times, it has _______ lines of rotational symmetry.
3. If a figure can be rotated *n* times, it has _______ lines of rotational symmetry.
4. To find the angle of rotation, divide 360° by the total number of _____________.
5. Every square has an angle of rotation of _________.

True or False

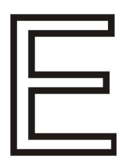
6. All right triangles have line symmetry.
7. All isosceles triangles have line symmetry.
8. Every rectangle has line symmetry.
9. Every rectangle has exactly two lines of symmetry.
10. Every parallelogram has line symmetry.
11. Every square has exactly two lines of symmetry.
12. Every regular polygon has three lines of symmetry.
13. Every sector of a circle has a line of symmetry.
14. Every parallelogram has rotational symmetry.
15. Every figure that has line symmetry also has rotational symmetry.

Draw the following figures.

16. A quadrilateral that has two pairs of congruent sides and exactly one line of symmetry.
17. A figure with infinitely many lines of symmetry.
18. A figure that has one line of symmetry and no rotational symmetry.

Find all lines of symmetry for the letters below.

19.

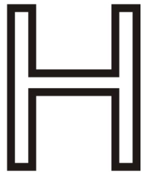


20.

21.

22.

23.



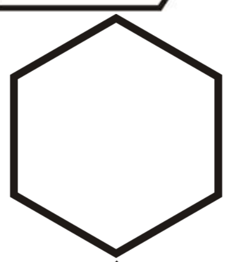
24. Do any of the letters above have rotational symmetry? If so, which one(s) and what are the angle of rotation?

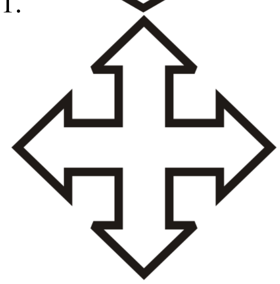
Determine if the words below have line symmetry or rotational symmetry.

25. OHIO
26. **MOW**
27. **WOW**
28. **KICK**
29. **pod**

Trace each figure and then draw in all lines of symmetry.

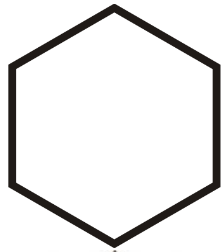
30. 

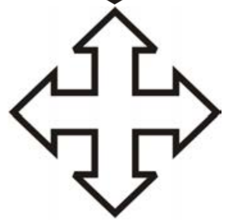
31. 

32. 

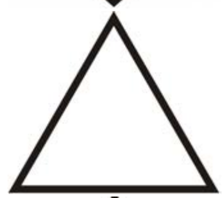
Find the angle of rotation and the number of times each figure can rotate.

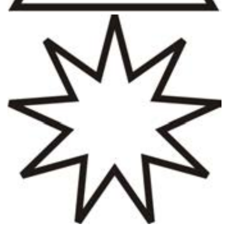
33. 

34. 

35. 

36. 

37. 

38. 

Determine if the figures below have line symmetry or rotational symmetry. Identify all lines of symmetry and the angle of rotation.

39. 

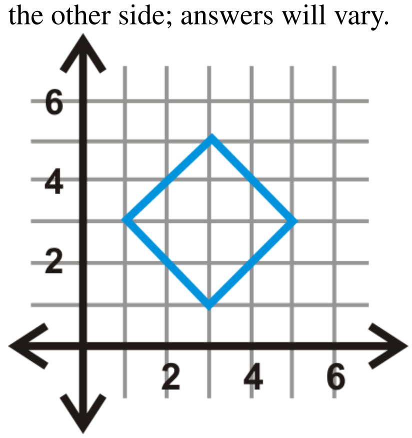
40. 

41. 

Review and Reflect

42. Can an object have both line symmetry and rotational symmetry?
43. If you said yes in the question above, draw an example. If you said no, explain why.

Warm-Up Answers

1. Where one side of an object matches the other side; answers will vary.
2. 

3. $m_{\overline{AD}} = 1, m_{\overline{AB}} = -1, m_{\overline{BC}} = 1, m_{\overline{CD}} = -1$
4. $m_{\overline{AC}} = 0, m_{\overline{BD}} = undefined$. The figure is a square.

11.2 Translations

TEKS G(3)A, G(3)C

Learning Objectives

- Graph a point, line, or figure and translate it x and y units
- Write a translation rule

Warm-Up

1. Find the equation of the line that contains (9, -1) and (5, 7).
2. What type of quadrilateral is formed by $A(1,-1), B(3,0), C(5,-5)$ and $D(-3,0)$?
3. Find the equation of the line parallel to #1 that passes through (4, -3).

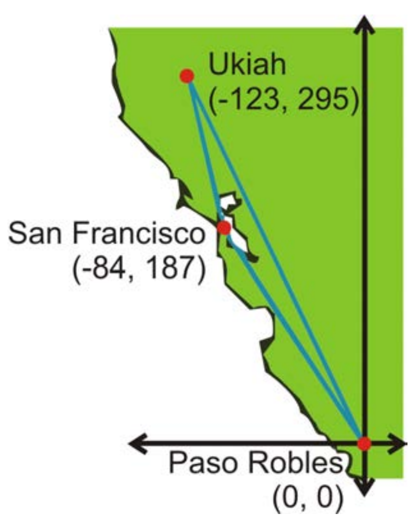
Know What? The distances between San Francisco, S, Paso Robles, P, and Ukiah, U, are given in miles the graph. Find:

a) The translation rule for P to S.

b) The translation rule for S to U.

c) The translation rule for P to U.

d) The translation rule for U to S. It is not the same as part b.



Transformations

Transformation: An operation that moves, flips, or changes a figure to create a new figure.

Rigid Transformation: A transformation that does not change the size or shape of a figure.

The rigid transformations are: translations, reflections, and rotations. The new figure created by a transformation is called the ***image***. The original figure is called the ***preimage***. Another word for a rigid transformation is an ***isometry*** or ***congruence transformations***.

In Lesson 7.6, we learned how to label an image. If the preimage is A, then the image would be A', said "a prime." If there is an image of A', that would be labeled A'', said "a double prime."

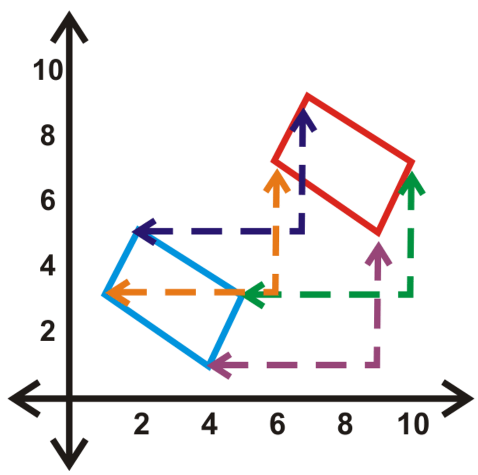
Watch This!!!

http://www.schooltube.com/video/4fcf9da17ae74cfea998/Transformations

Translations

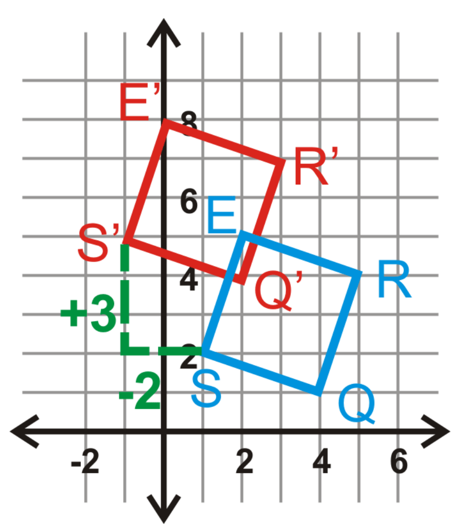
Translation: A transformation that moves every point in a figure the same distance in the same direction.

This transformation moves the parallelogram to the right 5 units and up 3 units. It is written $(x,y) \rightarrow (x+5,y+3)$.



Example 1: Graph square $S(1,2), Q(4,1), R(5,4)$ and $E(2,5)$. Find the image after the translation $(x,y) \rightarrow (x-2,y+3)$. Then, graph and label the image.

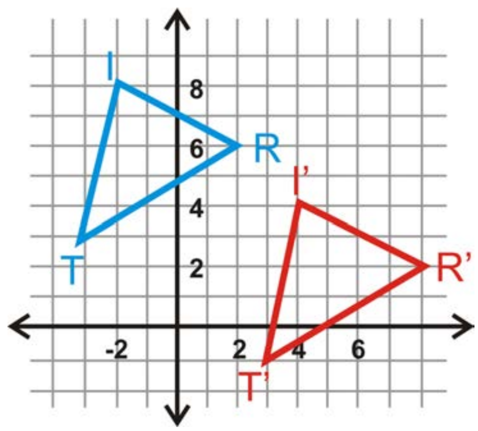
Solution: We are going to move the square to the left 2 and up 3.

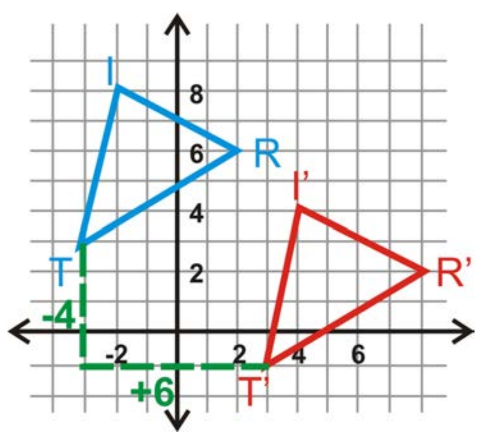


$$(x,y) \rightarrow (x-2,y+3)$$
$$S(1,2) \rightarrow S'(-1,5)$$
$$Q(4,1) \rightarrow Q'(2,4)$$
$$R(5,4) \rightarrow R'(3,7)$$
$$E(2,5) \rightarrow E'(0,8)$$

Example 2: Find the translation rule for $\triangle TRI$ to $\triangle T'R'I'$.

Solution: Look at the movement from T to T'. The translation rule is $(x,y) \rightarrow (x+6,y-4)$.





Example 3: Show $\triangle TRI \cong \triangle T'R'I'$ from Example 2.

Solution: Use the distance formula to find all the lengths of the sides of the two triangles.

$\underline{\triangle TRI}$

$TR = \sqrt{(-3-2)^2+(3-6)^2} = \sqrt{34}$

$RI = \sqrt{(2-(-2))^2+(6-8)^2} = \sqrt{20}$

$TI = \sqrt{(-3-(-2))^2+(3-8)^2} = \sqrt{26}$

$\underline{\triangle T'R'I'}$

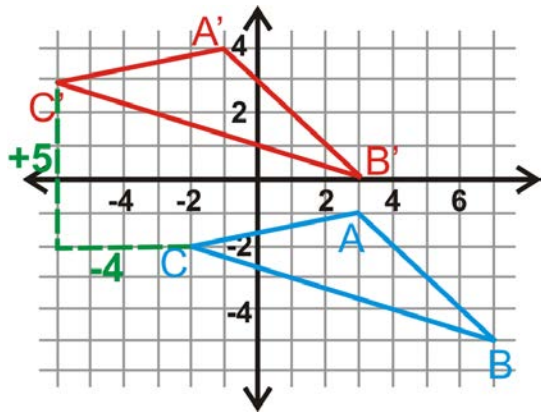
$T'R' = \sqrt{(3-8)^2+(-1-2)^2} = \sqrt{34}$

$R'I' = \sqrt{(8-4)^2+(2-4)^2} = \sqrt{20}$

$T'I' = \sqrt{(3-4)^2+(-1-4)^2} = \sqrt{26}$

This verifies our statement at the beginning of the section that ***a translation is an isometry*** or congruence translation.

Example 4: Triangle $\triangle ABC$ has coordinates $A(3,-1), B(7,-5)$ and $C(-2,-2)$. Translate $\triangle ABC$ to the left 4 units and up 5 units. Determine the coordinates of $\triangle A'B'C'$.



Solution: Graph $\triangle ABC$. To translate $\triangle ABC$, subtract 4 from each x value and add 5 to each y value.

$$A(3,-1) \rightarrow (3-4,-1+5) = A'(-1,4)$$
$$B(7,-5) \rightarrow (7-4,-5+5) = B'(3,0)$$
$$C(-2,-2) \rightarrow (-2-4,-2+5) = C'(-6,3)$$

The rule would be $(x,y) \rightarrow (x-4,y+5)$.

Know What? Revisited

a) $(x,y) \rightarrow (x-84,y+187)$

b) $(x,y) \rightarrow (x-39,y+108)$

c) $(x,y) \rightarrow (x-123,y+295)$

d) $(x,y) \rightarrow (x+39,y-108)$

Practice Problems

- Questions 1-13 are similar to Example 1.
- Questions 14-17 are similar to Example 2.
- Questions 18-20 are similar to Example 3.

- Questions 21-23 are similar to Example 1.
- Questions 24 and 25 are similar to Example 4.

Use the translation $(x,y) \rightarrow (x+5, y-9)$ for questions 1-7.

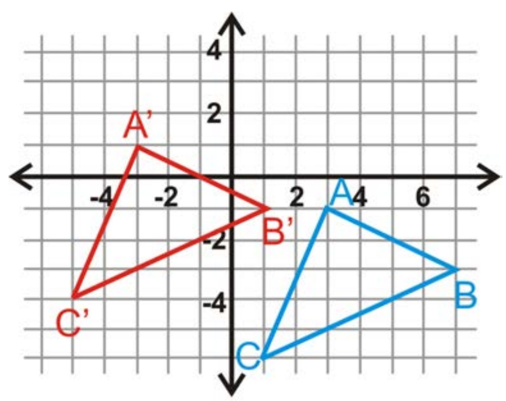
1. What is the image of $A(-6,3)$?
2. What is the image of $B(4,8)$?
3. What is the image of $C(5,-3)$?
4. What is the image of A'?
5. What is the preimage of $D'(12,7)$?
6. What is the image of A''?
7. Plot A, A', A'', and A''' from the questions above. What do you notice?

The vertices of $\triangle ABC$ are $A(-6,-7), B(-3,-10)$ and $C(-5,2)$. Find the vertices of $\triangle A'B'C'$, given the translation rules below.

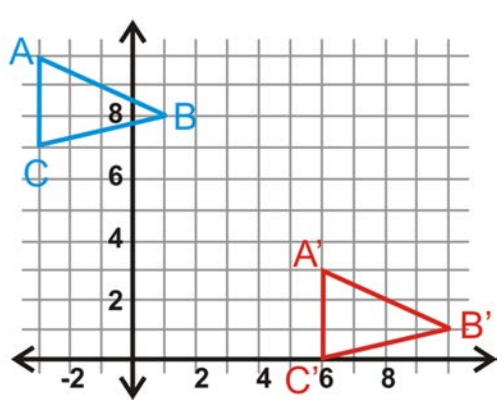
8. $(x,y) \rightarrow (x-2, y-7)$
9. $(x,y) \rightarrow (x+11, y+4)$
10. $(x,y) \rightarrow (x, y-3)$
11. $(x,y) \rightarrow (x-5, y+8)$
12. $(x,y) \rightarrow (x+1, y)$
13. $(x,y) \rightarrow (x+3, y+10)$

In questions 14-17, $\triangle A'B'C'$ is the image of $\triangle ABC$. Write the translation rule.

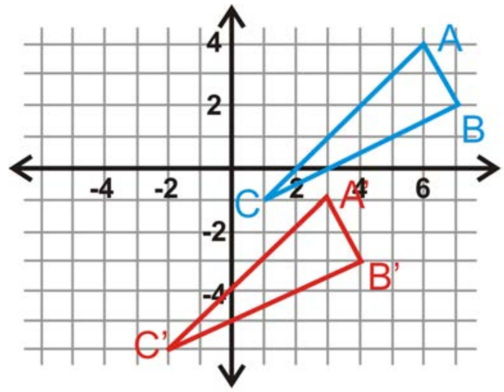
14.



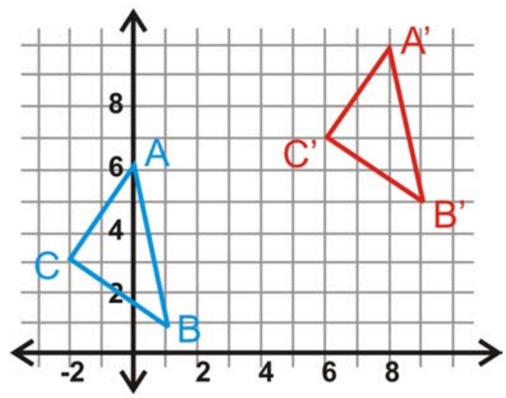
15.



16.



17.



Use the triangles from #17 to answer questions 18-20.

18. Find the lengths of all the sides of $\triangle ABC$.
19. Find the lengths of all the sides of $\triangle A'B'C'$.
20. What can you say about $\triangle ABC$ and $\triangle A'B'C'$? Can you say this for *any* translation?
21. If $\triangle A'B'C'$ was the *preimage* and $\triangle ABC$ was the image, write the translation rule for #14.
22. If $\triangle A'B'C'$ was the *preimage* and $\triangle ABC$ was the image, write the translation rule for #15.
23. Find the translation rule that would move A to $A'(0,0)$, for #16.
24. The coordinates of $\triangle DEF$ are $D(4,-2), E(7,-4)$ and $F(5,3)$. Translate $\triangle DEF$ to the right 5 units and up 11 units. Write the translation rule.
25. The coordinates of quadrilateral $QUAD$ are $Q(-6,1), U(-3,7), A(4,-2)$ and $D(1,-8)$. Translate $QUAD$ to the left 3 units and down 7 units. Write the translation rule.

Review and Reflect

26. The coordinates of $\triangle ABC$ are $A(2,1), B(2,6)$ and $C(4,4)$. The triangle was translated 4 units right and 5 units up, Then it was translated 2 units up and 3 units left. What additional translation would translate the Triangle back to its' original position?

Warm-Up Answers

1. $y = -2x + 17$

2. Kite
3. $y = -2x + 5$

11.3 Reflections

TEKS G(3)A, G(3)C

Learning Objectives

- Reflect a figure over a given line
- Find the rules for reflections

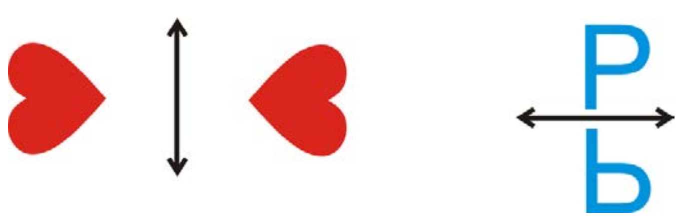
Warm-Up

1. Define reflection in your own words.
2. Plot $A(-3,2)$. Translate A such that $(x,y) \rightarrow (x+6,y)$.
3. What line is halfway between A and A'?

Know What? A lake can act like a mirror in nature. Describe the line of reflection in the picture to the right.

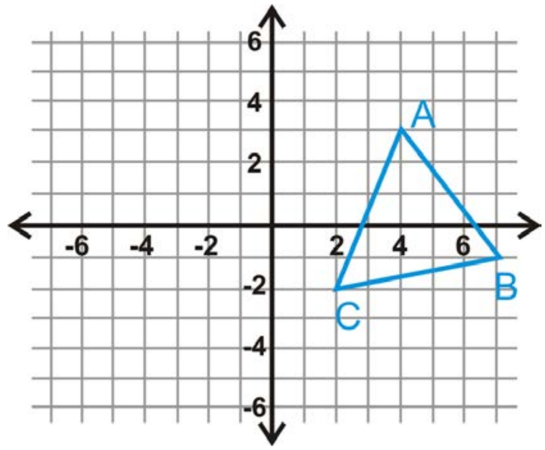
Reflections over an Axis

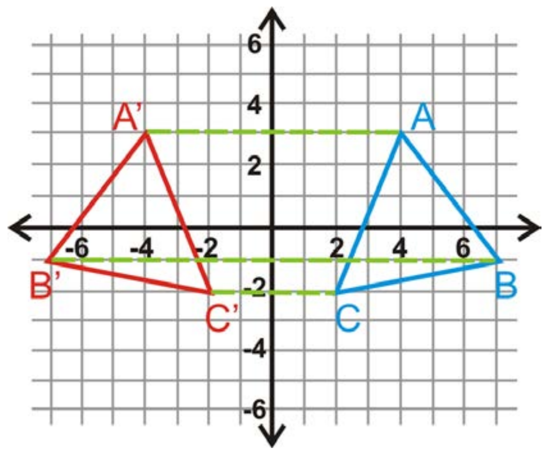
Reflection: A transformation that turns a figure into its mirror image by flipping it over a line.



Line of Reflection: The line that a figure is reflected over.

Example 1: Reflect $\triangle ABC$ over the $y-$axis. Find the coordinates of the image.



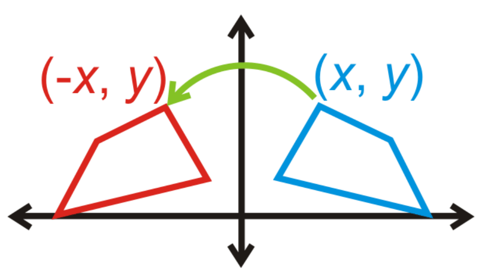


Solution: $\triangle A'B'C'$ will be the same distance away from the $y-$axis as $\triangle ABC$, but on the other side.

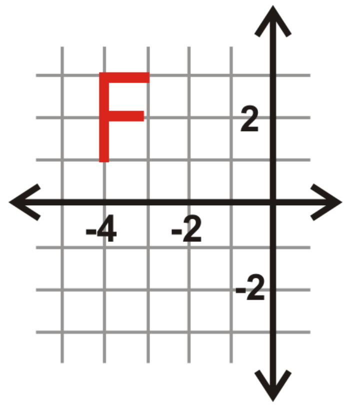
$$A(4,3) \rightarrow A'(-4,3)$$
$$B(7,-1) \rightarrow B'(-7,-1)$$
$$C(2,-2) \rightarrow C'(-2,-2)$$

From this example, we can generalize a rule for reflecting a figure over the $y-$axis.

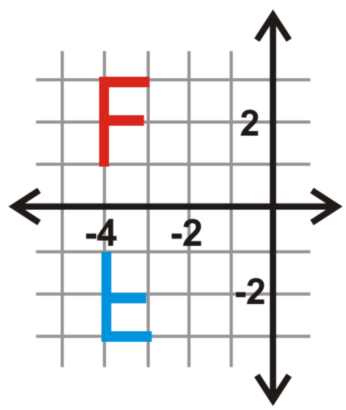
Reflection over the $y-$axis: $(x,y) \rightarrow (-x,y)$



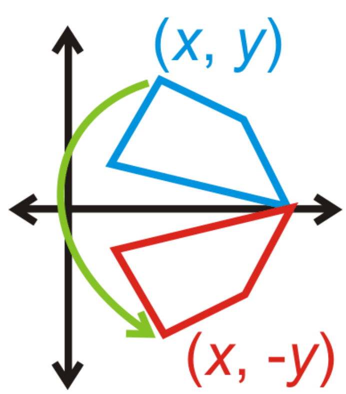
Example 2: Reflect the letter "F'' over the $x-$axis.



Solution: To reflect the letter F over the $x-$axis, the $y-$coordinates will be the same distance away from the $x-$axis, but on the other side of the $x-$axis.



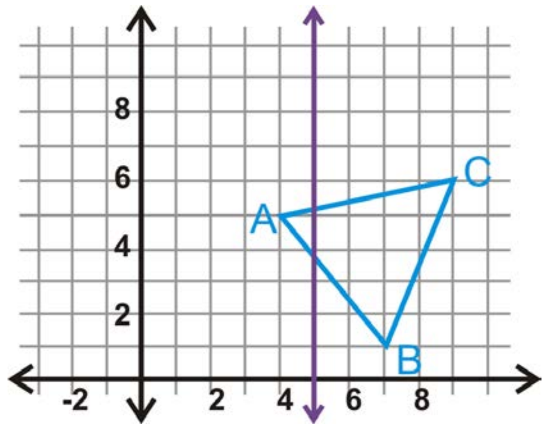
Reflection over the $x-$axis: $(x,y) \rightarrow (x,-y)$



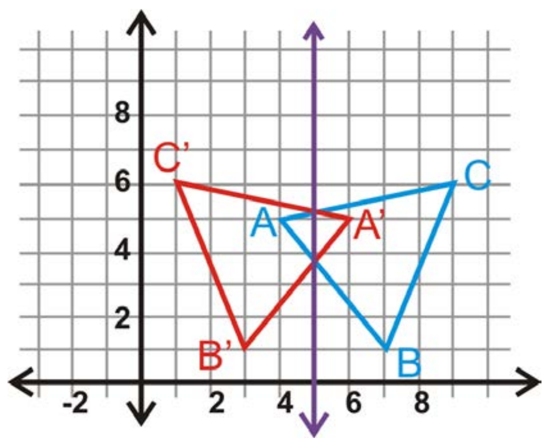
Reflections over Horizontal and Vertical Lines

We can also reflect a figure over any vertical or horizontal line.

Example 3: Reflect the triangle $\triangle ABC$ with vertices $A(4,5), B(7,1)$ and $C(9,6)$ over the line $x = 5$.



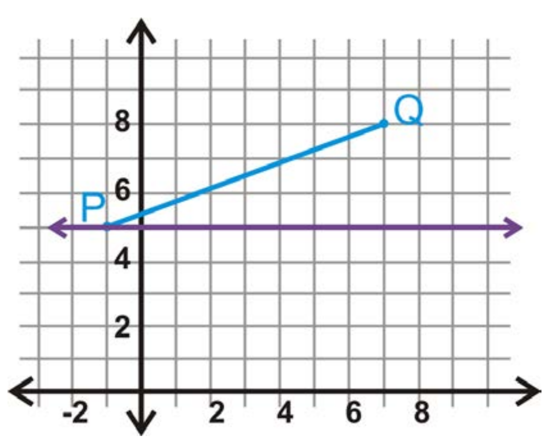
Solution: The image's vertices are the same distance away from $x = 5$ as the preimage.



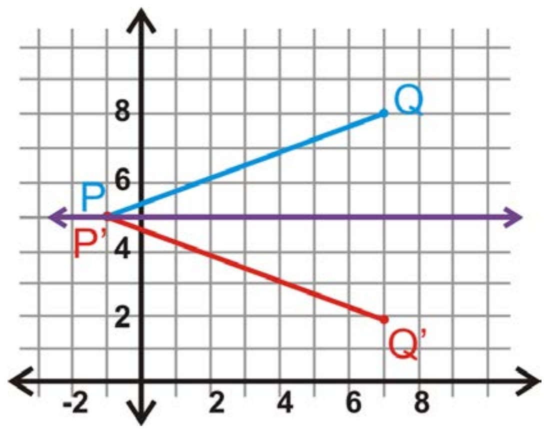
$$\begin{aligned}A(4,5) &\rightarrow A'(6,5)\\B(7,1) &\rightarrow B'(3,1)\\C(9,6) &\rightarrow C'(1,6)\end{aligned}$$

Example 4: Reflect the line segment $\overline{PQ}$ with endpoints $P(-1,5)$ and $Q(7,8)$ over the line $y = 5$.

Solution: The line of refection is on P, which means P' has the same coordinates. Q' is the same distance away from $y = 5$, but on the other side.

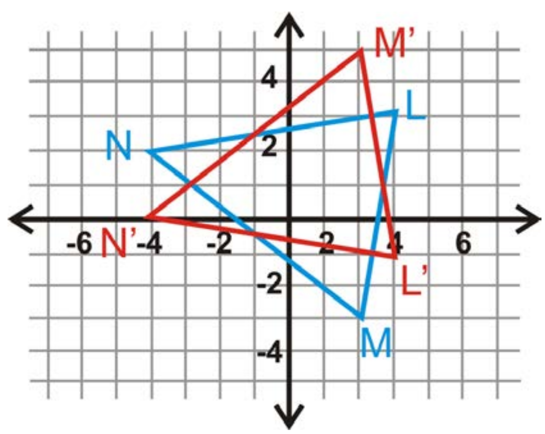


$P(-1,5) \rightarrow P'(-1,5)$
$Q(7,8) \rightarrow Q'(7,2)$



From these examples we have learned that ***if a point is on the line of reflection then the image is the same as the preimage.***

Example 5: A triangle $\triangle LMN$ and its reflection, $\triangle L'M'N'$ are to the left. What is the line of reflection?

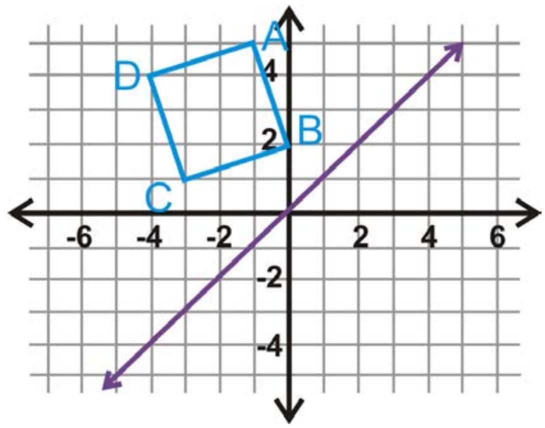


Solution: Looking at the graph, we see that the preimage and image intersect when $y = 1$. Therefore, this is the line of reflection.

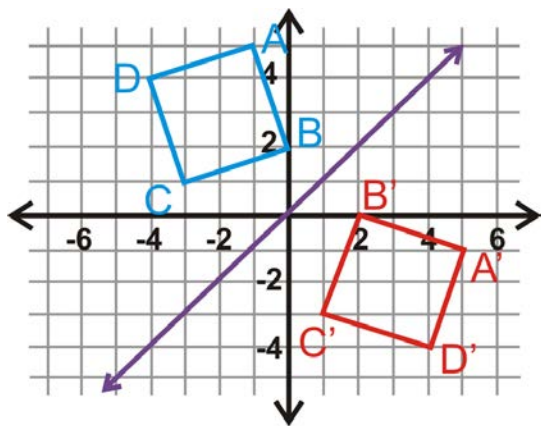
If the image does not intersect the preimage, find the midpoint between the preimage and its image. This point is on the line of reflection.

Reflections over $y = x$ **and** $y = -x$

Example 6: Reflect square $ABCD$ over the line $y = x$.



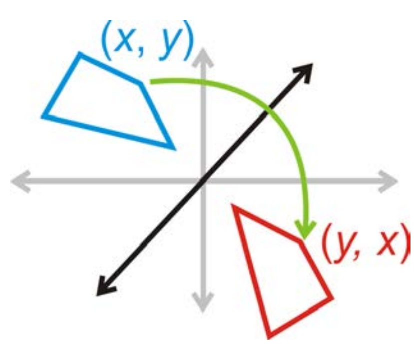
Solution: The purple line is $y = x$. Fold the graph on the line of reflection.



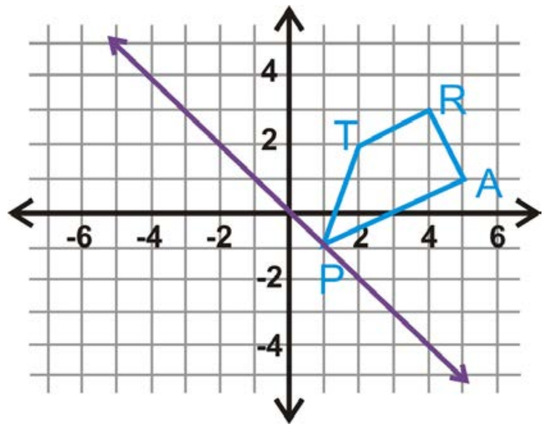
$$
\begin{aligned}
&A(-1,5) \rightarrow A'(5,-1)\\
&B(0,2) \rightarrow B'(2,0)\\
&C(-3,1) \rightarrow C'(1,-3)\\
&D(-4,4) \rightarrow D'(4,-4)
\end{aligned}
$$

From this example, we see that the x and y values are switched.

Reflection over $y = x$**:** $(x,y) \rightarrow (y,x)$

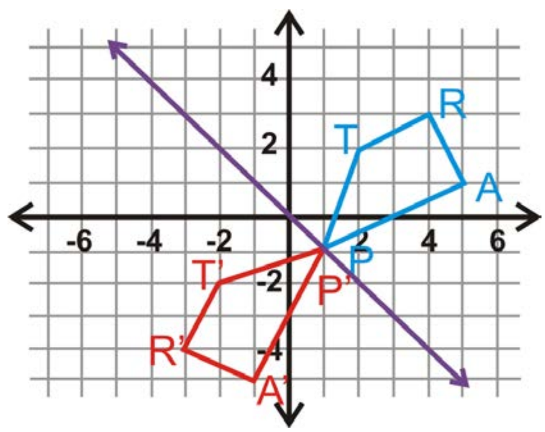


Example 7: Reflect the trapezoid $TRAP$ over the line $y = -x$.



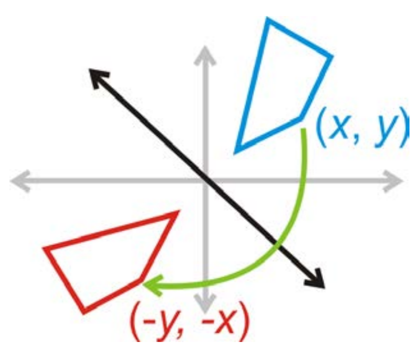
Solution: The purple line is $y=-x$. You can reflect the trapezoid over this line just like we did in Example 6.

$$T(2,2) \rightarrow T'(-2,-2)$$
$$R(4,3) \rightarrow R'(-3,-4)$$
$$A(5,1) \rightarrow A'(-1,-5)$$
$$P(1,-1) \rightarrow P'(1,-1)$$



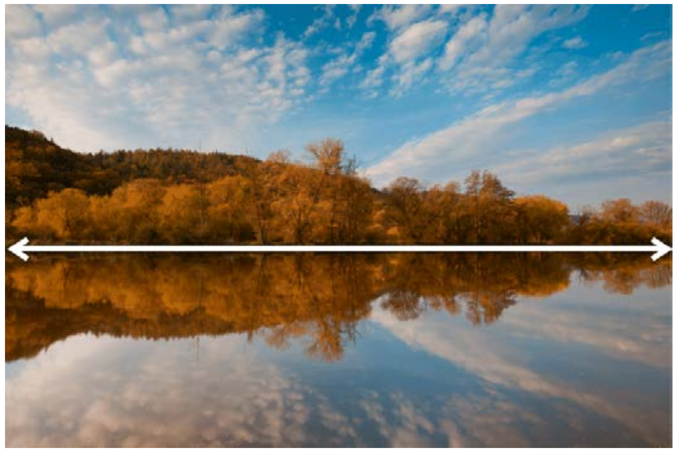
From this example, we see that the x and y values are switched with the opposite signs.

Reflection over $y=-x$**:** $(x,y) \rightarrow (-y,-x)$



From all of these examples, we notice that ***a reflection is an isometry***.

Know What? Revisited The white line in the picture is the line of reflection. This line coincides with the water's edge.



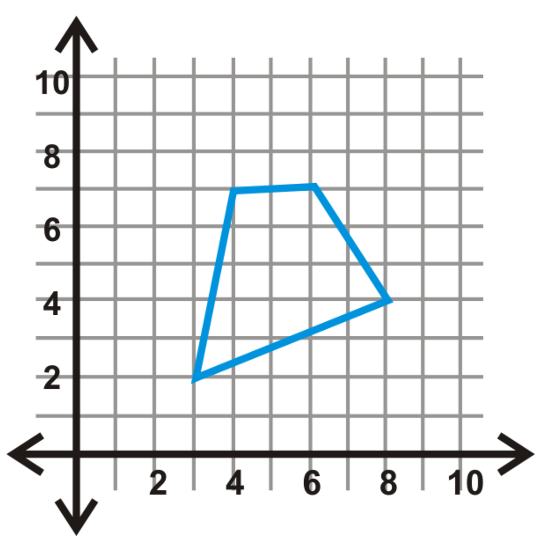
Practice Problems

- Questions 1-5 are similar to Examples 1, 3, 4, 6, and 7.
- Questions 6 and 7 are similar to Example 2.
- Questions 8-19 are similar to Examples 1, 3, 4, 6, and 7.
- Questions 20-22 are similar to Example 5.
- Questions 23-30 are similar to Examples 3 and 4.

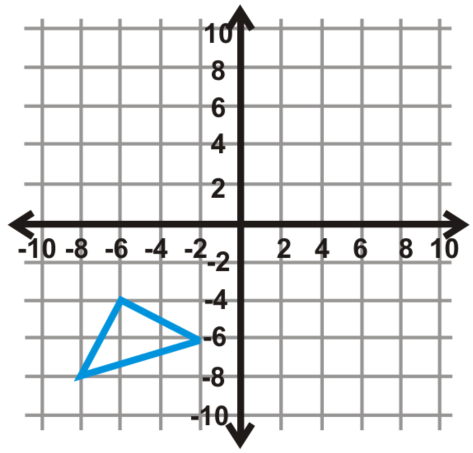
1. If (5, 3) is reflected over the $y-$axis, what is the image?
2. If (5, 3) is reflected over the $x-$axis, what is the image?
3. If (5, 3) is reflected over $y = x$, what is the image?
4. If (5, 3) is reflected over $y = -x$, what is the image?
5. Plot the four images. What shape do they make? Be specific.
6. Which letter is a reflection over a vertical line of the letter “b''?
7. Which letter is a reflection over a horizontal line of the letter “b''?

Reflect each shape over the given line

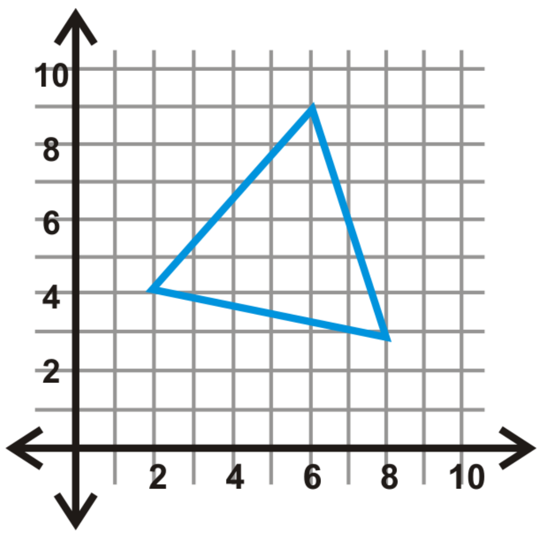
8.$.y-$axis



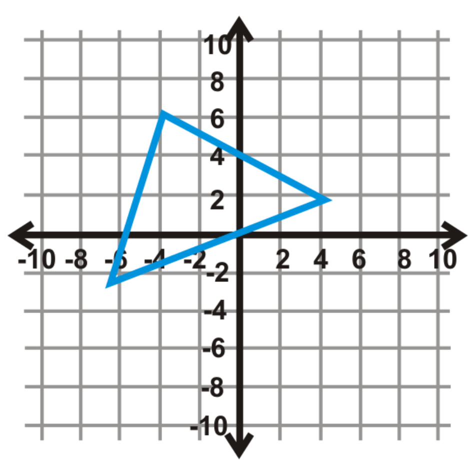
9.$x-$axis



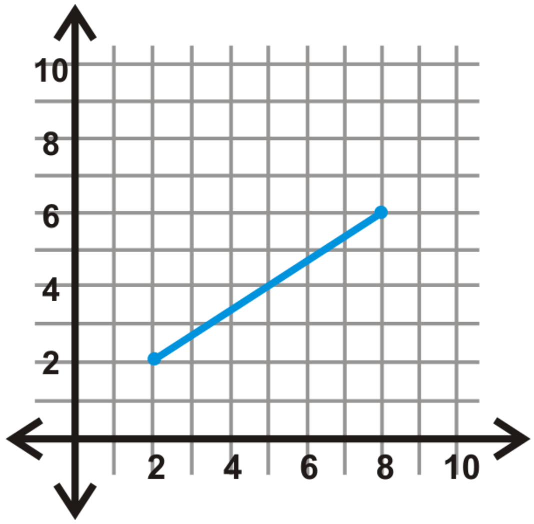
10. $y = 3$



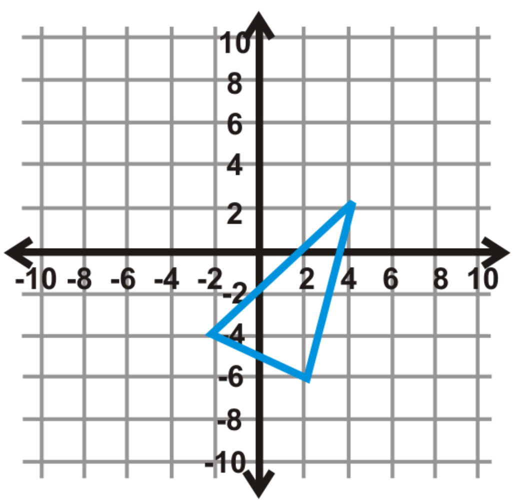
11. $x = -1$



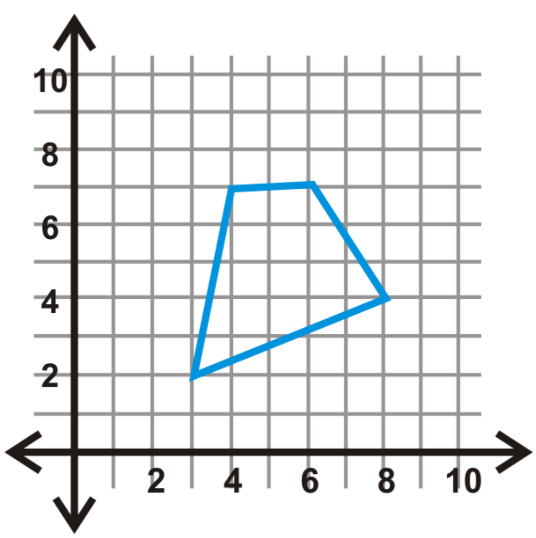
12. $x-$axis



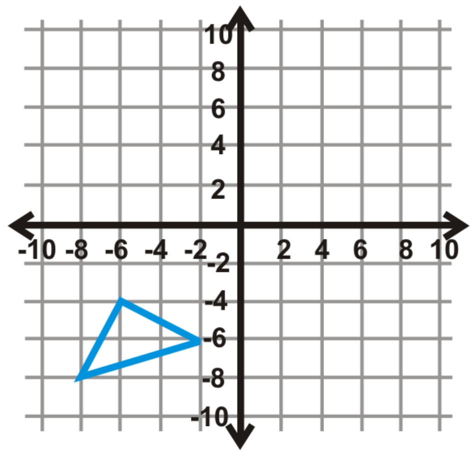
13. $y-$axis



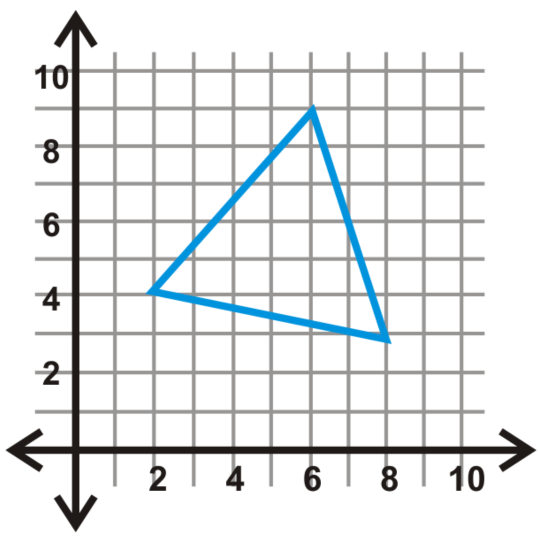
14. $y=x$



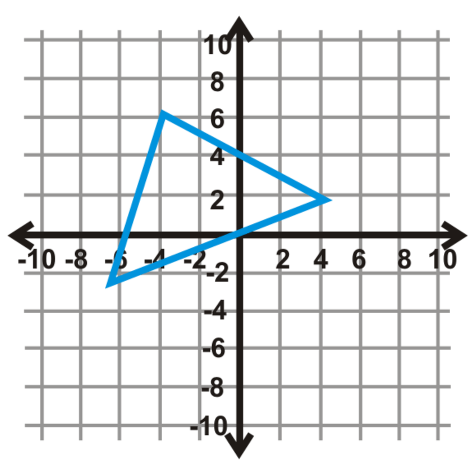
15. $y = -x$



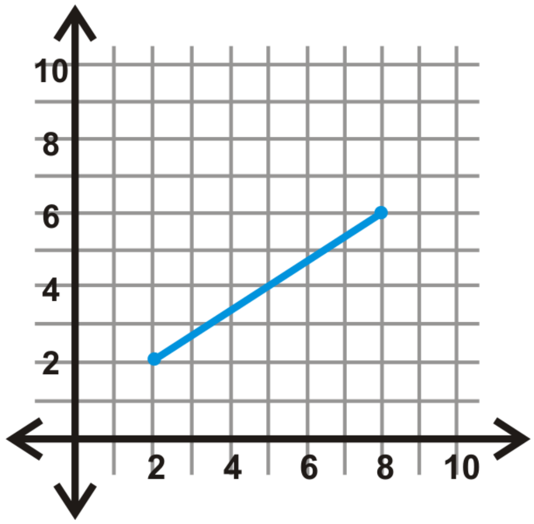
16. $x = 2$



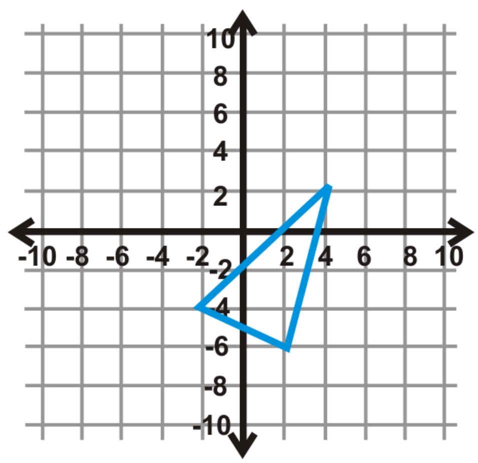
17. $y = -4$



18.$y = -x$

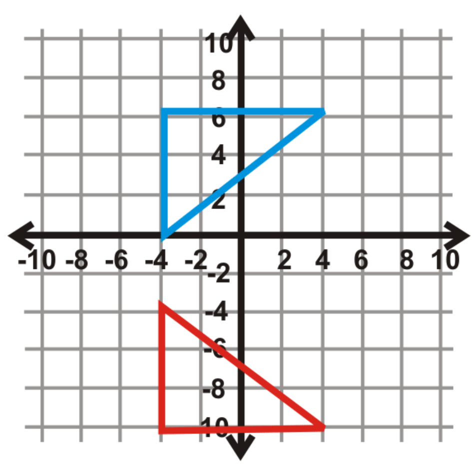


19. $y = x$

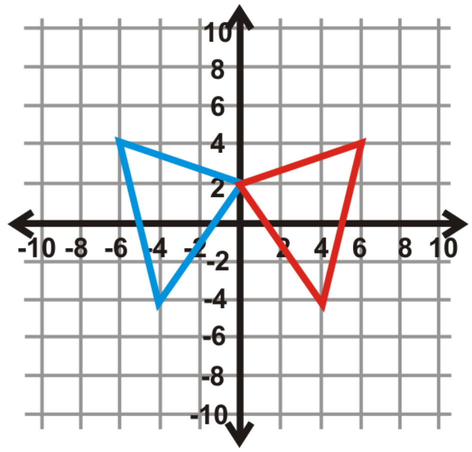


Find the line of reflection the blue triangle (preimage) and the red triangle (image).

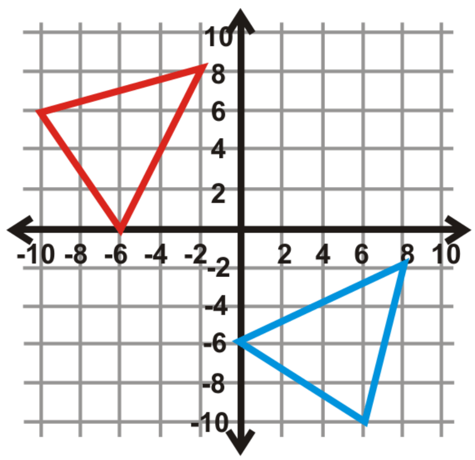
20.



21.



22.



Two Reflections The vertices of $\triangle ABC$ are $A(-5,1), B(-3,6)$, and $C(2,3)$. Use this information to answer questions 23-26.

23. Plot $\triangle ABC$ on the coordinate plane.
24. Reflect $\triangle ABC$ over $y = 1$. Find the coordinates of $\triangle A'B'C'$.
25. Reflect $\triangle A'B'C'$ over $y = -3$. Find the coordinates of $\triangle A''B''C''$.
26. What ***one*** transformation would be the same as this double reflection?

Two Reflections The vertices of $\triangle DEF$ are $D(6,-2), E(8,-4)$, and $F(3,-7)$. Use this information to answer questions 27-30.

27. Plot $\triangle DEF$ on the coordinate plane.
28. Reflect $\triangle DEF$ over $x = 2$. Find the coordinates of $\triangle D'E'F'$.
29. Reflect $\triangle D'E'F'$ over $x = -4$. Find the coordinates of $\triangle D''E''F''$.
30. What ***one*** transformation would be the same as this double reflection?

Review and Reflect

31. Can an object be reflected over a line and not move positions? If yes provide an example?

Warm-Up Answers

1. *Examples are:* To flip an image over a line; A mirror image.
2. $A'(3,2)$
3. the $y-$axis

11.4 Rotations

TEKS G(3)A, G(3)C, G(3)D

Learning Objectives

- Find the image of a figure in a rotation in a coordinate plane

Vocabulary

- Rotations
- Center of Rotation
- Angle of Rotation

Warm-Up

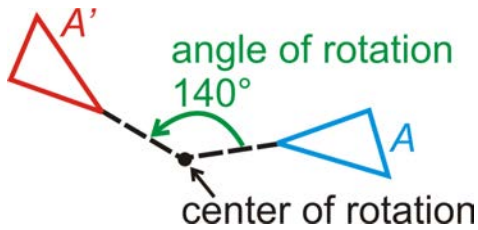
1. Reflect $\triangle XYZ$ with vertices $X(9,2), Y(2,4)$ and $Z(7,8)$ over the $y-$axis. What are the vertices of $\triangle X'Y'Z'$?
2. Reflect $\triangle X'Y'Z'$ over the $x-$axis. What are the vertices of $\triangle X''Y''Z''$?
3. How do the coordinates of $\triangle X''Y''Z''$ relate to $\triangle XYZ$?

Know What? The international symbol for recycling is to the right. It is three arrows rotated around a point. Let's assume that the arrow on the top is the preimage and the other two are its images. Find the center of rotation and the angle of rotation for each image.

Defining Rotations

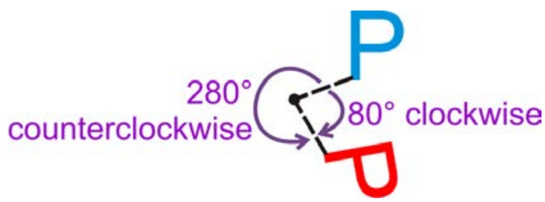
Rotation: A transformation where a figure is turned around a fixed point to create an image.

The lines drawn from the preimage to the ***center of rotation***, and from the center of rotation to the image form the ***angle of rotation***. In this section, we will only do ***counterclockwise rotations***.



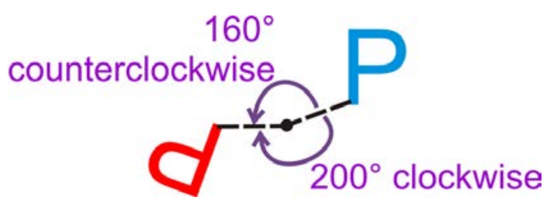
Example 1: A rotation of 80° clockwise is the same as what counterclockwise rotation?

Solution: There are 360° around a point. So, an 80° rotation clockwise is the same as a $360^\circ - 80^\circ = 280^\circ$ rotation counterclockwise.



Example 2: A rotation of 160° counterclockwise is the same as what clockwise rotation?

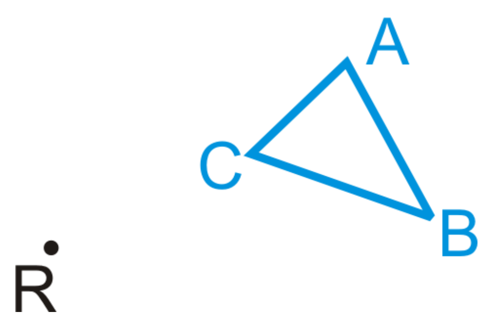
Solution: $360^\circ - 160^\circ = 200^\circ$ clockwise rotation.



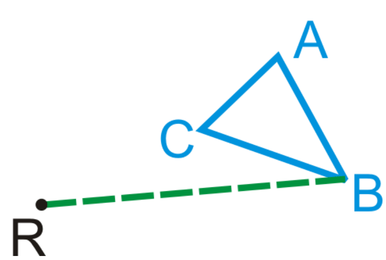
Investigation 11-1: Drawing a Rotation of 100°

Tools Needed: pencil, paper, protractor, ruler

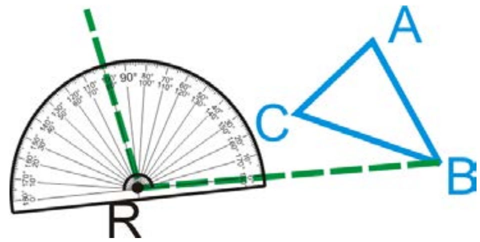
1. Draw $\triangle ABC$ and a point R.



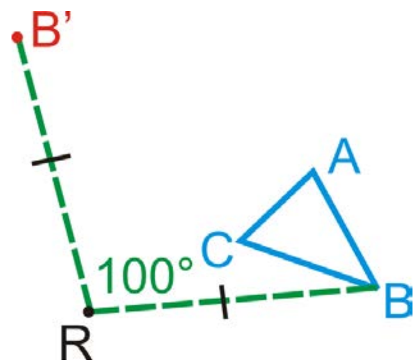
2. Draw $\overline{RB}$.



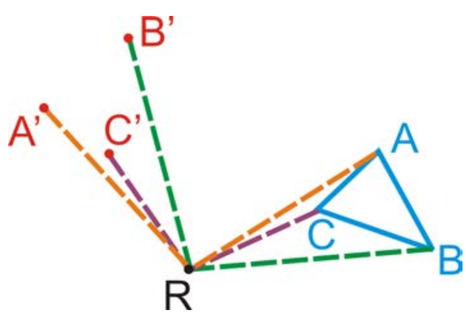
3. Place the center of a protractor on R and the 0° line on $\overline{RB}$. Mark a 100° angle.



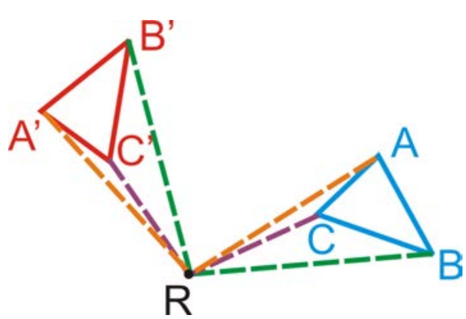
4. Mark B' on the 100° line so $RB = RB'$.



5. Repeat steps 2-4 with A and C.

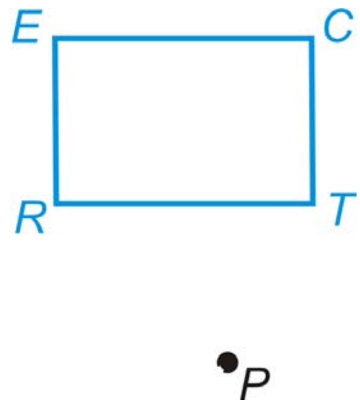


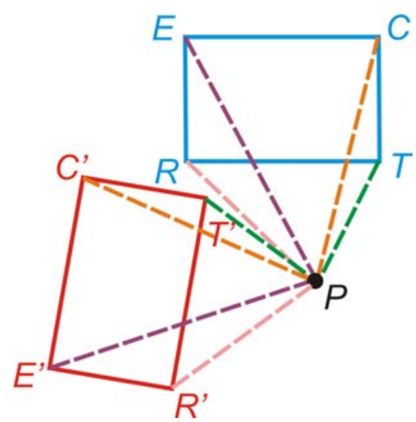
6. Make $\triangle A'B'C'$.



Use this process to rotate any figure.

Example 3: Rotate rectangle $RECT$ 80° counterclockwise around P.





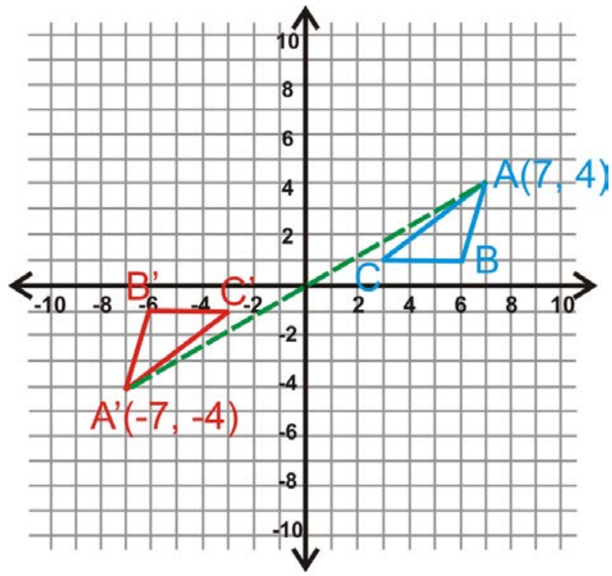
Solution: Use Investigation 11-1. In step 3, change the angle to 80°. Each angle of rotation is 80°.

$$m\angle RPR' = 80^\circ$$
$$m\angle EPE' = 80^\circ$$
$$m\angle CPC' = 80^\circ$$
$$m\angle TPT' = 80^\circ$$

180° **Rotation**

To rotate a figure 180°, in the $x-y$ plane, we ***use the origin as the center of the rotation***. A 180° angle is called a straight angle. So, an image rotated over the origin 180° will be on the same line and the same distance away from the origin as the preimage, but on the other side.

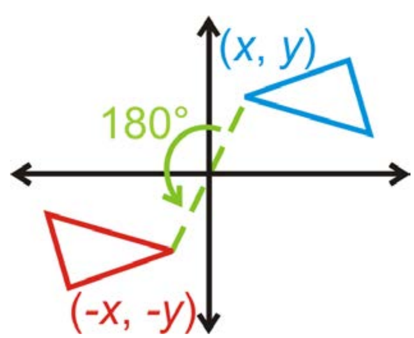
Example 4: Rotate $\triangle ABC$, with vertices $A(7,4), B(6,1)$, and $C(3,1)$, 180°. Find the coordinates of $\triangle A'B'C'$.



Solution: You can either use Investigation 11-1 or the hint given above to find $\triangle A'B'C'$. First, graph the triangle. If A is (7, 4), that means it is 7 units to the ***right*** of the origin and 4 units ***up***. A' would then be 7 units to the ***left*** of the origin and 4 units ***down***.

$$A(7,4) \rightarrow A'(-7,-4)$$
$$B(6,1) \rightarrow B'(-6,-1)$$
$$C(3,1) \rightarrow C'(-3,-1)$$

Rotation of 180°**:** $(x,y) \rightarrow (-x,-y)$

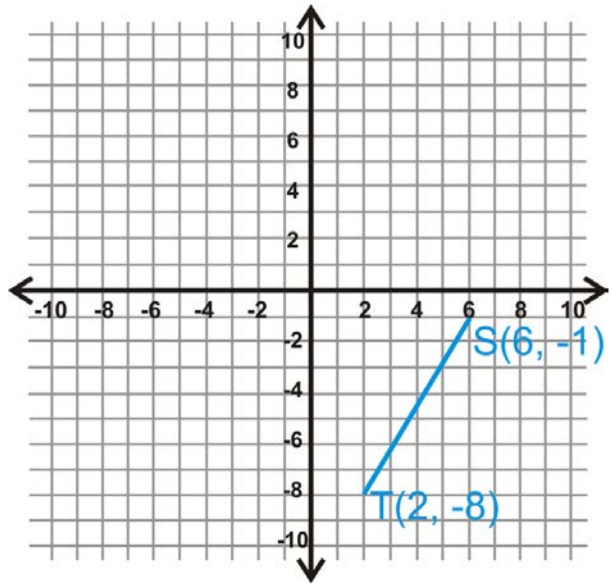


Recall from the second section that ***a rotation is an isometry***. This means that $\triangle ABC \cong \triangle A'B'C'$. You can use the distance formula to show this.

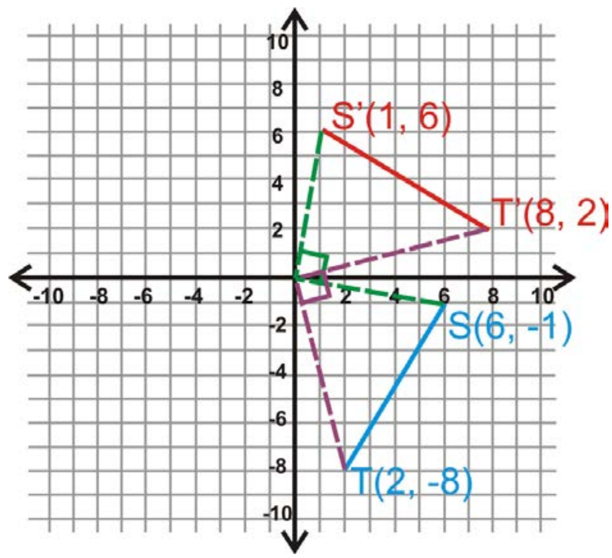
90° **Rotation**

Similar to the 180° rotation, the image of a 90° will be the same distance away from the origin as its preimage, but rotated 90°.

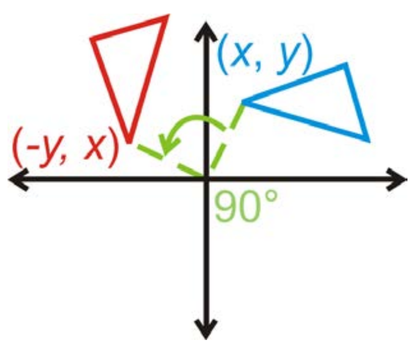
Example 5: Rotate $\overline{ST}$ 90°.



Solution: When rotating something 90°, use Investigation 12-1 to see if there is a pattern.



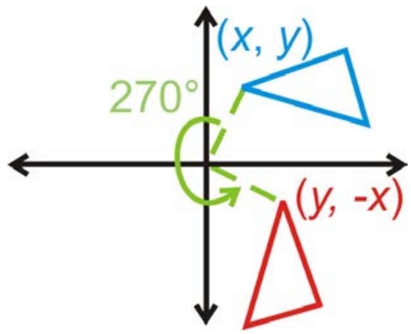
Rotation of 90°**:** $(x,y) \rightarrow (-y,x)$



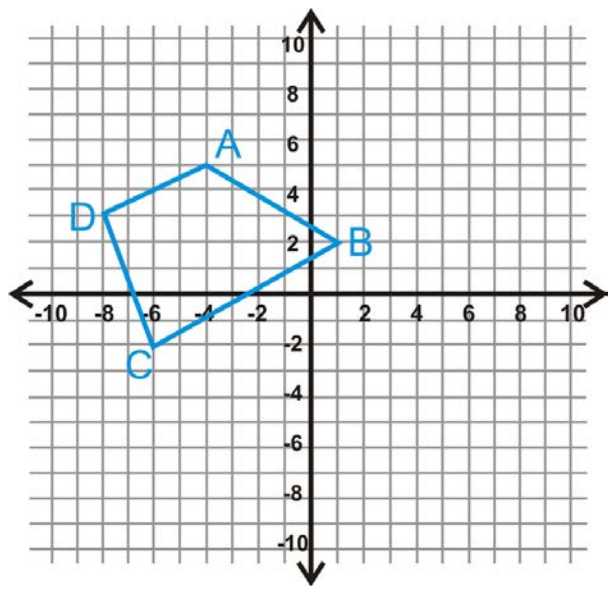
Rotation of 270°

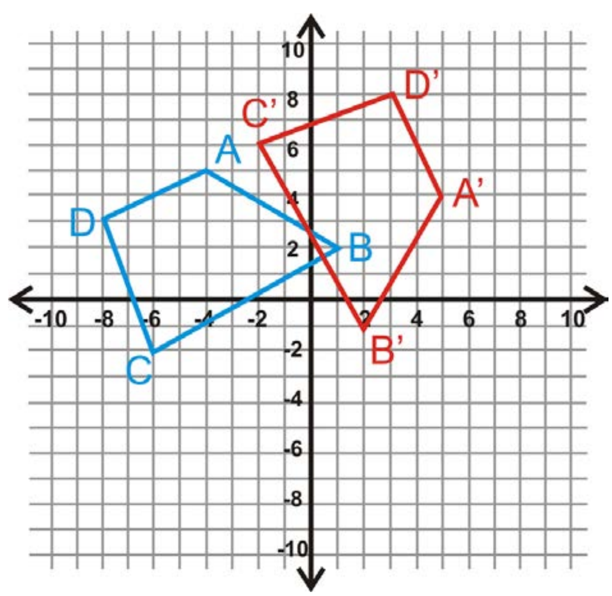
A rotation of 270° counterclockwise would be the same as a rotation of 90° *plus* a rotation of 180°. So, if the values of a 90° rotation are $(-y,x)$, then a 270° rotation would be the opposite sign of each, or $(y,-x)$.

Rotation of 270°**:** $(x, y) \rightarrow (y, -x)$



Example 6: Find the coordinates of *ABCD* after a 270° rotation.





Solution: Using the rule, we have:

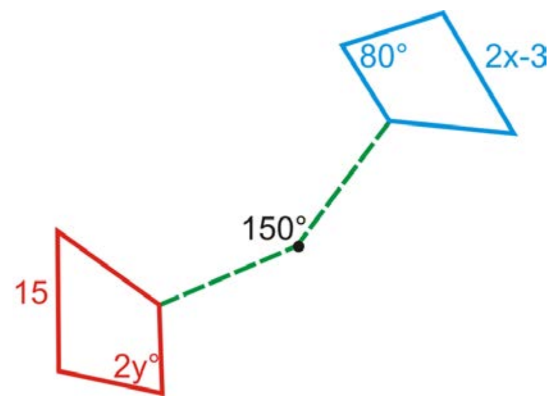
$$\begin{aligned}(x,y) &\rightarrow (y,-x)\\ A(-4,5) &\rightarrow A'(5,4)\\ B(1,2) &\rightarrow B'(2,-1)\\ C(-6,-2) &\rightarrow C'(-2,6)\\ D(-8,3) &\rightarrow D'(3,8)\end{aligned}$$

While we can rotate any image any amount of degrees, only $90^\circ, 180^\circ$ and 270° have special rules. To rotate a figure by an angle measure other than these three, you must use Investigation 12-1.

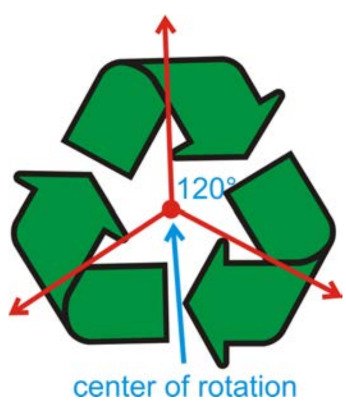
Example 7: ***Algebra Connection*** The rotation of a quadrilateral is shown below. What is the measure of x and y?

Solution: Because a rotation is an isometry, we can set up two equations to solve for x and y.

$$\begin{aligned}2y &= 80^\circ\\ y &= 40^\circ\end{aligned} \qquad \begin{aligned}2x-3 &= 15\\ 2x &= 18\\ x &= 9\end{aligned}$$



Know What? Revisited The center of rotation is shown in the picture to the right. If we draw rays to the same place in each arrow, the two images are a 120° rotation in either direction.



Practice Problems

- Questions 1-10 are similar to Examples 1 and 2.

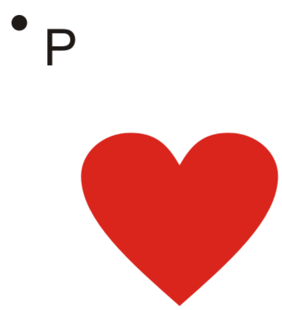
- Questions 11-16 are similar to Investigation 12-1 and Example 3.
- Questions 17-25 are similar to Examples 4-6.
- Questions 26-28 are similar to Example 7.
- Questions 29-34 are similar to Examples 4-6.
- Questions 34-37 are a review.
- Question 38 is similar to Example 4.

In the questions below, every rotation is *counterclockwise*, unless otherwise stated.

1. If you rotated the letter p 180° counterclockwise, what letter would you have?
2. If you rotated the letter p 180° *clockwise*, what letter would you have?
3. A 90° clockwise rotation is the same as what counterclockwise rotation?
4. A 270° clockwise rotation is the same as what counterclockwise rotation?
5. A 210° counterclockwise rotation is the same as what clockwise rotation?
6. A 120° counterclockwise rotation is the same as what clockwise rotation?
7. A 340° counterclockwise rotation is the same as what clockwise rotation?
8. Rotating a figure 360° is the same as what other rotation?
9. Does it matter if you rotate a figure 180° clockwise or counterclockwise? Why or why not?
10. When drawing a rotated figure and using your protractor, would it be easier to rotate the figure 300° counterclockwise or 60° clockwise? Explain your reasoning.

Using Investigation 11-1, rotate each figure counterclockwise around point P the given angle measure.

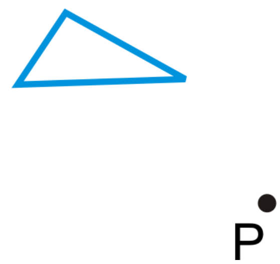
11. 50° rotation



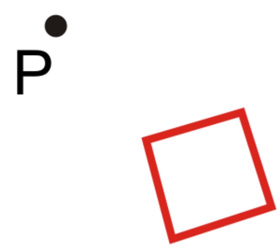
12. 120° rotation

13. 200° rotation

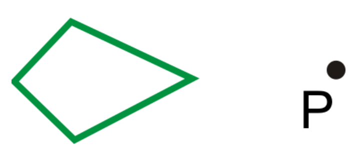
14. 330° rotation



15. 75° rotation

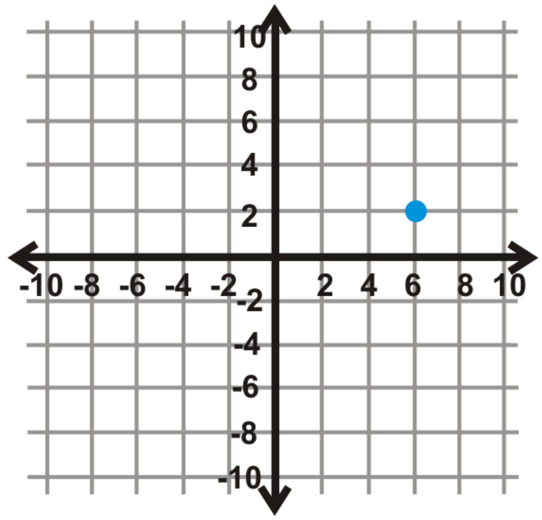


16. 170° rotation



Rotate each figure in the coordinate plane the given angle measure. The center of rotation is the origin. Give the coordinates and graph the image.

17. 180°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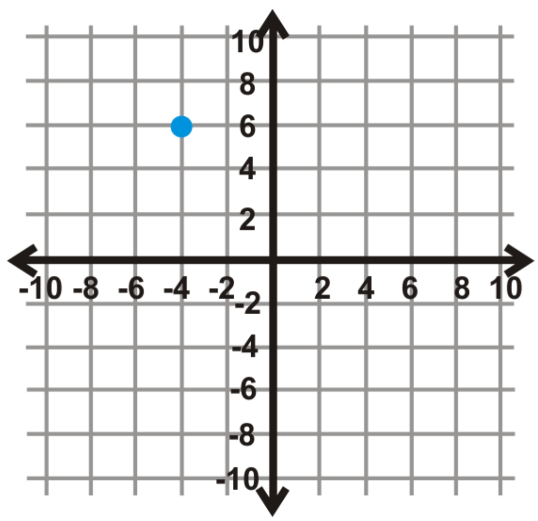
18. 90°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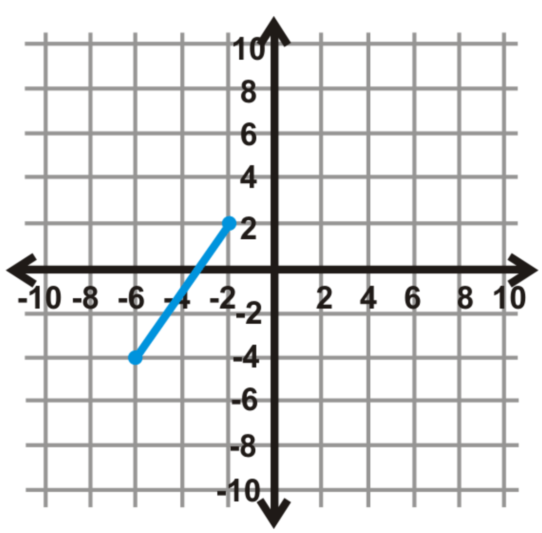
19. 180°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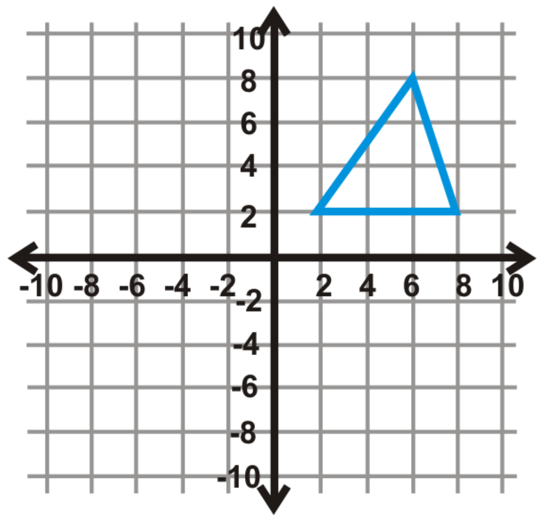
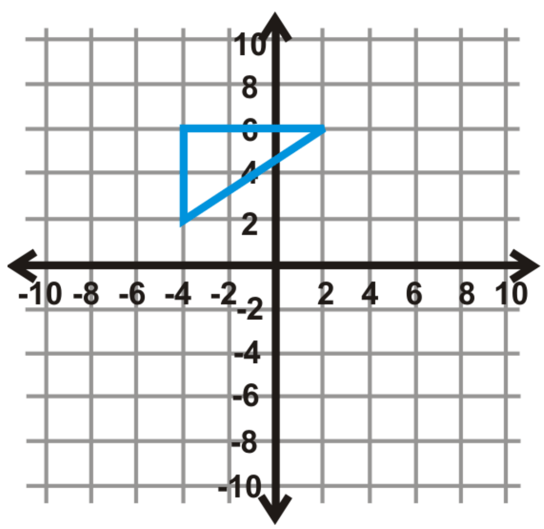
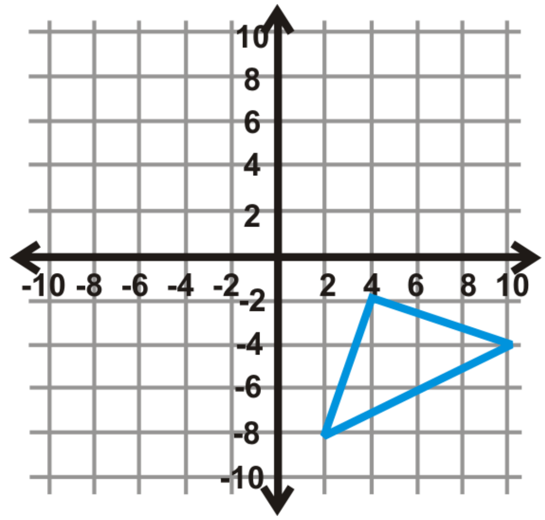
20. 270°



21. 90°



22. 270°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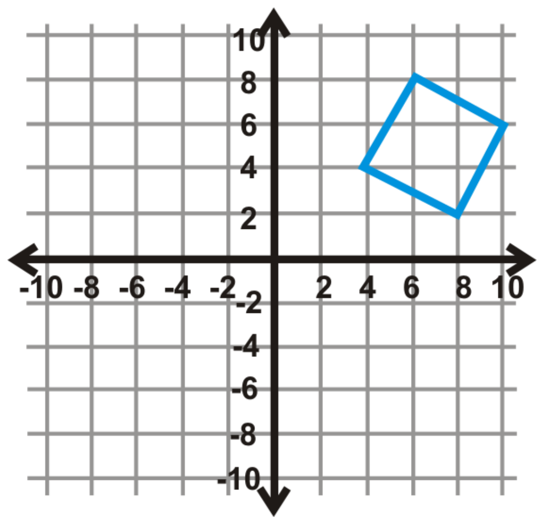
23. 180°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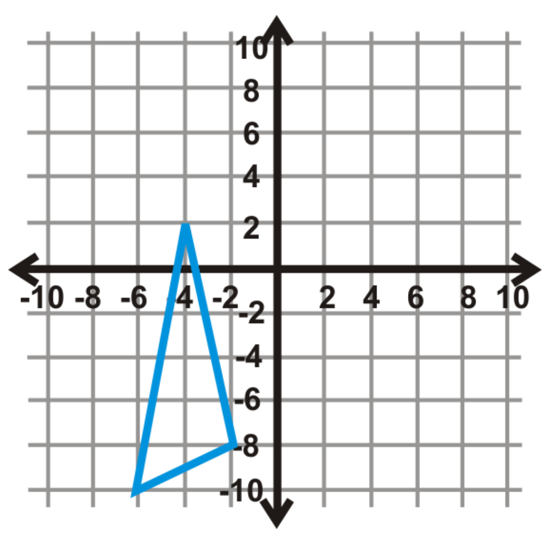
24. 270°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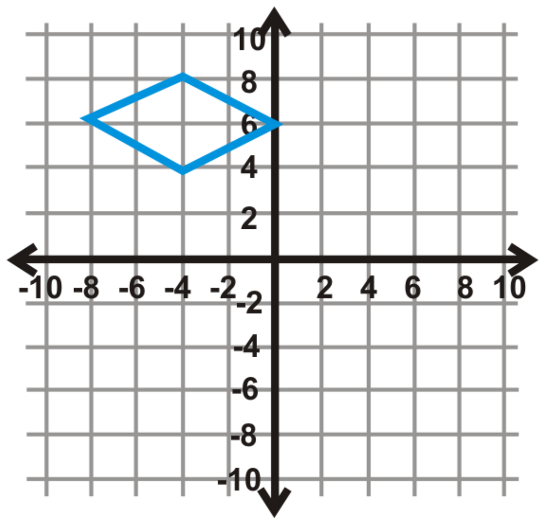
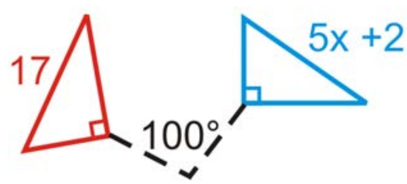
90°25.

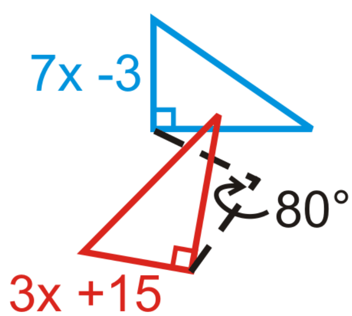


Algebra Connection Find the measure of x in the rotations below. The blue figure is the preimage.

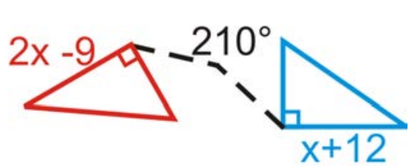
26.



27.

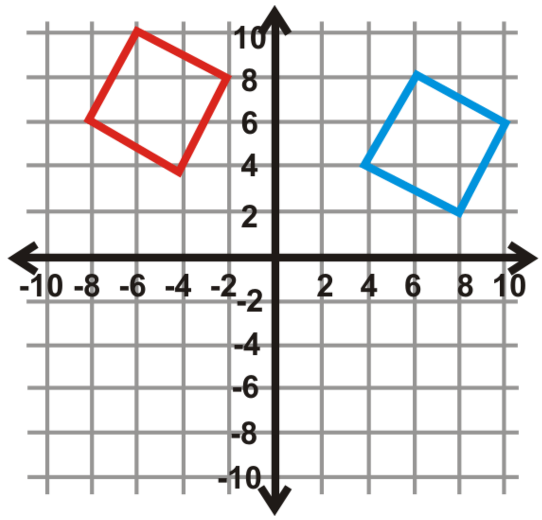


28.



Find the angle of rotation for the graphs below. The center of rotation is the origin and the blue figure is the preimage. Your answer will be $90^\circ, 270^\circ$, or 180°.

29.



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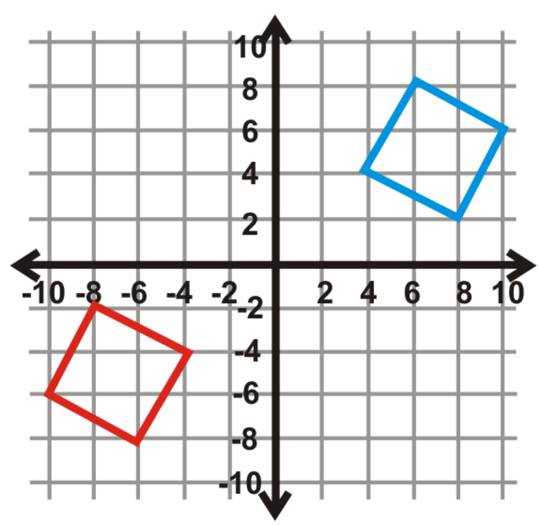
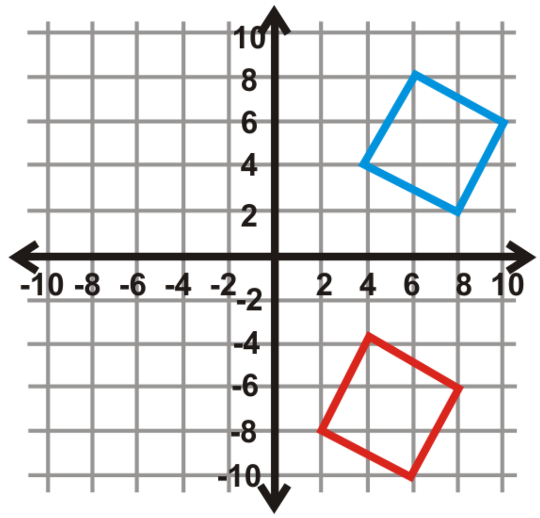


31.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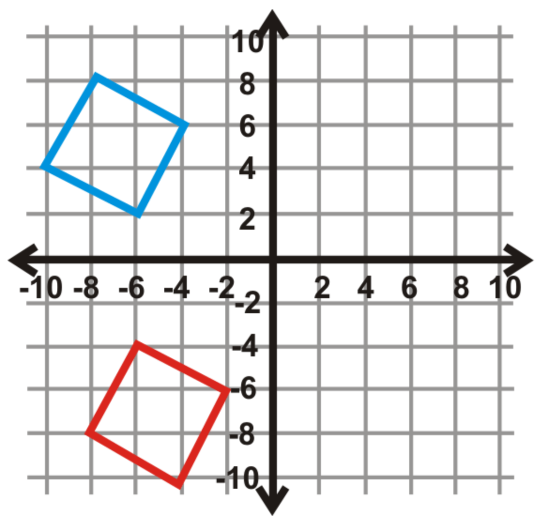
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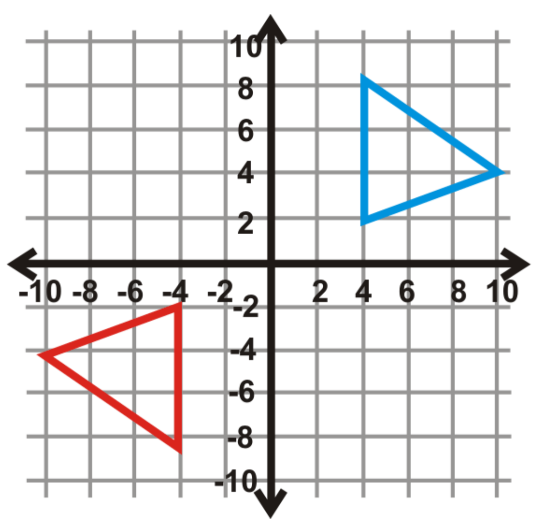
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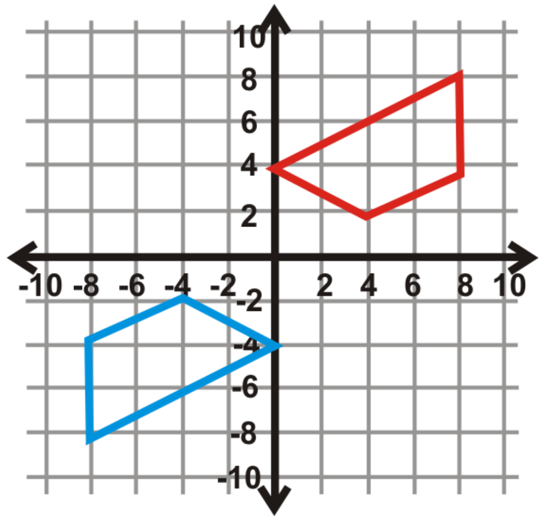
34.



Two Reflections The vertices of $\triangle GHI$ are $G(-2,2), H(8,2)$, and $I(6,8)$. Use this information to answer questions 24-27.

35. Plot $\triangle GHI$ on the coordinate plane.
36. Reflect $\triangle GHI$ over the $x-$axis. Find the coordinates of $\triangle G'H'I'$.
37. Reflect $\triangle G'H'I'$ over the $y-$axis. Find the coordinates of $\triangle G''H''I''$.
38. What ***one*** transformation would be the same as this double reflection?

Warm-Up Answers

1. $X'(-9,2), Y'(-2,4), Z'(-7,8)$
2. $X''(-9,-2), Y''(-2,-4), Z''(-7,-8)$
3. $\triangle X''Y''Z''$ is the double negative of $\triangle XYZ; (x,y) \rightarrow (-x,-y)$

11.5 Composition of Transformations

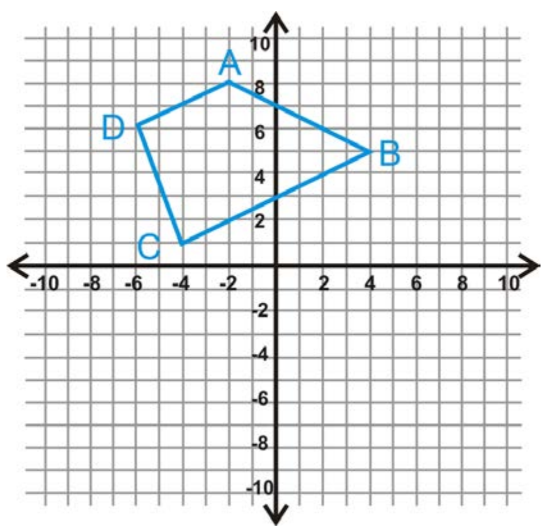
TEKS G(3)A, G(3)B, G(3)C, G(3)D

Learning Objectives

- Perform a glide reflection
- Perform a reflection over parallel lines and the axes
- Determine a single transformation that is equivalent to a composite of two transformations

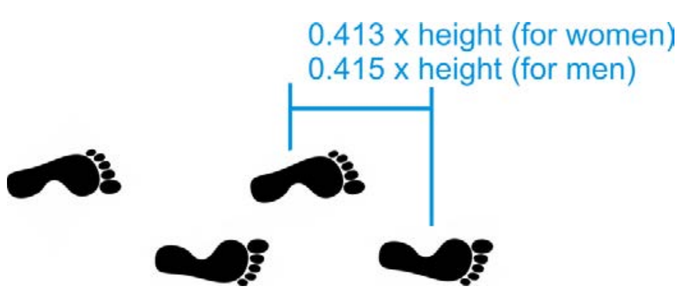
Warm-Up

1. Reflect $ABCD$ over the $x-$axis. Find the coordinates of $A'B'C'D'$.



2. Translate $A'B'C'D'$ such that $(x,y) \rightarrow (x+4,y)$. Find the coordinates of $A''B''C''D''$.

Know What? An example of a glide reflection is your own footprint. The equations to find your average footprint are in the diagram to the right. Find your average footprint and write the transformation rule for one stride.



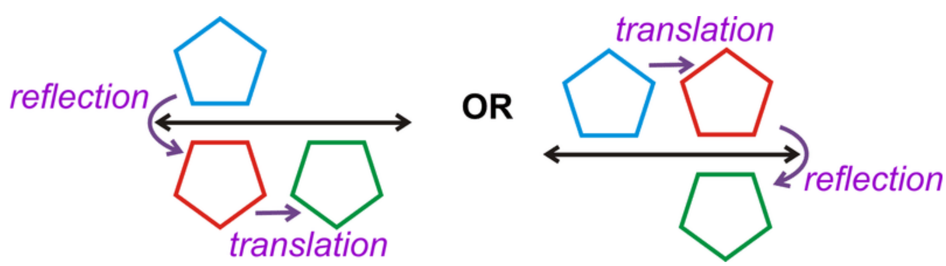
Glide Reflections

Now that we have learned all our rigid transformations, or isometries, we can perform more than one on the same figure.

Composition (of transformations): To perform more than one transformation on a figure.

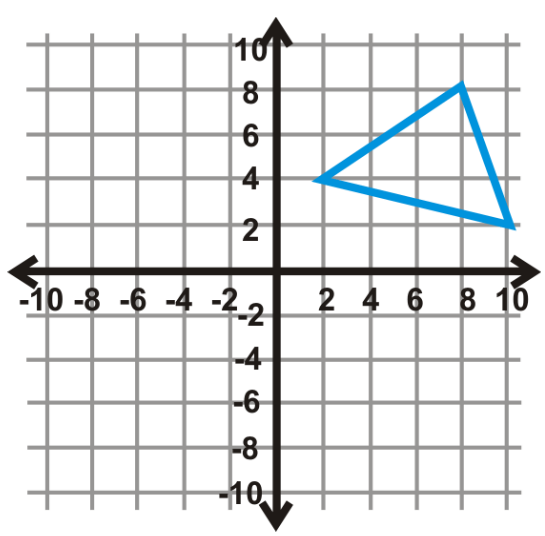
Glide Reflection: A composition of a reflection and a translation. The translation is in a direction parallel to the line of reflection.

For any glide reflection, order does not matter.

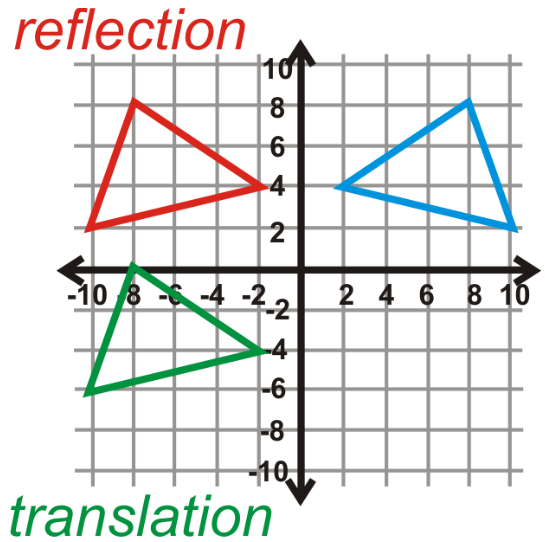


In the Review Queue above, you performed a glide reflection on $ABCD$. If you reflect over a vertical line, the translation will be up or down, and if you reflect over a horizontal line, the translation will be to the left or right.

Example 1: Reflect $\triangle ABC$ over the $y-$axis and then translate the image 8 units down.



Solution: The green image to the right is the final answer.



$$A(8,8) \to A''(-8,0)$$
$$B(2,4) \to B''(-2,-4)$$
$$C(10,2) \to C''(-10,-6)$$

Compositions can always be written as one rule.

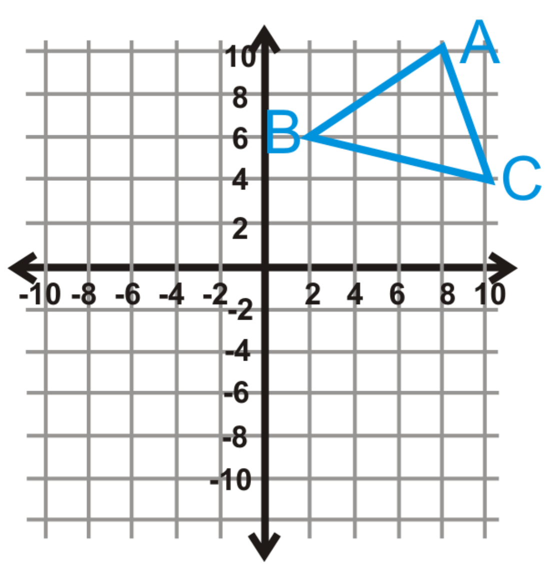
Example 2: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 1.

Solution: Looking at the coordinates of A to A'', the $x-$value is the opposite sign and the $y-$value is $y-8$. Therefore the rule would be $(x,y) \to (-x, y-8)$.

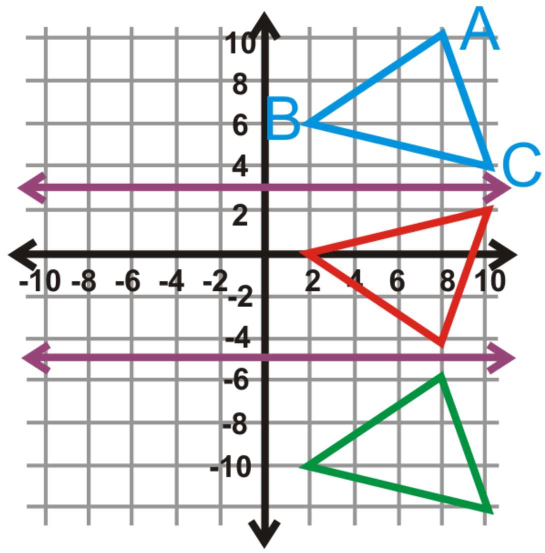
Reflections over Parallel Lines

The next composition we will discuss is a double reflection over parallel lines. For this composition, we will only use horizontal or vertical lines.

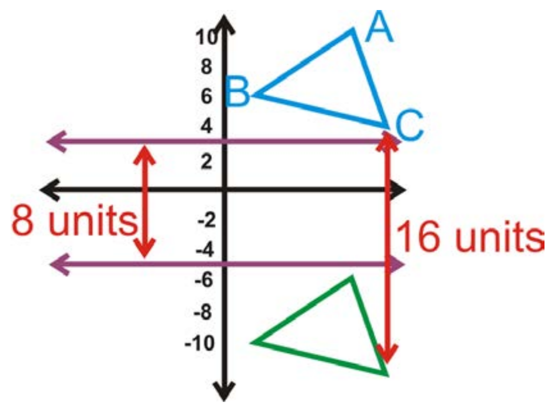
Example 3: Reflect $\triangle ABC$ over $y = 3$ and $y = -5$.



Solution: Unlike a glide reflection, ***order matters***. Therefore, you would reflect over $y = 3$ first, (red triangle) then a reflection over $y = -5$ (green triangle).



Example 4: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 3.

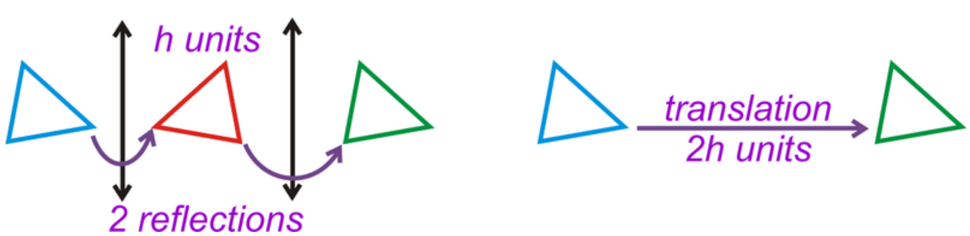


Solution: In the graph, the two lines are 8 units apart $(3-(-5)=8)$. The figures are 16 units apart. The ***double*** reflection is the same as a translation that is ***double*** the distance between the parallel lines.

$$(x,y) \rightarrow (x,y-16)$$

Reflections over Parallel Lines Theorem

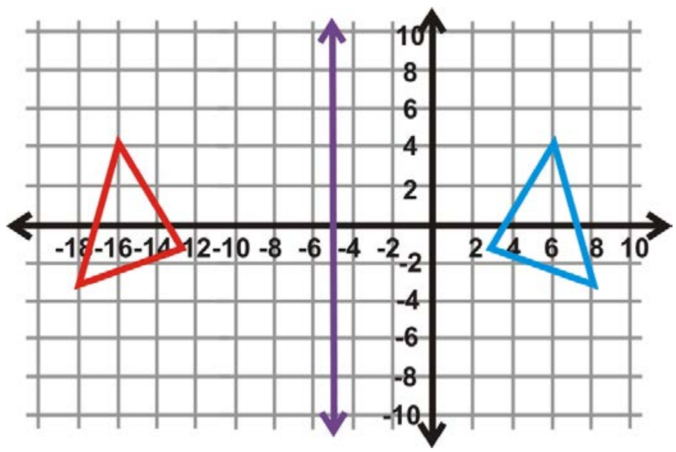
The composition of two reflections over parallel lines that are h units apart is the same as a translation of $2h$ units



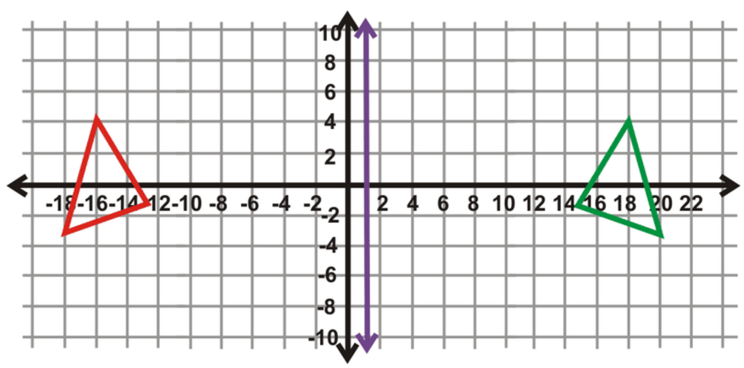
Be careful with this theorem because it does not say which direction the translation is in.

Example 5: $\triangle DEF$ has vertices $D(3,-1), E(8,-3)$, and $F(6,4)$. Reflect $\triangle DEF$ over $x=-5$ and $x=1$. Determine which one translation this double reflection would be the same as.

Solution: From the Reflections over Parallel Lines Theorem, we know that this double reflection is going to be the same as a single translation of $2(1-(-5))$ or 12 units.

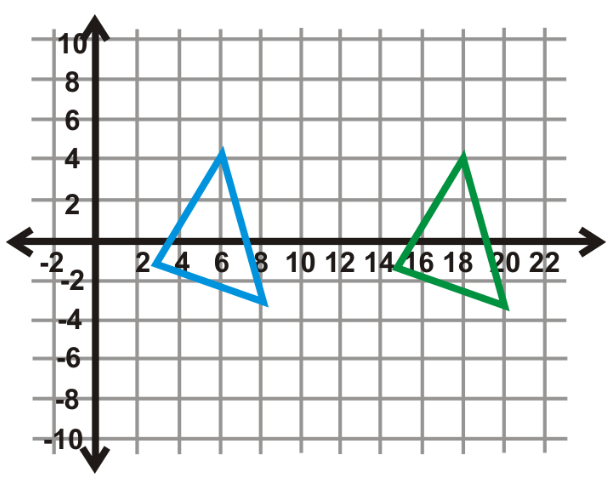


First, reflect over $x=-5$



Second, reflect over $x=1$

Comparing the preimage and image, this is a translation of 12 units to the right.



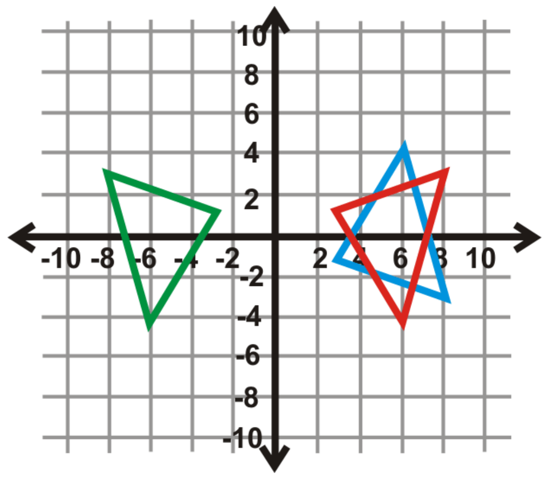
If the lines of reflection were switched, then it would have been a translation of 12 units to the ***left***.

Reflections over the x and y Axes

You can also reflect over intersecting lines. First, we will reflect over the x and y axes.

Example 6: Reflect $\triangle DEF$ from Example 5 over the $x-$axis, followed by the $y-$axis. Find the coordinates of $\triangle D''E''F''$ and the one transformation this double reflection is the same as.

Solution: $\triangle D''E''F''$ is the green triangle in the graph to the left. If we compare the coordinates of it to $\triangle DEF$, we have:

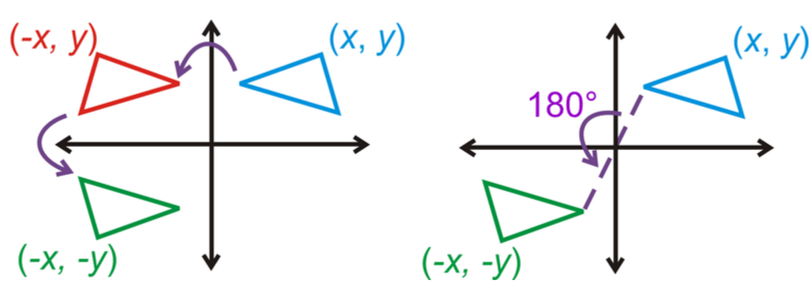


$$D(3,-1) \to D'(-3,1)$$
$$E(8,-3) \to E'(-8,3)$$
$$F(6,4) \to F'(-6,-4)$$

From the rules of rotations in the previous section, this is also an 180° rotation.

Reflection over the Axes Theorem

If you compose two reflection over each axis, then the final image is a rotation of 180° of the original

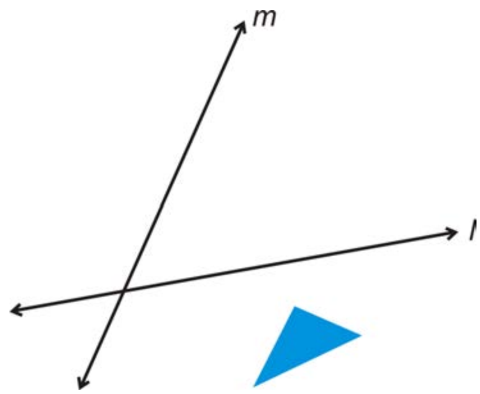


With this particular composition, order does not matter.

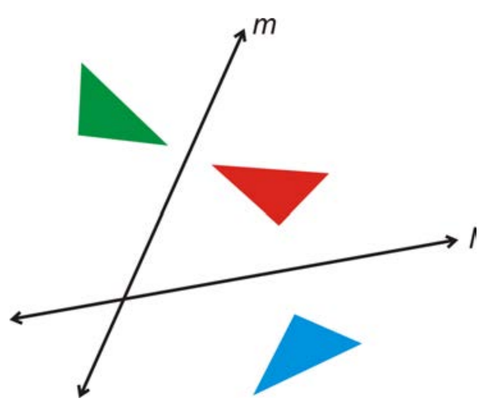
Reflections over Intersecting Lines

For this composition, we are going to take it out of the coordinate plane.

Example 7: Copy the figure below and reflect it over l, followed by m.



Solution: The easiest way to reflect the triangle is to fold your paper on each line of reflection and draw the image. It should look like this:



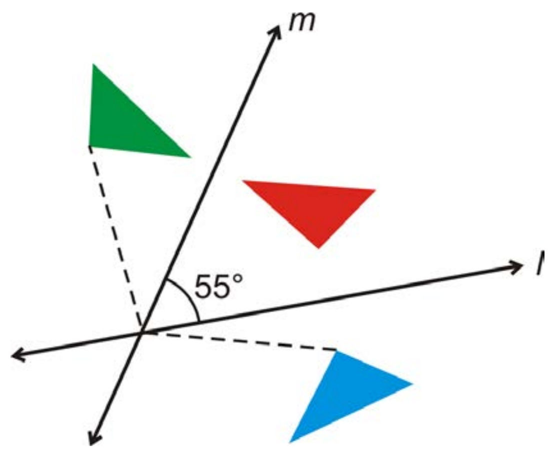
(Patty paper could be used here).

The green triangle is the final answer.

Investigation 11-2: Double Reflection over Intersecting Lines

Tools Needed: Example 7, protractor, ruler, pencil

1. Take your answer from Example 7 and measure the angle of intersection for lines l and m. If you copied it from the text, it is 55°.
2. Draw lines from the corresponding points on the blue triangle and the green triangle.
3. Measure this angle using your protractor. How does it related to 55°?

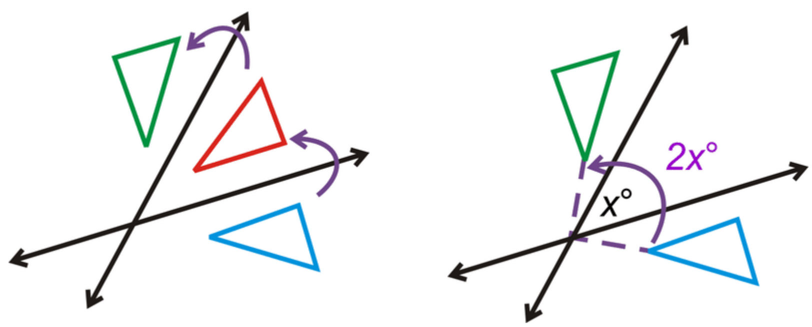


If you copied the image exactly from the text, the angle is 110° counterclockwise.

Notice that order would matter in this composition. If we had reflected the blue triangle over m followed by l, then the green triangle would be rotated 110° *clockwise*.

Reflections over Intersecting Lines Theorem

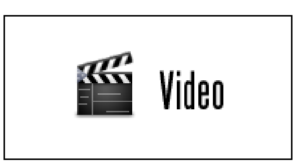
A composition of two reflections over lines that intersect at x°, then the resulting image is a rotation of 2x°. The center of rotation is the point of intersection.



Example 8: A square is reflected over two lines that intersect at a 79° angle. What one transformation will this be the same as?

Solution: From the theorem above, this is the same as a rotation of $2 \cdot 79^\circ = 178^\circ$.

Watch This!!!



MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/160648

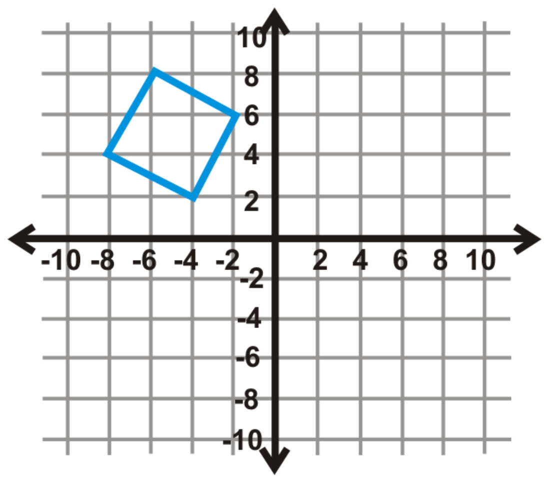
Know What? Revisited The average 6 foot tall man has a $0.415 \times 6 = 2.5$ foot stride. Therefore, the transformation rule for this person would be $(x, y) \rightarrow (-x, y + 2.5)$.

Practice Problems

- Questions 1-3 use the theorems learned in this section.
- Questions 4-12 are similar to Examples 1 and 2.
- Questions 13-19 are similar to Examples 3-5.
- Questions 20-22 are similar to Example 6.
- Questions 23-30 are similar to Example 8 and use the theorems learned in this section.

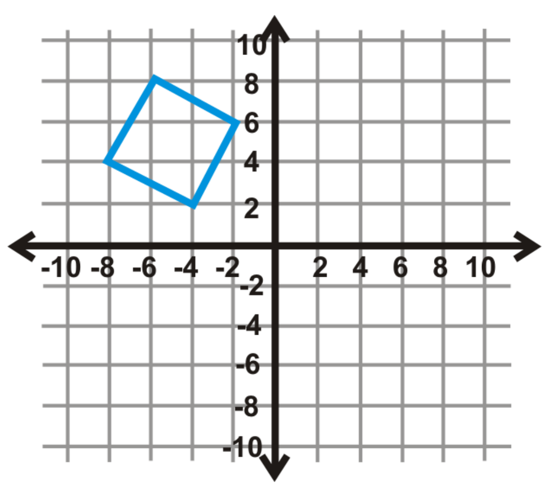
1. *Explain* why the composition of two or more isometries must also be an isometry.
2. What one transformation is the same as a reflection over two parallel lines?
3. What one transformation is the same as a reflection over two intersecting lines?

Use the graph of the square to the left to answer questions 4-6.



4. Perform a glide reflection over the $x-$axis and to the right 6 units. Write the new coordinates.
5. What is the rule for this glide reflection?
6. What glide reflection would move the image back to the preimage?

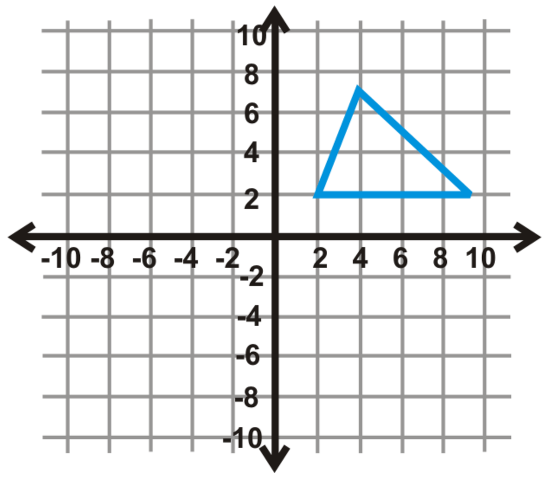
Use the graph of the square to the left to answer questions 7-9.



7. Perform a glide reflection to the right 6 units, then over the $x-$axis. Write the new coordinates.

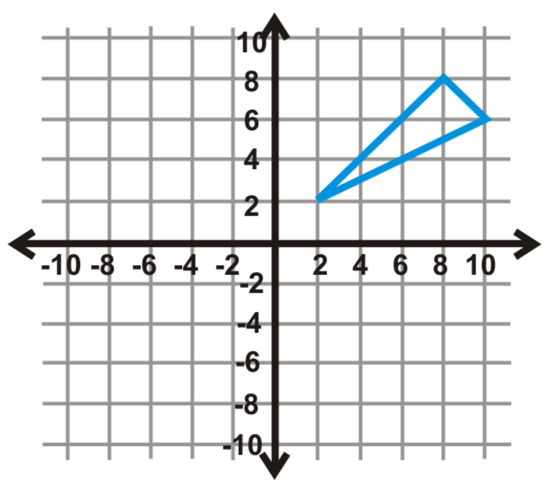
8. What is the rule for this glide reflection?
9. Is the rule in #8 different than the rule in #5? Why or why not?

Use the graph of the triangle to the left to answer questions 10-12.



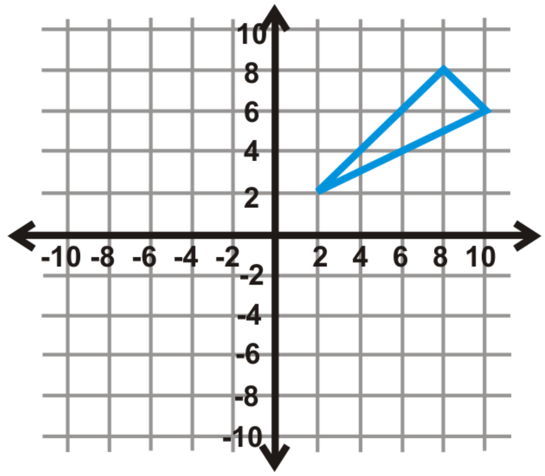
10. Perform a glide reflection over the $y-$axis and down 5 units. Write the new coordinates.
11. What is the rule for this glide reflection?
12. What glide reflection would move the image back to the preimage?

Use the graph of the triangle to the left to answer questions 13-15.



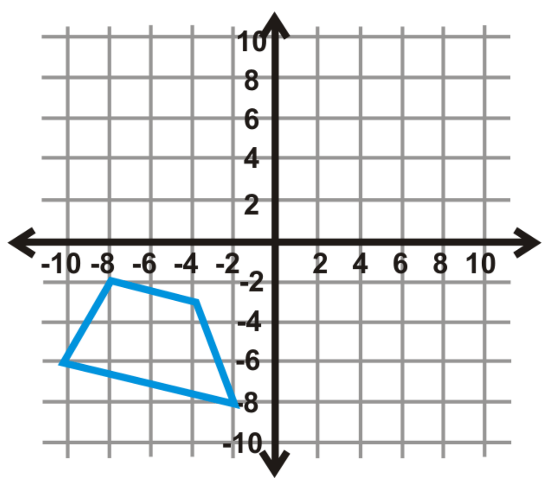
13. Reflect the preimage over $y = -1$ followed by $y = -7$. Draw the new triangle.
14. What one transformation is this double reflection the same as?
15. Write the rule.

Use the graph of the triangle to the left to answer questions 16-18.



16. Reflect the preimage over $y = -7$ *followed* by $y = -1$. Draw the new triangle.
17. What one transformation is this double reflection the same as?
18. Write the rule.
19. How do the final triangles in #13 and #16 differ?

Use the trapezoid in the graph to the left to answer questions 20-22.



20. Reflect the preimage over the $x-$axis then the $y-$axis. Draw the new trapezoid.
21. Now, start over. Reflect the trapezoid over the $y-$axis then the $x-$axis. Draw this trapezoid.
22. Are the final trapezoids from #20 and #21 different? Why do you think that is?

Answer the questions below. Be as specific as you can.

23. Two parallel lines are 7 units apart. If you reflect a figure over both how far apart with the preimage and final image be?
24. After a double reflection over parallel lines, a preimage and its image are 28 units apart. How far apart are the parallel lines?
25. Two lines intersect at a 165° angle. If a figure is reflected over both lines, how far apart will the preimage and image be?
26. What is the center of rotation for #25?

Review and Reflect

27. Two lines intersect at an 83° angle. If a figure is reflected over both lines, how far apart will the preimage and image be?
28. A preimage and its image are 244° apart. If the preimage was reflected over two intersecting lines, at what angle did they intersect?
29. A preimage and its image are 98° apart. If the preimage was reflected over two intersecting lines, at what angle did they intersect?
30. After a double reflection over parallel lines, a preimage and its image are 62 units apart. How far apart are the parallel lines?

Warm-Up Answers

1. $A'(-2,-8), B'(4,-5), C'(-4,-1), D'(-6,-6)$
2. $A''(2,-8), B''(8,-5), C''(0,-1), D''(-2,-6)$

11.6 Tessellating Polygons

TEKS G(5)A

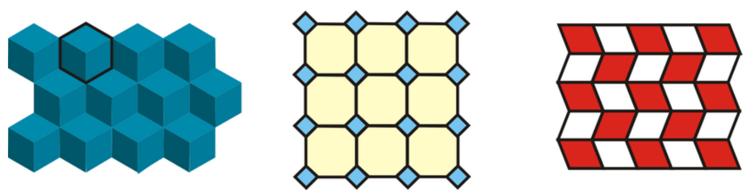
Learning Objectives

- Tessellating regular polygons

What is a Tessellation?

You have probably seen tessellations before. Examples of a tessellation are: a tile floor, a brick or block wall, a checker or chess board, and a fabric pattern.

Tessellation: A tiling over a plane with one or more figures such that the figures fill the plane with no overlaps and no gaps.

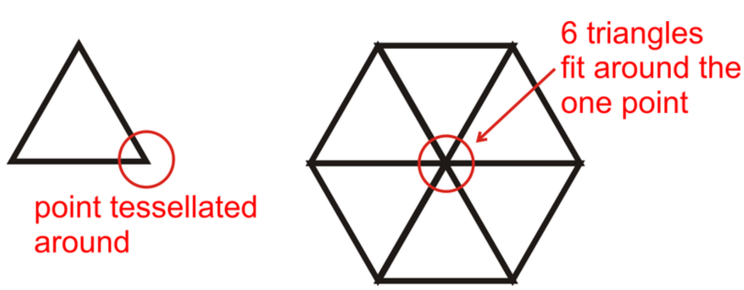


Notice the hexagon (cubes, first tessellation) and the quadrilaterals fit together perfectly. If we keep adding more, they will entirely cover the plane with no gaps or overlaps.

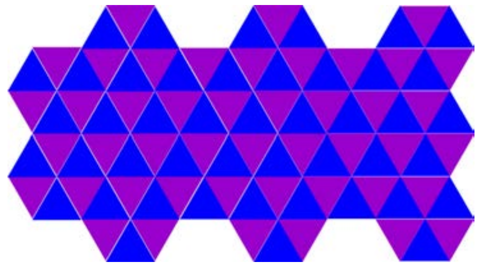
We are only going to worry about tessellating regular polygons. To tessellate a shape it must be able to exactly surround a point, or the sum of the angles around each point in a tessellation must be 360°. The only regular polygons with this feature are equilateral triangles, squares, and regular hexagons.

Example 1: Draw a tessellation of equilateral triangles.

Solution: In an equilateral triangle each angle is 60°. Therefore, six triangles will perfectly fit around each point.

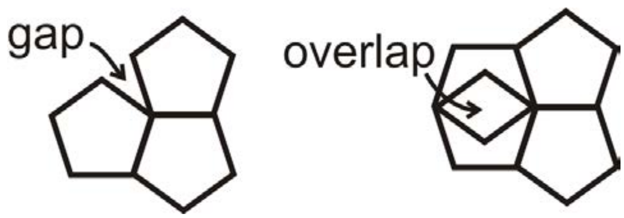


Extending the pattern, we have:



Example 2: Does a regular pentagon tessellate?

Solution: First, recall that there are 540° in a pentagon. Each angle in a regular pentagon is $540^\circ \div 5 = 108^\circ$. From this, we know that a regular pentagon will not tessellate by itself because 108° times 2 or 3 does not equal 360°.



Tessellations can also be much more complicated. Check out http://www.mathsisfun.com/geometry/tessellation.html to see other tessellations and play with the Tessellation Artist, which has a link at the bottom of the page.

Practice Problems

1. You were told that equilateral triangles, squares, and regular hexagons are the only regular polygons that tessellate. Tessellate a square. Add color to your design.
2. What is an example of a tessellated square in real life?
3. How many regular hexagons will fit around one point? (First, recall how many degrees are in a hexagon, and then figure out how many degrees are in each angle of a regular polygon. Then, use this number to see how many of them fit around a point.)
4. Using the information from #2, tessellate a regular hexagon. Add color to your design.
5. You can also tessellate two regular polygons together. Try tessellating a regular hexagon and an equilateral triangle. First, determine how many of each fit around a point and then repeat the pattern. Add color to your design.

11.7 Chapter 11 Review

Keywords & Theorems

Exploring Symmetry

- Line of Symmetry
- Line Symmetry
- Rotational Symmetry
- Center of Rotation
- Angle of Rotation

Translations

- Transformation
- Rigid Transformation
- Translation

Reflections

- Reflection
- Line of Reflection
- Reflection over the $y-$axis
- Reflection over the $x-$axis
- Reflection over $y = x$
- Reflection over $y = -x$

Rotations

- Rotation
- Center of Rotation
- Angle of Rotation
- Rotation of 180°
- Rotation of 90°
- Rotation of 270°

Compositions of Transformations

- Composition (of transformations)
- Glide Reflection
- Reflections over Parallel Lines Theorem
- Reflection over the Axes Theorem
- Reflection over Intersecting Lines Theorem

Extension: Tessellating Polygons

- Tessellation

Review Questions

Match the description with its rule.

1. Reflection over the $y-$axis - A. $(y, -x)$
2. Reflection over the $x-$axis - B. $(-y, -x)$
3. Reflection over $y = x$ - C. $(-x, y)$
4. Reflection over $y = -x$ - D. $(-y, x)$
5. Rotation of 180° - E. $(x, -y)$
6. Rotation of 90° - F. (y, x)
7. Rotation of 270° - G. (x, y)
8. Rotation of 360° - H. $(-x, -y)$

Texas Instruments Resources

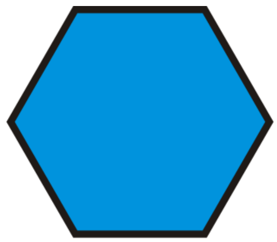
In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9697 .

11.8 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1^{st} **Section: Exploring Symmetry**

Line of Symmetry



Line Symmetry

Rotational Symmetry

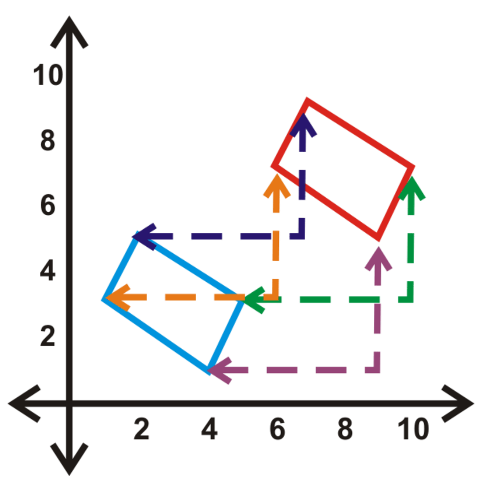
Center of Rotation

Angle of Rotation

Homework:

2^{nd} **Section: Translations**

Transformation



Rigid Transformation

Translation

Homework:

3^{rd} **Section: Reflections**

Reflection

Line of Reflection

Reflection over the $y-$axis

Reflection over the $x-$axis

Reflections over a horizontal line

Reflections over a vertical line

Reflection over $y=x$

Reflection over $y=-x$

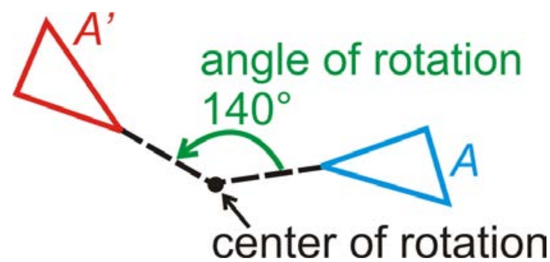
Homework:

4^{th} **Section: Rotations**

Rotation

Center of Rotation

Angle of Rotation



Rotation of 180°

Rotation of 90°

Rotation of 270°

Homework:

5^{th} **Section: Compositions of Transformations**

Composition (of transformations)

Glide Reflection

Reflections over Parallel Lines Theorem

Reflection over the Axes Theorem

Reflection over Intersecting Lines Theorem

Homework:

Extension: Tessellating Polygons

Tessellation

Homework:

CHAPTER 12 Surface Area and Volume

Chapter Outline

In this chapter we extend what we know about two-dimensional figures to three-dimensional shapes. First, we will define the different types of 3D shapes and their parts. Then, we will find the surface area and volume of prisms, cylinders, pyramids, cones, and spheres.

12.1 Exploring Solids

TEKS G(10)A, G(11)B

Learning Objectives

- Identify different types of solids and their parts
- Use Euler's formula and nets

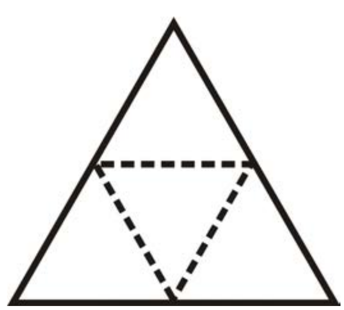
Vocabulary

- Polyhedron
- Face
- Edge
- Vertex
- Prism
- Pyramid

Warm-Up

1. Draw an octagon and identify the edges and vertices of the octagon. How many of each are there?
2. Find the area of a square with 5 cm sides.
3. Draw the following polygons.
    a. A convex pentagon.
    b. A concave nonagon.

Know What? Until now, we have only talked about two-dimensional, or flat, shapes. Copy the equilateral triangle to the right onto a piece of paper and cut it out. Fold on the dotted lines. What shape do these four equilateral triangles make?



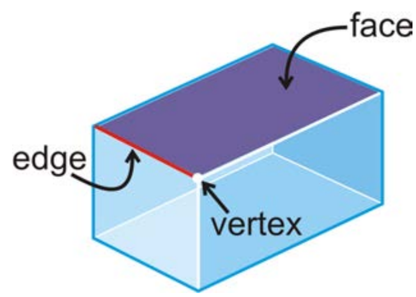
Polyhedrons

Polyhedron: A 3-dimensional figure that is formed by polygons that enclose a region in space.

Each polygon in a polyhedron is a ***face.***

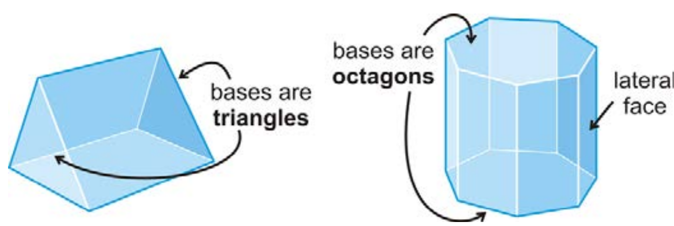
The line segment where two faces intersect is an ***edge.***

The point of intersection of two edges is a ***vertex.***

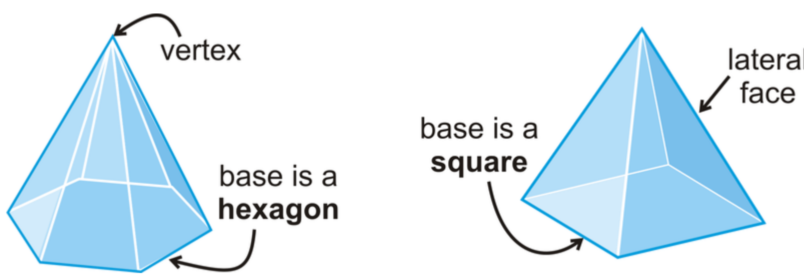


Examples of polyhedrons include a cube, prism, or pyramid. Non-polyhedrons are cones, spheres, and cylinders because they have sides that are not polygons.

Prism: A polyhedron with two congruent bases, in parallel planes, and the lateral sides are rectangles.



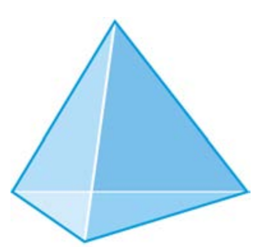
Pyramid: A polyhedron with one base and all the lateral sides meet at a common vertex.



All prisms and pyramids are named by their bases. So, the first prism would be a triangular prism and the first pyramid would be a hexagonal pyramid.

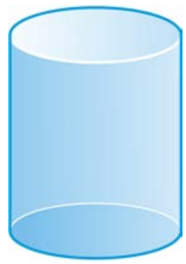
Example 1: Determine if the following solids are polyhedrons. If the solid is a polyhedron, name it and find the number of faces, edges and vertices each has.

a)



b)

c)



Solution:

a) The base is a triangle and all the sides are triangles, so this is a triangular pyramid. There are 4 faces, 6 edges and 4 vertices.

b) This solid is also a polyhedron. The bases are both pentagons, so it is a pentagonal prism. There are 7 faces, 15 edges, and 10 vertices.

c) The bases that are circles. Circles are not polygons, so it is not a polyhedron.

Euler's Theorem

Let's put our results from Example 1 into a table.

TABLE 12.1:

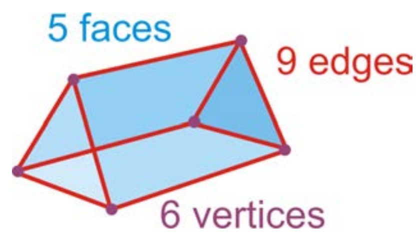
	Faces	*Vertices*	*Edges*
Triangular Pyramid	4	4	6
Pentagonal Prism	7	10	15

Notice that faces + vertices is two more that the number of edges. This is called Euler's Theorem, after the Swiss mathematician Leonhard Euler.

Euler's Theorem

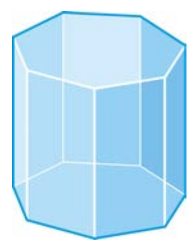
$F+V=E+2$

Euler's Theorem: $F+V=E+2$.



$$
\begin{aligned}
Faces + Vertices &= Edges + 2 \\
5 + 6 &= 9 + 2
\end{aligned}
$$

Example 2: Find the number of faces, vertices, and edges in the heptagonal prism.



Solution: There are 9 faces and 14 vertices. Use Euler's Theorem, to solve for E.

$$
\begin{aligned}
F + V &= E + 2 \\
9 + 14 &= E + 2 \\
21 &= E
\end{aligned}
$$

Example 3: In a six-faced polyhedron, there are 10 edges. How many vertices does the polyhedron have?

Solution: Solve for V in Euler's Theorem.

$$
\begin{aligned}
F + V &= E + 2 \\
6 + V &= 10 + 2 \\
V &= 6
\end{aligned}
$$

Example 4: A three-dimensional figure has 10 vertices, 5 faces, and 12 edges. Is it a polyhedron?

Solution: Plug in all three numbers into Euler's Theorem.

$$
\begin{aligned}
F + V &= E + 2 \\
5 + 10 &= 12 + 2 \\
15 &\neq 14
\end{aligned}
$$

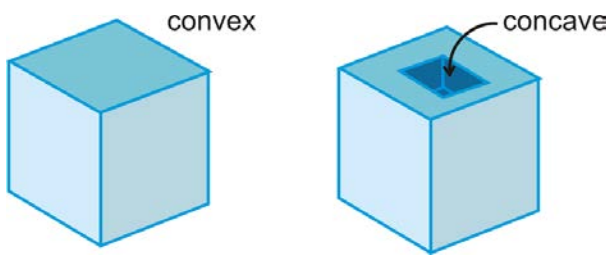
Because the two sides are not equal, this figure is not a polyhedron.

Regular Polyhedra

Regular Polyhedron: A polyhedron where all the faces are congruent regular polygons.

All regular polyhedron are ***convex.***

A ***concave*** polyhedron "caves in."



There are only ***five regular polyhedra, called the Platonic solids.***

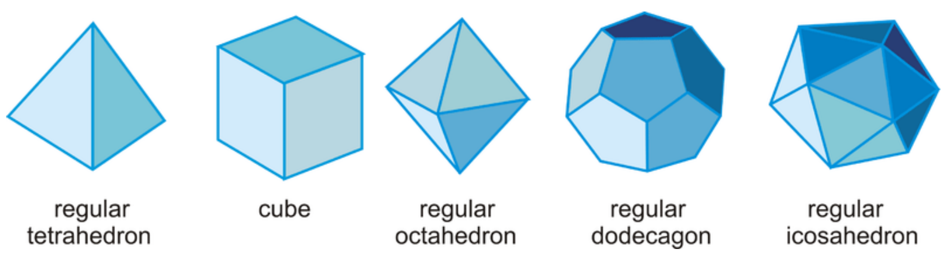
Regular Tetrahedron: A 4-faced polyhedron and all the faces are equilateral triangles.

Cube: A 6-faced polyhedron and all the faces are squares.

Regular Octahedron: An 8-faced polyhedron and all the faces are equilateral triangles.

Regular Dodecahedron: A 12-faced polyhedron and all the faces are regular pentagons.

Regular Icosahedron: A 20-faced polyhedron and all the faces are equilateral triangles.

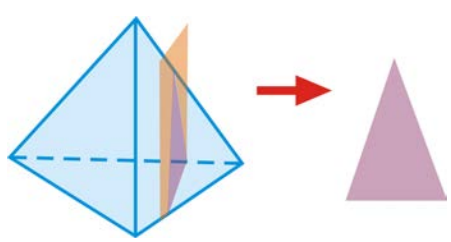


Cross-Sections

One way to "view" a three-dimensional figure in a two-dimensional plane, like in this text, is to use cross-sections.

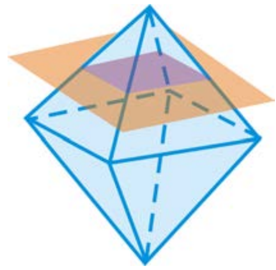
Cross-Section: The intersection of a plane with a solid.

The cross-section of the peach plane and the tetrahedron is a *triangle.*

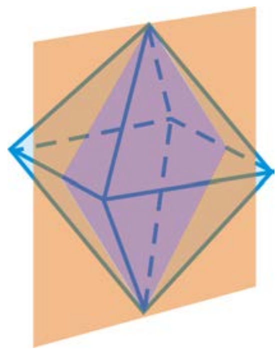


Example 5: What is the shape formed by the intersection of the plane and the regular octahedron?

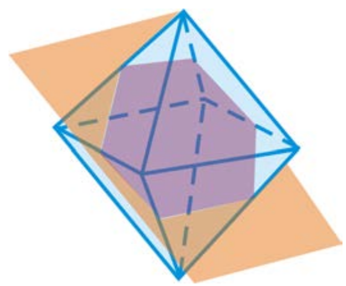
a)



b)



c)



Solution:

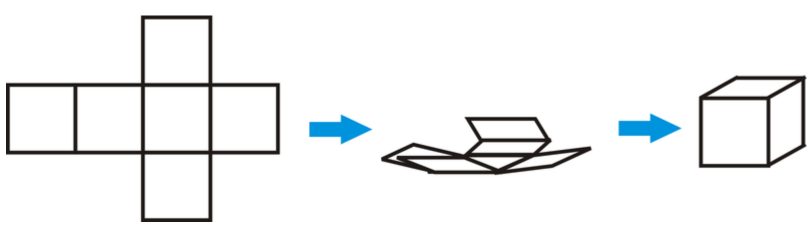
a) Square

b) Rhombus

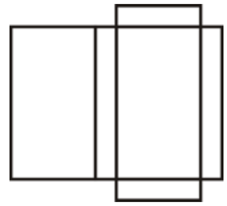
c) Hexagon

Nets

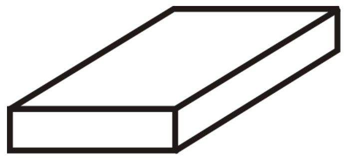
Net: An unfolded, flat representation of the sides of a three-dimensional shape.



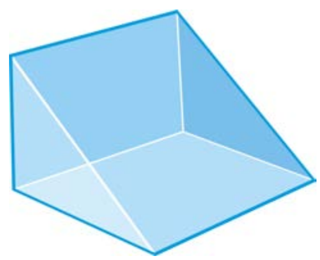
Example 6: What kind of figure does this net create?



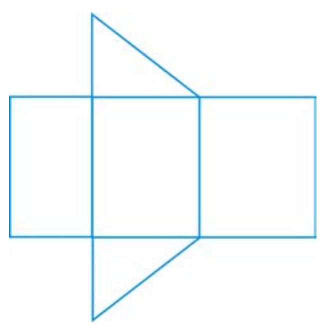
Solution: The net creates a rectangular prism.



Example 7: Draw a net of the right triangular prism below.



Solution: The net will have two triangles and three rectangles. The rectangles are different sizes and the two triangles are the same.



There are several different nets of any polyhedron. For example, this net could have the triangles anywhere along the top or bottom of the three rectangles. Click the site http://www.cs.mcgill.ca/~sqrt/unfold/unfolding.html to see a few animations of other nets.

Know What? Revisited The net of the shape is a regular tetrahedron.

Practice Problems

- Questions 1-8 are similar to Examples 2-4.
- Questions 9-14 are similar to Example 1.
- Questions 15-17 are similar to Example 5.

- Questions 18-23 are similar to Example 7.
- Questions 24-29 are similar to Example 6.
- Question 30 uses Euler's Theorem.

Complete the table using Euler's Theorem.

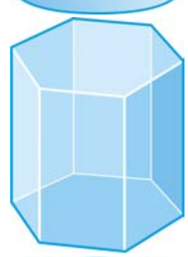
TABLE 12.2:

	Name	*Faces*	*Edges*	*Vertices*
1.	Rectangular Prism	6	12	
2.	Octagonal Pyramid		16	9
3.	Regular Icosahedron	20		12
4.	Cube		12	8
5.	Triangular Pyramid	4		4
6.	Octahedron	8	12	
7.	Heptagonal Prism		21	14
8.	Triangular Prism	5	9	

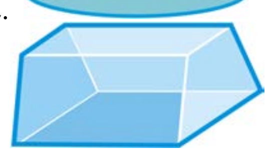
Determine if the following figures are polyhedra. If so, name the figure and find the number of faces, edges, and vertices.

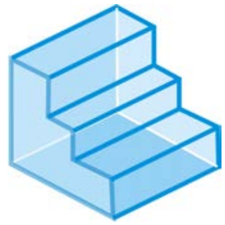
9. 

10. 

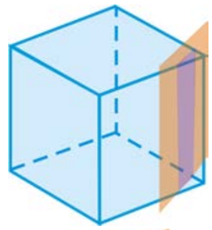
11. 

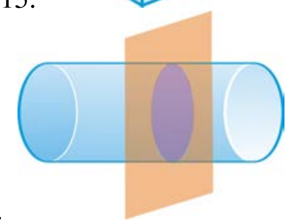
12. 

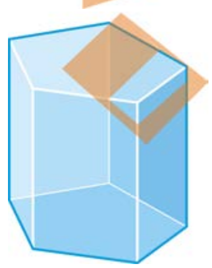
13. 

14. 

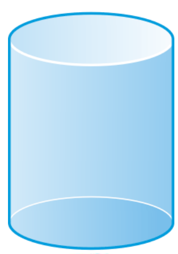
Describe the cross section formed by the intersection of the plane and the solid.

15. 

16. 

17. 

Draw the net for the following solids.

18. 

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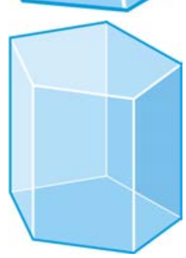
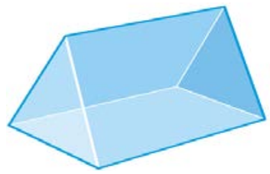
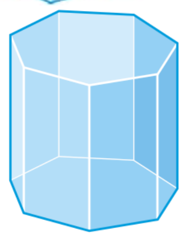
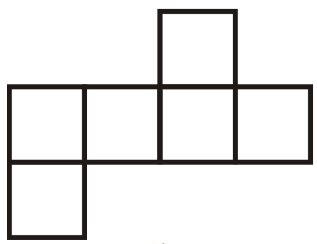
20. 

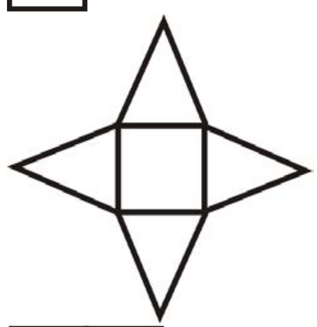
21. 

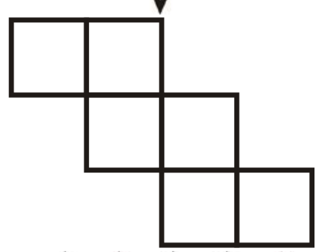
22. 

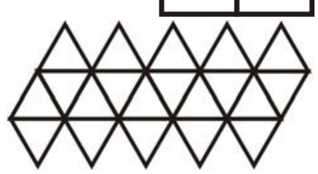
23. 

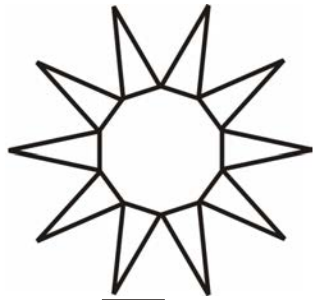
Determine what shape is formed by the following nets.

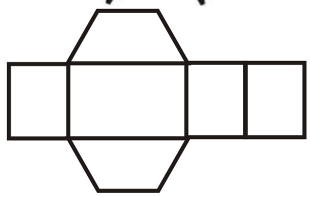
24. 

25. 

26. 

27. 

28. 

29. 

30. A ***truncated icosahedron*** is a polyhedron with 12 regular pentagonal faces and 20 regular hexagonal faces and 90 edges. This icosahedron closely resembles a soccer ball. How many vertices does it have? Explain your reasoning.

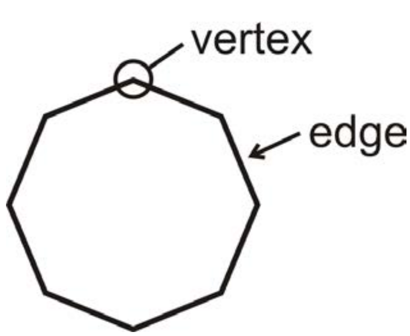
Review and Reflect

31. Given a cylinder, list all the possible cross sections that can be created.

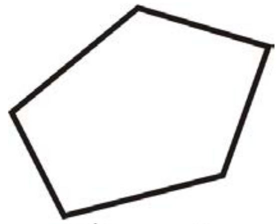
32. Describe why a cylinder is not a polyhedron.

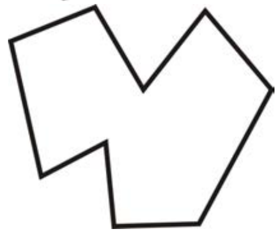
Warm-Up Answers

1. There are 8 vertices and 8 edges in an octagon.



2. $5^2 = 25\ cm^2$

a. 

b. 

12.2 Surface Area of Prisms and Cylinders

TEKS G(11)B, G(11)C

Learning Objectives

- Find the surface area of a prism and cylinder

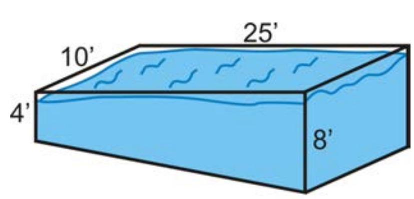
Vocabulary

- Lateral Face
- Lateral Edge
- Base Edge
- Right Prism
- Oblique Prism
- Surface Area
- Lateral Surface Area
- Cylinder

Warm-Up

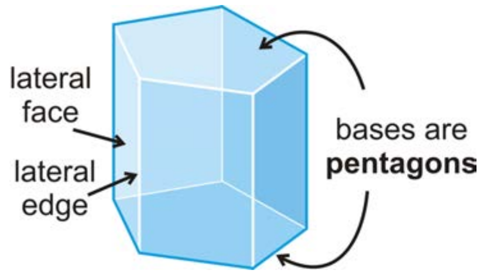
1. Find the area of a rectangle with sides:
   a. 6 and 9
   b. 11 and 4
   c. $5\sqrt{2}$ and $8\sqrt{6}$
2. If the area of a square is 36 $units^2$, what are the lengths of the sides?
3. If the area of a square is 45 $units^2$, what are the lengths of the sides?

Know What? Your parents decide they want to put a pool in the backyard. The shallow end will be 4 ft. and the deep end will be 8 ft. The pool will be 10 ft. by 25 ft. How much siding do they need to cover the sides and bottom of the pool?



Parts of a Prism

Prism: A 3-dimensional figure with 2 congruent bases, in parallel planes, and the other faces are rectangles.



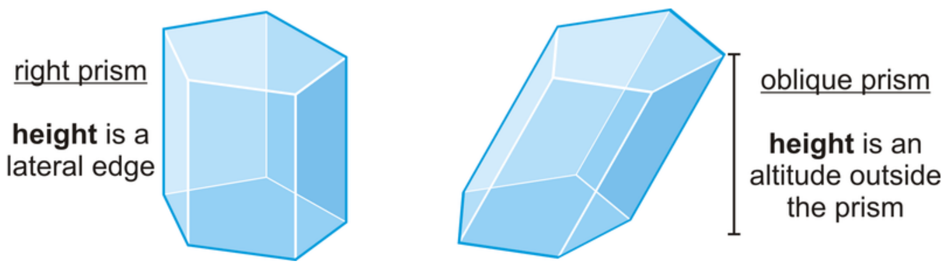
The non-base faces are ***lateral faces.***

The edges between the lateral faces are ***lateral edges.***

This is a ***pentagonal prism.***

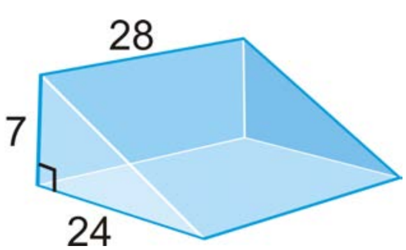
Right Prism: A prism where all the lateral faces are perpendicular to the bases.

Oblique Prism: A prism that leans to one side and the height is outside the prism.

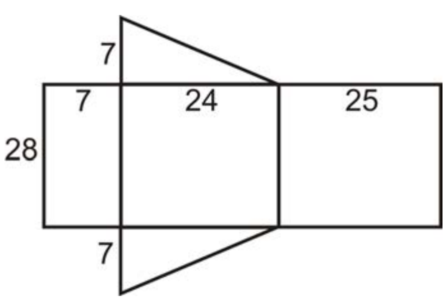


Surface Area of a Prism

Surface Area: The sum of the areas of the faces.



Solution: This is a right triangular prism. To find the surface area, we need to find the length of the hypotenuse of the base because it is the width of one of the lateral faces.



$$\begin{aligned} 7^2 + 24^2 &= c^2 \\ 49 + 576 &= c^2 \\ 625 &= c^2 \qquad c = 25 \end{aligned}$$

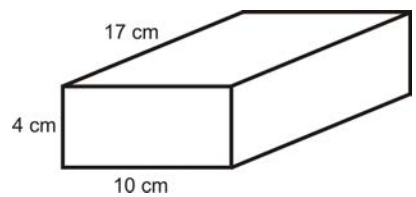
Looking at the net, the surface area is:

$$SA = 7(28)+24(28)+25(28)+2\left(\frac{1}{2}\cdot 7\cdot 24\right)$$
$$SA = (7+24+25)(28)+2\left(\frac{1}{2}\cdot 7\cdot 24\right)$$
$$SA = (56)(28)+2(84)$$
$$SA = 1,568+168 = 1736\ units^2$$

Notice that in the solution above $(7+24+25)$ is the perimeter of the triangular base, the (28) is the height of the prism, and the $2(\frac{1}{2}\cdot 7\cdot 24)$ is the area of the two bases.

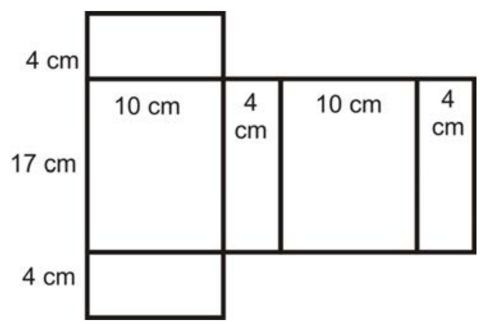
$SA_{Prism} = ph+2B$

Example 1: Find the surface area of the prism below.



Solution: Use the formula $SA_{Prism} = ph+2B$

$SA = (10+17+10+17)(4)+2(10\cdot 17)$
$SA = (54)(4)+2(170)$
$SA = 216+340 = 556 units^2$

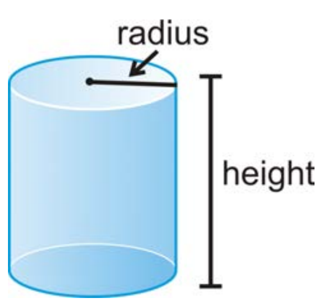


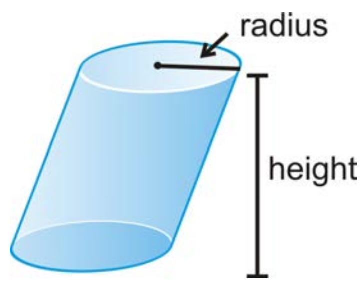
Cylinders

Cylinder: A solid with congruent circular bases that are in parallel planes. The space between the circles is enclosed.

A cylinder has a ***radius*** and a ***height.***

A cylinder can also be ***oblique***, like the one on the far right.

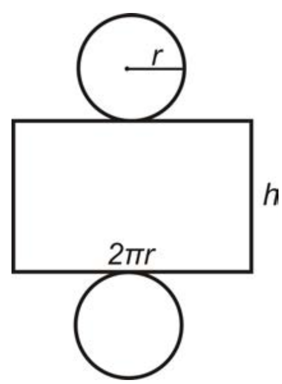




Surface Area of a Right Cylinder

Let's find the net of a right cylinder. One way to do this is to take the label off of a soup can. The label is a rectangle where the height is the height of the cylinder and the base is the circumference of the circle.

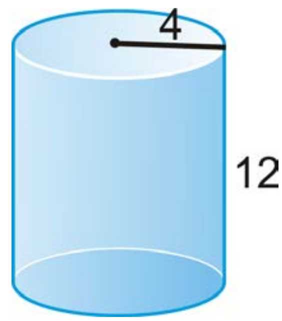
Surface Area of a Right Cylinder: $SA = (2\pi r)(h) + 2(\pi r^2)$.



$$\underbrace{2\pi r}_{\text{Circum of circle}} h + \underbrace{2\pi r^2}_{\text{Area of both circles}}$$

To see an animation of the surface area, click http://www.rkm.com.au/ANIMATIONS/animation-Cylinder-Surface-Area-Derivation.html , by Russell Knightley.

Example 2: Find the surface area of the cylinder.



Solution: $r = 4$ and $h = 12$.

$$\begin{aligned} SA &= 2\pi(4)(12) + 2\pi(4)^2 \\ &= 96\pi + 32\pi \\ &= 128\pi \ units^2 \end{aligned}$$

Example 3: The circumference of the base of a cylinder is 16π and the height is 21. Find the surface area of the cylinder.

Solution: We need to solve for the radius, using the circumference.

$$\begin{aligned} 2\pi r &= 16\pi \\ r &= 8 \end{aligned}$$

Now, we can find the surface area.

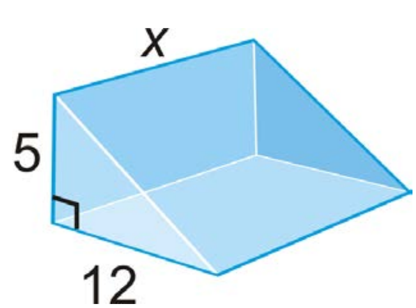
$$\begin{aligned} SA &= 2\pi(8)^2 + (16\pi)(21) \\ &= 128\pi + 336\pi \\ &= 464\pi \ units^2 \end{aligned}$$

Example 4: ***Algebra Connection*** The total surface area of the triangular prism is 540 $units^2$. What is x?

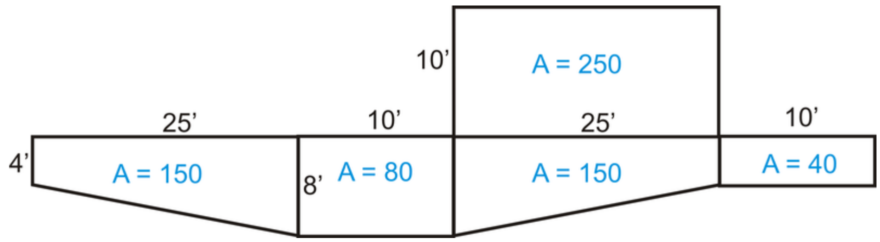
Solution: The total surface area is equal to:

$$ph + 2B = 540$$

The hypotenuse of the triangle bases is 13, $\sqrt{5^2 + 12^2} = \sqrt{169}$. Let's fill in what we know.



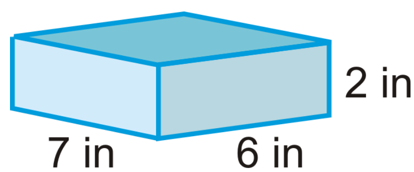
Know What? Revisited To the right is the net of the pool (minus the top). From this, we can see that your parents would need 670 square feet of siding.



Practice Problems

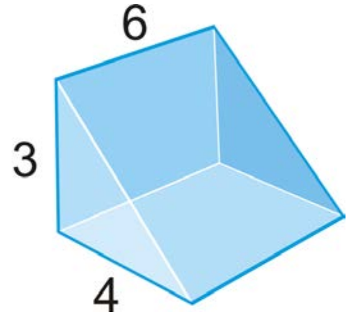
- Questions 1-9 are similar to Examples 1 and 2.
- Question 10 uses the definition of lateral and total surface area.
- Questions 11-18 are similar to Examples 1-3.
- Questions 19-21 are similar to Example 5.
- Questions 22-24 are similar to Example 4.
- Questions 25-30 use the Pythagorean Theorem and are similar to Examples 1-3.

1. What type of prism is this?



2. Draw the net of this prism.
3. Find the area of the bases.
4. Find the area of lateral faces, or the lateral surface area.
5. Find the total surface area of the prism.

Use the right triangular prism to answer questions 6-9.



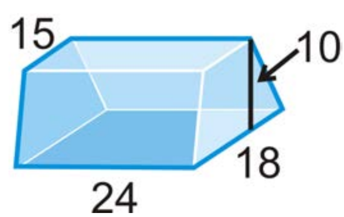
6. What shape are the bases of this prism? What are their areas?
7. What are the dimensions of each of the lateral faces? What are their areas?
8. Find the lateral surface area of the prism.
9. Find the total surface area of the prism.
10. ***Writing*** Describe the difference between *lateral* surface area and *total* surface area.
11. Fuzzy dice are cubes with 4 inch sides.

a. What is the surface area of one die?
b. Typically, the dice are sold in pairs. What is the surface area of two dice?

12. A right cylinder has a 7 cm radius and a height of 18 cm. Find the surface area.

Find the surface area of the following solids. Round your answer to the nearest hundredth.

13. Bases are isoceles trapezoids



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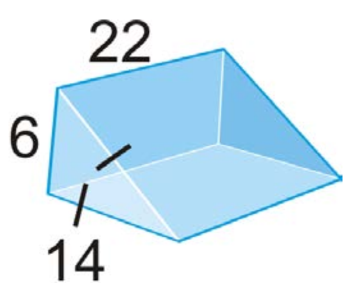

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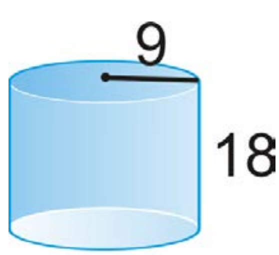

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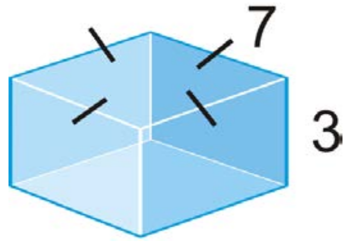
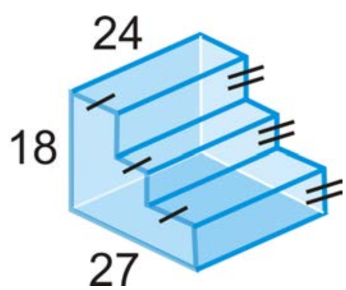


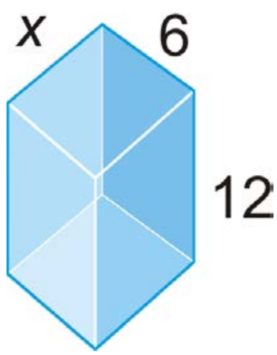
17.

18.

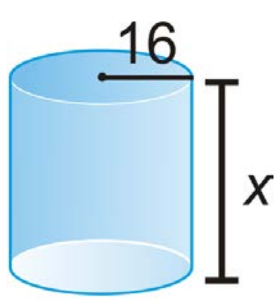


Algebra Connection Find the value of x, given the surface area.

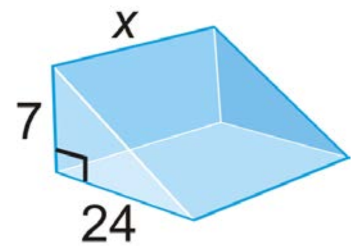
19. $SA = 432\ units^2$



20. $SA = 1536\pi\ units^2$



$SA = 1568\ units^2$21.

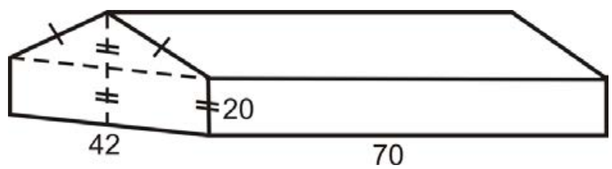


22. The area of the base of a cylinder is $25\pi\ in^2$ and the height is 6 in. Find the *lateral* surface area.

23. The circumference of the base of a cylinder is $80\pi\ cm$ and the height is 36 cm. Find the total surface area.

24. The lateral surface area of a cylinder is $30\pi\ m^2$ and the height is 5m. What is the radius?

Use the diagram below for questions 25-30. The barn is shaped like a pentagonal prism with dimensions shown in feet.



25. What is the width of the roof? (HINT: Use the Pythagorean Theorem)
26. What is the area of the roof? (Both sides)
27. What is the floor area of the barn?
28. What is the area of the rectangular sides of the barn?
29. What is the area of the two pentagon sides of the barn? (HINT: Find the area of two congruent trapezoids for each side)
30. Find the total surface area of the barn (Roof and sides).

Review and Reflect

31. Describe a real-life example where it would be important to find the lateral area of the cone.
32. Explain how a cylinder and a prism are similar and how they are different.

Warm-Up Answers

1. a. 54
   b. 44
   c. $80\sqrt{3}$
2. $s = 6$
3. $s = 3\sqrt{5}$

12.3 Surface Area of Pyramids and Cones

TEKS G(11)C

Learning objectives

- Find the surface area of pyramids and cones

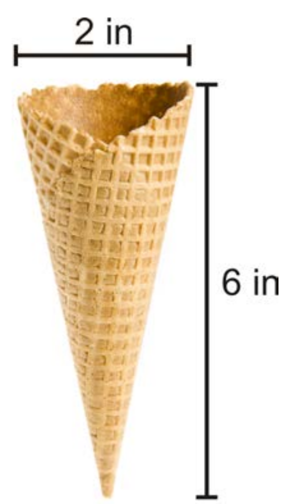
Vocabulary

- Pyramid
- Cone
- Slant Height

Warm-Up

1. A rectangular prism has sides of 5 cm, 6 cm, and 7 cm. What is the surface area?
2. A cylinder has a diameter of 10 in and a height of 25 in. What is the surface area?
3. A cylinder has a circumference of $72\pi\ ft.$ and a height of 24 ft. What is the surface area?
4. Draw the net of a square pyramid.

Know What? A typical waffle cone is 6 inches tall and has a diameter of 2 inches. What is the surface area of the waffle cone? (You may assume that the cone is straight across at the top)

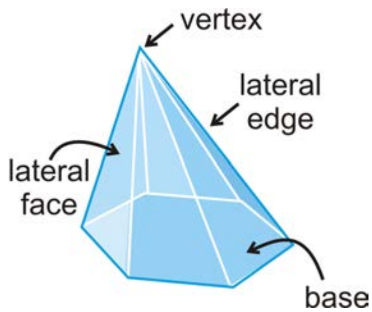


Parts of a Pyramid

Pyramid: A solid with one ***base*** and the ***lateral faces*** meet at a common ***vertex.***

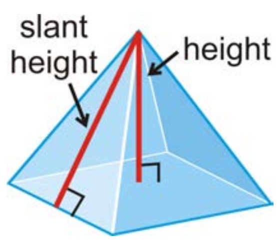
The edges between the lateral faces are ***lateral edges.***

The edges between the base and the lateral faces are ***base edges.***

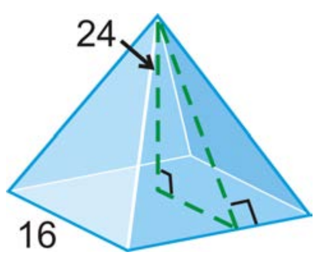


Regular Pyramid: A pyramid where the base is a regular polygon.

All regular pyramids also have a ***slant height*** which is the height of a lateral face. A non-regular pyramid does not have a slant height.



Example 1: Find the slant height of the square pyramid.



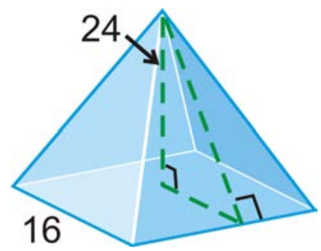
Solution: The slant height is the hypotenuse of the right triangle formed by the height and half the base length. Use the Pythagorean Theorem.

$$\begin{aligned} 8^2 + 24^2 &= l^2 \\ 640 &= l^2 \\ l = \sqrt{640} &= 8\sqrt{10} \end{aligned}$$

Surface Area of a Regular Pyramid

Finding the surface area of a regular pyramid is similar to a prism. First we must find the area of the lateral faces. Using the slant height, which is labeled l, the area of each triangular face is $A = \frac{1}{2}bl$.

Example 2: Find the surface area of the pyramid from Example 1.



Solution:

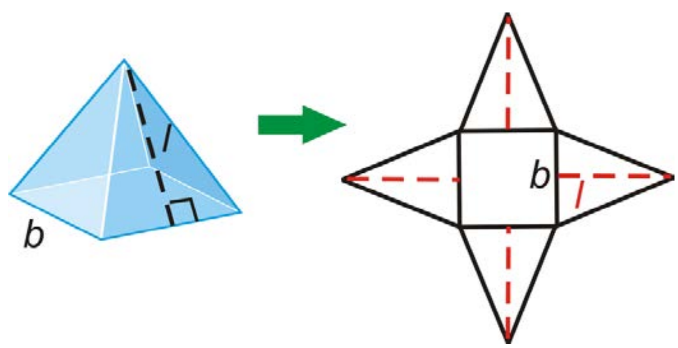
The four triangular faces are $\frac{1}{2}\cdot 16\cdot 8\sqrt{10}+\frac{1}{2}\cdot 16\cdot 8\sqrt{10}+\frac{1}{2}\cdot 16\cdot 8\sqrt{10}+\frac{1}{2}\cdot 16\cdot 8\sqrt{10}$
$\frac{1}{2}(16+16+16+16)\cdot 8\sqrt{10}=256\sqrt{10}$
$\frac{1}{2}(Perimeter)\cdot(SlantHeight)$. To find the total surface We also need the area of the base, which is $16^2=256$. The total surface area is$256\sqrt{10}+256\approx 1065.54\ units^2$.

Surface Area of a Regular Pyramid: We see that the formula for a pyramid is:

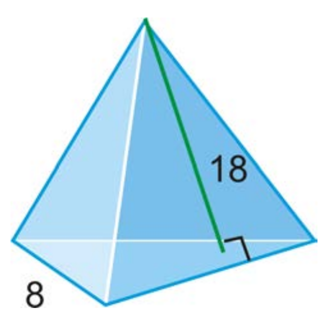
$\frac{1}{2}pl$

B is the area of the base.

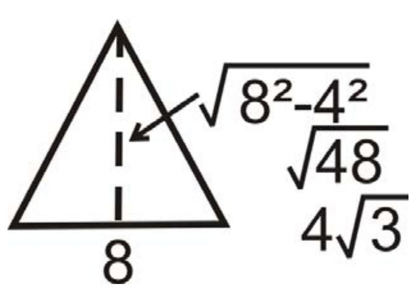
The net shows the surface area of a pyramid. If you ever forget the formula, use the net.



Example 3: Find the area of the ***regular*** triangular pyramid.



Solution: "Regular" tells us the base is an equilateral triangle. Let's draw it and find its area.



$B = \frac{1}{2} \cdot 8 \cdot 4\sqrt{3} = 16\sqrt{3}$

The surface area is:

$SA = \frac{1}{2}(24) \cdot 18 + 16\sqrt{3} = 216 + 16\sqrt{3} \approx 243.71$

Example 4: If the lateral surface area of a square pyramid is 72 ft^2 and the base edge is equal to the slant height. What is the length of the base edge?

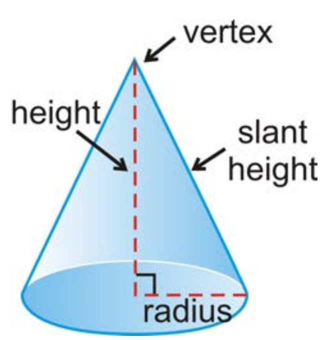
Solution: In the formula for surface area, the lateral surface area is $\frac{1}{2}pl$. We know that the perimeter is 4 times the length of one base edge and $l = b$. Let's solve for b.

$$\begin{aligned}
\frac{1}{2}(4b) \cdot b &= 72\ ft^2 \\
\frac{1}{2}(4)b^2 &= 72 \\
2b^2 &= 72 \\
b^2 &= 36 \\
b &= 6
\end{aligned}$$

Surface Area of a Cone

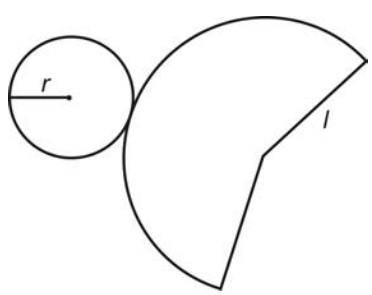
Cone: A solid with a circular base and sides taper up towards a vertex.

A cone has a slant height, just like a pyramid.



A cone is generated from rotating a right triangle, around one leg, in a circle.

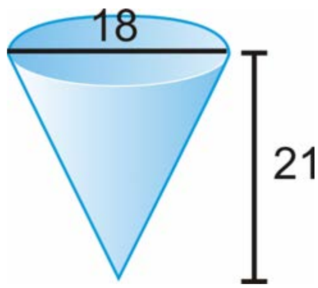
Surface Area of a Right Cone: $SA = \frac{1}{2}(\text{Circumference})(\text{Slant Height}) + \text{Area of Base}$
$SA = \frac{1}{2}(2\pi rl) + \pi r^2$
$SA = \pi rl + \pi r^2$.



Area of the base: πr^2

Area of the sides: πrl

Example 5: What is the surface area of the cone?



Solution: First, we need to find the slant height. Use the Pythagorean Theorem.

$$\begin{aligned} l^2 &= 9^2 + 21^2 \\ &= 81 + 441 \\ l &= \sqrt{522} \approx 22.85 \end{aligned}$$

The surface area would be $SA = \pi(9)(22.85) + \pi 9^2 \approx 900.54\ units^2$.

Example 6: The surface area of a cone is 36π and the radius is 4 units. What is the slant height?

Solution: Plug in what you know into the formula for the surface area of a cone and solve for l.

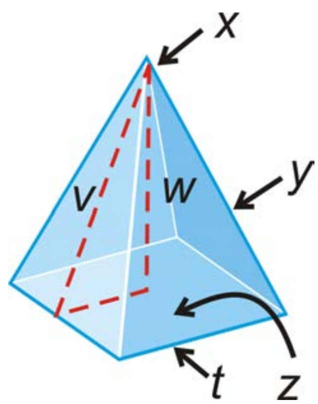
$$\begin{aligned} 36\pi &= \pi 4l + \pi 4^2 \\ 36 = 4l + 16 \qquad \textit{When each term has a } \pi, &\ \textit{they cancel out.} \\ 20 &= 4l \\ 5 &= l \end{aligned}$$

Know What? Revisited The standard cone has a surface area of $\pi + \sqrt{35}\pi \approx 21.73\ in^2$.

Practice Problems

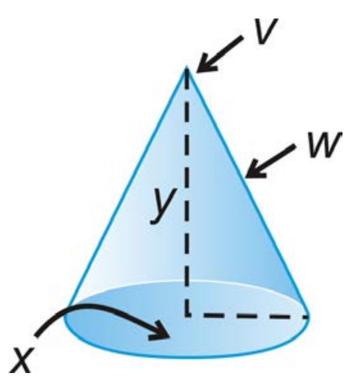
- Questions 1-10 use the definitions of pyramids and cones.
- Questions 11-19 are similar to Example 1.
- Questions 20-26 are similar to Examples 2, 3, and 5.
- Questions 27-31 are similar to Examples 4 and 6.
- Questions 32-25 are similar to Example 5.

Fill in the blanks about the diagram to the left.



1. x is the ___________.
2. The slant height is ________.
3. y is the ___________.
4. The height is ________.
5. The base is _______.
6. The base edge is ________.

Use the cone to fill in the blanks.



7. v is the ___________.
8. The height of the cone is ______.
9. x is a __________ and it is the ___________ of the cone.
10. w is the _____________ ____________.

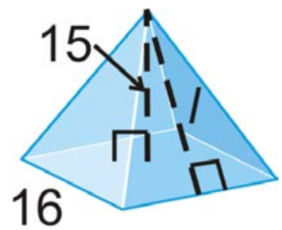
For questions 11-13, sketch each of the following solids and answer the question. Your drawings should be to scale, but not one-to-one. Leave your answer in simplest radical form.

11. Draw a right cone with a radius of 5 cm and a height of 15 cm. What is the slant height?
12. Draw a square pyramid with an edge length of 9 in and a 12 in height. Find the slant height.
13. Draw an equilateral triangle pyramid with an edge length of 6 cm and a height of 6 cm. What is the height of the base?

Find the slant height, l, of one lateral face in each pyramid or of the cone. Round your answer to the nearest hundredth.

14.

15.



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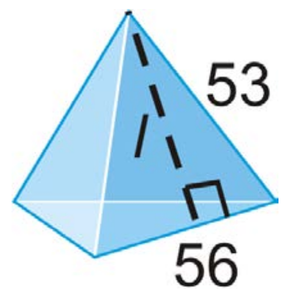
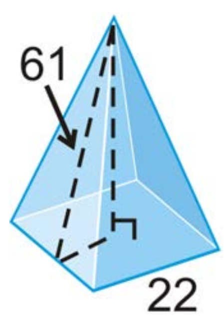


Find the area of a lateral face of the regular pyramid. Round your answers to the nearest hundredth.

17.



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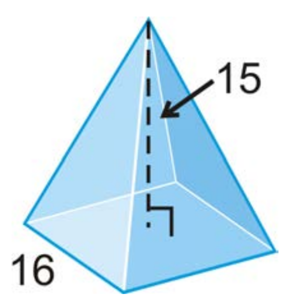

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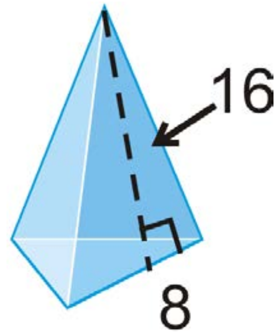
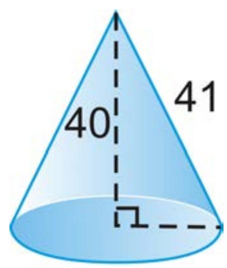


Find the surface area of the regular pyramids and right cones. Round your answers to 2 decimal places.

20.



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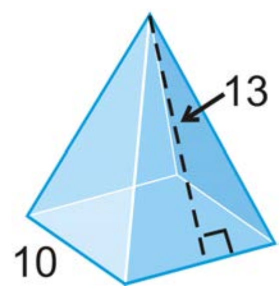

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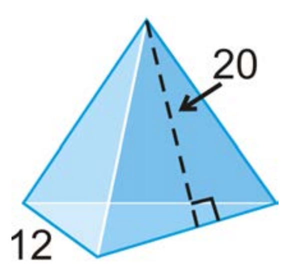

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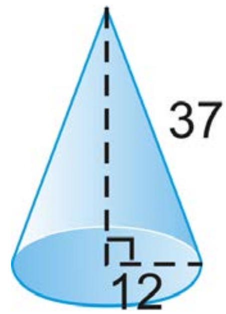

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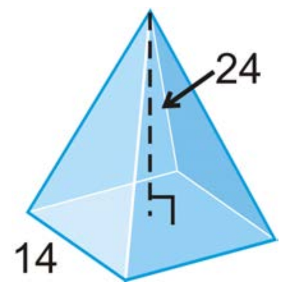

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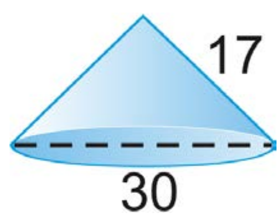
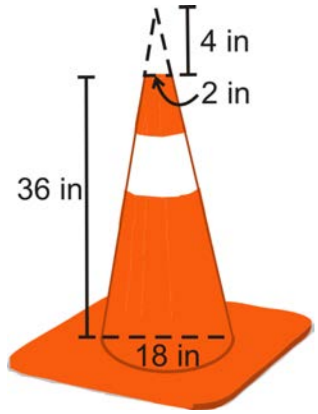


26. A regular tetrahedron has four equilateral triangles as its faces

a. Find the height of one of the faces if the edge length is 6 units.
b. Find the area of one face.
c. Find the total surface area of the regular tetrahedron.

27. If the lateral surface area of a cone is $30\pi\ cm^2$ and the radius is 5cm, what is the slant height? 28. If the surface area of a cone is $105\pi\ cm^2$ and the slant height is 8 cm, what is the radius? 29. If the surface area of a square pyramid is 40 ft^2 and the base edge is 4 ft, what is the slant height? 30. If the lateral area of a square pyramid is 800 in^2 and the slant height is 16 in, what is the length of the base edge? 31. If the lateral area of a regular triangle pyramid is 252 in^2 and the base edge is 8 in, what is the slant height?

The traffic cone is cut off at the top and the base is a square with 24 in sides. Round answers to the nearest hundredth.

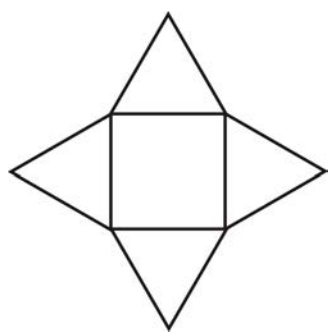


32. Find the area of the entire square. Then, subtract the area of the base of the cone.
33. Find the lateral area of the cone portion (include the 4 inch cut off top of the cone).
34. Subtract the cut-off top of the cone, to only have the lateral area of the cone portion of the traffic cone.
35. Combine your answers from #27 and #30 to find the entire surface area of the traffic cone.

Review and Reflect

36. When finding the surface area of pyramids and cones describe why we use the slant height instead of the height.
37. Describe any other differences between of the formulas for surface area of prisms and pyramids.

Warm-Up Answers

1. $2(5 \cdot 6) + 2(5 \cdot 7) + 2(6 \cdot 7) = 214\ cm^2$
2. $2(15 \cdot 18) + 2(15 \cdot 21) + 2(18 \cdot 21) = 1926\ cm^2$
3. $2 \cdot 25\pi + 250\pi = 300\pi\ in^2$
4. $36^2(2\pi) + 72\pi(24) = 4320\pi\ ft^2$
5. 

12.4 Volume of Prisms and Cylinders

TEKS G(11)D

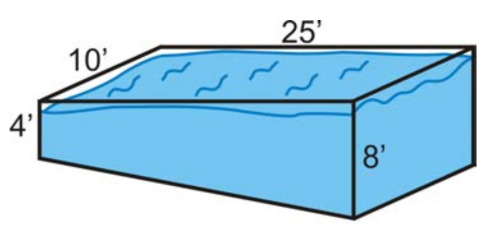
Learning Objectives

- Find the volume of prisms and cylinders

Warm-Up

1. Define volume in your own words.
2. What is the surface area of a cube with 3 inch sides?
3. A regular octahedron has 8 congruent equilateral triangles as the faces.
   a. If each edge is 4 cm, what is the slant height for one face?
   b. What is the surface area of one face?
   c. What is the total surface area?

Know What? Let's fill the pool it with water. The shallow end is 4 ft. and the deep end is 8 ft. The pool is 10 ft. wide by 25 ft. long. How many cubic feet of water is needed to fill the pool?



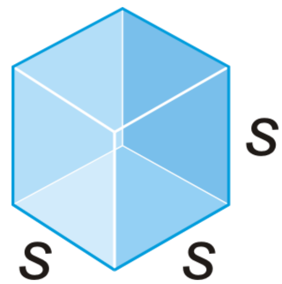
Volume of a Rectangular Prism

Volume: The measure of how much space a three-dimensional figure occupies.

Another way to define volume would be how much a three-dimensional figure can hold. The basic unit of volume is the cubic unit: cubic centimeter (cm^3), cubic inch (in^3), cubic meter (m^3), cubic foot (ft^3).

Volume of a Cube Postulate: $V = s^3$.

$V = s \cdot s \cdot s = s^3$

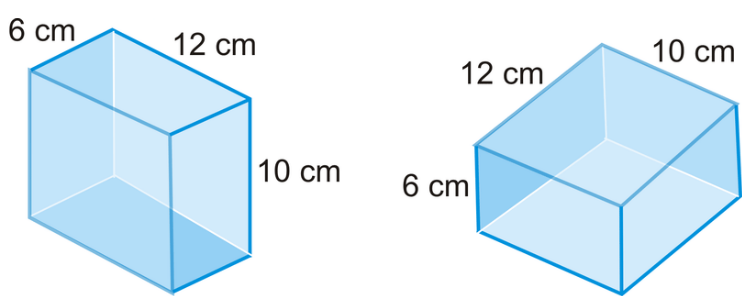


What this postulate tells us is that every solid can be broken down into cubes. For example, if we wanted to find the volume of a cube with 9 inch sides, it would be $9^3 = 729\ in^3$.

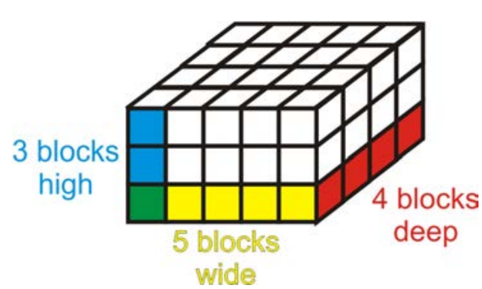
Volume Congruence Postulate

If two solids are congruent, then their volumes are congruent

These prisms are congruent, so their volumes are congruent.



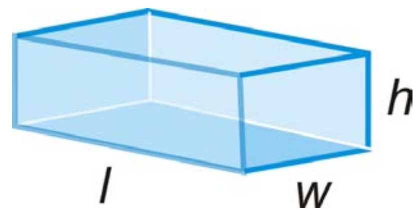
Example 1: Find the volume of the right rectangular prism below.



Solution: Count the cubes. The bottom layer has 20 cubes, or 4×5, and there are 3 layers. There are 60 cubes. The volume is also 60 $units^3$.

Each layer in Example 1 is the same as the area of the base and the number of layers is the same as the height. This is the formula for volume.

Volume of a Rectangular Prism: $V = l \cdot w \cdot h$.



Example 2: A typical shoe box is 8 in by 14 in by 6 in. What is the volume of the box?

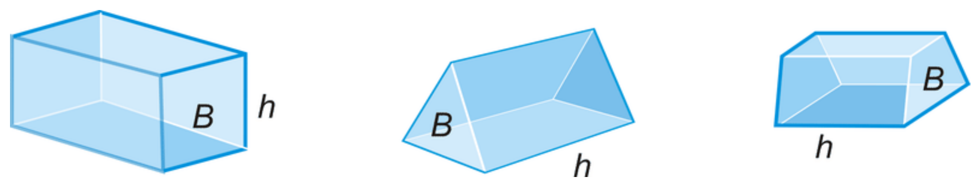
Solution: We can assume that a shoe box is a rectangular prism.

$V = (8)(14)(6) = 672\ in^3$

Volume of any Prism

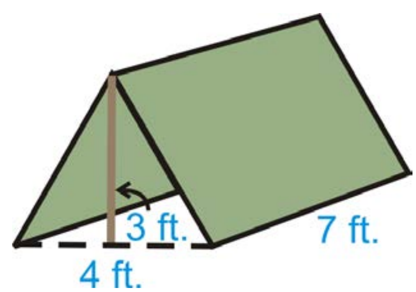
Notice that $l \cdot w$ is equal to the area of the base of the prism, which we will re-label B.

Volume of a Prism: $V = B \cdot h$.



"B" is not always going to be the same. So, to find the volume of a prism, you would first find the area of the base and then multiply it by the height.

Example 3: You have a small, triangular prism shaped tent. How much volume does it have, once it is set up?



Solution: First, we need to find the area of the base.

$$B = \frac{1}{2}(4)(3) = 6\ ft^2.$$
$$V = Bh = 6(7) = 42\ ft^3$$

Even though the height in this problem does not look like a "height," it is. The height of the triangular prism is the distance between the two triangular bases. Usually, the height of a prism is going to be the last length you need to use.

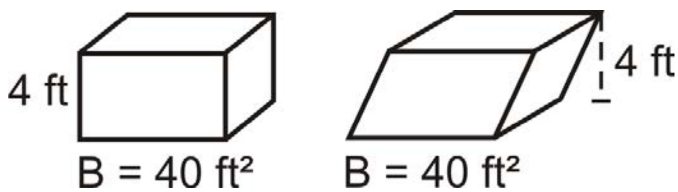
Oblique Prisms

Recall that oblique prisms are prisms that lean to one side and the height is outside the prism. What would be the volume of an oblique prism? Consider to piles of books below.

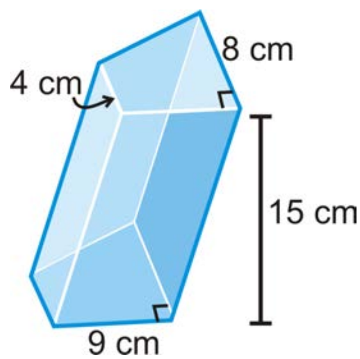
Both piles have 15 books, which means they will have the same volume. Cavalieri's Principle says that leaning does not matter, the volumes are the same.

Cavalieri's Principle: If two solids have the same height and the same cross-sectional area at every level, then they will have the same volume.

If an oblique prism and a right prism have the same base area and height, then they will have the same volume.



Example 4: Find the area of the oblique prism below.



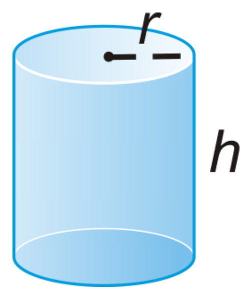
Solution: This is an oblique right trapezoidal prism. Find the area of the trapezoid.

$$B = \frac{1}{2}(9)(8+4) = 9(6) = 54\ cm^2$$
$$V = 54(15) = 810\ cm^3$$

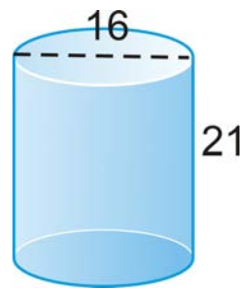
Volume of a Cylinder

If we use the formula for the volume of a prism, $V = Bh$, we can find the volume of a cylinder. In the case of a cylinder, the base is the area of a circle. Like a prism, Cavalieri's Principle holds.

Volume of a Cylinder: $V = \pi r^2 h$.



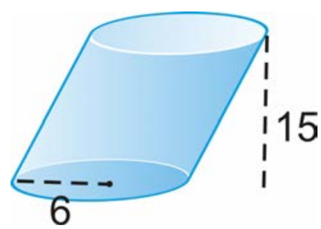
Example 5: Find the volume of the cylinder.



Solution: If the diameter is 16, then the radius is 8.

$V = \pi \cdot (8)^2 \cdot (21) = 1344\pi \ units^3$

Example 6: Find the volume of the cylinder.



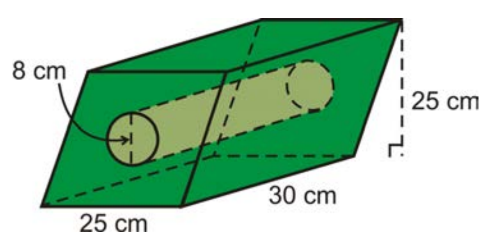
Solution: $V = \pi \cdot (6)^2 \cdot (15) = 540\pi \ units^3$

Example 7: If the volume of a cylinder is $484\pi \ in^3$ and the height is 4 in, what is the radius?

Solution: Solve for r.

$$\begin{aligned} 484\pi &= \pi r^2(4) \\ 121 &= r^2 \\ 11 &= r \end{aligned}$$

Example 8: Find the volume of the solid below.



Solution: This solid is a parallelogram-based prism with a cylinder cut out of the middle.

$$V_{prism} = (25 \cdot 25)30 = 18750\ cm^3$$
$$V_{cylinder} = \pi(4)^2(30) = 480\pi\ cm^3$$

The total volume is $18750 - 480\pi \approx 17242.04\ cm^3$.

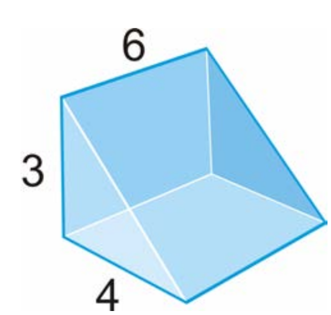
Know What? Revisited Even though it doesn't look like it, the trapezoid is the base of this prism. The area of the trapezoids are $\frac{1}{2}(4+8)25 = 150\ ft^2$. $V = 150(10) = 1500\ ft^3$

Practice Problems

- Question 1 uses the volume formula for a cylinder.
- Questions 2-4 are similar to Example 1.
- Questions 5-18 are similar to Examples 2-6.
- Questions 19-24 are similar to Example 7.
- Questions 25-30 are similar to Example 8.

1. Two cylinders have the same surface area. Do they have the same volume? How do you know?
2. How many one-inch cubes can fit into a box that is 8 inches wide, 10 inches long, and 12 inches tall? Is this the same as the volume of the box?
3. A cereal box in 2 inches wide, 10 inches long and 14 inches tall. How much cereal does the box hold?
4. A can of soda is 4 inches tall and has a diameter of 2 inches. How much soda does the can hold? Round your answer to the nearest hundredth.
5. A cube holds 216 in^3. What is the length of each edge?
6. A cube has sides that are 8 inches. What is the volume?
7. A cylinder has $r = h$ and the radius is 4 cm. What is the volume?
8. A cylinder has a volume of $486\pi\ ft.^3$. If the height is 6 ft., what is the diameter?

Use the right triangular prism to answer questions 9 and 10.



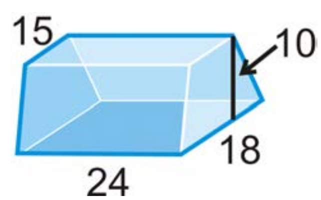
9. What is the length of the third base edge?
10. Find the volume of the prism.
11. Fuzzy dice are cubes with 4 inch sides.

a. What is the volume of one die?
b. What is the volume of both dice?

12. A right cylinder has a 7 cm radius and a height of 18 cm. Find the volume.

Find the volume of the following solids. Round your answers to the nearest hundredth.

13.



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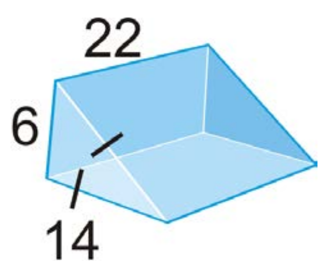

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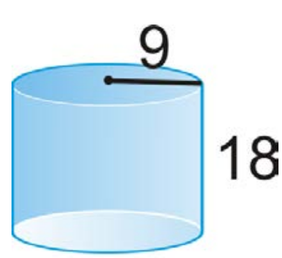

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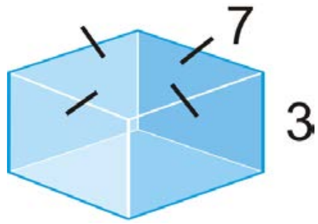
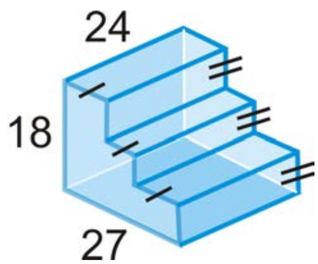


17.

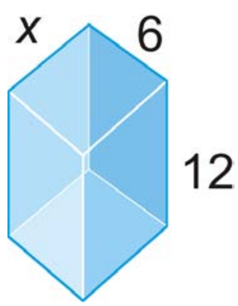
18.



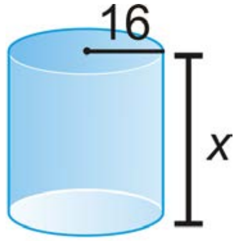
Algebra Connection Find the value of x, given the surface area.

19.

$V = 504\ units^3$

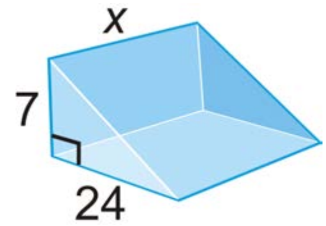


20. $V = 6144\pi\ units^3$



21.

$V = 2688\ units^3$

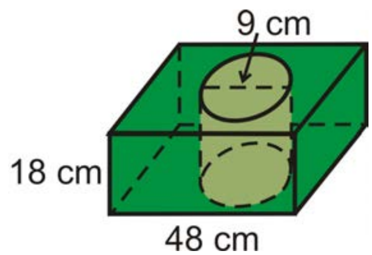


22. The area of the base of a cylinder is $49\pi\ in^2$ and the height is 6 in. Find the volume.

23. The circumference of the base of a cylinder is $80\pi\ cm$ and the height is 15 cm. Find the volume.

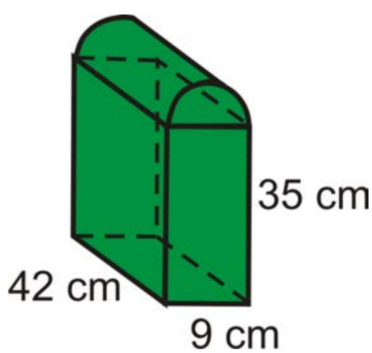
24. The lateral surface area of a cylinder is $30\pi\ m^2$ and the circumference is $10\pi\ m$. What is the volume of the cylinder?

The bases of the prism are squares and a cylinder is cut out of the center.



25. Find the volume of the prism.
26. Find the volume of the cylinder in the center.
27. Find the volume of the figure.

This is a prism with half a cylinder on the top.



28. Find the volume of the prism.
29. Find the volume of the half-cylinder.
30. Find the volume of the entire figure.

Review and Reflect

31. In your own words, describe why and oblique prism has the same volume as a right prism.
32. How is the volume of a prism affected if the height is doubled?Tripled?

Warm-Up Answers

1. The amount a three-dimensional figure can hold.
2. $54\ in^2$
   a. $2\sqrt{3}$
   b. $\frac{1}{2} \cdot 4 \cdot 2\sqrt{3} = 4\sqrt{3}$
   c. $8 \cdot 4\sqrt{3} = 32\sqrt{3}$

12.5 Volume of Pyramids and Cones

TEKS G(11)D

Learnign Objectives

- Find the volume of pyramids and cones

Warm-Up

1. Find the volume of a square prism with 8 inch base edges and a 12 inch height.
2. Find the volume of a cylinder with a *diameter* of 8 inches and a height of 12 inches.
3. Find the surface area of a square pyramid with 10 inch base edges and a height of 12 inches.

Know What? The Khafre Pyramid is a pyramid in Giza, Egypt. It is a square pyramid with a base edge of 706 feet and an original height of 407.5 feet. What was the original volume of the Khafre Pyramid?

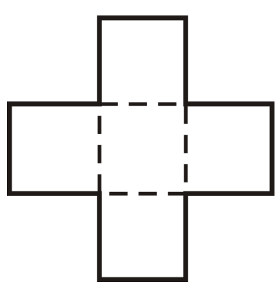
Volume of a Pyramid

The volume of a pyramid is closely related to the volume of a prism with the same sized base.

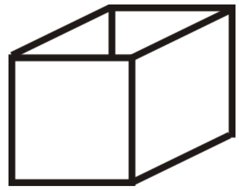
Investigation 10-1: Finding the Volume of a Pyramid

Tools needed: pencil, paper, scissors, tape, ruler, dry rice.

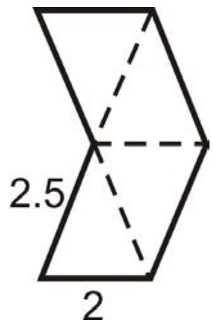
1. Make an open net (omit one base) of a cube, with 2 inch sides.



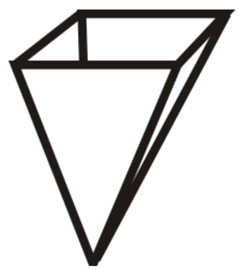
2. Cut out the net and tape up the sides to form an open cube.



3. Make an open net (no base) of a square pyramid, with lateral edges of 2.5 inches and base edges of 2 inches.



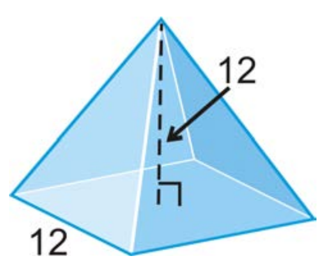
4. Cut out the net and tape up the sides to form an open pyramid.



5. Fill the pyramid with dry rice and dump the rice into the open cube. Repeat this until you have filled the cube?

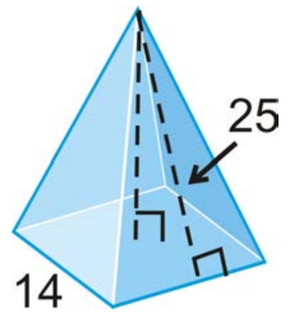
Volume of a Pyramid: $V = \frac{1}{3}Bh$.

Example 1: Find the volume of the pyramid.



Solution: $V = \frac{1}{3}(12^2)12 = 576\ units^3$

Example 2a: Find the height of the pyramid.



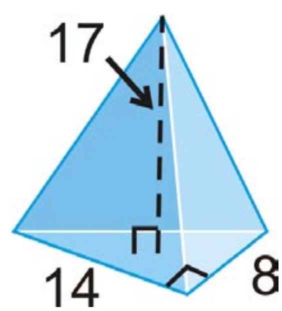
Solution: In this example, we are given the *slant* height. Use the Pythagorean Theorem.

$$\begin{aligned} 7^2 + h^2 &= 25^2 \\ h^2 &= 576 \\ h &= 24 \end{aligned}$$

Example 2b: Find the volume of the pyramid in Example 2a.

Solution: $V = \frac{1}{3}(14^2)(24) = 1568\ units^3$.

Example 3: Find the volume of the pyramid.



Solution: The base of the pyramid is a right triangle. The area of the base is $\frac{1}{2}(14)(8) = 56\ units^2$.

$V = \frac{1}{3}(56)(17) \approx 317.33\ units^3$

Example 4: A rectangular pyramid has a base area of 56 cm^2 and a volume of 224 cm^3. What is the height of the pyramid?

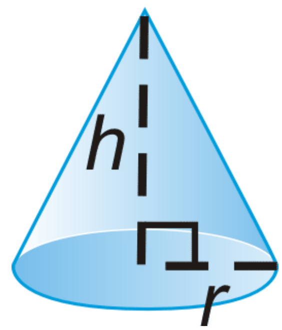
Solution:

$$\begin{aligned} V &= \frac{1}{3}Bh \\ 224 &= \frac{1}{3} \cdot 56h \\ 12 &= h \end{aligned}$$

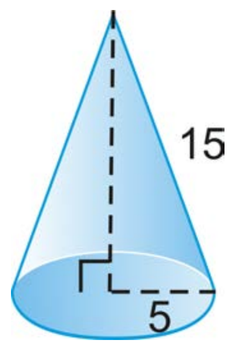
Volume of a Cone

Volume of a Cone: $V = \frac{1}{3}\pi r^2 h$.

This is the same relationship as a pyramid's volume with a prism's volume.



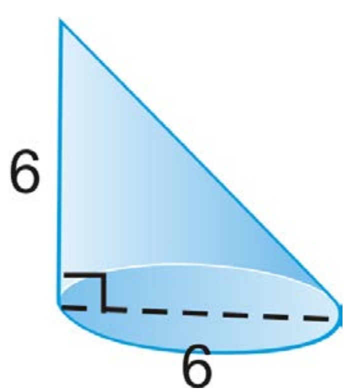
Example 5: Find the volume of the cone.



Solution: First, we need the height. Use the Pythagorean Theorem.

$$\begin{aligned} 5^2 + h^2 &= 15^2 \\ h &= \sqrt{200} = 10\sqrt{2} \\ V &= \frac{1}{3}(5^2)\left(10\sqrt{2}\right)\pi \approx 370.24 \end{aligned}$$

Example 6: Find the volume of the cone.



Solution: We can use the same volume formula. Find the *radius*.

$$V = \frac{1}{3}\pi(3^2)(6) = 18\pi \approx 56.55$$

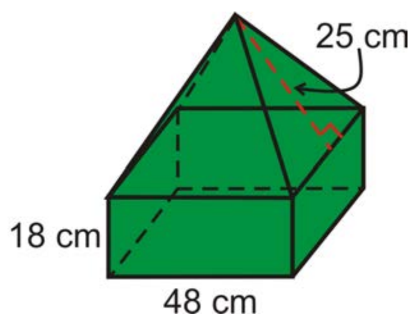
Example 7: The volume of a cone is $484\pi\ cm^3$ and the height is 12 cm. What is the radius?

Solution: Plug in what you know to the volume formula.

$$\begin{aligned} 484\pi &= \frac{1}{3}\pi r^2(12) \\ 121 &= r^2 \\ 11 &= r \end{aligned}$$

Composite Solids

Example 8: Find the volume of the composite solid. All bases are squares.



Solution: This is a square prism with a square pyramid on top. First, we need the height of the pyramid portion. Using the Pythagorean Theorem, we have, $h = \sqrt{25^2 - 24^2} = 7$.

$$\begin{aligned} V_{prism} &= (48)(48)(18) = 41472\ cm^3 \\ V_{pyramid} &= \frac{1}{3}(48^2)(7) = 5376\ cm^3 \end{aligned}$$

The total volume is $41472 + 5376 = 46,848\ cm^3$.

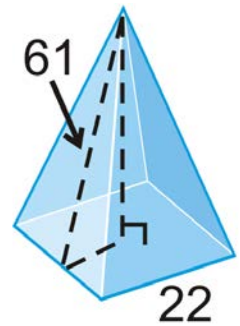
Know What? Revisited The original volume of the pyramid is $\frac{1}{3}(706^2)(407.5) \approx 67,704,223.33\ ft^3$.

Practice Problems

- Questions 1-13 are similar to Examples 1-3, 5 and 6.
- Questions 14-22 are similar to Examples 4 and 7.
- Questions 23-31 are similar to Example 8.

Find the volume of each regular pyramid and right cone. Round any decimal answers to the nearest hundredth. The bases of these pyramids are either squares or equilateral triangles.

1.



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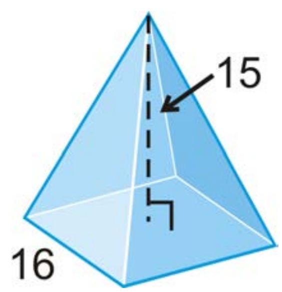
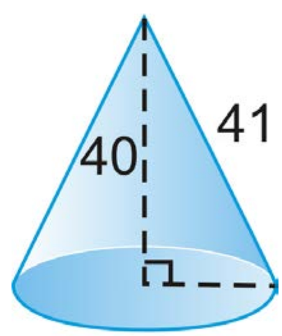


3.



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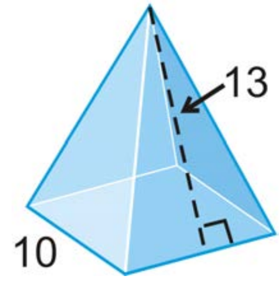

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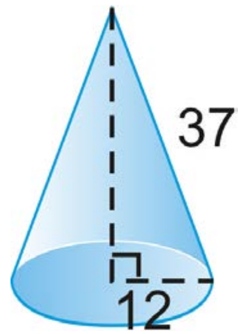

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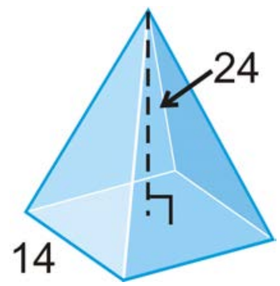

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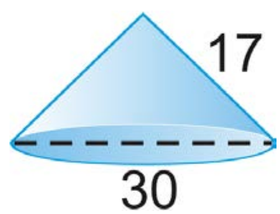
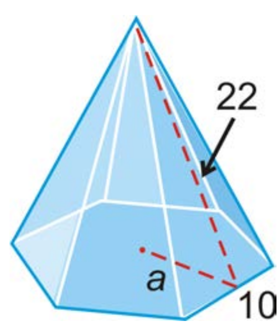
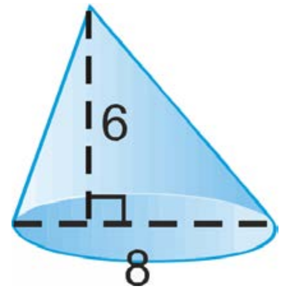


8.

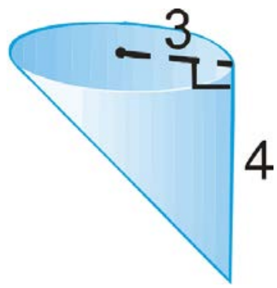


Find the volume of the following non-regular pyramids and cones. Round any decimal answers to the nearest hundredth.

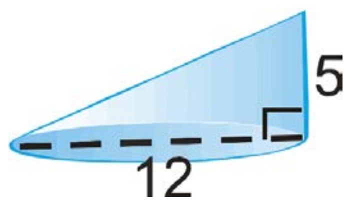
9.



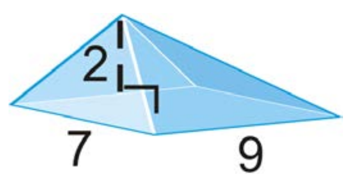
10.



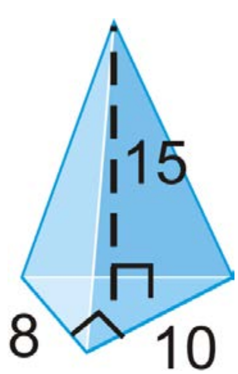
11.



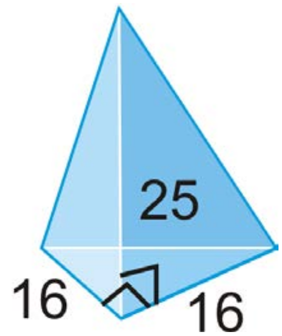
12. Base is a rectangle



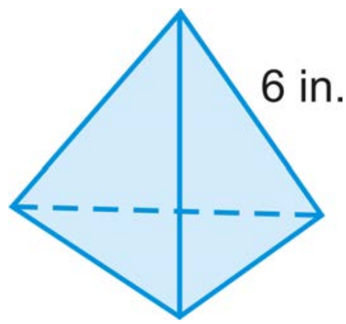
13.



14.

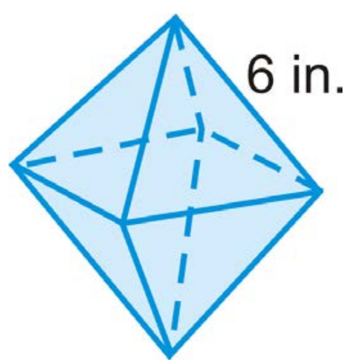


A ***regular tetrahedron*** has four equilateral triangles as its faces. Use the diagram to answer questions 15-17. Round your answers to the nearest hundredth.



15. What is the area of the base of this regular tetrahedron?
16. What is the height of this figure? Be careful!
17. Find the volume.

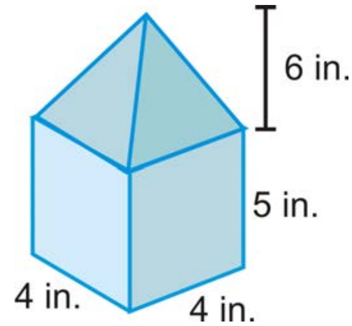
A ***regular octahedron*** has eight equilateral triangles as its faces. Use the diagram to answer questions 18-22. Round your answers to the nearest hundredth.



18. *Describe* how you would find the volume of this figure.
19. Find the volume.
20. The volume of a square pyramid is 72 square inches and the base edge is 4 inches. What is the height?
21. If the volume of a cone is $30\pi\ cm^3$ and the radius is 5 cm, what is the height?
22. If the volume of a cone is $105\pi\ cm^3$ and the height is 35 cm, what is the radius?
23. The volume of a triangle pyramid is 170 in^3 and the base area is 34 in^2. What is the height of the pyramid?

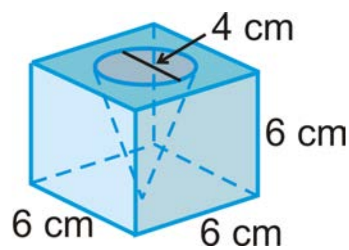
For questions 24-31, round your answer to the nearest hundredth.

24. Find the volume of the base prism.



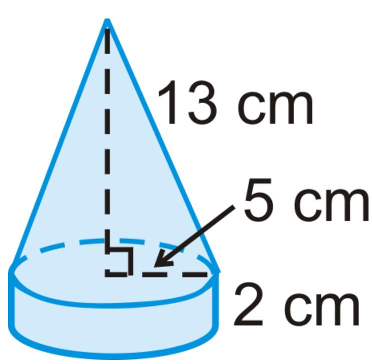
25. Find the volume of the pyramid.
26. Find the volume of the entire solid.

The solid to the right is a cube with a cone cut out.



26. Find the volume of the cube.
27. Find the volume of the cone.
28. Find the volume of the entire solid.

The solid to the left is a cylinder with a cone on top.



29. Find the volume of the cylinder.
30. Find the volume of the cone.
31. Find the volume of the entire solid.

Review and Reflect

32. Describe two different methods you could use to triple the volume of a cone.
33. Describe the differences between the formulas for volume of a cone and volume of a pyramid.

Warm-Up Answers

1. $(8^2)(12) = 768\ in^3$
2. $(4^2)(12)\pi = 192\pi \approx 603.19$
3. Find slant height, $l = 13$. $SA = 100 + \frac{1}{2}(40)(13) = 360\ in^2$

12.6 Surface Area and Volume of Spheres

TEKS G(11)C, G(11)D

Learning Objectives

- Find the surface area of a sphere
- Find the volume of a sphere

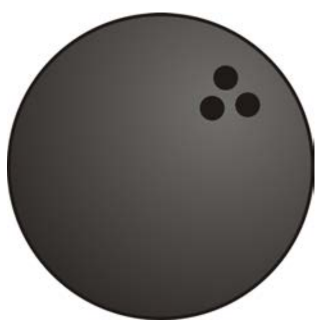
Vocabulary

- Sphere
- Great Circle

Warm-Up

1. List three spheres you would see in real life.
2. Find the area of a circle with a 6 cm radius.
3. Find the volume of a cylinder with the circle from #2 as the base and a height of 5 cm.

Know What? A regulation bowling ball is a sphere with a circumference of 27 inches. Find the radius of a bowling ball, its surface area and volume. You may assume the bowling ball does not have any finger holes. Round your answers to the nearest hundredth.

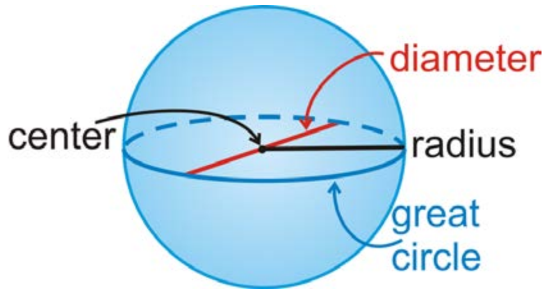


Defining a Sphere

A sphere is the last of the three-dimensional shapes that we will find the surface area and volume of. Think of a sphere as a three-dimensional circle.

Sphere: The set of all points, in three-dimensional space, which are equidistant from a point.

The ***radius*** has an endpoint on the sphere and the other endpoint is the center.

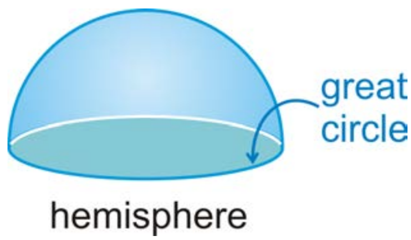


The ***diameter*** must contain the center.

Great Circle: A cross section of a sphere that contains the diameter.

A great circle is the largest circle cross section in a sphere. ***The circumference of a sphere is the circumference of a great circle***.

Every great circle divides a sphere into two congruent ***hemispheres.***



Example 1: The circumference of a sphere is $26\pi\ feet$. What is the radius of the sphere?

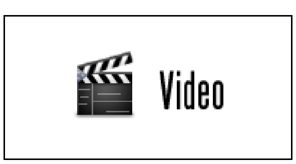
Solution: The circumference is referring to the circumference of a great circle. Use $C = 2\pi r$.

$$\begin{aligned} 2\pi r &= 26\pi \\ r &= 13\ ft. \end{aligned}$$

Surface Area of a Sphere

The best way to understand the surface area of a sphere is to watch the link by Russell Knightley, http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Surface-Area-Derivation.html .

Watch this!



MEDIA

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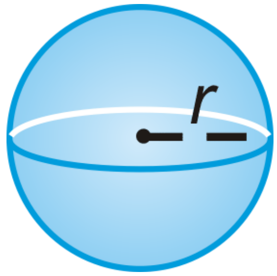
URL: https://www.ck12.org/flx/render/embeddedobject/161355

Surface Area of a Sphere: $SA = 4\pi r^2$.

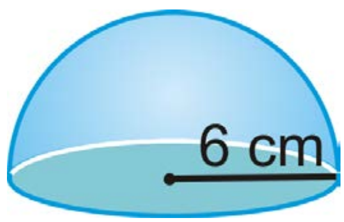


Example 2: Find the surface area of a sphere with a radius of 14 feet.

Solution:

$$\begin{aligned} SA &= 4\pi(14)^2 \\ &= 784\pi \ ft^2 \end{aligned}$$

Example 3: Find the surface area of the figure below.



Solution: Be careful when finding the surface area of a hemisphere because you need to include the area of the base.

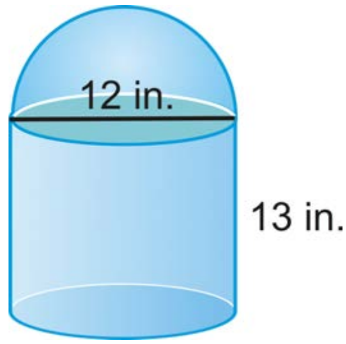
$$\begin{aligned} SA &= \pi r^2 + \frac{1}{2}4\pi r^2 \\ &= \pi(6^2) + 2\pi(6^2) \\ &= 36\pi + 72\pi = 108\pi \ cm^2 \end{aligned}$$

Example 4: The surface area of a sphere is $100\pi \ in^2$. What is the radius?

Solution:

$$\begin{aligned} SA &= 4\pi r^2 \\ 100\pi &= 4\pi r^2 \\ 25 &= r^2 \\ 5 &= r \end{aligned}$$

Example 5: Find the surface area of the following solid.



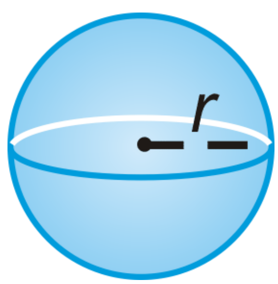
Solution: This solid is a cylinder with a hemisphere on top. It is one solid, so do not include the bottom of the hemisphere or the top of the cylinder.

$$\begin{aligned} SA &= LA_{cylinder} + LA_{hemisphere} + A_{base\ circle} \\ &= \pi rh + \frac{1}{2}4\pi r^2 + \pi r^2 \\ &= \pi(6)(13) + 2\pi 6^2 + \pi 6^2 \\ &= 78\pi + 72\pi + 36\pi \\ &= 186\pi\ in^2 \qquad LA \text{ stands for } lateral\ area. \end{aligned}$$

Volume of a Sphere

To see an animation of the volume of a sphere, see http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Volume-Derivation.html by Russell Knightley.

Volume of a Sphere: $V = \frac{4}{3}\pi r^3$.



Example 6: Find the volume of a sphere with a radius of 6 m.

Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi 6^3 \\ &= \frac{4}{3}\pi(216) \\ &= 288\pi\ m^3 \end{aligned}$$

Example 7: A sphere has a volume of 14137.167 ft^3, what is the radius?

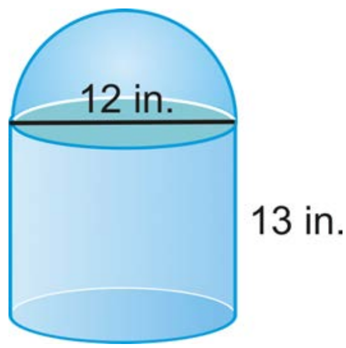
Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ 14137.167 &= \frac{4}{3}\pi r^3 \\ \frac{3}{4\pi} \cdot 14137.167 &= r^3 \\ 3375 &= r^3 \end{aligned}$$

At this point, you will need to take the ***cubed root*** of 3375. Ask your teacher how to do this on your calculator.

$$\sqrt[3]{3375} = 15 = r$$

Example 8: Find the volume of the following solid.



Solution:

$$\begin{aligned} V_{cylinder} &= \pi 6^2(13) = 78\pi \\ V_{hemisphere} &= \frac{1}{2}\left(\frac{4}{3}\pi 6^3\right) = 36\pi \\ V_{total} &= 78\pi + 36\pi = 114\pi \ in^3 \end{aligned}$$

Know What? Revisited The radius would be $27 = 2\pi r$, or $r = 4.30$ *inches*. The surface area would be $4\pi 4.3^2 \approx 232.35 \ in^2$, and the volume would be $\frac{4}{3}\pi 4.3^3 \approx 333.04 \ in^3$.

Practice Problems

- Questions 1-3 look at the definition of a sphere.
- Questions 4-17 are similar to Examples 1, 2, 4, 6 and 7.
- Questions 18-21 are similar to Example 3 and 5.
- Questions 22-25 are similar to Example 8.
- Question 26 is a challenge.
- Questions 27-29 are similar to Example 8.
- Question 30 analyzes the formula for the surface area of a sphere.

1. Are there any cross-sections of a sphere that are not a circle? Explain your answer.

2. List all the parts of a sphere that are the ***same*** as a circle.
3. List any parts of a sphere that a circle does not have.

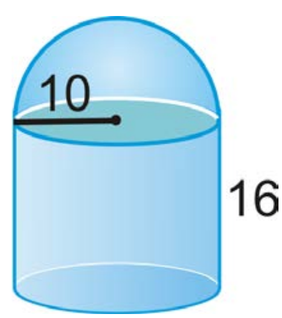
Find the surface area and volume of a sphere with: (Leave your answer in terms of π)

4. a radius of 8 in.
5. a diameter of 18 cm.
6. a radius of 20 ft.
7. a diameter of 4 m.
8. a radius of 15 ft.
9. a diameter of 32 in.
10. a circumference of $26\pi\ cm$.
11. a circumference of $50\pi\ yds$.
12. The surface area of a sphere is $121\pi\ in^2$. What is the radius?
13. The volume of a sphere is $47916\pi\ m^3$. What is the radius?
14. The surface area of a sphere is $4\pi\ ft^2$. What is the volume?
15. The volume of a sphere is $36\pi\ mi^3$. What is the surface area?
16. Find the radius of the sphere that has a volume of $335\ cm^3$. Round your answer to the nearest hundredth.
17. Find the radius of the sphere that has a surface area $225\pi\ ft^2$.

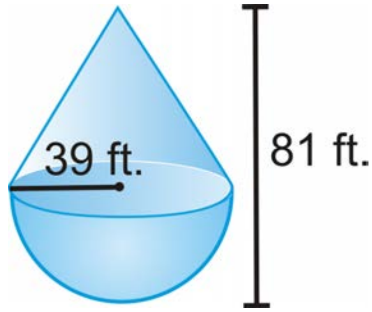
Find the surface area of the following shapes. Leave your answers in terms of π.

18.

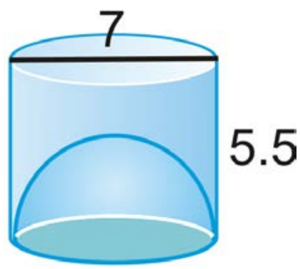
19.



20.

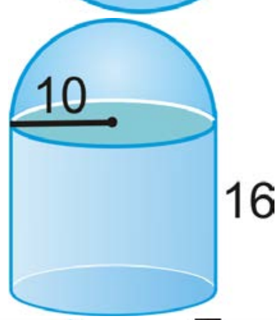


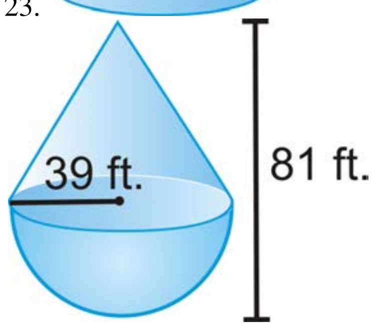
21. You may assume the bottom is open.

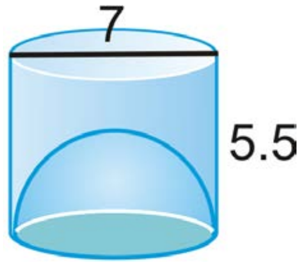


Find the volume of the following shapes. Round your answers to the nearest hundredth.

22. 

23. 

24. 

25. 

26. A sphere has a radius of 5 cm. A right cylinder has the same radius and volume. Find the height of the cylinder.

Review and Reflect

Tennis balls with a 3 inch diameter are sold in cans of three. The can is a cylinder. Round your answers to the nearest hundredth.

27. What is the volume of one tennis ball?
28. What is the volume of the cylinder?
29. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space *not* occupied by the tennis balls?
30. How does the formula of the surface area of a sphere relate to the area of a circle?

Warm-Up Answers

1. *Answers will vary.* Possibilities are any type of ball, certain lights, or the 76/Unical orb.
2. 36π
3. 180π

12.7 Exploring Similar Solids

TEKS G(10)B

Learning Objectives

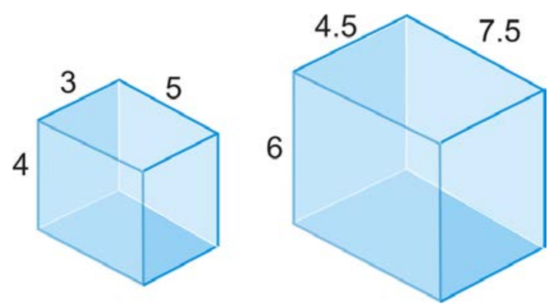
- Find the relationship between similar solids and their surface areas and volumes

Similar Solids

Recall that two shapes are similar if all the corresponding angles are congruent and the corresponding sides are proportional.

Similar Solids: Two solids are similar if they are the same type of solid and their corresponding radii, heights, base lengths, widths, etc. are proportional.

Example 1: Are the two rectangular prisms similar? How do you know?

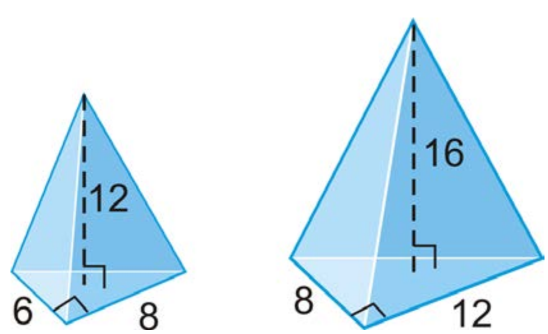


Solution: Match up the corresponding heights, widths, and lengths.

$$\frac{small\ prism}{large\ prism} = \frac{3}{4.5} = \frac{4}{6} = \frac{5}{7.5}$$

The congruent ratios tell us the two prisms are similar.

Example 2: Determine if the two triangular pyramids similar.



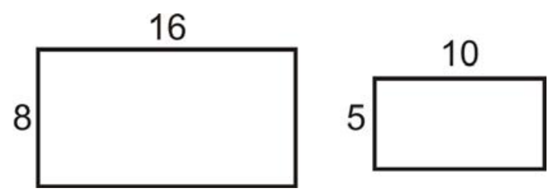
Solution: Just like Example 1, let's match up the corresponding parts.

$\frac{6}{8} = \frac{12}{16} \neq \frac{8}{12}$.

These triangle pyramids are not similar.

Surface Areas of Similar Solids

If two shapes are similar, then the ratio of the area is a square of the scale factor.

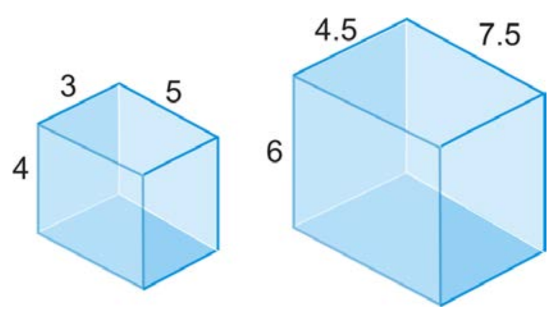


For example, the two rectangles are similar because their sides are in a ratio of 5:8. The area of the larger rectangle is $8(16) = 128\ units^2$. The area of the smaller rectangle is $5(10) = 50\ units^2$.

Comparing the areas in a ratio, it is $50 : 128 = 25 : 64 = 5^2 = 8^2$.

So, what happens with the surface areas of two similar solids?

Example 3: Find the surface area of the two similar rectangular prisms.



Solution:

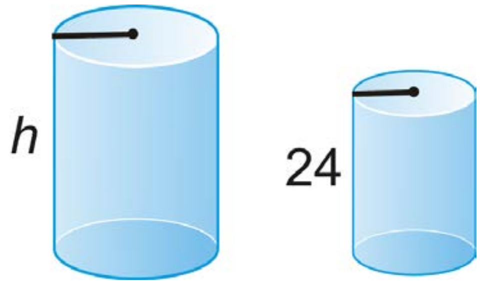
$$\begin{aligned} SA_{smaller} &= 2(4 \cdot 3) + 2(4 \cdot 5) + 2(3 \cdot 5) \\ &= 24 + 40 + 30 = 94\ units^2 \\ SA_{larger} &= 2(6 \cdot 4.5) + 2(4.5 \cdot 7.5) + 2(6 \cdot 7.5) \\ &= 54 + 67.5 + 90 = 211.5\ units^2 \end{aligned}$$

Now, find the ratio of the areas. $\frac{94}{211.5} = \frac{4}{9} = \frac{2^2}{3^2}$. The sides are in a ratio of $\frac{4}{6} = \frac{2}{3}$, so the surface areas are in a ratio of $\frac{2^2}{3^2}$.

Surface Area Ratio

If two solids are similar with a scale factor of $\frac{a}{b}$, then the surface areas are in the ratio of $\left(\frac{a}{b}\right)^2$

Example 4: Two similar cylinders are below. If the ratio of the areas is 16:25, what is the height of the taller cylinder?



Solution: First, we need to take the square root of the area ratio to find the scale factor, $\sqrt{\frac{16}{25}} = \frac{4}{5}$. Set up a proportion to find h.

$$\frac{4}{5} = \frac{24}{h}$$
$$4h = 120$$
$$h = 30$$

Example 5: Using the cylinders from Example 4, if the area of the smaller cylinder is $1536\pi\ cm^2$, what is the area of the larger cylinder?

Solution: Set up a proportion using the ratio of the areas, 16:25.

$$\frac{16}{25} = \frac{1536\pi}{A}$$
$$16A = 38400\pi$$
$$A = 2400\pi\ cm^2$$

Volumes of Similar Solids

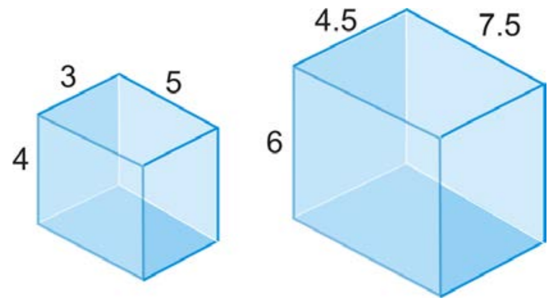
Let's look at what we know about similar solids so far.

TABLE 12.3:

	Ratios	*Units*
Scale Factor	$\frac{a}{b}$	in, ft, cm, m, etc.
Ratio of the Surface Areas	$\left(\frac{a}{b}\right)^2$	in^2, ft^2, cm^2, m^2, etc.
Ratio of the Volumes	**??**	in^3, ft^3, cm^3, m^3, etc.

If the ratio of the volumes follows the pattern from above, it should be the ***cube*** of the scale factor.

Example 6: Find the volume of the following rectangular prisms. Then, find the ratio of the volumes.



Solution:

$$V_{smaller} = 3(4)(5) = 60$$
$$V_{larger} = 4.5(6)(7.5) = 202.5$$

The ratio is $\frac{60}{202.5}$, which reduces to $\frac{8}{27} = \frac{2^3}{3^3}$.

Volume Ratio

If two solids are similar with a scale factor of $\frac{a}{b}$, then the volumes are in a ratio of $\left(\frac{a}{b}\right)^3$

Example 7: Two spheres have radii in a ratio of 3:4. What is the ratio of their volumes?

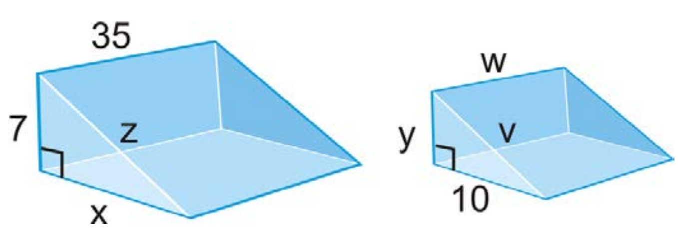
Solution: If we cube 3 and 4, we will have the ratio of the volumes. $3^3 : 4^3 = 27 : 64$.

Example 8: If the ratio of the volumes of two similar prisms is 125:8, what is the scale factor?

Solution: Take the ***cubed root*** of 125 and 8 to find the scale factor.

$\sqrt[3]{125} : \sqrt[3]{8} = 5 : 2$

Example 9: Two similar right triangle prisms are below. If the ratio of the volumes is 343:125, find the missing sides in both triangles.



Solution: The scale factor is 7:5, the cubed root. With the scale factor, we can now set up several proportions.

$$\frac{7}{5} = \frac{7}{y} \quad y = 5 \qquad \frac{7}{5} = \frac{x}{10} \quad x = 14 \qquad \frac{7}{5} = \frac{35}{w} \quad w = 25$$

$$7^2 + x^2 = z^2$$
$$7^2 + 14^2 = z^2$$
$$z = \sqrt{245} = 7\sqrt{5}$$

$$\frac{7}{5} = \frac{z}{v}$$
$$\frac{7}{5} = \frac{7\sqrt{5}}{v} \rightarrow v = 5\sqrt{5}$$

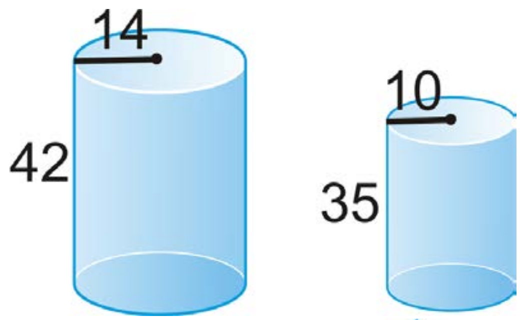
Example 10: The ratio of the surface areas of two similar cylinders is 16:81. What is the ratio of the volumes?

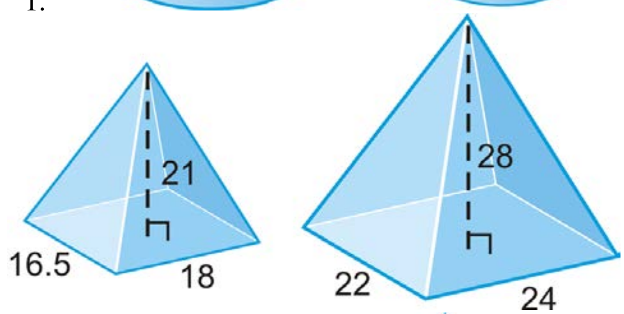
Solution: First, find the scale factor. If we take the square root of both numbers, the ratio is 4:9. Now, cube this to find the ratio of the volumes, $4^3 : 9^3 = 64 : 729$.

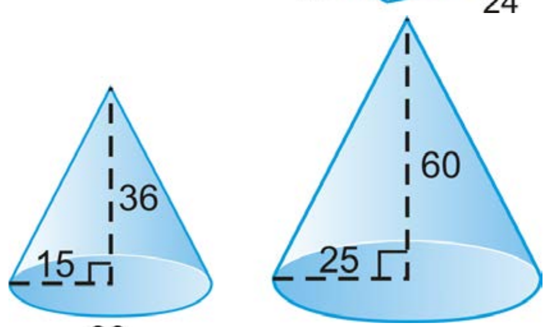
Practice Problems

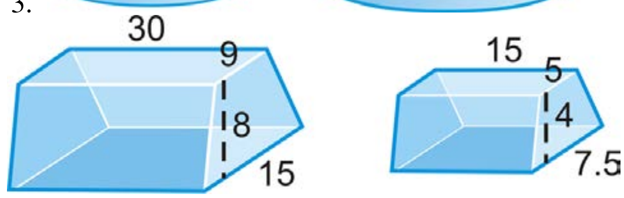
- Questions 1-4 are similar to Examples 1 and 2.
- Questions 5-14 are similar to Examples 3-8 and 10.
- Questions 15-18 are similar to Example 9.
- Questions 19 and 20 are similar to Example 1.

Determine if each pair of right solids are similar.

1. 

2. 

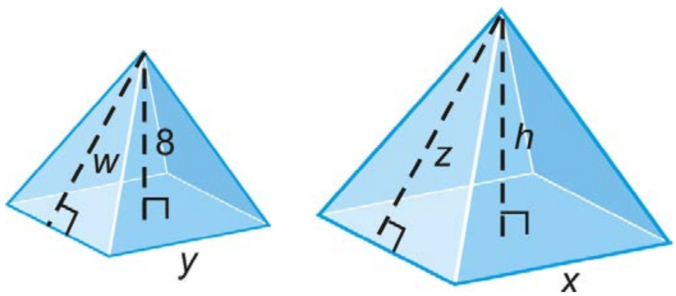
3. 

4. 

5. Are all cubes similar? Why or why not?
6. Two prisms have a scale factor of 1:4. What is the ratio of their surface areas?
7. Two pyramids have a scale factor of 2:7. What is the ratio of their volumes?
8. Two spheres have radii of 5 and 9. What is the ratio of their volumes?
9. The surface area of two similar cones is in a ratio of 64:121. What is the scale factor?
10. The volume of two hemispheres is in a ratio of 125:1728. What is the scale factor?

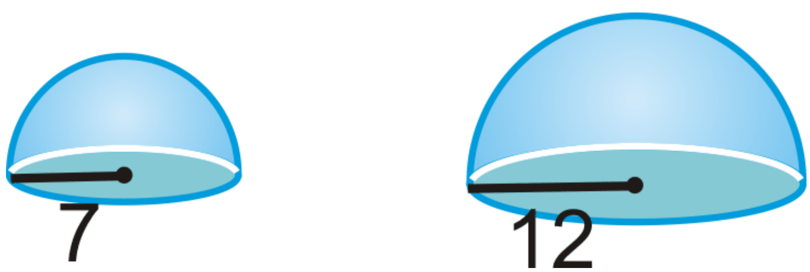
11. A cone has a volume of 15π and is similar to another larger cone. If the scale factor is 5:9, what is the volume of the larger cone?
12. The ratio of the volumes of two similar pyramids is 8:27. What is the ratio of their total surface areas?
13. The ratio of the volumes of two tetrahedrons is 1000:1. The smaller tetrahedron has a side of length 6 cm. What is the side length of the larger tetrahedron?
14. The ratio of the surface areas of two cubes is 64:225. What is the ratio of the volumes?

Below are two similar square pyramids with a volume ratio of 8:27. The base lengths are equal to the heights. Use this to answer questions 15-18.



15. What is the scale factor?
16. What is the ratio of the surface areas?
17. Find h, x and y.
18. Find the volume of both pyramids.

Use the hemispheres below to answer questions 19-20.



19. Are the two hemispheres similar? How do you know?
20. Find the ratio of the surface areas and volumes.

Review and Reflect

21. If you have a cube and wanted to create a similar cube with double the volume, what would you multiply the sides of the cube by.
22. What if you wanted to double the surface area of the cube.

12.8 Chapter 12 Review

Keywords, Theorems, & Formulas

Exploring Solids

- Polyhedron
- Face
- Edge
- Vertex
- Prism
- Pyramid
- Euler's Theorem
- Regular Polyhedron
- Regular Tetrahedron
- Cube
- Regular Octahedron
- Regular Dodecahedron
- Regular Icosahedron
- Cross-Section
- Net

Surface Area of Prisms Cylinders

- Lateral Face
- Lateral Edge
- Base Edge
- Right Prism
- Oblique Prism
- Surface Area
- Lateral Area
- Surface Area of a Right Prism
- Cylinder
- Surface Area of a Right Cylinder

Surface Area of Pyramids Cones

- Surface Area of a Regular Pyramid
- Cone
- Slant Height
- Surface Area of a Right Cone

Volume of Prisms Cylinders

- Volume
- Volume of a Cube Postulate

- Volume Congruence Postulate
- Volume of a Rectangular Prism
- Volume of a Prism
- Cavalieri's Principle
- Volume of a Cylinder

Volume of Pyramids Cones

- Volume of a Pyramid
- Volume of a Cone

Surface Area and Volume of Spheres

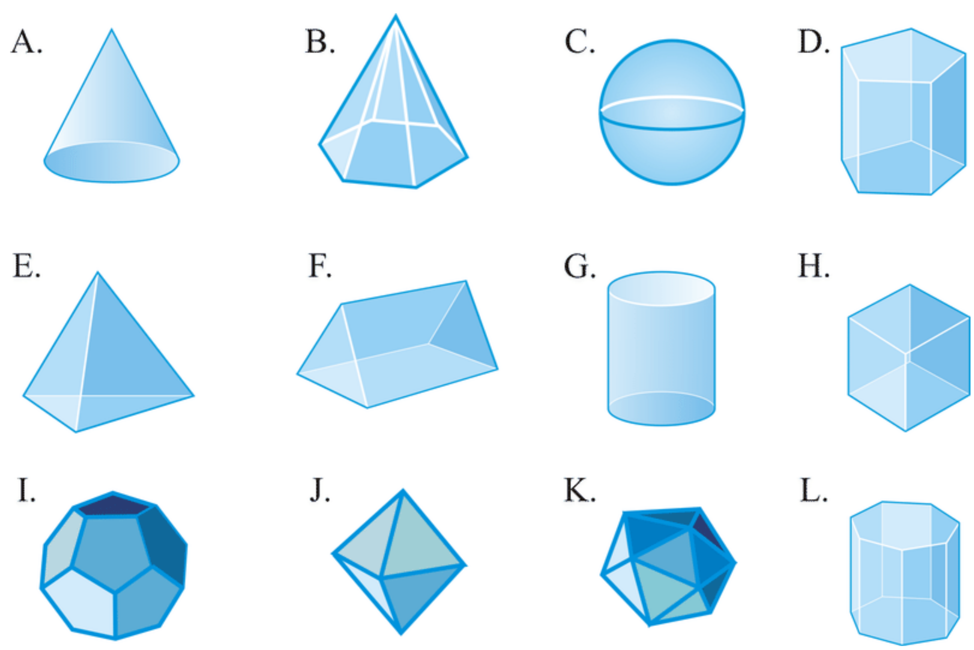
- Sphere
- Great Circle
- Surface Area of a Sphere
- Volume of a Sphere

Extension: Similar Solids

- Similar Solids
- Surface Area Ratio
- Volume Ratio

Review Questions

Match the shape with the correct name.



1. Triangular Prism
2. Icosahedron
3. Cylinder

4. Cone
5. Tetrahedron
6. Pentagonal Prism
7. Octahedron
8. Hexagonal Pyramid
9. Octagonal Prism
10. Sphere
11. Cube
12. Dodecahedron

Match the formula with its description.

13. Volume of a Prism - A. $\frac{1}{3}\pi r^2 h$
14. Volume of a Pyramid - B. $\pi r^2 h$
15. Volume of a Cone - C. $4\pi r^2$
16. Volume of a Cylinder - D. $\frac{4}{3}\pi r^3$
17. Volume of a Sphere - E. $\pi r^2 + \pi rl$
18. Surface Area of a Prism - F. $2\pi r^2 + 2\pi rh$
19. Surface Area of a Pyramid - G. $\frac{1}{3}Bh$
20. Surface Area of a Cone - H. Bh
21. Surface Area of a Cylinder - I. $B + \frac{1}{2}Pl$
22. Surface Area of a Sphere - J. The sum of the area of the bases and the area of each rectangular lateral face.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9696 .

12.9 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

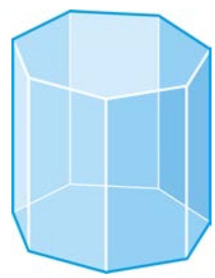
1^{st} **Section: Exploring Solids**

Polyhedron

Face

Edge

Vertex

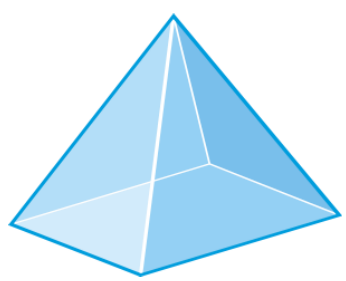


Prism

Pyramid

Euler's Theorem

Regular Polyhedron



Regular Tetrahedron

Cube

Regular Octahedron

Regular Dodecahedron

Regular Icosahedron

Cross-Section

Net

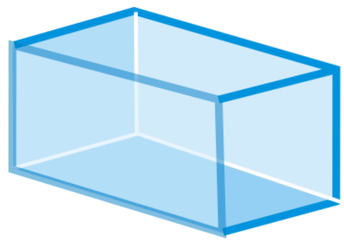
Homework:

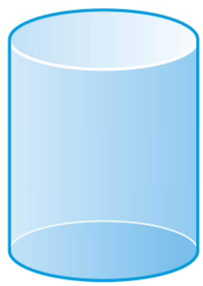
2^{nd} **Section: Surface Area of Prisms Cylinders**

Lateral Face

Lateral Edge

Base Edge





Right Prism

Oblique Prism

Surface Area

Lateral Area

Surface Area of a Right Prism

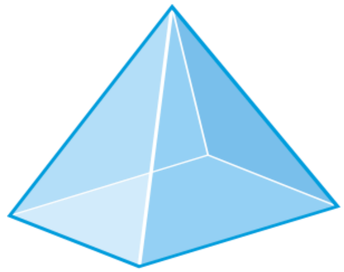
Cylinder

Surface Area of a Right Cylinder

Homework:

3rd **Section: Surface Area of Pyramids Cones**

Surface Area of a Regular Pyramid



Cone

Slant Height

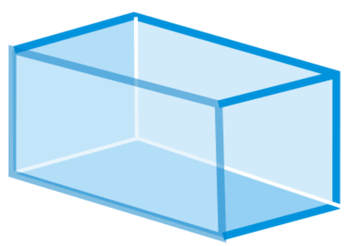
Surface Area of a Right Cone

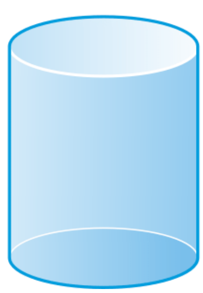
Homework:

4^{th} **Section: Volume of Prisms Cylinders**

Volume

Volume of a Cube Postulate





Volume Congruence Postulate

Volume Addition Postulate

Volume of a Rectangular Prism

Volume of a Prism

Cavalieri's Principle

Volume of a Cylinder

Homework:

5^{th} **Section: Volume of Pyramids Cones**

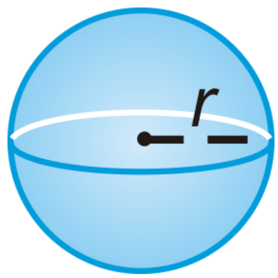
Volume of a Pyramid

Volume of a Cone

Homework:

6^{th} **Section: Surface Area and Volume of Spheres**

Sphere



Great Circle

Surface Area of a Sphere

Volume of a Sphere

Homework:

Extension: Similar Solids

Similar Solids

Surface Area Ratio

Volume Ratio

Homework:

CHAPTER 13 Probability

Chapter Outline

13.1 Geometric Probability

TEKS G(1)B, G(13)B

Learning Objectives

- find geometric probability
- make predictions involving geometric probabilty

Vocabulary

- complementary events

Warm-Up

The Bike Trick

"Look at this," Carey said showing Telly a picture of a stunt bike rider.

The picture was of a bicyclist riding down a piece of pipe. In fact, the pipe had been split into two separate pieces and the bicyclist was able to ride on both pieces of the pipe.

"I made one of those once," Ms. Kelley said looking at the article.

"How did you do that?" Telly asked.

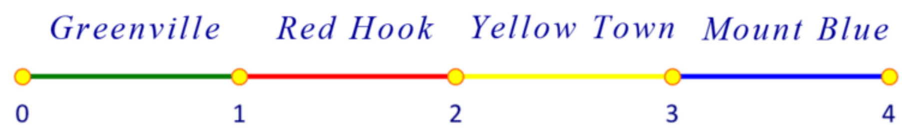
"Well, you start with a 10 foot piece of pipe. Then you metal saw it into two separate pieces. You hope that your work is accurate and that you end up with a piece that is about 7 feet or longer. That way you can really ride down before the quick turn," Ms. Kelly said walking away.

"What are the chances that the pipe is split like that?" Telly asked.

This is another place for probability. This is geometric probability though, so pay close attention and you will be able to figure out this problem.

The best way to begin thinking about geometric probability is to look at an example. Let's start there.

Example 1: A bus is traveling on Four-Color Road through the towns of Greenville, Red Hook, Yellow Town, and Mt. Blue. Each town takes up exactly 1 mile of distance on the Four-Color Road.



If a rider gets on the bus at some random point along the 4-mile route, what is the probability that she will board the bus in Greenville?

Solution: Let's think about how we can solve this problem. To solve it, imagine that not a single rider gets on the bus, but 100 riders. If each rider enters at some random point, you would expect to collect the riders to be evenly distributed over the 4 different towns.

TABLE 13.1:

Day	Greenville	Red Hook	Yellow town	Mt.blue
expected riders	25	25	25	25

Of course, in real life, the data might be slightly different, but overall you would expect 25 of 100, or $\frac{1}{4}$ of the riders, to get on in Greenville. In probability terms:

$$P(\text{Greenville}) = \frac{1}{4}$$

This is because there is one spot out of four possible points that a person could be picked up on the bus.

In general:

The probability of a randomly selected point to be located in a given "favorable" section of a distance is equal to the ratio of the length of the favorable section to the entire distance.

$$P(\text{point in section}) = \frac{\text{Length of "favorable" section}}{\text{Total distance}}$$

Let's look at another problem where we will use the same diagram of the four color towns.

Example 2: In the bus problem above, what is the probability that a rider will randomly board the bus in Red Hook or Yellow Town?

To solve this problem represent the rider as a point that can appear anywhere along the route.

$$\begin{aligned} P(\text{Red Hook or Yellow Town}) &= \frac{\text{length of “favorable” section}}{\text{total distance}} \\ &= \frac{1\ mile + 1\ mile}{4\ miles} \\ &= \frac{1}{2} \end{aligned}$$

You would expect the rider to board the bus in Red Hook or Yellow Town about $\frac{1}{2}$ of the time.

These two examples are related to geometric probability-space and possible outcomes. The space of the road is four roads. There are four possible outcomes within the four roads.

Think back to what you just learned above, you learned how to determine where one point can be randomly located within other given sections. Here we have possible outcomes and we have our favorable outcome identified. It is identified physically in space and not just in number form.

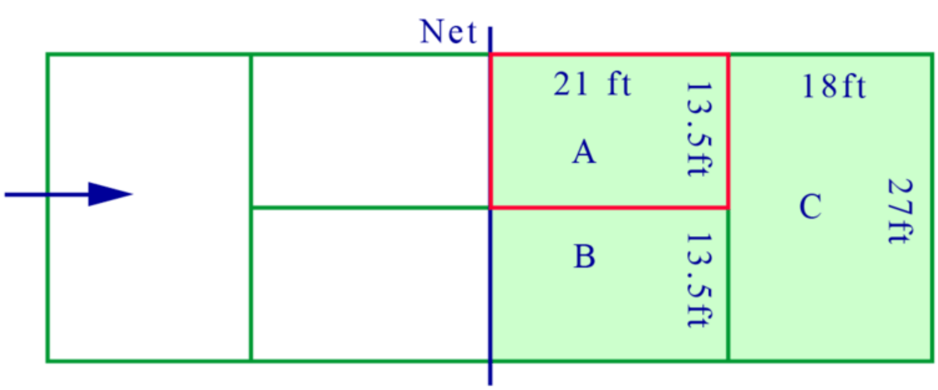
$$P(\text{point in section}) = \frac{\text{length of “favorable” section}}{\text{total distance}}$$

A similar rule applies for area.

The probability of a randomly selected point to be located in a section of an **area** is equal to the ratio of the area of the "favorable" section to the entire **area**.

$$P(\text{point in section}) = \frac{\textbf{area} \text{ of “favorable” section}}{\text{total } \textbf{area}}$$

Example 3: As a joke, a tennis player on the left side of the court who was attempting to place his serve into service box A hit a serve straight up in the air as high as he could. The ball landed on the green side of the court in some random location.



What is the probability that the serve landed in the service box A?

Solution: To find the probability that the ball will land in the box, just find the ratio of the area of box A to the area of the entire green side of the court.

$$\begin{aligned} P(\text{ball lands in box}) &= \frac{\text{area of favorable section}}{\text{total area}} \\ &= \frac{\text{area of box } A}{\text{area box } A + \text{box } B + \text{box } C} \end{aligned}$$

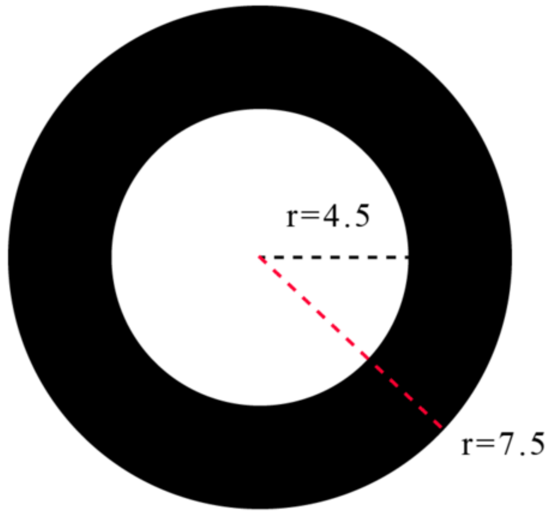
Now calculate the area of box A, box B, and box C. You can look at the diagram to determine this. Remember that the formula for the area of a rectangle is $A = lw$ and the units are measured in square units. In this case, it will be square feet.

$$\begin{aligned} \text{Box } A &= 21\ ft \cdot 13.5\ ft \\ &= 283.5\ sq\ ft \\ \text{Box } B &= 21\ ft \cdot 13.5\ ft \\ &= 283.5\ sq\ ft \\ \text{Box } C &= 18\ ft \cdot 27\ ft \\ &= 486\ sq\ ft \end{aligned}$$

So:

$$\begin{aligned} P(\text{ball lands in box}) &= \frac{\text{area of ``favorable'' section}}{\text{total area}} \\ &= \frac{\text{area } A}{\text{area } B + \text{area } B + \text{area } C} \\ &= \frac{283.5}{1053} \\ &= 26.9\% \end{aligned}$$

We just looked at geometric probability involving rectangular regions. To solve these problems, we used the formula for the area of a rectangle and then we used information tha we have already learned about ratios and percentages to determine our outcomes. We can also work with circular regions. This lesson will teach you how to figure out the geometric probability of circular regions. Let's look at an example.



Example 4: What is the probability for a random point being located in the white circle? black circle?

$$P(\text{white}) = \frac{\text{Area of white}}{\text{Total area}}$$

Solution: Use the formula for the area of a circle, $A = \pi r^2$ to find the area of the white circle and the total area of the figure.

$$\begin{aligned}
A(\text{white}) &= \pi r^2 \\
&= (3.14) \cdot (4.5) \cdot (4.5) \\
&= 63.6 \ sq \ in \\
A(\text{large}) &= \pi r^2 \\
&= (3.14) \cdot (7.5) \cdot (7.5) \\
&= 176.6 \ sq \ in
\end{aligned}$$

So:

$$\begin{aligned}
P(\text{white}) &= \frac{\text{area of white}}{\text{total area}} \\
&= \frac{63.6}{176.6} \\
&= 36\%
\end{aligned}$$

To find the probability of a point being randomly located in the black region, first find the area of the black region.

$$\begin{aligned} A(\text{black}) &= \text{total area} - \text{white area} \\ &= 176.6 - 63.6 \\ &= 113.0 \end{aligned}$$

So:

$$\begin{aligned} P(\text{black}) &= \frac{\text{area of black}}{\text{total area}} \\ &= \frac{113.0}{176.6} \\ &= 64\% \end{aligned}$$

Notice that an easy way to find the $P(\text{black})$ is to recognize that $P(\text{black})$ and $P(\text{white})$ are ***complementary*** events.

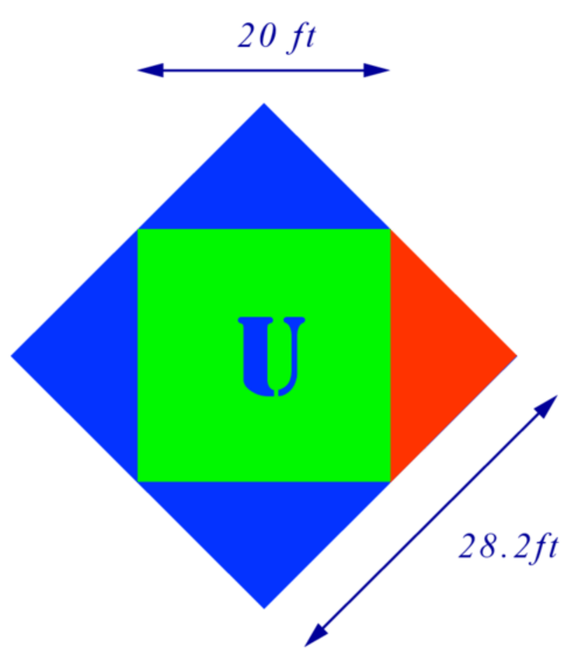
The point must be either in the black area or the white area, so the two probabilities must add up to 100 percent.

$$\begin{aligned} P(\text{black}) + P(\text{white}) &= 100\% \\ P(\text{black}) + 36\% &= 100\% \\ 64\% + 36\% &= 100\% \end{aligned}$$

This is a way that we can check our work. Notice that you will need to use the formula for the area of a circle to determine geometric probability related to area**.**

Now that you understand geometric probability, we can also make predictions involving geometric probability. This will help us to understand how to work with geometric probability. Let's look at an example.

Example 5: The Ultra Company displays this giant logo on a downtown billboard. The logo is lit up by thousands of small high definition LCD pixels. The pixels are often damaged or burn out. Predict where the next 60 damaged pixels will be located.



Solution: Before making predictions, we must figure out the probability of each section. To find the probability of where a damaged pixel will be located, first find the area of each section. As the diagram shows, each side of green square measures 20 feet, while each side of the blue diamond measures 28.2 feet

$$\begin{aligned}A(\text{green}) &= 20 \cdot 20\\ &= 400\ sq\ ft\\ A(\text{diamond}) &= 28.2 \cdot 28.2\\ &= 795.2\ sq\ ft\end{aligned}$$

Now the total area of all 4 equal-sized triangles is equal to the area of the entire figure minus the central green square.

$$\begin{aligned}A(\text{diamond}) - A(\text{green}) &= A(4\text{ triangles})\\ 795.2 - 400 &= 395.2\ sq\ ft\end{aligned}$$

Since there are 4 equal-sized triangles, each triangle has the following areas.

$$\begin{aligned}\text{Area of all triangles} \div 4 &= \text{area of single triangle}\\ 395.4 \div 4 &= 98.9\ sq\ ft\end{aligned}$$

So:

$$\begin{aligned}A(\text{red}) &= 98.9\ sq\ ft\\ A(\text{blue}) &= 3 \cdot 98.9\\ &= 296.6\ sq\ ft\end{aligned}$$

To find the probability of each area:

$$\begin{aligned}P(\text{green}) &= \frac{\text{area of green}}{\text{area of diamond}}\\ &= \frac{400}{795.2}\\ &= 50.2\%\end{aligned}$$

$$\begin{aligned}P(\text{blue}) &= \frac{\text{area of blue}}{\text{area of diamond}}\\ &= \frac{296.6}{795.2}\\ &= 37.3\%\end{aligned}$$

$$\begin{aligned}P(\text{red}) &= \frac{\text{area of red}}{\text{area of diamond}}\\ &= \frac{98.9}{795.2}\\ &= 12.5\%\end{aligned}$$

Now that we have all of the probabilities figured out, we can make predictions. Remember that before you make a prediction, you will need to figure out the probability first. The probability will help you to figure out each prediction.

Problem: Predict where the next 64 damaged pixels will be located in the figure above.

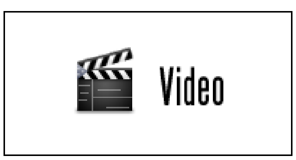
Step 1: Find the probabilities. (You already found them above.)

$$\begin{aligned}P(\text{green}) &= 50.2\%\\ P(\text{blue}) &= 37.3\%\\ P(\text{green}) &= 12.5\%\end{aligned}$$

Step 2: Multiply each probability by the number of events. In this case, the total number of events is 64, the number of damaged pixels. Round off where necessary

$$\begin{aligned}\text{green} &= 0.502 \cdot 64\\ &= 32\\ \text{blue} &= 0.373 \cdot 64\\ &= 24\\ \text{red} &= 0.125 \cdot 64\\ &= 8\end{aligned}$$

Watch this!



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Practice Problems

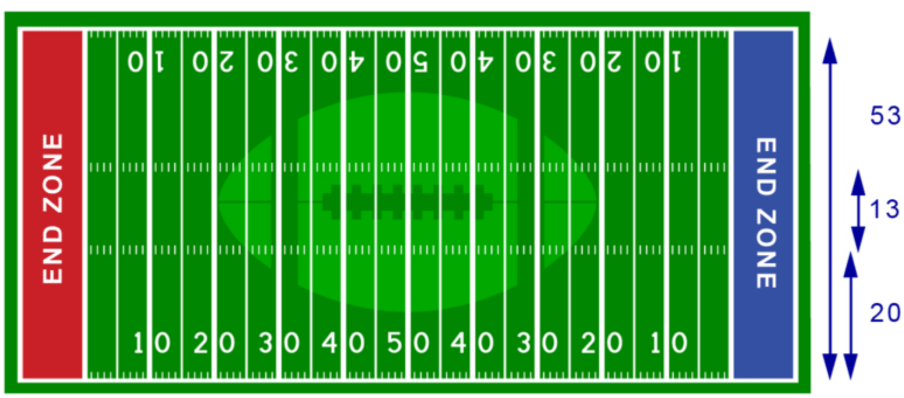
- Questions 1-5 are similar to Examples 1 and 2.
- Questions 6-11 are similar to Example 3.
- Questions 12-19 are similar to Example 4.
- Questions 20-24 are similar to Example 5.

Directions: Draw diagrams to solve the problems.

1. Geoff rode his bike along an 8 mile path and lost his cell phone at some random location somewhere along the way. What is the probability that Geoff's phone dropped during the first mile of the path?
2. In the phone problem above, what is the probability that Geoff dropped his phone during the first mile or the last 2 miles?
3. In the phone problem above, Geoff searched from mile 4.5 to mile 7. What is the phone probability that he found the phone?
4. In the phone problem above, a cell phone tower is located at mile 4 in the exact center of the path. The tower has a range of 2.75 miles. If Geoff uses a second cell phone to call his lost phone, what is the probability that the lost phone will ring?
5. In the phone problem above, what range would the tower need to have to be sure that Geoff's lost phone would ring?

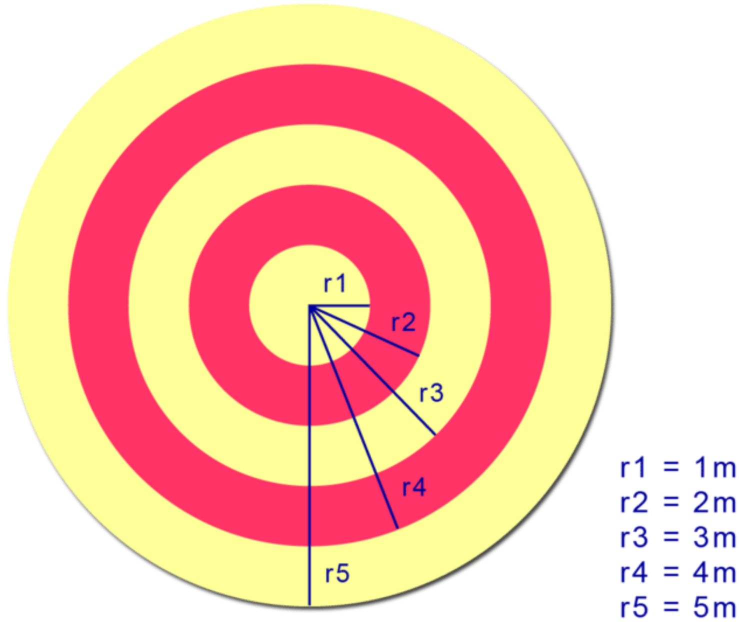
Directions: Use this information on the football field to answer the following questions.

A football field is 120 yards long - 100 yards (green) plus two end zones (shown in red and blue) and 53 yards wide. The two hash mark lines that run across the center of the field are 13 yards apart and 20 yards from the sidelines. A pigeon flies over the stadium and lands at some random location on the football field.



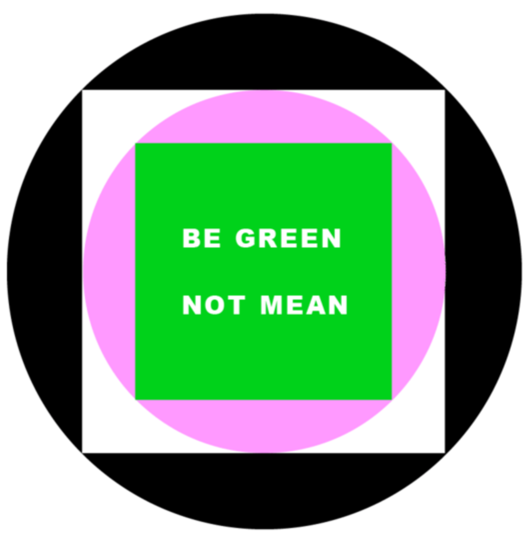
6. What is the probability that the pigeon will land in one of the end zones?
7. What is the probability that the pigeon will land between the goal line and the 20 yard line on either side of the field?
8. What is the probability that the pigeon will land within 5 yards of the 50-yard line?
9. What is the probability that the pigeon will land between the two hashmarks somewhere on the green part of the field?
10. What is the probability that the pigeon will land somewhere on the green part of the field outside of the hashmarks (not between both hashmarks)?
11. What is the probability that the pigeon will land between the two hashmarks and between the two 40-yard lines?

Directions: Look at the diagram and then answer each question as it is related to geometric probabilities.



12. The radius of circle 1 (the inner-most yellow circle) is 1 meter. Each radius thereafter increases by 1 m, as shown. What is the probability of a randomly thrown dart landing on circle 1?
13. What is the probability that the dart will land in circle 2?
14. What is the probability that the dart will land in circle 3?
15. What is the probability that the dart will land in circle 4?
16. What is the probability that the dart will land in circle 5?
17. What pattern describes how the probability changes for each circle?
18. What is the probability that the dart will land in a yellow area?
19. What is the probability that the dart will land in an red area?

Directions: Now use what you have learned to make predictions.



Measurements of the **be *green*, not mean** sign are:

$$\text{black circle radius} = 13.8\ ft$$
$$\text{pink circle radius} = 10.0\ ft$$
$$\text{white square side} = 20.0\ ft$$
$$\text{green square side} = 14.1\ ft$$

20. Of the next 240 pixels in the *be green, not mean* sign to be damaged, predict how many will be green.
21. Of the next 500 pixels in the *be green, not mean* sign to be damaged, predict how many will be white.
22. Of the next 60 pixels in the *be green, not mean* sign to be damaged, predict how many will be black.
23. Of the next 200 pixels in the *be green, not mean* sign to be damaged, predict how many will be black or white.
24. Of the next 81 pixels in the *be green, not mean* sign to be damaged, predict how many will not be green.

Review and Reflect

25. Describe in your own words what *geometric probability* means.

26. Write you own geometric probability problems involving area of a regular polygon.

Warm-Up Answers

The Bike Trick

Here is the problem from the introduction. Use the information to figure out the probability that one piece of the pipe is greater than or equal to 7 feet in length.

"Look at this," Carey said showing Telly a picture of a stunt bike rider.

The picture was of a bicyclist riding down a piece of pipe. In fact, the pipe had been split into two separate pieces and the bicyclist was able to ride on both pieces of the pipe.

"I made one of those once," Ms. Kelley said looking at the article.

"How did you do that?" Telly asked.

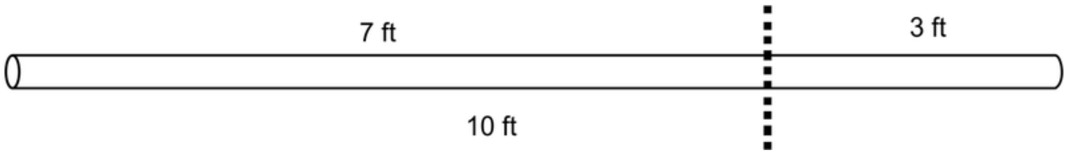
"Well, you start with a 10 foot piece of pipe. Then you metal saw it into two separate pieces. You hope that your work is accurate and that you end up with a piece that is about 7 feet or longer.That way you can really ride down before the quick turn," Ms. Kelly said walking away.

"What are the chances that the pipe is split like that?" Telly asked.

Now work through the solution.

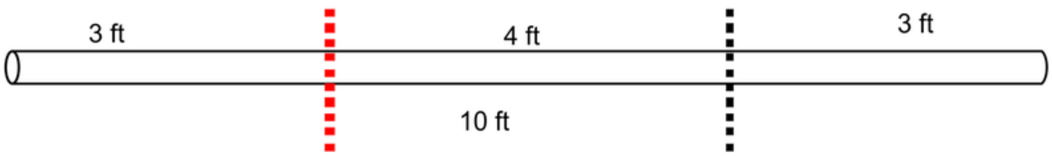
Solution to Real - Life Example

Step 1: The minimum size for the larger piece would be 7 feet. That would make the smaller piece 3 feet in length. Mark this off on your diagram.

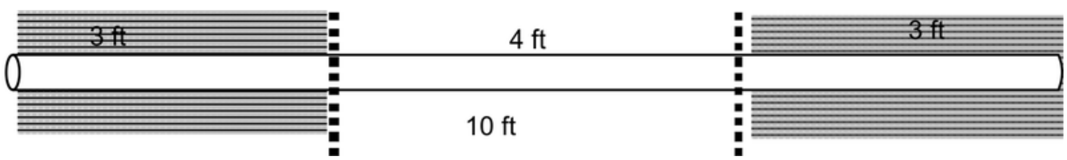


Step 2: Now notice that the pipe can also break on the other side. This is shown in red.

With 3 feet on either end, that leaves a 4 foot length in the center of the pipe.



Step 3: Now think of the pipe as separate areas. The shaded areas show regions where one of the pieces will be greater than 7 feet in length. These are your favorable sections.



So the probability of breaking into a piece that is 7 feet in length or greater is:

$$\begin{aligned}
P(>7\ ft) &= \frac{\text{length of favorable sections}}{\text{total distance}}\\
&= \frac{3\ ft + 3\ ft}{10\ ft}\\
&= \frac{6}{10}\\
&= \frac{3}{5}
\end{aligned}$$

The probability is 60% that one piece of the pipe will be 7 feet long or greater.

13.2 Counting with Permutations and Combinations

Here you will review counting using decision charts, permutations and combinations.

TEKS G(1)A, G(13)A

Learning Objectives

- Develop strategies to use permutations and combinations to solve problem

Vocabulary

- Factorial
- Permutation
- Combination
- Decision chart

Warm-Up

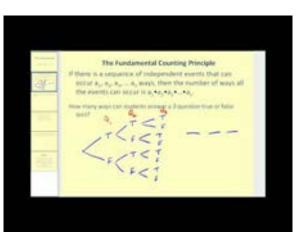
1. There are 20 hockey players on a pro NHL team, two of which are goalies. In how many different ways can 5 skaters and 1 goalie be on the ice at the same time?

2. In how many different ways could you score a 70% on a 10 question test where each question is weighted equally and is either right or wrong?

3. How many different 4 digit ATM passwords are there? Assume you can repeat digits.

Know what? Sometimes it makes sense to count the number of ways for an event to occur by looking at each possible outcome. However, when there are a large number of outcomes this method quickly becomes inefficient. If someone asked you how many possible regular license plates there are for the state of California, it would not be feasible to count each and every one. Instead, you would need to use the fact that on the typical California license plate there are four numbers and three letters. Using this information, about how many license plates could there be?

Watch This



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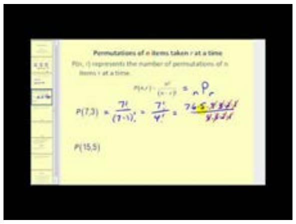
http://www.youtube.com/watch?v=qJ7AYDmHVRE James Sousa: The Counting Principle



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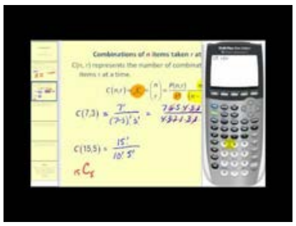
http://www.youtube.com/watch?v=JyRKTesp6fQ James Sousa: Permutations



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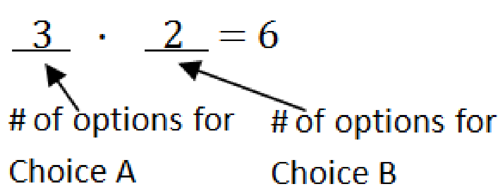
URL: https://www.ck12.org/flx/render/embeddedobject/62279

http://www.youtube.com/watch?v=SGn1913lOYM James Sousa: Combinations

Consider choice A with three options (A_1, A_2, A_3) and choice B with two options (B_1, B_2). If you had to choose an option from A and then an option from B, the overall total number of options would be $3 \cdot 2 = 6$. The options are $A_1B_1, A_1B_2, A_2B_1, A_2B_2, A_3B_1, A_3B_2$.

You can see where the six comes from by making a decision chart and using the Fundamental Counting Principle. First, determine how many decisions you are making. Here, there are only two decisions to make (1: choose an option from A; 2: choose an option from B), so you will have two "slots" in your decision chart. Next, think about how many possibilities there are for the first choice (in this case there are 3) and how many possibilities there are for the second choice (in this case there are 2). The Fundamental Counting Principle says that you can multiply those numbers together to get the total number of outcomes.



Another type of counting question is when you have a given number of objects, you want to choose some (or all) of them, and you want to know how many ways there are to do this. For example, a teacher has a classroom of 30 students, she wants 5 of them to do a presentation, and she wants to know how many ways this could happen. These types of questions have to do with **combinations** and **permutations.** The difference between combinations and permutations has to do with whether or not the order that you are choosing the objects matters.

- A teacher choosing a group to make a presentation would be a **combination** problem, because **order does not matter.**
- A teacher choosing 1^{st}, 2^{nd}, and 3^{rd} place winners in a science fair would be a **permutation** problem, because the **order matters** (a student getting 1^{st} place vs. 2^{nd} place are different outcomes).

Recall that the **factorial** symbol, **!**, means to multiply every whole number up to and including that whole number together.

Example 1: Simplify $5!$.

Solution: $.5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

The factorial symbol is used in the formulas for permutations and combinations

Combination Formula: The number of ways to choose k objects from a group of n objects is –

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Permutation Formula: The number of ways to choose **and arrange** k objects from a group of n objects is –

$${}_nP_k = k!\binom{n}{k} = k! \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!}$$

Notice that in both permutation and combination problems you are not allowed to repeat your choices. Any time you are allowed to repeat and order does not matter, you can use a decision chart. (Problems with repetition where order does not matter are more complex and are not discussed in this text.)

Whenever you are doing a counting problem, the first thing you should decide is if the problem is a decision chart problem, a permutation problem, or a combination problem. You will find that permutation problems can also be solved with decision charts. The opposite is not true. There are many decision chart problems (ones where you are allowed to repeat choices) that could not be solved with the permutation formula.

Note: Here you have only begun to explore counting problems. For more information about combinations, permutations, and other types of counting problems, consult a Probability text.

Example 2: You are going on a road trip with 4 friends in a car that fits 5 people. How many different ways can everyone sit if you have to drive the whole way?

Solution: A decision chart is a great way of thinking about this problem. You have to sit in the driver's seat. There are four options for the first passenger seat. Once that person is seated there are three options for the next passenger seat. This goes on until there is one person left with one seat.

$1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Example 3: How many different ways can the gold, silver and bronze medals be awarded in an Olympic event with 12 athletes competing?

Solution: Since the order does matter with the three medals, this is a permutation problem. You will start with 12 athletes and then choose and arrange 3 different winners.

$${}_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot \ldots}{9 \cdot \ldots} = 12 \cdot 11 \cdot 10 = 1320$$

Note that you could also use a decision chart to decide how many possibilities are there for gold (12) how many possibilities are there for silver (11 since one already has gold) and how many possibilities are there for bronze (10). You can use a decision chart for any permutation problem.

$12 \cdot 11 \cdot 10 = 1320$

Example 4: You are deciding which awards you are going to display in your room. You have 8 awards, but you only have room to display 4 awards. Right now you are not worrying about how to arrange the awards so the order does not matter. In how many ways could you choose the 4 awards to display?

Solution: Since order does not matter, this is a combination problem. You start with 8 awards and then choose 4.

$${}_8C_4 = \binom{8}{4} = \frac{8!}{4!(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70$$

Note that if you try to use a decision chart with this question, you will need to do an extra step of reasoning. There are 8 options I could choose first, then 7 left, then 6 and lastly 5.

$8 \cdot 7 \cdot 6 \cdot 5 = 1680$

This number is so big because it takes into account order, which you don't care about. It is the same result you would get if you used the permutation formula instead of the combination formula. To get the right answer, you need to divide this number by the number of ways 4 objects can be arranged, which is $4! = 24$. This has to do with the connection between the combination formula and the permutation formula.

Know what Revisited

A license plate that has 3 letters and 4 numbers can be represented by a decision chart with seven spaces. You can use a decision chart because order definitely does matter with license plates. The first spot is a number, the next three spots are letters and the last three spots are numbers. Note that when choosing a license plate, repetition is allowed.

$$10 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^4 = 175,760,000$$

This number is only approximate because in reality there are certain letter and number combinations that are not allowed, some license plates have extra symbols, and some commercial and government license plates have more numbers, fewer letters or blank spaces.

Practice Problems

- Questions 1-3 are similar to Examples 1.
- Questions 4-5 are similar to Example 2 and 3.
- Questions 6-13 are similar to Example 4.

Simplify each of the following expressions so that they do not have a factorial symbol.

1. $\frac{7!}{3!}$
2. $\frac{110!}{105!5!}$
3. $\frac{52!}{49!}$

4. In how many ways can you choose 3 objects from a set of 9 objects?

5. In how many ways can you choose and arrange 4 objects from a set of 15 objects?

First, state whether each problem is a **permutation/decision chart** problem or a **combination** problem. Then, solve.

6. Suppose you need to choose a new combination for your combination lock. You have to choose 3 numbers, each different and between 0 and 40. How many combinations are there?

7. You just won a contest where you can choose 2 friends to go with you to a concert. You have five friends who are available and want to go. In how many ways can you choose the friends?

8. You want to construct a 3 digit number from the digits 4, 6, 8, 9. How many possible numbers are there?

9. There are 12 workshops at a conference and Sam has to choose 3 to attend. In how many ways can he choose the 3 to attend?

10. 9 girls and 5 boys are finalists in a contest. In how many ways can 1^{st}, 2^{nd}, and 3^{rd} place winners be chosen?

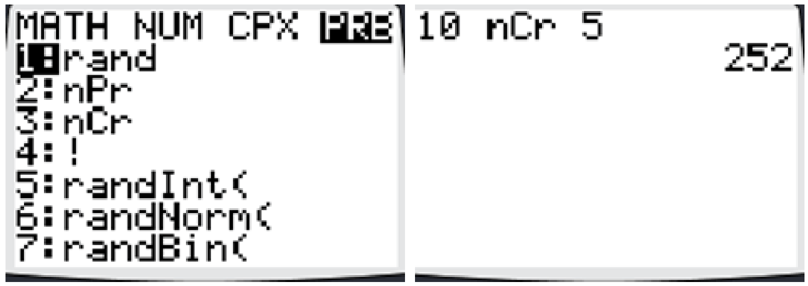
11. For the special at a restaurant you can choose 3 different items from the 10 item menu. How many different combinations of meals could you get?

12. You visit 12 colleges and want to apply to 4 of them. In how many ways could you choose the four to apply to?

13. For the 12 colleges you visited, you want to rank your top five. In how many ways could you rank your top 5?

14. Your graphing calculator has the combination and permutation formulas built in. Push the MATH button and scroll to the right to the PRB list. You should see ${}_nP_r$ and ${}_nC_r$ as options. In order to use these: 1) On your home

screen type the value for n; 2) Select ${}_nP_r$ or ${}_nC_r$; 3) Type the value for k (r on the calculator). Use your calculator to verify that ${}_{10}C_5 = 252$.



Review and Reflect

15. *Explain why the following problem is not strictly a permutation or combination problem*: The local ice cream shop has 12 flavors. You decide to buy 2 scoops in a dish. In how many ways could you do this if you are allowed to get two of the same scoop?

16. Explain when it is necessary to permutations and combinations.

Warm-Up Answers

1. The question asks for how many on the ice, implying that order does not matter. This is combination problem with two combinations. You need to choose 1 goalie out of a possible of 2 and choose 5 skaters out of a possible 18.

$\binom{2}{1}\binom{18}{5} = 2 \cdot \frac{18!}{5! \cdot 13!} = 17136$

2. The order of the questions you got right does not matter, so this is a combination problem.

$\binom{10}{7} = \frac{10!}{7!3!} = 120$

3. Order does matter. There are 10 digits and repetition is allowed. You can use a decision chart for each of the four options.

$10 \cdot 10 \cdot 10 \cdot 10 = 10,000$

13.3 Probability of Independent Events

Here you'll learn to find the probability of two independent events both occurring.

TEKS G(1)A, G(13)C

Learning Objectives

- identify independent events
- compute probability with and without replacement

Vocabulary

- independent event

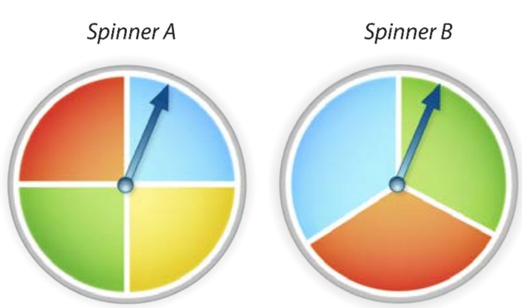
Warm-Up

Have you ever wondered if two things can happen at once?

Jana has two decks of cards. Each deck has ten cards in it. There are three face cards in the first deck and four in the second. What are the chances that Jana will draw a face card from both decks?

An **independent even** is an event that does not impact the result of a second event. For example, there are two different spinners A and B. The result of spinning spinner A does not affect the result of spinning spinner B.

But now we ask a new question. What is the probability of two completely independent events *both* occurring? For example, what is the probability of spinner A landing on red and spinner B landing on blue?



We could create a tree diagram to show all of the possible options and figure out the probability, but that is very complicated. There is a simpler way.

Notice that this probability equals the product of the two independent probabilities.

$$\begin{aligned} P(\text{red-blue}) &= P(\text{red}) \cdot P(\text{blue}) \\ &= \frac{1}{4} \cdot \frac{1}{3} \\ &= \frac{1}{12} \end{aligned}$$

Where did these fractions come from?

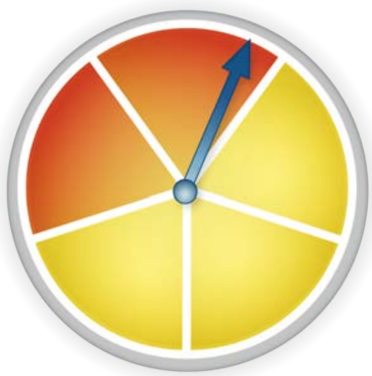
They came from the probability of the sample space of each spinner. The first spinner has four possible options, so the probability is $\frac{1}{4}$. The second spinner has three possible options, so the probability is $\frac{1}{3}$. The Probability Rule takes care of the rest.

In fact, this method works for any independent events as summarized in this rule.

Probability Rule

The probability that two independent events A and B, will occur is: $P(A \text{ and } B) = P(A) \cdot P(B)$

What is the probability that if you spin the spinner two times, it will land on yellow on the first spin and red on the second spin?



To find the solution, use the rule.

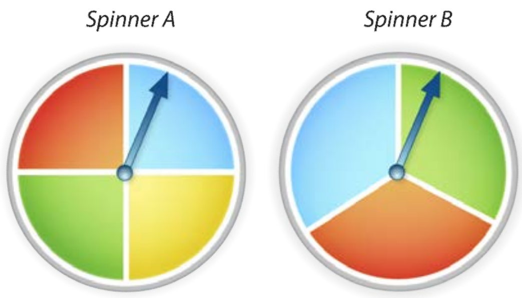
$$\begin{aligned} P(\text{yellow and red}) &= P(\text{yellow}) \cdot P(\text{red}) \\ P(\text{yellow}) &= \frac{3}{5} \\ P(\text{red}) &= \frac{2}{5} \end{aligned}$$

So:

$$\begin{aligned} P(\text{yellow and red}) &= \frac{3}{5} \cdot \frac{2}{5} \\ &= \frac{6}{25} \end{aligned}$$

The probability of both of these events occurring is $\frac{6}{25}$.

Now it's time for you to try a few on your own.



Example 1:

What is the probability of spinner A landing on red and spinner B landing on red?

Solution: $\frac{1}{12}$

Example 2:

What is the probability of spinner A landing on blue or yellow and spinner B landing on blue?

Solution: $\frac{1}{6}$

Example 3:

What is the probability of spinner A landing on yellow and spinner B landing on red or green?

Solution: $\frac{1}{6}$

Example 4:

The probability of rain tomorrow is 40 percent. The probability that Jeff's car will break down tomorrow is 3 percent. What is the probability that Jeff's car will break down in the rain tomorrow?

Solution: To find the solution, use the rule.

$$P(\text{rain and break}) = P(\text{rain}) \cdot P(\text{break})$$
$$P(\text{rain}) = 40\% = \frac{40}{100} = \frac{2}{5}$$
$$P(\text{break}) = 3\% = \frac{3}{100}$$

So:

$$P(\text{rain and break}) = \frac{2}{5} \cdot \frac{3}{100}$$
$$= \frac{3}{250}$$

Probability can also be calculated with and without replacement. The following video will show this type of probability.

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Example 5: A bag contains 12 coins: 2 quarters, 3 dimes, and 7 nickels. What is the probability without replacement that 3 coins chosen are all nickels?

Solution:

$P(3nickels) = \frac{7}{12} \cdot \frac{6}{11} \cdot \frac{5}{10}$
$P(3nickels) = \frac{7}{44}$

Example 6: Find the probability of flipping a coin 4 times and getting all 4 heads in a row.

Solution:

$P(4heads) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$
$P(4heads) = \frac{1}{16}$

Video Review

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This is a Khan Academy video on the probability of independent events.

Practice Problems

- Problems 1-4 are similar to Example 1-3
- Problems 5-7 are similar to Example 5
- Problems 8-10 are similar to the Warm-Up
- Problems 11-15 are similar to Example 6

1. Mia spins the spinner two times. What is the probability that the arrow will land on 2 both times?

2. Mia spins the spinner two times. What is the probability that the arrow will land on 2 on the first spin and 3 on the second spin?

3. Mia spins the spinner two times. What is the probability that the arrow will land on an even number on the first spin and an odd number on the second spin?

4. Mia spins the spinner two times. What is the probability that the arrow will land on an odd number on the first spin and a number less than 4 on the second spin?

5. A laundry bag has 8 black socks and 2 white socks. If you pull out a sock, then put it back in the bag and pull out a second sock, what is the probability that both socks will be black? Write your answer as a decimal.

6. A laundry bag has 8 black socks and 2 white socks. If you pull out a sock, then put it back in the bag and pull out a second sock, what is the probability that both socks will be white? Write your answer as a decimal.

7. A laundry bag has 8 black socks and 2 white socks. If you pull out a sock, then put it back in the bag and pull out a second sock, what is the probability that the first sock will be black and the second sock will be white? Write your answer as a decimal.

8. Dirk has two 52-card decks. Each deck has 4 Aces, 4 Kings, 4 Queens, and so on. What is the probability that Dirk will pick an Ace out of each deck?

9. Dirk has two 52-card decks. Each deck has 4 Aces, 4 Kings, 4 Queens, and so on. What is the probability that Dirk will pick a face card (Jack, Queen, King) out of each deck?

10. Dirk has two 52-card decks. Each deck has 4 Aces, 4 Kings, 4 Queens, and so on. What is the probability that Dirk will pick a card lower than a Jack out of each deck?

11. Karina flips a coin 3 times. What is the probability that she will flip heads 3 times in a row?

12. Karina flips a coin 4 times. What is the probability that she will flip heads 4 times in a row?

13. Karina flips a coin 4 times. What is the probability that she will NOT flip heads 4 times in a row?

14. Karina flips a coin 5 times. What is the probability that she will flip heads once?

15. Karina flips a coin 5 times. What is the probability that she will flip tails once?

Review and Reflect

16. What makes an event independent?

17. Describe the sample space for flipping a coin 4 times.

Warm-Up Answers

Here is the original problem once again.

Jana has two decks of cards. Each deck has ten cards in it. There are three face cards in the first deck and four in the second. What are the chances that Jana will draw a face card from both decks?

To figure this out, let's write the probability of picking a face card from the first deck.

$\frac{3}{10}$

Now let's write the probability of picking a face card from the second deck.

$\frac{4}{10} = \frac{2}{5}$

Now we can multiply and simplify.

$\frac{3}{10} \times \frac{2}{5}$

$\frac{3}{25}$

13.4 Understanding Conditional Probability

Here you'll recognize and apply the definition of conditional probability to find probabilities in finite sample spaces.

TEKS G(1)A, G(13)C, G(13)D

Learning Objectives

- identify whether to events are independent or dependent
- compute probability with and without replacement

Vocabulary

- conditional probability
- independent event
- dependent event

Warm-Up

Have you ever worked in a bike shop? Take a look at this dilemma.

On Thursday, Carey was in charge of answering the phones and booking appointments for bike repairs. The bike shop repairs bikes on Monday, Tuesday and Wednesday mornings and on Thursday and Friday afternoons. All appointments are booked randomly. The person making the appointment can choose or the person answering the phone can choose.

Carey booked two appointments right away.

What are the chances that both of the these appointments were booked on a Monday, Tuesday or Wednesday morning?

Know what?

Sometimes the outcome that you get when figuring out a probability is what we call "conditional." This means that we will only find an outcome if the condition is designed to cause a specific result.

Take a look at this situation.

Consider a jar with 4 black marbles and 6 white marbles. If you pull out 2 marbles from the jar randomly, one at a time, without replacing the first marble, what is the probability that both marbles will be white?

Start by approaching the problem the same as you would with independent events.

The probability of the first marble being white is:

$$P(\text{white 1st marble}) = \frac{6}{10} = \frac{3}{5}$$

What about the second marble?

Having removed the first marble from the bag, now instead of 6 white marbles out of 10 total marbles, there are only 5 white marbles left out of 9 total marbles:

$$P(\text{white 2nd marble}) = \frac{5}{9}$$

This gives a probability of both events occurring as:

$$\begin{aligned} P(\text{white then white}) &= P(\text{white 1st marble}) \cdot P(\text{white 2nd marble}) \\ &= \frac{3}{5} \cdot \frac{5}{9} \\ &= \frac{1}{3} \end{aligned}$$

The same general method works for calculating any two (or more) dependent events.

Now let's look at conditional probability and outcomes.

Conditional probability involves situations in which you determine the probability of an event based on another event having occurred.

Example 1: Suppose you roll two number cubes on a table. The first cube lands face up on 4. The second cube falls off of the table so you can't see how it landed. Given what you know so far, what is the probability that the sum of the number cubes will be 9?

To solve this problem, consider the entire sample space for rolling two number cubes.

66	56	46	36	26	16
65	55	45	35	25	15
64	54	44	34	24	14
63	53	43	33	23	13
62	52	42	32	22	12
61	51	41	31	21	11

You already know that the first number cube landed on 4, so now you need to consider only those outcomes marked in red. Only 1 of those 6 results in a 9, so:

$$P(9|4) = \frac{favorable\ outcomes}{total\ outcomes} = \frac{1}{6}$$

Notice that we write the conditional probability as $P(9|4)$. You can read this as:

$P(9|4) \Longleftarrow$ the probability of 9, given 4

Here are some other ways to read this notation.

$P(B|A) \Longleftarrow$ the probability of B, given A

$P(7|3) \Longleftarrow$ the probability of 7, given 3

$P(\text{heads}|\text{tails}) \Longleftarrow$ the probability of heads, given tails

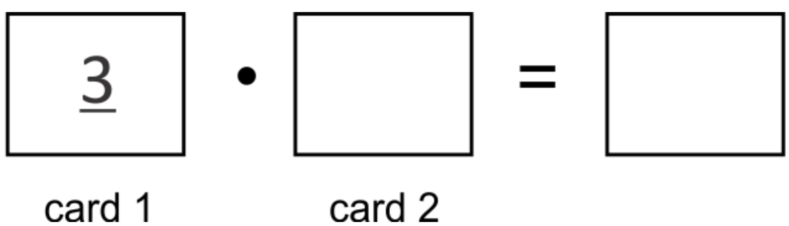
$P(\text{red}|\text{blue}) \Longleftarrow$ the probability of red, given blue

The probability is determined because certain factors are in place.

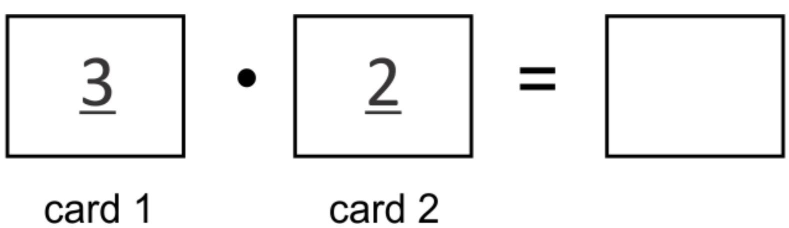
We can use conditional probability to determine probabilities, but also to make predictions.

Example 2:A stack of 12 cards has the Ace, King, and Queen of all 4 suits, spades, hearts, diamonds, and clubs. What is the probability that if you draw 2 cards randomly, they will both be hearts? Make a prediction.

Step 1: Draw the first card. The probability of it being a heart is 3 of 12.



Step 2: Now draw the second card. Since the first card was a heart, there are only 11 cards left and only 2 of them are hearts.



Step 3: Calculate the final probability.

$P(heart) = \frac{3}{12}$
$P(anotherheart) = \frac{2}{11}$
$P(2hearts) = \frac{3}{12} \cdot \frac{2}{11}$
$P(2hearts) = \frac{1}{22}$

So $P(\text{heart and heart}) = \frac{1}{22}$. You would predict that both cards would be hearts $\frac{1}{22}$ of the time.

Example 3: A jar contains four blue marbles and eight red marbles.

What is the probability of taking out a blue, then a red marble without replacing the first marble?

Solution: $P((Red|Blue)) = \frac{4}{12} \cdot \frac{8}{11}$
$P(Red|Blue) = \frac{8}{33}$

Example 4: Jack's Catering Service is accepting weekday appointments for Monday through Thursday, and weekend appointments for Friday through Sunday. If appointment dates are made randomly, what is the probability that 2 weekdays will be the first 2 days to be booked?

Solution: The probability that the first day will be a weekday is:

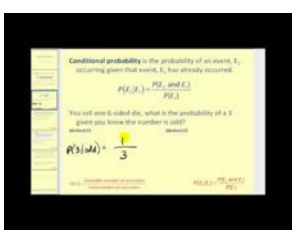
$$P(\text{weekday 1st}) = \frac{4}{7}$$

The probability that the second booked day will also be a weekday is:

$$P(\text{weekday 2nd}) = \frac{3}{6} = \frac{1}{2}$$

$$\begin{aligned} P(\text{weekday and weekday}) &= P(\text{weekday 1st}) \cdot P(\text{weekday 2nd}) \\ &= \frac{4}{7} \cdot \frac{1}{2} \\ &= \frac{2}{7} \end{aligned}$$

Watch this!



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Conditional Probability

Practice Problems

- Questions 1-3 are similar to Example 2
- Questions 4-11 are similar to Example 3
- Questions 12-15 are similar to Example 1

Directions: Solve the problems.

1. A stack of 12 cards has 4 Aces, 4 Kings, and 4 Queens. What is the probability of picking 2 Aces from the stack at random?
2. What is the probability of picking an Ace then a King from the stack above?
3. What is the probability of picking 3 Queens from the stack above?
4. Stoyko's shirt drawer has 4 colored t-shirts and 4 white t-shirts. If Stoyko picks out 2 shirts at random, what is the probability that they will both be colored?
5. If Stoyko picks out 2 shirts at random from the drawer above, what is the probability that the first one will be colored and the second one will be white?
6. On a game show, there are 16 questions: 8 easy, 5 medium-hard, and 3 hard. If contestants are given questions randomly, what is the probability that the first two contestants will get easy questions?
7. On the game show above, what is the probability that the first contestant will get an easy question and the second contestant will get a hard question?
8. On the game show above, what is the probability that both of the first two contestants will get hard questions?
9. For a single toss of a number cube, what is the probability that the cube will land on a number that is both odd and greater than 2?
10. For a single toss of a number cube, what is the probability that the cube will land on a number that is greater than 2 and less than 6?
11. For a single toss of a number cube, what is the probability that the cube will land on a number that is greater than 1 and less than 6?
12. What is the probability that a sum of a pair of number cubes will be 11 if the first cube lands on 5?
13. What is the probability that a sum of a pair of number cubes will be odd if the first cube lands on 2?
14. What is the probability that a sum of a pair of number cubes will be even and greater than 6 if the first cube lands on 4?
15. If you toss a number cubes, predict how likely is it to roll a number less than 6?

Review and Reflect

16. What is the difference between and independent and a dependent event?

17. How does with replacement and without replacement effect probability?

Warm-Up Answers

First we must determine the probability of the first appointment booked being on a Monday, Tuesday or Wednesday. There are five possible days for appointments, but three favorable outcomes.

Probability of first appointment being Mon, Tues or Weds $= \frac{3}{5}$

Probability of second appointment being Mon, Tues or Weds $= \frac{2}{4}$ or $\frac{1}{2}$

Now we can multiply them for the conditional probability.

$\frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}$ or 30%

There is a 30% chance that the first two appointments booked would be on a Monday, Tuesday or Wednesday morning.

13.5 References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA