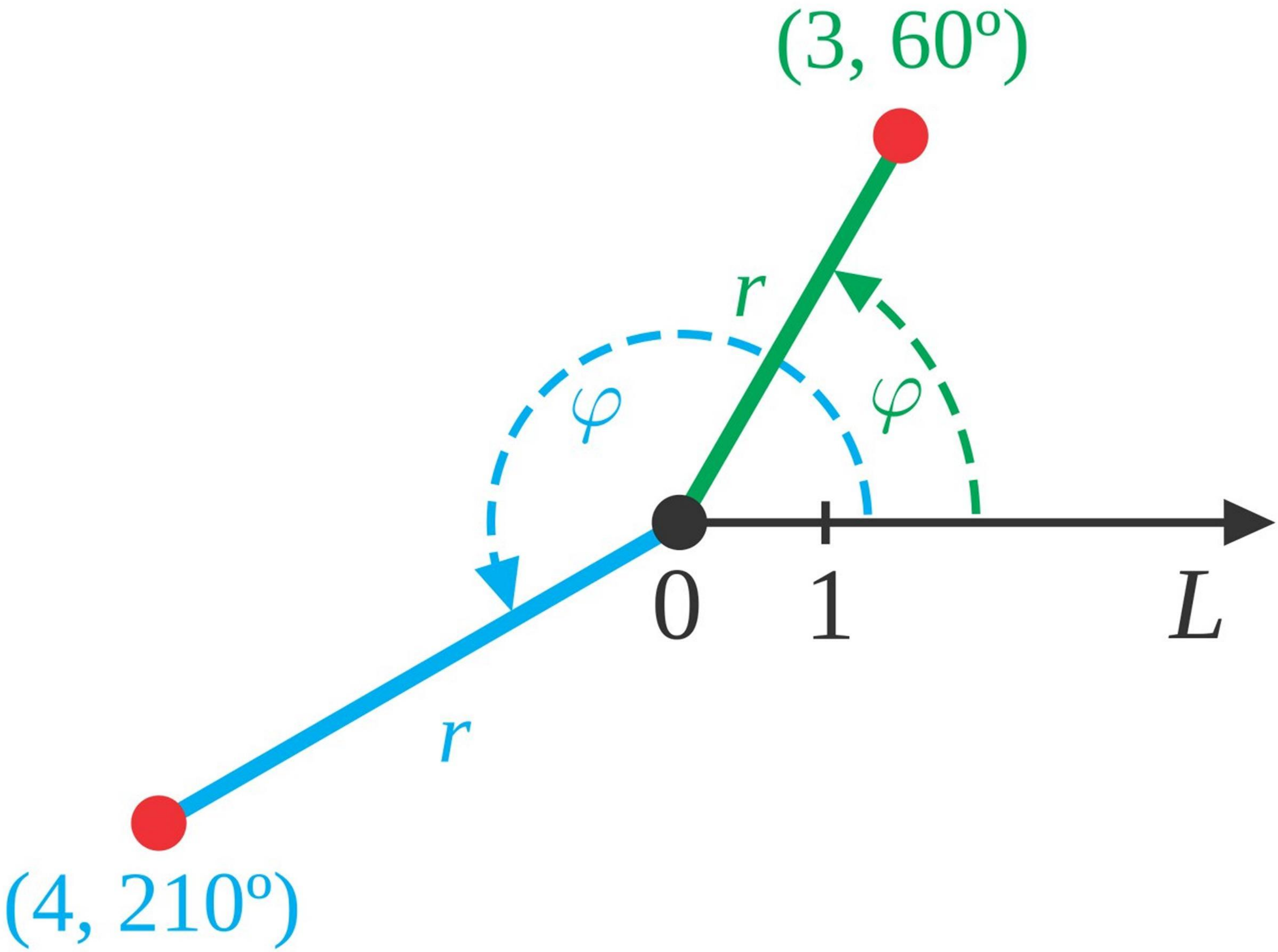
SFDRCISD PreCalculus



SFDRCISD PreCalculus

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CHAPTER 1 Optional Topics (Algebra Review)

Chapter Outline

1.1 Algebraic Expressions Basics

TEKS

1. P.1.F

Lesson Objectives

In this section you will review:

1. Evaluating algebraic expressions.
2. Properties of real numbers.
3. Simplifying algebraic expressions.

Introduction

When an expression looks complex or overcomplicated there are ways to simplify them. In the following examples you will learn skills associated with simplifying overcomplicated expressions.

Vocabulary

algebraic expression, evaluate, real number, commutative property, associative property, distributive property, identity property, inverse property, like terms

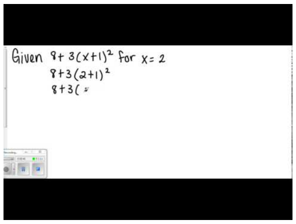
1. **Evaluating Algebraic Expressions:**

When asked to evaluate an expression we are being asked to substitute a given value into the variable and compute using order of operations.

PEMDAS: Parenthesis, Exponents, Multiplication, Division, Addition and Subtraction

Example: Evaluate.

A. Given $8+3(x+1)^2$ for $x=2$



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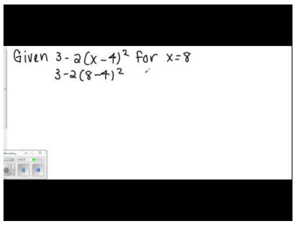
B. Given $3-2(x-4)^2$ for $x=8$



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2. **Properties of Real Numbers:**

Real Numbers have operators that allow us to simplify expressions. In Algebra I and Algebra II you learned the commutative, associative, distributive, identity and inverse properties of addition and multiplication. The table below shows a summary of these properties. Given that a, b, and c are real numbers:

TABLE 1.1: Properties of Real Numbers

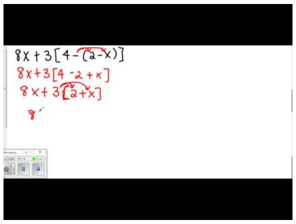
Property	Algebraic Example	Numeric Example
Commutative Property of Addition	$a+b=b+a$	$5+3=3+5$
Associative Property of Addition	$(a+b)+c=a+(b+c)$	$(2+3)+5=2+(3+5)$
Identity Property of Addition	$a+0=a$	$6+0=6$
Inverse Property of Addition	$a+(-a)=0$	$7+(-7)=0$
Commutative Property of Multiplication	$a\times b=b\times a$	$2\times 4=4\times 2$
Associative Property of Multiplication	$(a\times b)\times c=a\times(b\times c)$	$(2\times 3)\times 4=2\times(3\times 4)$
Identity Property of Multiplication	$a\times 1=a$	$9\times 1=9$
Inverse Property of Multiplication	$a\times\frac{1}{a}=1$	$3\times\frac{1}{3}=1$
Distributive Property	$a\times(b+c)=a\times b+a\times c$	$2(4+5)=2\times 4+2\times 5$

3. **Simplifying Algebraic Expressions:**

Using what we have reviewed, we can now use these properties and the order of operations to simplify expressions that are overcomplicated.

Example: Simplify.

A. $8x+3[4-(2-x)]$



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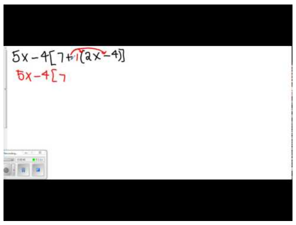
B. $5x-4[7+(2x-4)]$



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Vocabulary

Algebraic Expression

An expression made up of integer constants, variables and the algebraic operations (addition, subtraction, multiplication and division)

Evaluate

To perform the operations to obtain the value of the Algebraic Expression.

Real Number

Any number of the set of real numbers being the union of the rational and irrational numbers.

Like Terms

Terms whose variables and their exponents are exactly the same.

The following properties are defined in the table above.

commutative property, associative property, distributive property, identity property, inverse property

In Summary

In this section we have learned to use the properties of real numbers to evaluate and simplify algebraic expressions.

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1.2 Exponents, Radicals and Rational Exponents

TEKS

1. P.1.F

Lesson Objectives

In this section you will review:

1. Properties of Exponents.
2. Simplify Expressions with Exponents.
3. Rules for Square Roots.
4. Rationalizing Denominators.
5. Radicals and Other Roots.
6. Rational Exponents.

Introduction

Sometimes when simplifying algebraic expressions we find variables with exponents. In this section you will review the ways we can simplify such expressions and rules associated with all types of exponents.

Vocabulary

base, exponent, principle root, radical, radicand, index, rational exponent

1. **Properties of Exponents**

Below is a table summarizing the properties of exponents taught in Algebra I and Algebra II.

TABLE 1.2:

Property	**Algebraic Example**	**Numeric Example**
Product Property	$a^m \times a^n = a^{m+n}$	$5^2 \times 5^3 = 5^5$

TABLE 1.2: (continued)

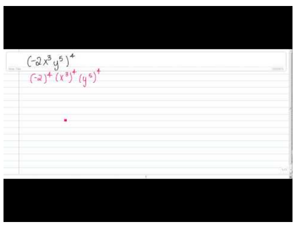
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{5^3}{5^2} = 5^1$
Power to Power Property	$(a^m)^n = a^{m \times n}$	$(5^2)^3 = 5^6$
Negative Property	$a^{-m} = \frac{1}{a^m}$	$3^{-2} = \frac{1}{3^2}$
Zero Property	$a^0 = 1$	$4^0 = 1$

2. **Simplifying Expressions with Exponents**

We can use any combination of the above properties and our knowledge of simplifying algebraic expressions to simplify more complex expressions.

Example: Simplify.

A. $\left(-2x^3y^5\right)^4$



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B. $\left(-8xy^3\right)\left(-2x^5y\right)$

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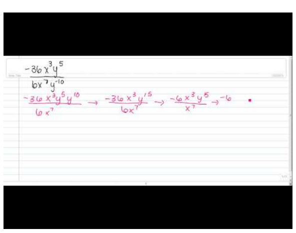
C. $\frac{-36x^3y^5}{6x^7y^{-10}}$



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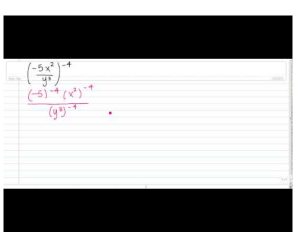
D. $\left(\frac{-5x^2}{y^3}\right)^{-4}$



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3. **Rules for Square Roots**

Principle Square Root:

If a is a non negative number and $b^2 = a$, then $\sqrt{a} = b$, $b \geq 0$.

Example: Evaluate the principle square root.

A. $\sqrt{25}$

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B. $\sqrt{64}$

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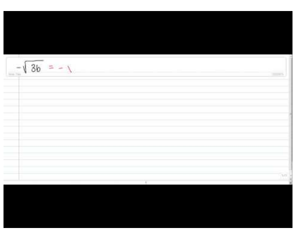
C. $-\sqrt{36}$



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D. $\sqrt{\frac{1}{16}}$

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E. $\sqrt{9+16}$

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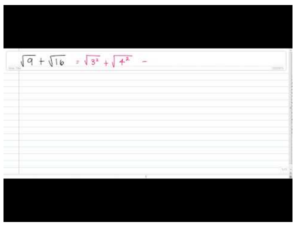
F. $\sqrt{9}+\sqrt{16}$



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Properties/Rules for Radicals:

The table below summarizes the rules/properties for radicals.

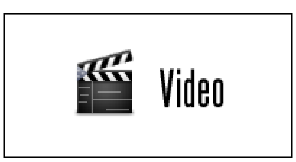
TABLE 1.3:

Property/Rule	**Algebraic Example**	**Numeric Example**
Product Rule	$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$	$\sqrt{3 \times 4} = \sqrt{3} \times \sqrt{4} = 2\sqrt{3}$
Quotient Rule	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$	$\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}$
Addition/Subtraction Rule	If $a = b$ then $\sqrt{a} \pm \sqrt{b} = 2\sqrt{a}$	$\sqrt{3}+\sqrt{3} = 2\sqrt{3}$

Example: Use the rules for square roots to simplify.

A. $\sqrt{5x} \times \sqrt{10x}$

B. $\frac{\sqrt{150x^3}}{\sqrt{3x}}$



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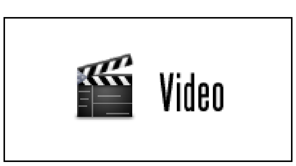
C. $8\sqrt{13}+7\sqrt{13}$



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D. $\sqrt{2}-\sqrt{8}$

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4. **Rationalizing Denominators**

When working with radicals it is improper to have a radical in the denominator of a rational expression (fraction). In order to "fix" this or simplify this, we use multiplication to rationalize the denominator.

Example: Rationalize the denominator using multiplication.

A. $\frac{3}{\sqrt{8}}$

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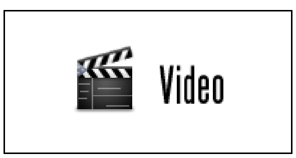
B. $\frac{5}{\sqrt{2}}$



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5. **Radicals of Other Roots**

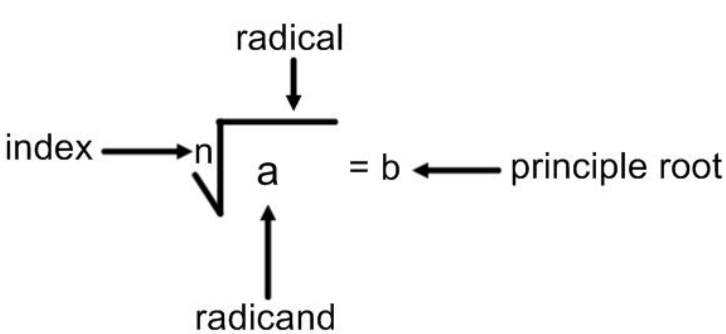

FIGURE 1.1

Radicals with indices greater than 2 have the same rules and properties as the square root.

Example: Simplify using the properties of square roots.

A. $\sqrt[3]{24}$

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B. $\sqrt[4]{8} \times \sqrt[4]{4}$

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C. $\sqrt[3]{\frac{125}{27}}$

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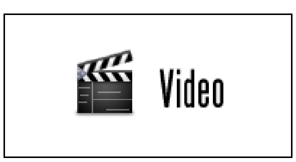
6. **Rational Exponents**

Let $\sqrt[n]{a^m}$ represent a real number where $n \geq 2$, then $a^{\frac{m}{n}} = \sqrt[n]{a^n}$.

Example: Simplify using properties of exponents and radicals.

A. $27^{\frac{2}{3}}$



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B. $4^{\frac{3}{2}}$

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C. $32^{-\frac{2}{5}}$

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Vocabulary

Base

The number being raised to a power. For example $3^2$the number 3 is the base.

Exponent

The power of a number or a variable. In the previous definition the 2 is the exponent.

Radical

The house-like symbol that is used to represent a root of a number. Many times it is is called the square root symbol, but it can be used to determine any root like cube root, fourth root, etc.

Radicand

The number inside the radical symbol.

Index

The number outside the radical symbol to the left that represents the kind of root you are trying to find.

Rational Exponent

An exponent that is a fraction.

In Summary

We have learned the properties of exponents and to simplify expressions with exponents. We have also learned to combine and simplify expressions containing radicals, to rationalize and to work with rational exponents.

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1.3 Basics of Polynomials

TEKS

1. P.1.F
2. A2.7.B

Lesson Objectives

In this lesson you will review:

1. Finding the Degree of a Polynomial.
2. Adding and Subtracting Polynomials.
3. Multiplying Polynomials.

Introduction

Combining expressions with variables that have exponents with variables that do not we get a more advanced form of expressions called a polynomial, an algebraic expression containing one or more terms.

Vocabulary

polynomial, degree, leading coefficient, constant, monomial, binomial, trinomial

1. **Finding the Degree of a Polynomial**

If $a \neq 0$ we say the degree of ax^n is n.

If $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0x^0$ is a polynomial with $a_n, a_{n-1}, a_{n-2}, \ldots a_0$ real numbers and $a_n \neq 0$, then the degree of the polynomial is n, a_n is called the leading coefficient, and $a_0$is called the constant term.

Example: State the degree, leading coefficient and the constant term of each polynomial.

A. $5x + 3x^3 - 4x^2 + 16x^4 + 10$

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B. $4x^3 + 2x^2 - 6x^5 - 2$

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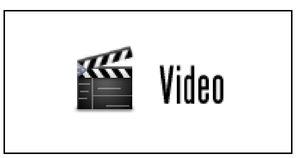
2. **Adding and Subtracting Polynomials**

When combining two or more polynomials with addition or subtraction or a combination of the two we follow the order of operations to combine like terms. Like terms in a higher degree polynomial are terms that have the same degree.

Example: Add or subtract as indicated.

A. $(-17x^3 + 5x^2 - 12x + 5) + (13x^3 - 4x^2 + 2x - 15)$



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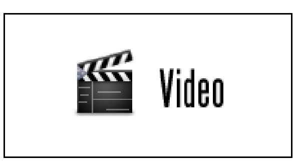
B. $(5x^4 + 3x^3 - 4x^2 + 2) - (3x^4 + 2x^3 - 2x^2 - 5)$



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3. **Multiplying Polynomials**

To multiply polynomials we need to make sure to use the properties of real numbers, more specifically the distributive property and the properties of exponents.

Example: Multiply.

A. $(-8x^3)(3x^5)$

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B. $(4x^2)(x^2+2x+6)$

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C. $(3x+2)(x^2+4x+3)$

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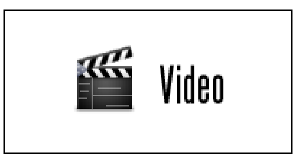
D. $(2x+3)(5x+4)$



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Vocabulary

Polynomial

A mathematical expression that contains terms in which the variables in the terms have powers that are positive integers.

Degree

It is the value of the largest exponent in a polynomial.

Leading Coefficient

The coefficient of the term containing the degree of the polynomial.

Constant

The term in a polynomial that has no variable.

Monomial

An algebraic expression containing only one term.

Binomial

An algebraic expression containing two terms.

Trinomial

An algebraic expression containing three terms

In Summary

In this lesson we have learned to find the degrees of polynomials, about adding and subtracting polynomials and to multiply polynomials.

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1.4 Factoring

TEKS

1. P.1.F
2. A2.7.E

Lesson Objectives

In this section you will review:

1. Finding the Greatest Common Factor.
2. Factoring by Grouping.
3. Factoring Trinomials.
4. Difference of Two Squares.
5. Sum or Difference of Two Cubes.

Introduction

Sometimes we need to break a polynomial down into its factors. We do this by undoing multiplication, called factoring. Once a polynomial is factored, we can do many other operations.

Vocabulary

Factor, Difference of squares, Difference of Cubes, Sum of cubes.

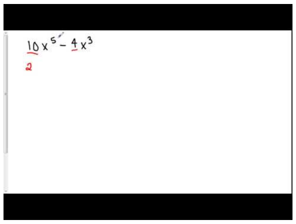
1. **Finding the Greatest Common Factor**

From the distributive property we have that $ab+ac=a(b+c)$

To factor out the greatest common factor we are in a sense undoing the distributive property.

Example:

A. $10x^5-4x^3$



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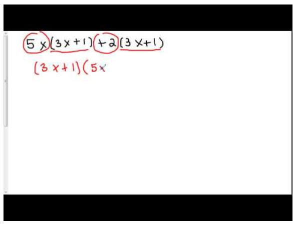
B. $5x(3x+1)+2(3x+1)$



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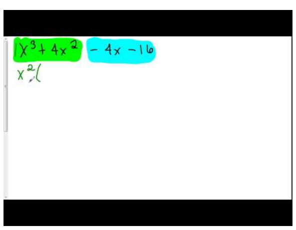
2. **Factoring by Grouping**

When we don't have an obvious greatest common factor we resort to other forms of factoring. We can group our polynomial that is in standard form int two groups and factor out the greatest common factor for each group, then factor again if there is a greatest common factor for the two groups.

Example:

A. $x^3+4x^2-4x-16$



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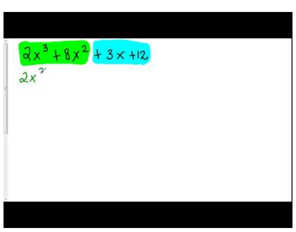
B. $2x^3+8x^2+3x+12$



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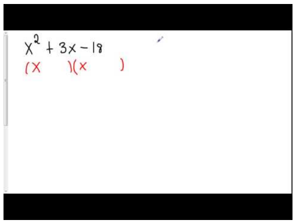
3. **Factoring Trinomials**

A trinomial is a polynomial containing three terms, typically in the form ax^2+bx+c. There are essentially many different ways to factor. Here is one way to factor:

Example:

A. $x^2+3x-18$



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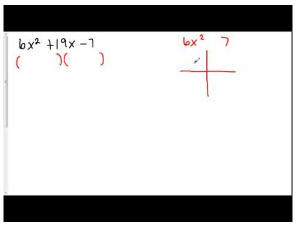
B. $6x^2 + 19x - 7$



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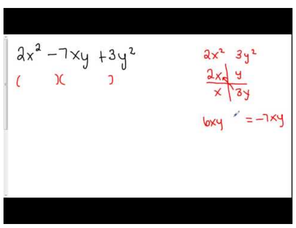
C. $2x^2 - 7xy + 3y^2$



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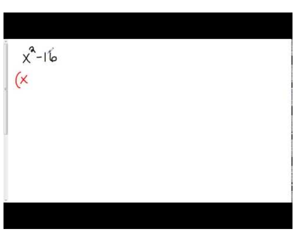
4. **Difference of Two Squares**

Given that A and B are real, $A^2 - B^2 = (A+B)(A-B)$

Example:

A. $x^2 - 16$



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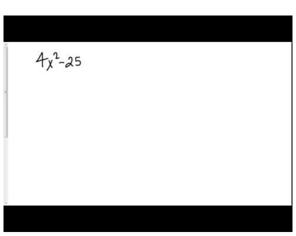
B. $4x^2 - 25$



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5. **Sum or Difference of Two Cubes**

Sum of Two Cubes

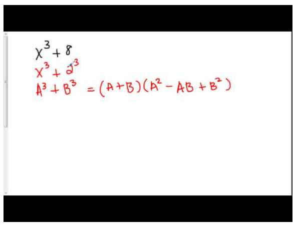
$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

Difference of Two Cubes

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

Example:

A. $x^3 + 8$



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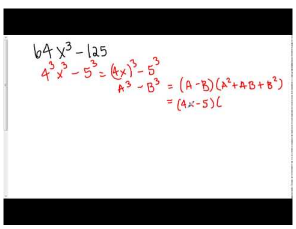
B. $64x^3 - 125$



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Vocabulary

Factor

To break up into simpler expressions (roots) that can be multiplied to obtain the original expression.

Difference of Squares

Given that A and B are real, $A^2 - B^2 = (A+B)(A-B)$

Sum and Difference of Cubes

$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

In Summary

We have learned how to factor an expression by factoring the greatest common factor, we also learned to factor by grouping and factor trinomials, and finally we learned to recognize and factor the difference of squares, the sum of cubes and the difference of cubes.

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1.5 Rational Expression Basics

TEKS

1. P.1.F
2. A2..7.F
3. A2.7.G

Lesson Objectives

In this section you will review:

1. Restricting Values of "x".
2. Simplify Rational Expressions.
3. Multiply and Divide Rational Expressions.
4. Add and Subtract Rational Expressions.

Introduction

When two polynomials are being divided by one another we call it a rational expression. The numerator is a polynomial and the denominator is a polynomial. Like other rational numbers these expressions can be reduced, added, subtracted, multiplied and divided.

Vocabulary

rational expression, numerator, denominator

1. **Restricting Values of "x"**

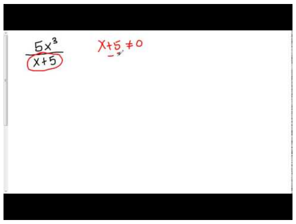
   Because a rational expression contains variables in the denominator we have to check and make sure the denominator is not zero because we cannot divide by zero.

   Given $p(x)$ and $q(x)$ two polynomials then the rational expression is

$$\frac{p(x)}{q(x)}, q(x) \neq 0$$

Example: State the values of x that must be excluded.

A. $\frac{5x^2}{x+5}$



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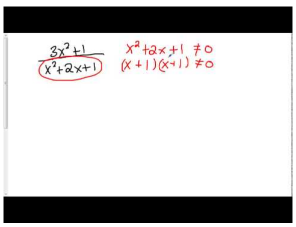
B. $\frac{3x^2+1}{x^2+2x+1}$



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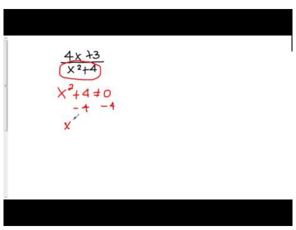
C. $\frac{4x+3}{x^2+4}$



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2. **Simplify Rational Expressions**

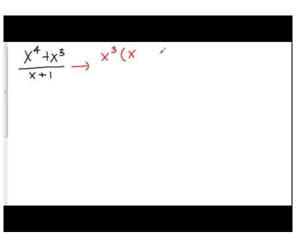
Recall from basic arithmetic:

$$\frac{10}{20} = \frac{10 \times 1}{10 \times 2} = \frac{1}{2}$$

The same can be done for a rational expression, factor the polynomials (both the numerator and denominator) completely then check to find common factors that eliminate.

Example: Simplify.

A. $\frac{x^4+x^3}{x+1}$



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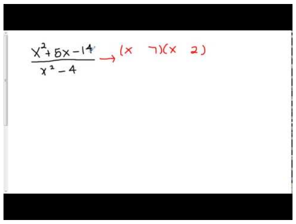
B. $\frac{x^2+5x-14}{x^2-4}$



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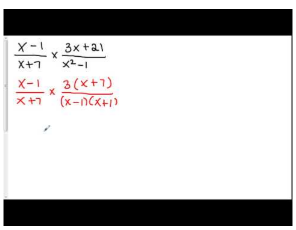
3. **Multiply and Divide Rational Expressions**

Recall multiplying rational numbers:

$$\frac{2}{3} \times \frac{5}{2} = \frac{2 \times 5}{3 \times 2} = \frac{5}{3}$$

Example: Multiply each set of rational expressions.

A. $\frac{x-1}{x+7} \times \frac{3x+21}{x^2-1}$



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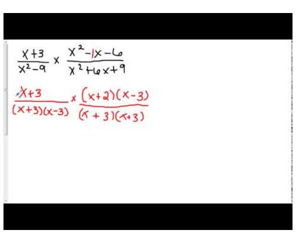
B. $\frac{x+3}{x^2-9} \times \frac{x^2-x-6}{x^2+6x+9}$



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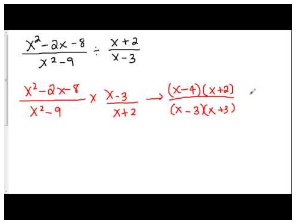
Recall dividing rational numbers: (Keep Change Flip or Copy Change Flip)

$$\frac{2}{3} \div \frac{5}{2} = \frac{2}{3} \times \frac{2}{5} = \frac{2 \times 2}{3 \times 5} = \frac{4}{15}$$

Example: Divide each set of rational expressions.

A. $\frac{x^2-2x-8}{x^2-9} \div \frac{x+2}{x-3}$



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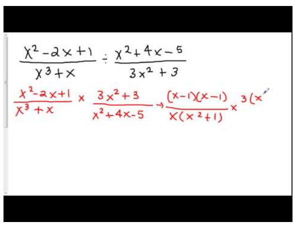
B. $\frac{x^2-2x+1}{x^3+x} \div \frac{x^2+4x-5}{3x^2+3}$



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4. **Add and Subtract Rational Expressions**

Recall adding and subtracting rational numbers:

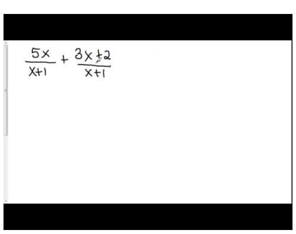
In order to add or subtract two or more rational numbers, the denominators must match. We call this common denominators.

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{2}{2}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

$$\frac{1}{2}-\frac{1}{3}=\frac{3}{3}\times\frac{1}{2}-\frac{1}{3}\times\frac{2}{2}=\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$$

Example: Add or subtract as indicated.

A. $\frac{5x}{x+1}+\frac{3x+2}{x+1}$



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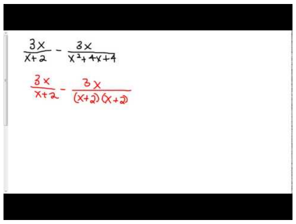
B. $\frac{3x}{x+2}-\frac{3x}{x^2+4x+4}$



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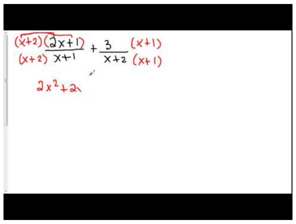
C. $\frac{2x+1}{x+1}+\frac{3}{x+2}$



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Vocabulary

Rational Expression

A mathematical expression that is a fraction in which the numerator and the denominator are polynomials.

Denominator

The number or expression in the bottom of a fraction.

Numerator

The number or expression in the top of a fraction.

In Summary

We have learned what a rational expression is and how to exclude values of x that make the expression undefined. We also learned how to multiply divide, add and subtract rational expressions.

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1.6 Equations

TEKS

1. P.1.F
2. A2.4.F

Lesson Objectives

In this section you will review

1. Solving Polynomials of Degree 1.
2. Solving Polynomials of Degree 2.

Introduction

Now that we know the different ways to manipulate polynomial expressions we can go from variables (where the x values can represent any number) to unknowns (where the x is a fixed number). Now we will be looking for a value of an unknown that will make our equation a true statement.

Vocabulary

equation, completing the square, zero multiplication rule, quadratic formula

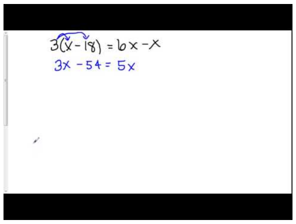
1. **Solving Polynomials of Degree 1**

A polynomial of degree 1 can also be called an equation in one variable and written in the form $ax+b=0$ where $a \neq 0$.

We find the value of the unknown x by moving all the unknowns to one side of the equation and all the constants to the other side.

Example: Solve for the unknown.

A. $3(x-18) = 6x-x$



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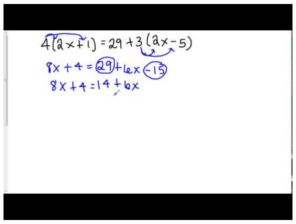
B. $4(2x+1) = 29+3(2x-5)$



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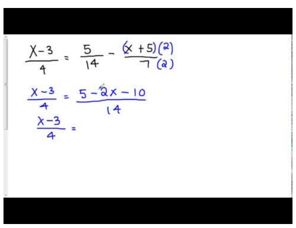
C. $\frac{x-3}{4} = \frac{5}{14} - \frac{x+5}{7}$



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2. **Solving Polynomials of Degree 2**

Another name for a second degree polynomial is a quadratic. Quadratics are written in different forms, the two most common being general and vertex/standard form.

General: $ax^2+bx+c=0$

Vertex/Standard: $a(x-h)^2+k=0$

There are four different ways to solve quadratics we will start with using the square root.

The square root is used when our quadratic can be written in the form $ax^2+b=0$.

Example: Solve.

A. $4x^2-16=0$

B. $x^2-49=0$

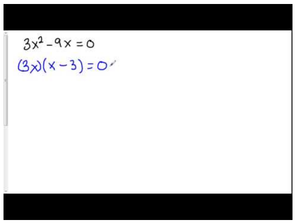
Zero Multiplication Property:

If A and B are two algebraic expressions that when multiplied equal zero, then either $A=0$ or $B=0$.

This property is used for our second method, factoring. To factor a quadratic we are breaking it down into its two factors. If we can separate a quadratic into these two factors we can apply the zero multiplication property and solve for each factor independently.

Example: Solve.

A. $3x^2-9x=0$



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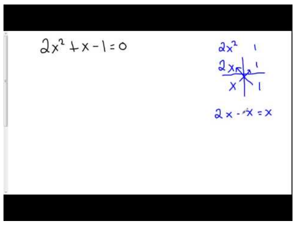
B. $2x^2 + x - 1 = 0$



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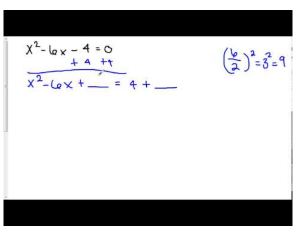
The third method of solving quadratics is when we cannot separate the quadratic into its two factors, it is unfactorable or prime. When this happens we can use completing the square.

Recall to complete the square you must add $\left(\frac{b}{2}\right)^2$ *to both sides of the equation.*

Example: Solve.

A. $x^2 - 6x - 4 = 0$



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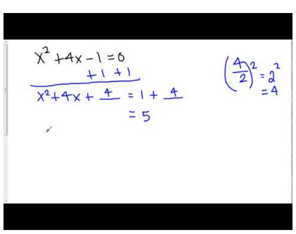
B. $x^2 + 4x - 1 = 0$



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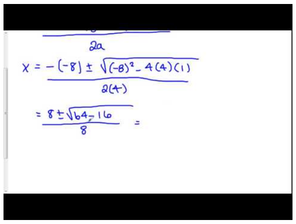
The final way to solve a quadratic is the generalized form of using completing the square. The quadratic formula will always work to get the exact solutions of any polynomial of degree 2.

Quadratic Formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example: Solve.

A. $2x^2 - 6x + 1 = 0$



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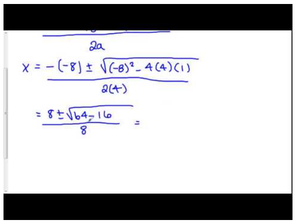
B. $4x^2 - 8x + 1 = 0$



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Vocabulary

Equation

A mathematical expression that involves an equal sign.

Completing the Square,

A technique used to solve quadratic equations and to graph quadratic functions.

Zero Multiplication Rule

If A and B are two algebraic expressions that when multiplied equal zero, then either $A = 0$ or $B = 0$.

Quadratic Formula

A formula that gives the solutions to a quadratic equation. The formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In Summary We have learned how to solve polynomials/equations of degree 1 and the different methods to solve polynomials of degree two. [U+EFFE]

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1.7 Inequalities

TEKS

1. P.1.F

Lesson Objectives

In this section you will review

1. Interval Notation.
2. Intersection and Union of Sets.
3. Solving Inequalities.
4. Compound Inequalities.

Introduction

When we are comparing expressions we use inequality symbols, our values of the unknown become a range of values, not just one value.

Vocabulary

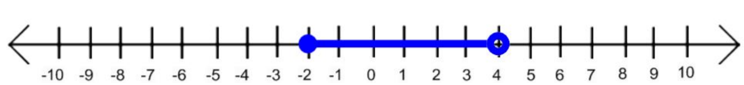
inequality, compound inequality, intersection, union, interval

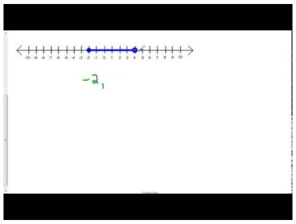
1. **Interval Notation**

We can represent a range of numbers using intervals. When an endpoint is included we use either [or]. If an endpoint is not included we use (or), infinity also uses (or) because infinity is not a value it is a concept of "never ending."

Example: Use the number line to write the interval in interval notation.

A.





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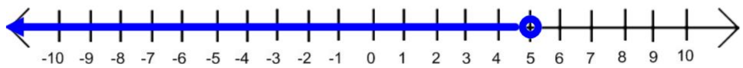
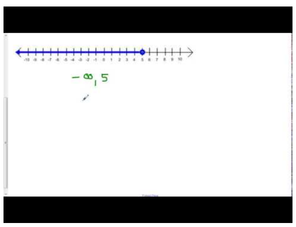
B.





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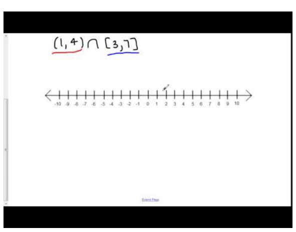
2. **Intersection and Union of Sets**

When we want to combine intervals or sets we can use the intersection to describe what the two sets have in common using $\bigcap$ or we can use the union to describe what the new set comprised of the two or more sets is using $\bigcup$.

Example: Graph the intervals using number lines.

A. $(1,4)\bigcap[3,7]$



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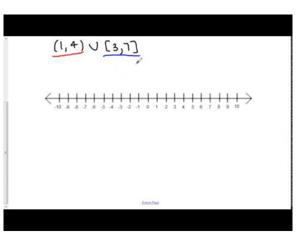
B. $(1,4)\bigcup[3,7]$



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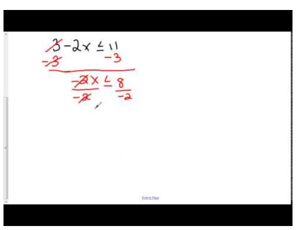
3. **Solving Inequalities**

TABLE 1.4:

Symbolic	Meaning	End Point
	Less Than	
$>$	Greater Than	
$\leq$	Less Than or Equal	
$\geq$	Greater Than or Equal	
$\neq$	Not Equal	

Example: Solve and graph the solution set.

A. $3-2x \leq 11$



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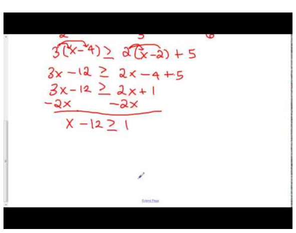
B. $\frac{x-4}{2} \geq \frac{x-2}{3} + \frac{5}{6}$



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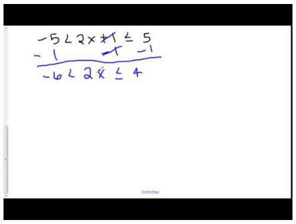
4. **Compound Inequalities**

Sometimes an inequality can satisfy more than one condition for example if $-3 < 3x+4$ and $3x+4 \leq 3$ we can write these two inequalities with the common expression in the middle. $-3 < 3x+4 \leq 3$. This inequality can also be solved by isolating the x variable in the middle. In order to do this we must pay close attention to solving on both sides of the expression.

Example: Solve and graph the solution set.

A. $-5 < 2x + 1 \leq 5$



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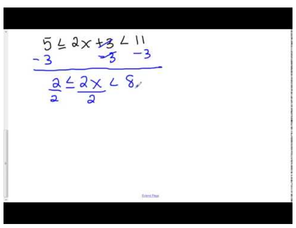
B. $5 \leq 2x + 3 < 11$



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Vocabulary

Inequality

An inequality is a mathematical expression that uses the symbols $<, >, \leq, \geq$

Compound Inequality

An expression of two or more inequalities joined together for example $-5 \leq 2x + 3 < 7$

Intersection

The intersection of two sets A and B is the set of all elements that are in both set A and set B.

Union

The union of two sets A and B is the set of all elements that are are in set A or set B

Interval Notation

A notation that represents an interval as a set of points where parenthesis exclude the endpoints and brackets include the endpoints.

In Summary

We have learned about interval notation and about the intersection and union of sets. We have also learned about solving simple inequalities and compound inequalities. [U+EFFE]

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CHAPTER 2 Relations, Functions and Their Graphs

Chapter Outline

Here you will review and extend concepts about functions and graphing. You will learn how to transform basic functions and write these transformations using the correct notation. You will learn to describe a function in terms of its domain, range, extrema, symmetry, intercepts, asymptotes, and continuity. Finally, you will learn about function composition and inverses of functions.

2.1 Relations and Basics of Functions

TEKS

1. P.1.D
2. P.1.E

Lesson Objectives

In this section you will learn about:

1. Ordered Pairs and Point Plotting.
2. Graphing Relations using Point Plotting.
3. Identifying Intercepts.
4. Domain and Range.
5. Relations as Functions.
6. Evaluating Functions.
7. Obtaining Information from a Graph.

Introduction

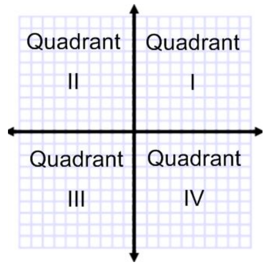
A number line is a one dimensional representation of numbers, if we take two we can represent numbers in a two dimensional plane. Rene Descartes envisioned a way to bring Geometry and algebra together creating the branch of mathematics called analytic geometry. The marriage of algebra and geometry bred the rectangular coordinate system, also called the Cartesian coordinate system after Descartes. In this section you will learn about relations and functions represented on this plane.

Vocabulary

relation, ordered pair, intercepts, domain, range, function, vertical line test, independent variable, dependent variable

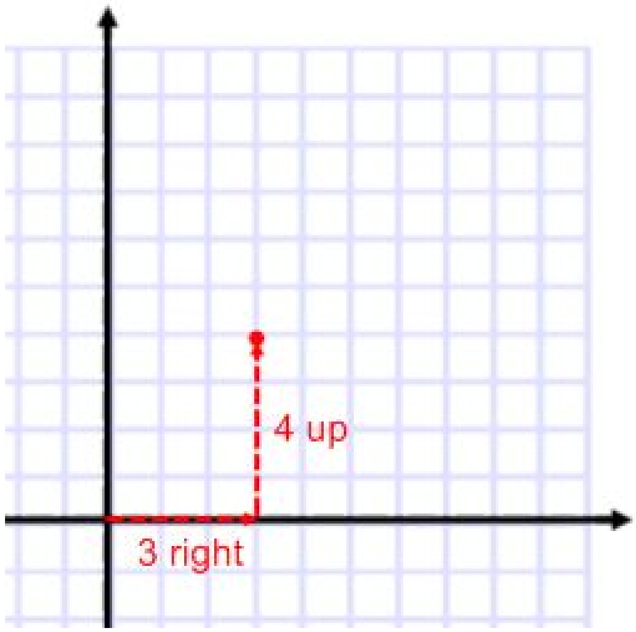
1. **Ordered Pairs and Plotting Points**

A relation is a pairing of any two things, expressions, numbers, letters etc. This pairing creates a set of ordered pairs that can be plotted on the coordinate grid. The Cartesian (rectangular) Coordinate grid contains two axes, a vertical most commonly named y, and a horizontal most commonly named x.

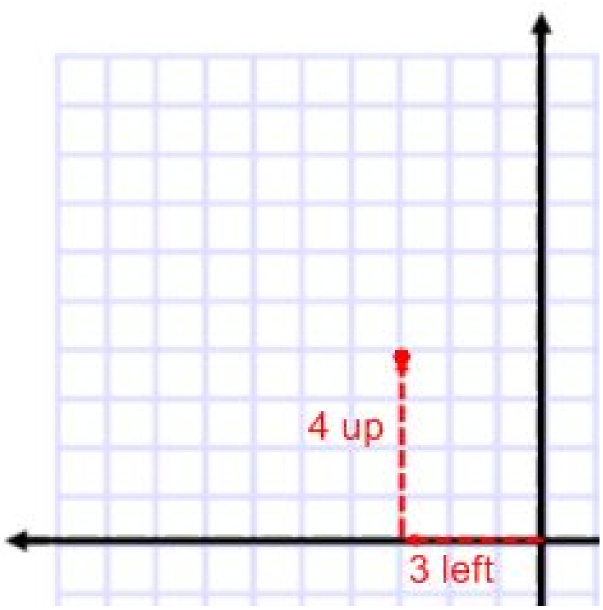


Example: Plot the points and identify the quadrant the point lies in.

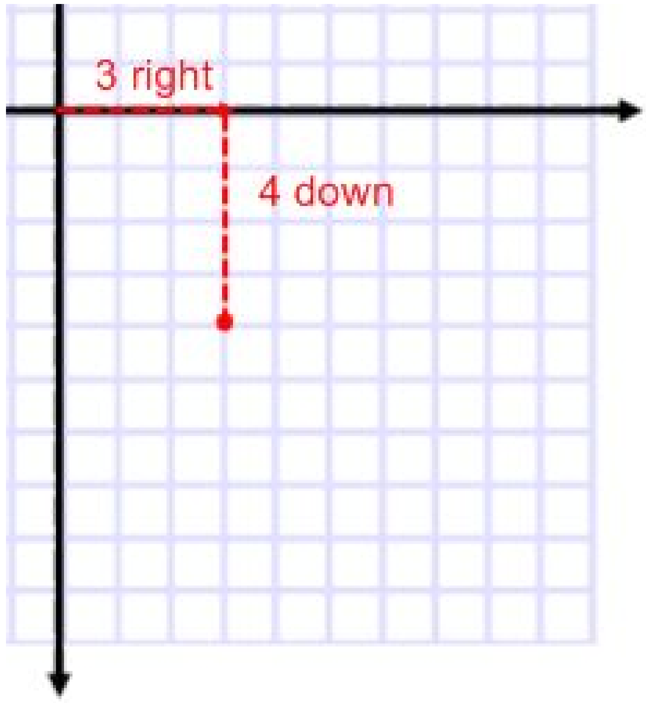
A. $(3,4)$



B. $(-3,4)$



C. $(3,-4)$



2. **Graphing Relations Using Point Plotting**

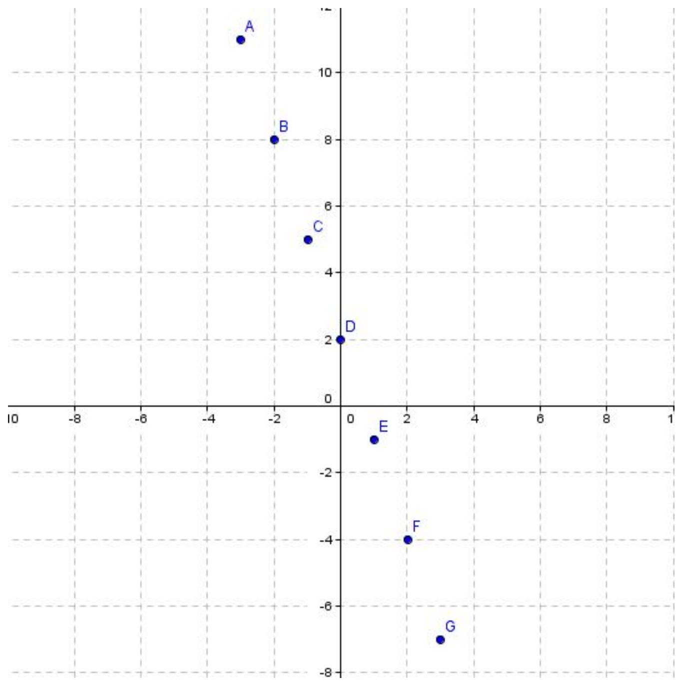
Now that we know how to plot points, we can look at equations of relations that represent sets of ordered pairs and plot them on the Cartesian coordinate grid to make a graph.

Example: Plot each relation on the interval $-3 \leq x \leq 3$ by using a table and point plotting.

A. $y = -3x+2$

TABLE 2.1:

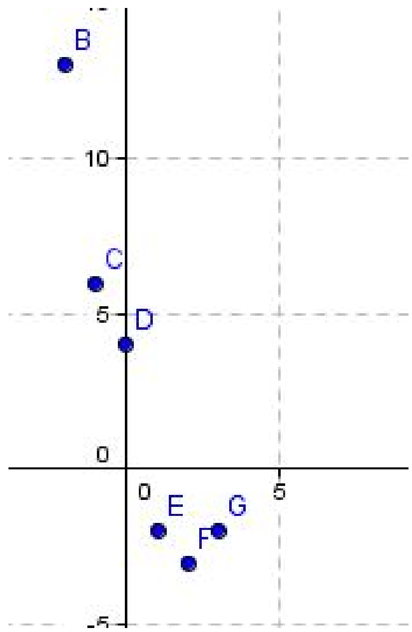
x	$y = -3x+2$	(x,y)
-3	$-3(-3)+2 = 9+2 = 11$	$(-3,11)$
-2	$-3(-2)+2 = 6+2 = 8$	$(-2,8)$
-1	$-3(-1)+2 = 3+2 = 5$	$(-1,5)$
0	$-3(0)+2 = 0+2 = 2$	$(0,2)$
1	$-3(1)+2 = -3+2 = -1$	$(1,-1)$
2	$-3(2)+2 = -6+2 = -4$	$(2,-4)$
3	$-3(3)+2 = -9+2 = -7$	$(3,-7)$



B. $y = x^2 - 4x + 1$

TABLE 2.2:

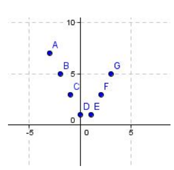
x	$y = x^2 - 4x + 1$	(x, y)
-3	$(-3)^2 - 4(-3) + 1 = 9 + 12 + 1 = 22$	$(-3, 22)$
-2	$(-2)^2 - 4(-2) + 1 = 4 + 8 + 1 = 13$	$(-2, 13)$
-1	$(-1)^2 - 4(-1) + 1 = 1 + 4 + 1 = 6$	$(-1, 6)$
0	$(0)^2 - 4(0) + 4 = 0 + 0 + 4 = 4$	$(0, 4)$
1	$(1)^2 - 4(1) + 1 = 1 - 4 + 1 = -2$	$(1, -2)$
2	$(2)^2 - 4(2) + 1 = 4 - 8 + 1 = -3$	$(2, -3)$
3	$(3)^2 - 4(3) + 1 = 9 - 12 + 1 = -2$	$(3, -2)$



C. $y = |2x-1|$

TABLE 2.3:

x	$y = \|2x-1\|$	(x,y)
-3	$\|2(-3)-1\| = \|-6-1\| = \|-7\| = 7$	$(-3,7)$
-2	$\|2(-2)-1\| = \|-4-1\| = \|-5\| = 5$	$(-2,5)$
-1	$\|2(-1)-1\| = \|-2-1\| = \|-3\| = 3$	$(-1,3)$
0	$\|2(0)-1\| = \|0-1\| = \|-1\| = 1$	$(0,1)$
1	$\|2(1)-1\| = \|2-1\| = \|1\| = 1$	$(1,1)$
2	$\|2(2)-1\| = \|4-1\| = \|3\| = 3$	$(2,3)$
3	$\|2(3)-1\| = \|6-1\| = \|5\| = 5$	$(3,5)$

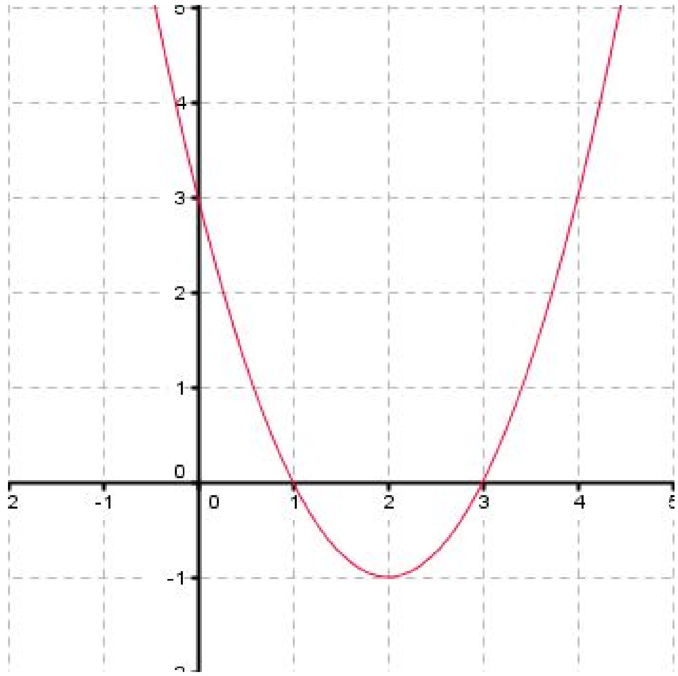


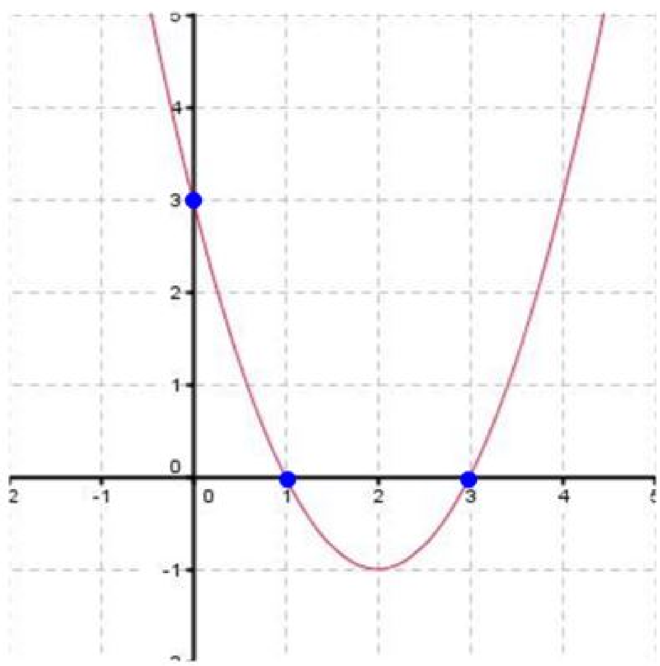
3. **Identifying Intercepts**

An intercept is a point or points where the graph of a relation crosses either the x or y axis. The x-intercept is where the y value of the relation is zero, the y-intercept is where the x-value of the relation is zero.

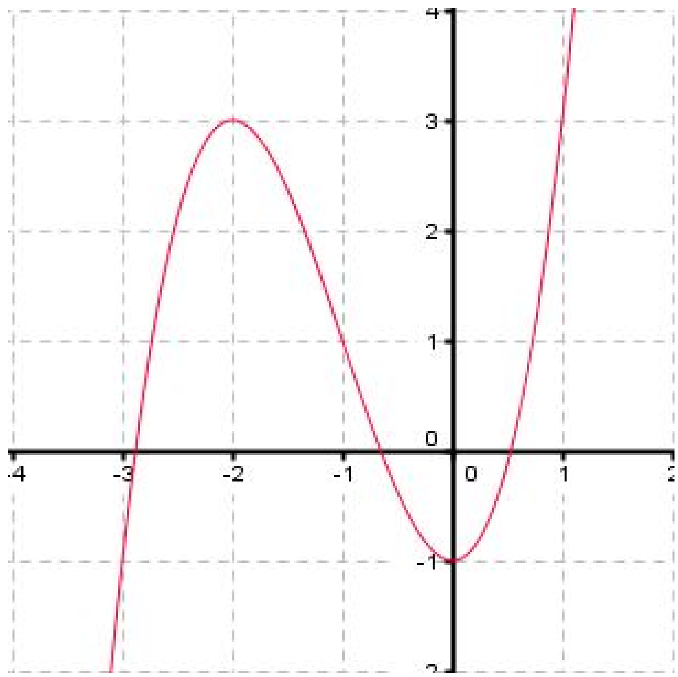
Example: Identify the x and y intercepts.

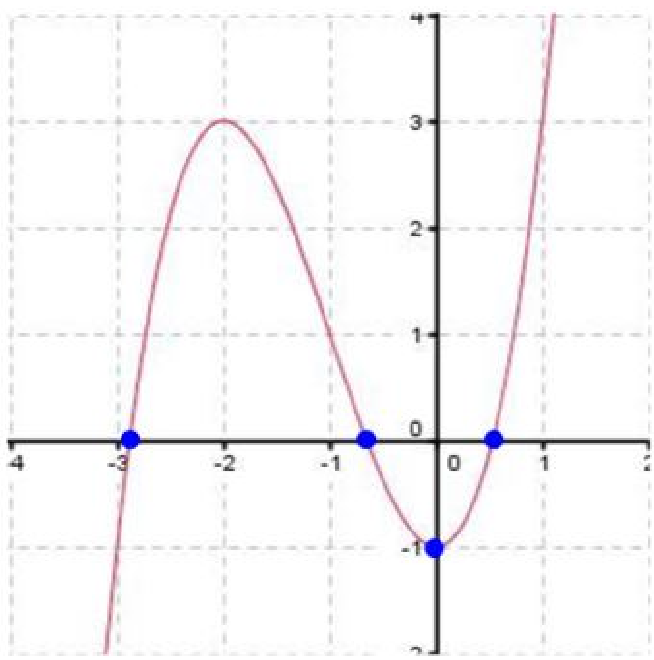
A.



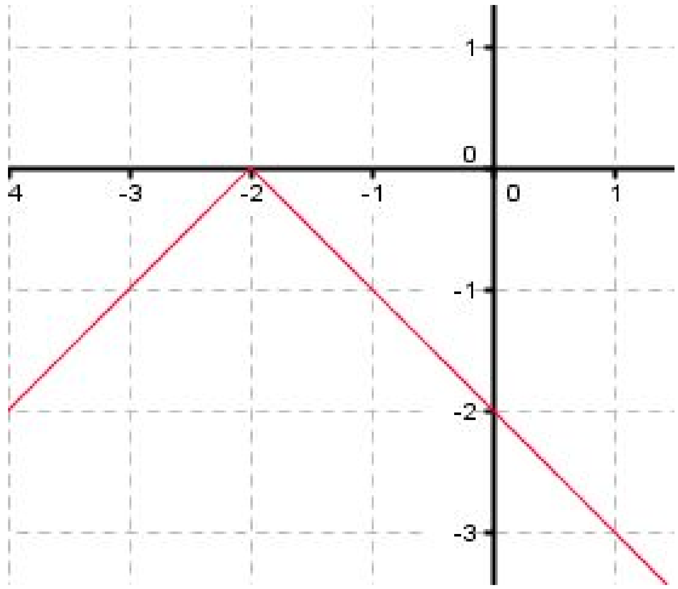


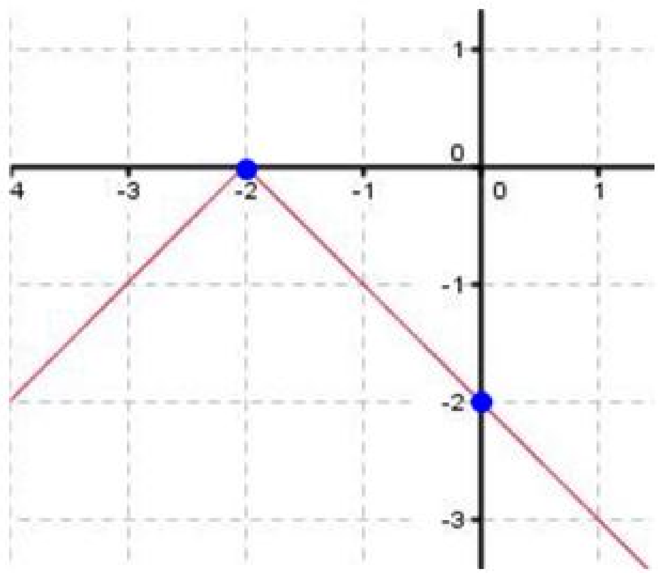
B.





C.





4. **Domain and Range**

The set of ordered pairs make a relation. The subset comprised of only the first coordinates of the ordered pairs is called the domain. The subset comprised of only the second coordinates of the ordered pairs is called the range.

Example: State the domain and range of each relation.

A. $\{(-3,2),(-2,1),(-1,0),(-1,4),(1,2)\}$

The Domain of this relation is: $\{-3,-2,-2,1\}$

The Range of this relation is: $\{0,1,2,4\}$

B. $\{(5,4),(4,5),(3,2),(2,3),(0,0)\}$

The Domain of this relation is: $\{2,3,4,5\}$

The Range of this relation is: $\{0,2,3,4,5\}$

5. **Relations as Functions**

A relation is a function if for every input there exists an unique output. This means that each x value has exactly one and only one y value. When looking at a set of ordered pairs the x value should not repeat, because if it does then there are more than one y values for that x value.

Example: Determine whether the relation is a function.

A. $\{(-3,4),(-2,4),(2,4),(3,4)\}$

This function is a function for each x value there is exactly one unique y value.

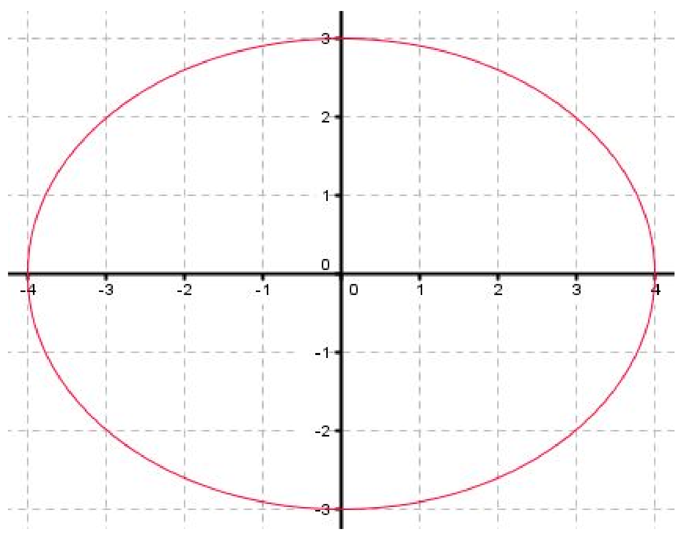
B.$\{(-3,2),(2,2),(4,3),(-3,-2)\}$

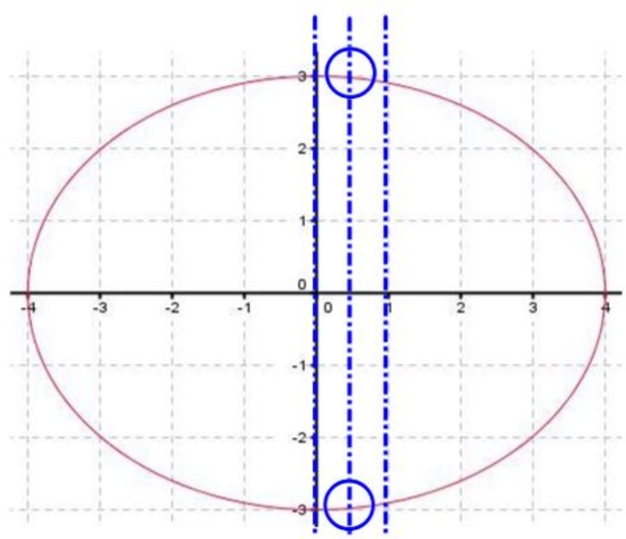
This is not a function for the value x=-3 there are two possible y values.

If you plot a set of points of a relation to get a graph you can determine if the graph represents a function by using the vertical line test. In the vertical line test you draw many vertical lines through the graph. If the graph crosses any one of the lines more than once it fails and the graph does not represent a function.

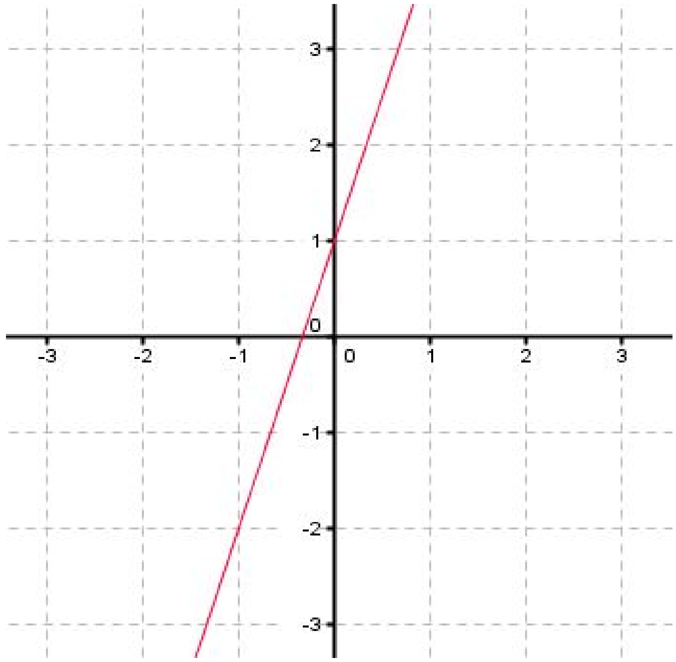
Example: Determine if the graph represents a function.

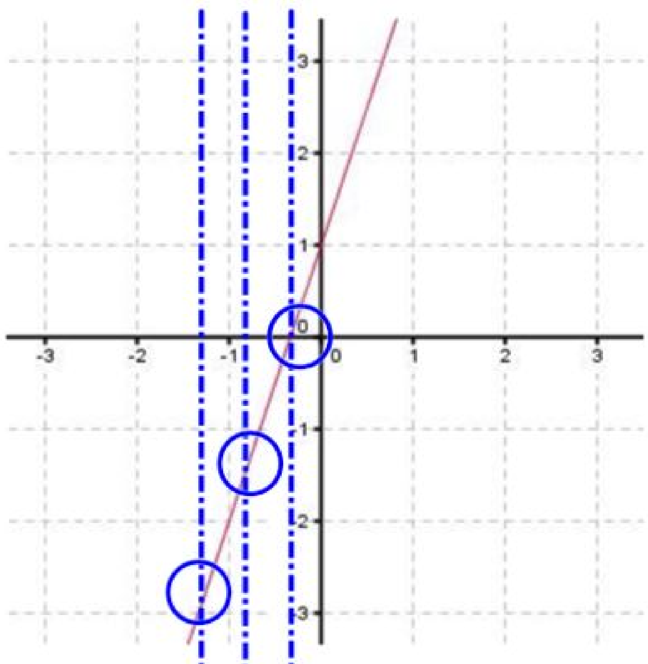
A.





B.





6. **Evaluating a Function**

When a relation is a function we use function notation to define it.

If y is a function then it can be defined as $y = f(x)$ using function notation. Any value inside the parenthesis are the independent variables. These variables can be used to determine the values of the dependent variable $f(x)$ using substitution.

Example: Evaluate.

A. $f(x) = -2x^3 - 4x - 13; f(-3)$

$$\begin{aligned} f(x) &= -2x^3 - 4x - 13; f(-3) \\ f(-3) &= -2(-3)^2 - 4(-3) - 13 \\ &= -2(9) + 12 - 13 \\ &= -18 + 12 - 13 \\ &= -19 \end{aligned}$$

B. $f(x) = -|-2x+5|; f(2)$

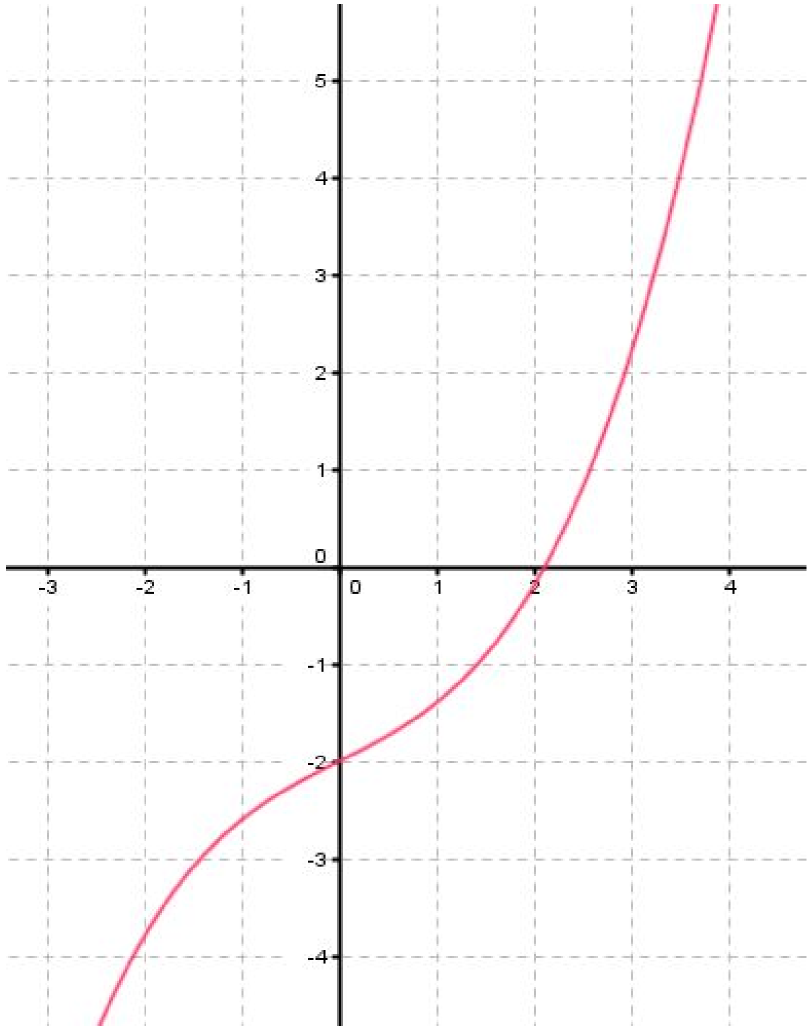
$$\begin{aligned} f(x) &= -|-2x+5|; f(2) \\ f(2) &= -|-2(2)+5| \\ &= -|-4+5| \\ &= -|1| \\ &= -1 \end{aligned}$$

7. **Obtaining Information from a Graph**

Given a graph we can determine if a relation ins a function using the vertical line test, find the intercepts by finding where the graph crosses the axes, and the domain and range.

Example: Given the graph identify the intercepts, domain, range, $f(-1)$, and $f(3)$

A.



x-intercept: $(2,0)$

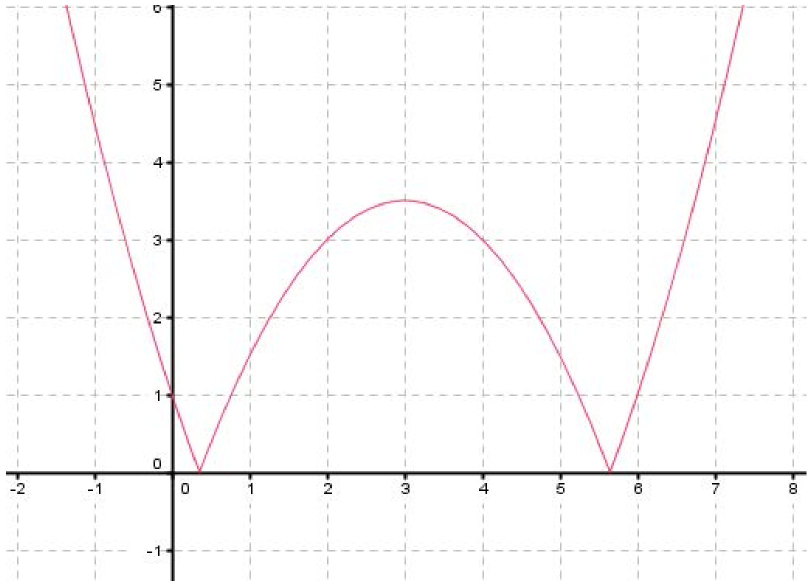
y-intercept: $(0,-2)$

domain: $(-\infty,\infty)$

range: $(-\infty,\infty)$

$f(-1) = -2.5$
$f(3) = 2$

B.



x-intercept: $(2,0)$ and $(5.5,0)$

y-intercept: $(0,1)$

domain: $(-\infty,\infty)$

range: $(-\infty,\infty)$

$f(-1) = 4.5$
$f(3) = 3.5$

Vocabulary

Relation

Is the pairing of any two things, in this section we will pair the x coordinate to the y coordinate obtaining a point.

Ordered Pair

Is a pair of numbers used to locate a point in the coordinate plane, usually denoted (x, y).

Intercepts

Points in which a function crosses the either the x or y axis. If it crosses the x axis the point has the form (x,0) and if it crosses the y axis it has the form (0, y)

Domain

The set of values of the independent variable for which a function is defined.

Range

The set of output values of a function obtained from the domain.

Function

Is a relation from a set of inputs to a set of outputs in which each input is related to exactly one output. In other words for one input, there cannot exist two different outputs.

Vertical Line Test

A visual test that is used to identify if a graph of a relation is a function.

Independent Variable

The variable in a relation that determines the values of other variables in the relation.

Dependent Variable

The variable in a relation that is determined whose value is determined by other variables in the relation.

In Summary

We have learned about functions and their characteristics. We have learned about relations, domain, range, intercepts, ordered pairs, graphing by point plotting, the vertical line test we have defined the independent variable and the dependent variable.

Check for Understanding:

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2.2 Piecewise Functions

TEKS

1. P.1.D
2. P.1.E
3. P.2.F

Lesson Objectives

In this section you will learn about:

1. Evaluate Piecewise Functions.
2. Graph the Piecewise Functions.
3. Applications of Piecewise Functions.

Introduction

Some functions are comprised of parts of other functions defined over specific intervals. These are so appropriately named piecewise functions.

Vocabulary

interval, piecewise

1. **Evaluating Piecewise Functions**

Because piecewise functions are very specifically defined over set intervals to evaluate you must check which interval the value lies in.

Example: Evaluate.

A. $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$ $; f(-2), f(0), f(3)$

Since $-2 < 0$ we use the first equation to evaluate $f(-2)$

$f(-2) = (-2)^2 = 4$

Since $0 \geq 0$ we use the second equation to evaluate $f(0)$

$f(0) = (0)^3 = 0$

Since $3 \geq 0$ we use the second equation again to evaluate $f(3)$

$f(3) = (3)^3 = 27$

B. $f(x) = \begin{cases} \frac{x+1}{x-4} & : x \neq 4 \\ x^2 & : x = 4 \end{cases}$; $f(-2), f(0), f(4)$

Since $-2 \neq 4$ we use the first equation to find $f(-2)$

$f(-2) = \frac{-2+1}{-2-4} = \frac{-1}{-6} = \frac{1}{6}$

Since $0 \neq 4$ we use the first equation to find $f(0)$

$f(0) = \frac{0+1}{0-4} = \frac{1}{-4} = -\frac{1}{4}$

Since $4 = 4$ we use the second equation to find $f(4)$

$f(4) = 4^2 = 16$

2. **Graphing Piecewise Functions**

The most basic way to graph a piecewise function is to graph each piece separately over its specified intervals.

Recall: Inequalities

TABLE 2.4:

	
$>$	
$\leq$	
$\geq$	
$\neq$	

Example: Graph each piecewise functions

A. $f(x) = \begin{cases} x^2 & : x \leq 0 \\ x+3 & : x > 0 \end{cases}$

First note that the first equation starts at 0 and is decreasing x values with a closed circle as an endpoint. The second equation also starts at 0 and is increasing x values with an open circle as an endpoint.

$f(x) = x^2$

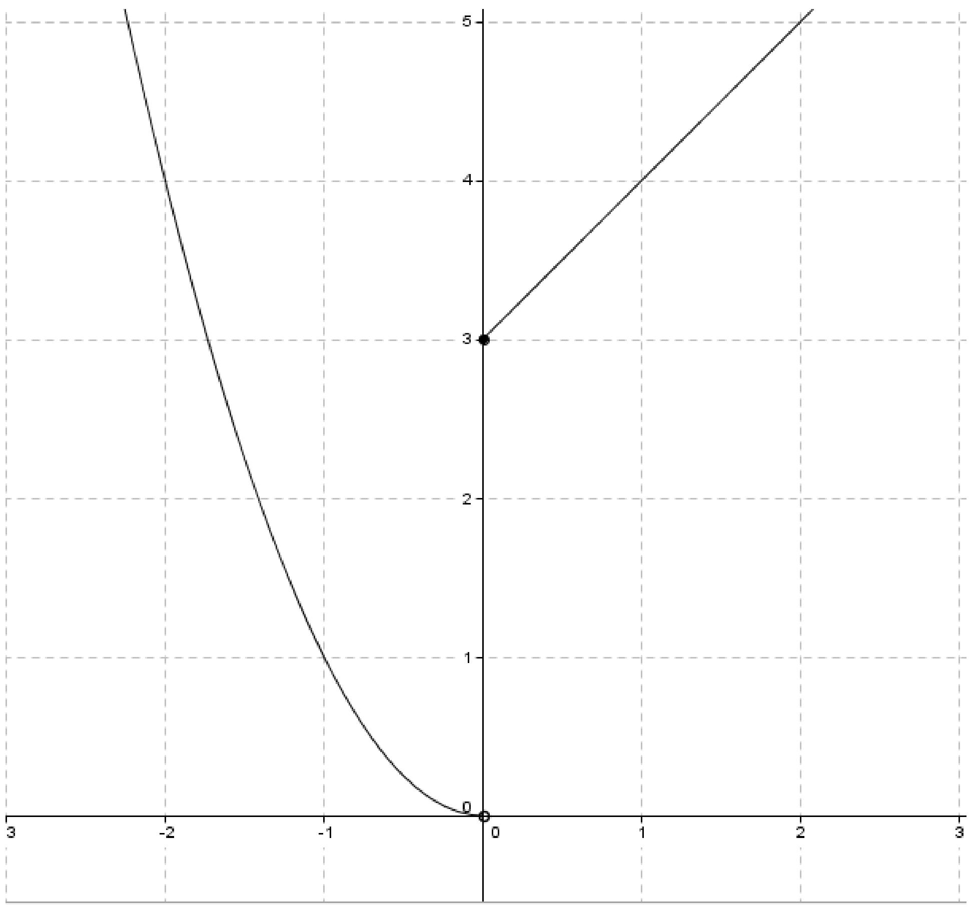
TABLE 2.5:

x	x^2	(x,y)
0	$0^2 = 0$	$(0,0)$
-1	$(-1)^2 = 1$	$(-1,1)$
-2	$(-2)^2 = 4$	$(-2,4)$

$f(x) = x+3$

TABLE 2.6:

x	$f(x) = x+3$	(x,y)
0	$0+3 = 3$	$(0,3)$
1	$1+3 = 4$	$(1,4)$
2	$2+3 = 5$	$(2,5)$



B. $f(x) = \begin{cases} 2x+1 & : x < 1 \\ x^2+2 & : x \geq 1 \end{cases}$

The first equation starts at 1 with decreasing x values and an open circle as an endpoint. The second equation also starts at 1 but with increasing x values and a closed circle as an endpoint.

$f(x) = 2x+1$

TABLE 2.7:

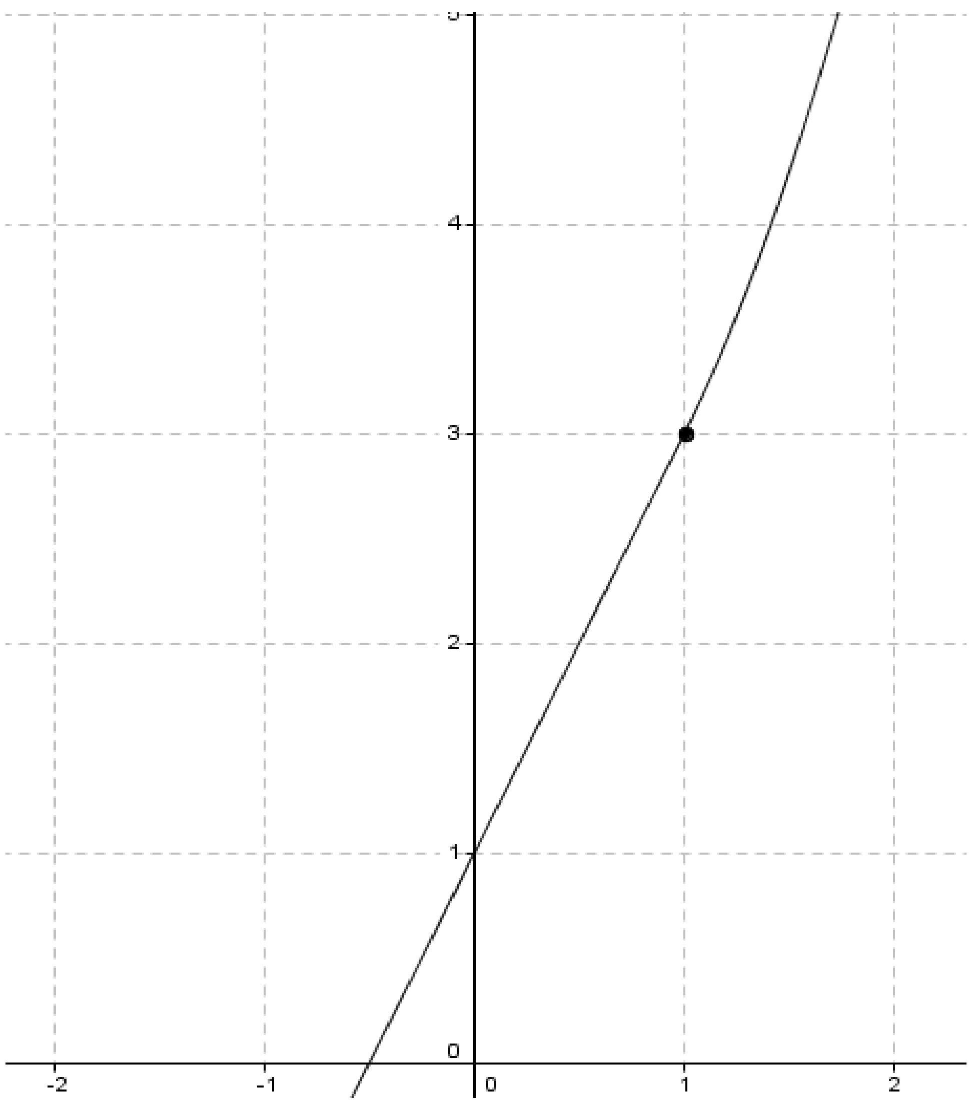
TABLE 2.7: (continued)

x	$2x+1$	(x,y)
1	$2(1)+1=2+1=3$	$(1,3)$
0	$2(0)+1=0+1=1$	$(0,1)$
-1	$2(-1)+1=-2+1=-1$	$(-1,-1)$

$f(x)=x^2+2$

TABLE 2.8:

x	x^2+2	(x,y)
1	$1^2+2=1+2=3$	$(1,3)$
2	$2^2+2=4+2=6$	$(2,6)$
3	$3^2+2=9+2=11$	$(3,11)$



3. **Applications of Piecewise Function**

Most commonly used in finance there are many other applications for piecewise functions.

Example: Solve the problem as indicated.

A. The speed v of a car in miles per hour can be represented by the following piecewise function where t is the time in seconds. Find the speed of the car at 15 seconds, 245 seconds and 5 seconds.

$$v(t) = \begin{cases} 4t & : 0 \leq t \leq 15 \\ 60 & : 15 < t < 240 \\ -6t + 1500 & : 240 \leq t \leq 250 \end{cases}$$

$t = 15$
$v(t) = 4t$
$v(15) = 4(15)$
$= 60 \quad t = 245$
$v(t) = -6t + 1500$
$v(245) = -6(245) + 1500$
$= -1470 + 1500$
$= 30 \quad t = 5$
$v(t) = 4t$
$v(5) = 4(5)$
$= 20$

B. Cell Communications charges a plan of \$20 for 200 minutes and once the customer goes over 200 minutes they are charged \$0.50 for each minute. Define a piecewise function $C(t)$ for the cost of a cell plan after t minutes. How much does a customer pay if they use no minutes? 205 minutes?

The piecewise function can defined using two pieces, one for minutes up to 200 minutes and the second piece for any minutes after the 200.

$$C(t) = \begin{cases} 20 & : 0 \leq t \leq 200 \\ 20 + 0.50(200 - t) & : t > 200 \end{cases}$$

If a customer uses no minutes they are under the first piece of the function so they pay \$20.00.

If a customer uses 205 minutes they are under the second piece of the function so they pay:

$20 + (200 - 205) = 20 + 0.50(5) = 20 + 2.50 = 22.50$

Vocabulary

Interval

The set of real numbers that lie between two points called the endpoints. For example (a,b) means all the points between a and b excluding the endpoints a <x <b, or [a,b] means all the points between a and b including the endpoints a $\leq$ x $\leq$ b

Piecewise Function

Is a function that is defined by more than one function. Each of the functions that form the piecewise function has its own interval.

In Summary

We have learned what a piecewise function is. We have learned how to evaluate a piecewise function and how to graph a piecewise function.

Check for Understanding:

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2.3 Information From Graphs

TEKS

1. P.2.I

Lesson Objectives

In this section you will learn about:

1. Define Intervals of Increasing, Decreasing or Constant.
2. Identify Relative Maxima and Minima.
3. Symmetry of Functions.
4. Difference Quotient.

Introduction

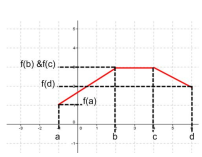
The graphs of functions have many different attributes that make each graph/function unique or different from other graphs/functions.

Vocabulary

increasing, decreasing, constant relative maxima, relative minima, even, odd, origin symmetric, y-axis symmetric, difference quotient

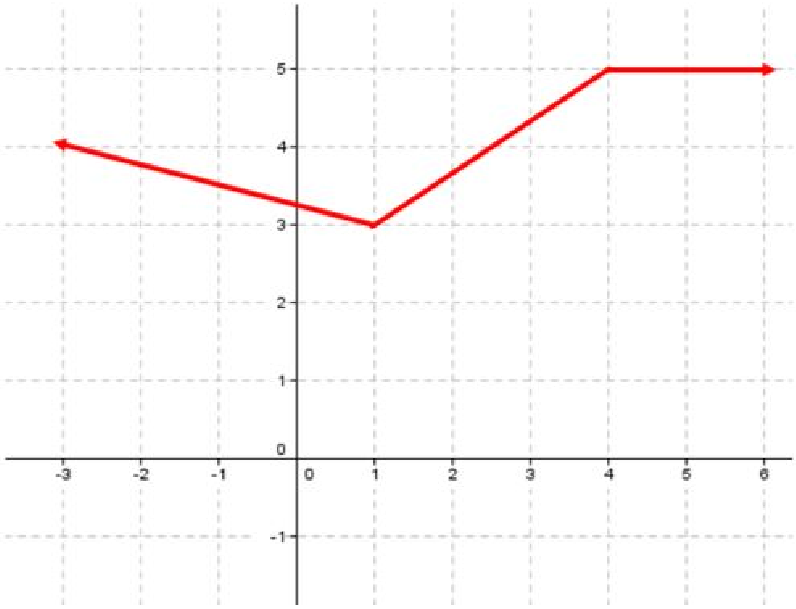
1. **Defining Intervals of Increasing, Decreasing, and Constant**

TABLE 2.9:

	• A function is increasing on an interval if • A function is constant on a an interval if • A function is decreasing on an interval if

Example: Define the intervals were the function is increasing, decreasing or constant.

A.

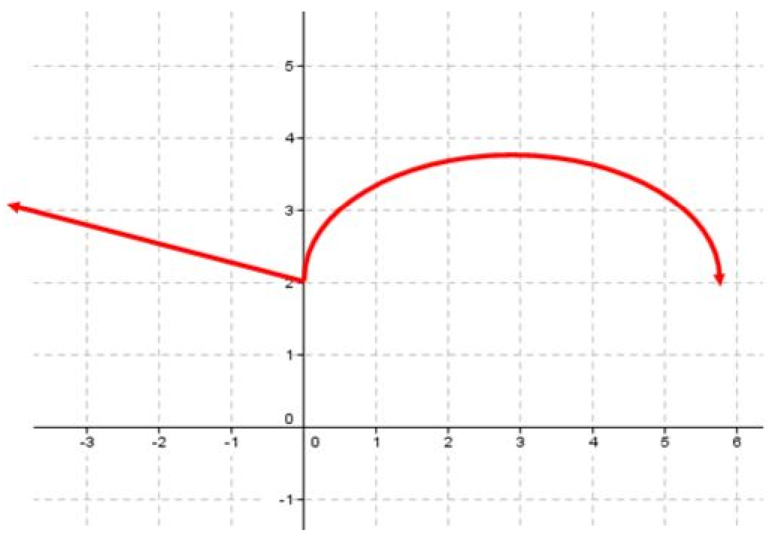


Increasing: $[1,4]$

Decreasing: $(-\infty,1]$

Constant: $[4,\infty+)$

B.



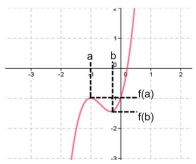
Increasing: $[0,3]$

Decreasing: $(-\infty,0]\bigcup[3,\infty+)$

Constant: None

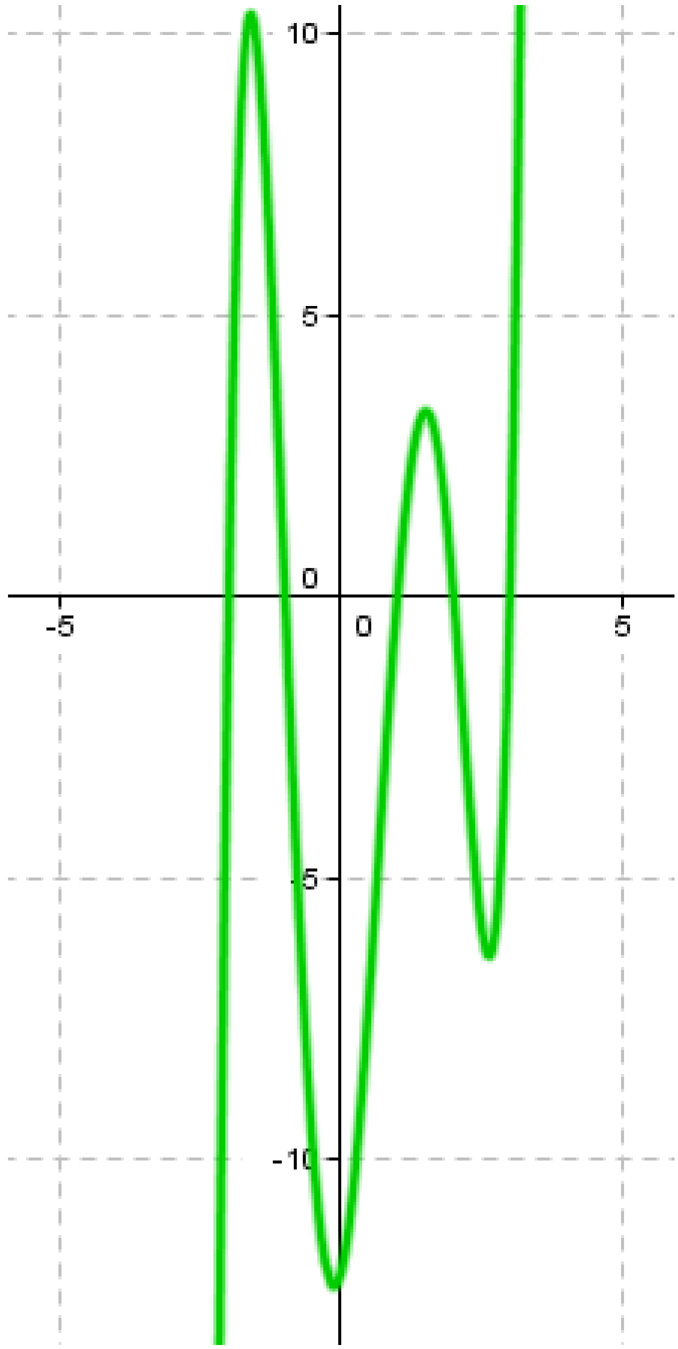
2. **Identifying Relative Maxima and Minima**

TABLE 2.10:

	• The relative maxima is $f(a)$ at $x = a$ • The relative minima is $f(b)$ at $x = b$

Example: Identify the relative maxima and minima.

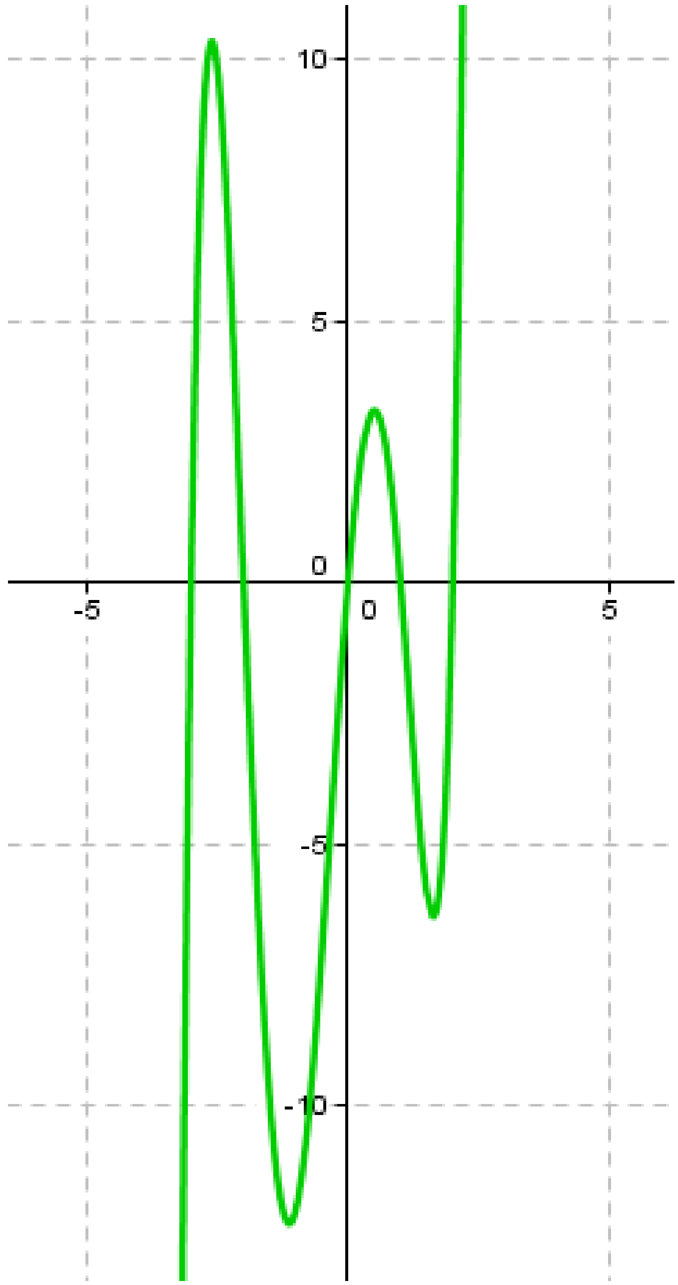
A.



Relative Maximum is about 10.5 at about -1.5 (-1.5, 10.5)

Relative Minimum is about -12 at about -0.5 (-0.5, -12)

B.



Relative Maximum is about 10.5 at about -2.5 (-2.5, 10.5)

Relative Minimum is about -12 at about -2 (-2, -12)

3. **Symmetry of Functions**

Before we discuss the symmetry of functions, we need to make the following clear:

Having an **EVEN degree** is not the same as being an **EVEN function.**

Having an **ODD degree** is not the same as being an **ODD function.**

Algebraically, the symmetry of functions is given by the following

Given a function $f(x)$ if:

$f(-x) = f(x)$ for all x in the domain, then $f(x)$ is an even function.

$f(-x) = -f(x)$ for all x in the domain, then $f(x)$ is an odd function.

If neither of the above happen, then $f(x)$ is neither even nor odd.

Example: Evaluate $f(-x)$ to determine if $f(x)$ is even, odd or neither.

A. $f(x) = x^3 - 10x$

$$\begin{aligned} f(-x) &= (-x)^3 - 10(-x) \\ &= -x^3 + 10x \\ &= -(x^3 - 10x) \end{aligned}$$

Since $f(-x) = -f(x)$this function is **odd**.

B. $f(x) = 2x^4 - 4x^2$

$$\begin{aligned} f(-x) &= 2(-x)^4 - 4(-x)^2 \\ &= 2x^4 - 4x^2 \end{aligned}$$

Since $f(-x) = f(x)$ this function is **even.**

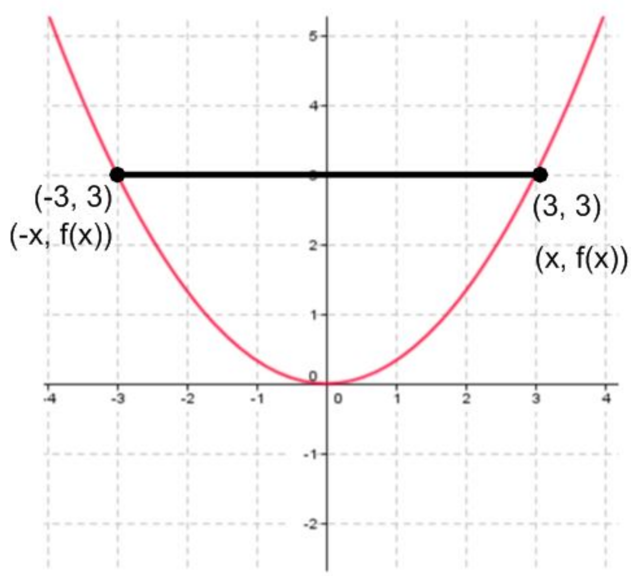
C. $f(x) = -x^2 - 2x + 1$

$$\begin{aligned} f(-x) &= -(-x)^2 - 2(-x) + 1 \\ &= -x^2 + 2x + 1 \end{aligned}$$

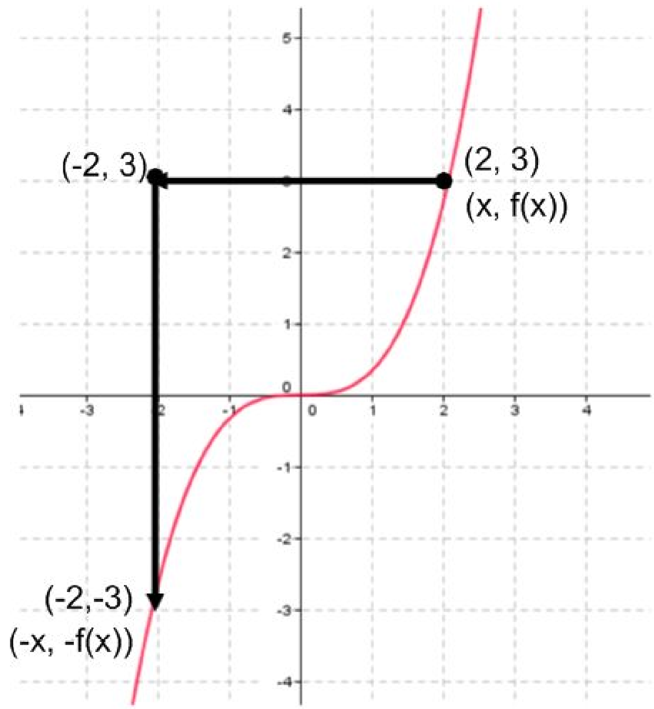
Since neither of the above happen, this function is **neither** even nor odd.

Graphically:

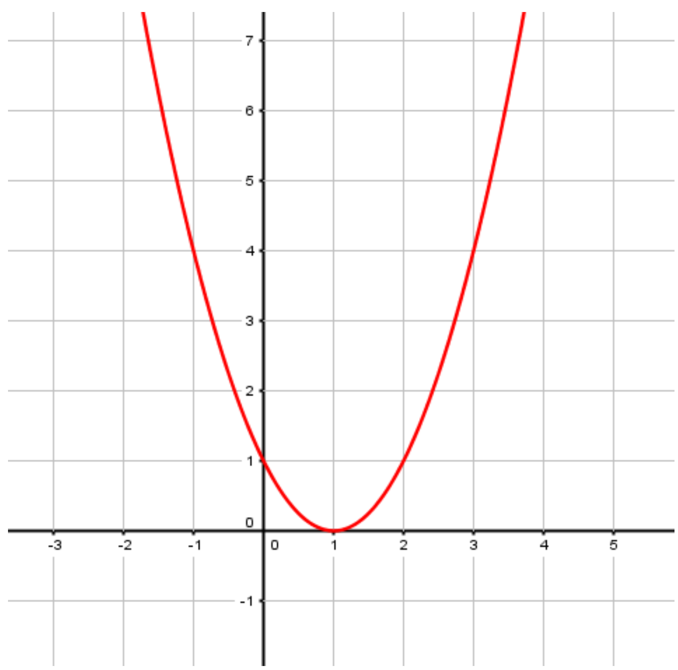
An ***even* function** is symmetric to the ***y-axis*.**



An ***odd* function** is symmetric to the ***origin*.**



The following graph is the graph of the $y = (x-1)^2 \rightarrow x^2 - 2x + 1$



This function even though its of even degree, since it is not symmetrical about the y-axis is not an even

4. **Difference Quotient**

The difference quotient is the quotient between the difference $f(x+h) - f(x)$ and h, where h is the horizontal distance between two points $(x, f(x))$ and $(x+h, f(x+h))$. It is written as:

$$\frac{f(x+h) - f(x)}{h}$$

The difference quotient represents a rate of change between these two points.

Example: Find/calculate the difference quotient for the given functions.

A. $f(x) = 3x+1$

First we find $f(x+h)$:

$$\begin{aligned} f(x+h) &= 3(x+h)+1 \\ &= 3x+3h+1 \end{aligned}$$

Next we take the difference $f(x+h)-f(x)$:

$$\begin{aligned} f(x+h)-f(x) &= 3x+3h+1-(3x+1) \\ &= 3x+3h+1-3x-1 \\ &= 3h \end{aligned}$$

Then the final step is to divide by h

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{3h}{h} \\ &= 3 \end{aligned}$$

B. $f(x) = x^2$

First we find $f(x+h)$:

$$\begin{aligned} f(x+h) &= (x+h)^2 \\ &= x^2+2xh+h^2 \end{aligned}$$

Next we take the difference $f(x+h)-f(x)$:

$$\begin{aligned} f(x+h)-f(x) &= x^2+2xh+h^2-x^2 \\ &= 2xh+h^2 \\ &= h(2x+h) \end{aligned}$$

Then the final step is to divide by h:

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{h(2x+h)}{h} \\ &= 2x+h \end{aligned}$$

C. $f(x) = -x^2 + 2x - 1$

Find $f(x+h)$:

$$\begin{aligned} f(x+h) &= -(x+h)^2 + 2(x+h) - 1 \\ &= -(x^2 + 2xh + h^2) + 2x + 2h - 1 \\ &= -x^2 - 2xh - h^2 + 2x + 2h - 1 \end{aligned}$$

Then $f(x+h) - f(x)$:

$$\begin{aligned} f(x+h) - f(x) &= -x^2 - 2xh - h^2 + 2x + 2h - 1 - (-x^2 + 2x - 1) \\ &= -x^2 - 2xh - h^2 + 2x + 2h - 1 + x^2 - 2x + 1 \\ &= -2xh - h^2 + 2h \\ &= h(2 - 2x - h) \end{aligned}$$

And finally $\frac{f(x+h)-f(x)}{h}$:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{h(2-2x-h)}{h} \\ &= 2 - 2x - h \end{aligned}$$

Vocabulary

Increasing

A function is increasing in the interval [a,b] if for all b>a then f(b)>f(a).

Decreasing

A function is decreasing in the interval [a,b] if for all b>a then f(b)<f(a)

Constant

A function is constant in the interval [a,b] if for all b>a then f(b)=f(a)

Relative maxima

A point that is the highest point with respect to the surrounding points.

Relative minima

A point that is the lowest point with respect to the surrounding points.

Even Function

A function is even if $f(-x) = f(x)$. It is an even function if the graph of the function reflects over the y-axis.

Odd Function

A function is odd if $f(-x) = -f(x)$. It is an odd function if the graph of the function reflects over the origin.

Difference Quotient

Is the formula that computes the slope between two point in a function the points (x, f(x)) and (x+h, f(x+h)) the formula is $\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$

In summary

We have learned about increasing and decreasing intervals in functions, how to identify relative maximums and relative minimums. We also leaned about even and odd functions and about the different quotient.

Check for Understanding:

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2.4 Rates of Change

TEKS

1. P.2.P

Lesson Objectives

In this section you will learn about:

1. Slopes and Linear Functions.
2. Point Slope form, Slope Intercept form, and General form.
3. Average Rate of Change for Non Linear Functions.
4. Average Velocity of an Object.

Introduction

In the previous section we learned about the difference quotient which is the rate of change between any two given points of any function with a horizontal difference of h. Functions that have a constant rate of change form what we call linear functions.

Vocabulary

slope, secant line, slope intercept form, tangent line, standard form, general form average rate of change, point slope form

1. **Slopes and Linear Functions**

There are many ways to define slope (the rate of change) for a linear function. If we take the difference quotient $\frac{f(x+h)-f(x)}{h}$ and use (x_1,y_1) and (x_2,y_2) in place of $(x,f(x))$ and $(x+h,f(x+h))$ we have $\frac{y_2-y_1}{x_2-x_1}$. Which is the formula for finding slope given two points. Below are the most common forms of describing the rate of change of a linear function:

$\frac{rise}{run} = \frac{changeiny}{changeinx} = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = m$

Example: Calculate the slope for each of the following sets of points

A. $(2,4)$ and $(-3,2)$

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2-4}{-3-2} \\
&= \frac{-2}{-5} \\
&= \frac{2}{5}
\end{aligned}
$$

B. $(2,3)$ and $(2,5)$

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5-3}{2-2} \\
&= \frac{2}{0} \\
&= undefined
\end{aligned}
$$

C. $(2,4)$ and $(3,4)$

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{4-4}{3-2} \\
&= \frac{0}{1} \\
&= 0
\end{aligned}
$$

2. **Point Slope form, Slope Intercept form, and General Form**

Point Slope Form: $y - \ \ y_1 = m(x - x_1)$
$\underline{+\ \ y_1 \qquad mx - mx_1 + y_1}$
$y = mx - mx_1 + y_1$

Slope Intercept Form: $y = mx + b$

Note: In the **slope intercept form** being obtained from the **point slope form** the value of $b = (-mx_1 + y_1)$

General Form: $Ax + By + C = 0$

where x intercept: $\frac{-C}{A}$ and y intercept: $\frac{-C}{B}$ and the slope: $\frac{-A}{B}$

Example: Write the equation in Point Slope and Slope Intercept Form.

A. $m = -5$ and $(-2, -5)$

Point Slope Form:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-5) &= -5(x - (-2)) \\ y + 5 &= -5(x + 2) \end{aligned}$$

Slope Intercept form:

$$\begin{aligned} y + 5 &= -5(x + 2) \\ y + 5 &= -5x - 10 \\ y &= -5x - 15 \end{aligned}$$

B. $(2,4)$ and $(-2,5)$

Slope:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5-4}{-2-2} \\ &= \frac{1}{-4} \\ &= -\frac{1}{4} \end{aligned}$$

Point Slope Form:

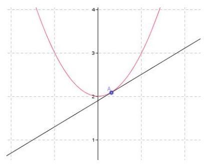
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{1}{4}(x - 2) \\ y - 4 &= -\frac{1}{4}x + \frac{2}{4} \\ y - 4 &= -\frac{1}{4}x + \frac{1}{2} \end{aligned}$$

Slope Intercept Form:

$$\begin{aligned} y - 4 &= -\frac{1}{4}x + \frac{1}{2} \\ y &= -\frac{1}{4}x + \frac{1}{2} + 4 \\ &= -\frac{1}{4}x + \frac{1}{2} + \frac{8}{2} \\ &= -\frac{1}{4}x + \frac{9}{2} \end{aligned}$$

The tangent line is the line that touches a graph of a non linear function at one and only point.

TABLE 2.11:

	To find the tangent line you need the slope, the function rule and an x value that exists on the graph. Use the function rule to find the corresponding y value and then use point slope form.

Example: Find the equation of the line tangent to the given function at the given x value with the given slope.

$f(x) = x^2$ at $x = 3$ with slope $m = 6$

First we need to identify the y value that corresponds to the x value given.

$f(3) = (3)^2$
$= 9$

Therefore we have the point **(3, 9)**

Then we use point slope form to get the equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 9 &= 6(x - 3) \\ y - 9 &= 6x - 18 \\ y &= 6x - 9 \end{aligned}$$

3. **Average Rate of Change for Non Linear Functions**

Given f is a function, the average rate of change between x_1 and x_2 is defined as:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example: Find the average rate of change given the following functions and x values.

A. $f(x) = x^2 + 1$ at $x_1 = 3$ and $x_2 = 4$

First find the corresponding y values for the given x values:

$f(x_1) = (x_1)^2 + 1$
$f(3) = 3^2 + 1$
$= 9 + 1$
$= 10 \quad f(x_2) = (x_2)^2 + 1$
$f(4) = 4^2 + 1$
$= 16 + 1$
$= 17$

Therefore we have the points **(3,10)** and **(4,17)**. Now we use the average rate of change formula:

$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{17 - 10}{4 - 3} = \frac{7}{1} = 7$

Therefore the average rate of change is 7.

B. $f(x) = 2x^2 + x - 1$ at $x_1 = -1$ and $x_2 = 3$

First find the corresponding y values for the given x values:

$f(x_1) = 2(x_1)^2 - x_1 - 1$
$f(-1) = 2((-1)^2) - (-x) - 1$
$= 2(1) + 1 - 1$
$= 2 \quad f(x_2) = 2(x_2)^2 - x_2 - 1$
$f(3) = 2(3^2) - 3 - 1$
$= 2(9) - 3 - 1$
$= 14$

Therefore we have the points **(-1,2)** and **(3,14)**. Now we use the average rate of change formula:

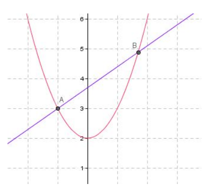
$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{14-2}{3--1} = \frac{12}{4} = 3$

Therefore the average rate of change is 3.

When you are finding the average rate of change you are actually finding the slope of the secant line look at the next example.

The secant line is the line that connects any two points on any graph of any given function

TABLE 2.12:

	First you must find the average rate of change between the two points A and B and use point slope form to get the equation of the line.

Example: Find the equation of the secant line for the given function at the given x values. A. $f(x) = x^2 + 1$ with $x_1 = 0$ and $x_2 = 3$

First we have to find the average rate of change by finding the corresponding y values:

$f(x_1) = (x_1)^2 + 1$
$f(0) = 0^2 + 1$
$= 0 + 1$
$= 1 \quad f(x_2) = (x_2)^2 + 1$
$f(3) = 3^2 + 1$
$= 9 + 1$
$= 10$

Therefore we have the points **(0, 1)** and **(3,10)**. Then use the average rate of change formula:

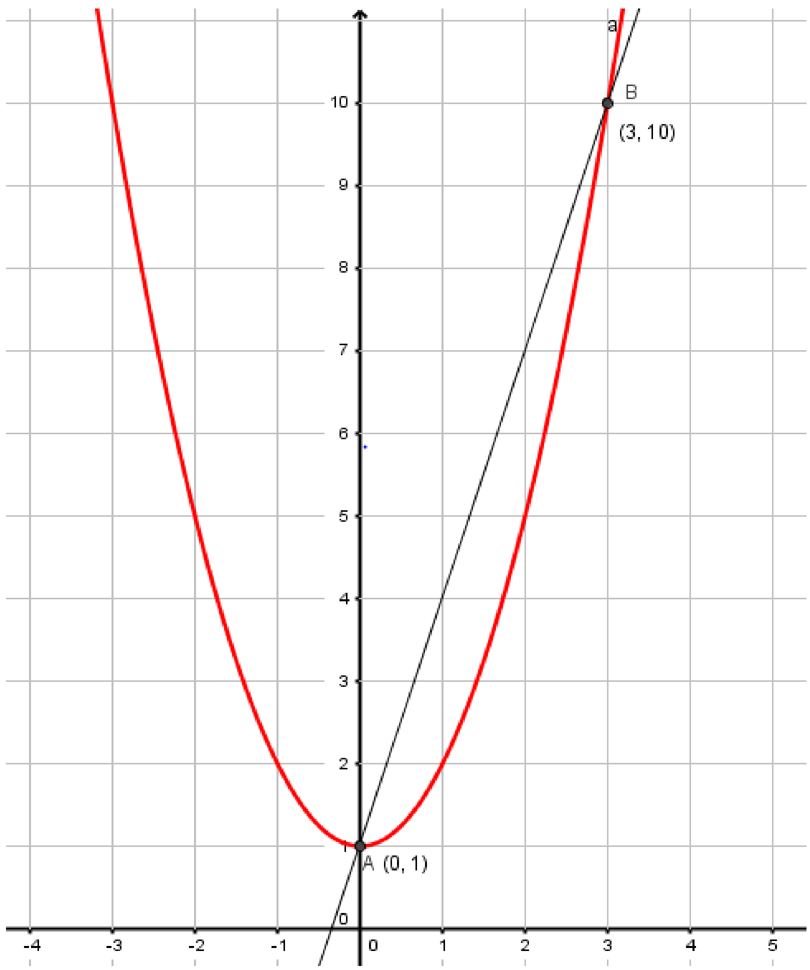
$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{10-1}{3-0} = \frac{9}{3} = 3$

Therefore the slope of the secant line passing through the points **(0, 1)** and **(3,10)** is 3.

Finally use point slope formula in this case we will also use the point (0, 1).

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 1 &= 3(x - 0) \\
y - 1 &= 3x - 0 \\
y &= 3x + 1
\end{aligned}
$$

The following graph shows the function and the secant line.



3. **Average Velocity of an Object**

The average velocity of an object is found by taking the quotient of the distance traveled and the time traveled.

Example: Find the average velocity of an object whose distance in feet is defined by $d(t) = 16t^2$ between 2 and 4 seconds.

First we need to find the distance at 4 and 2 seconds.

$amp; d(4) = 16(4)^2 = 16(16) = 256$
$amp; d(2) = 16(2)^2 = 16(4) = 64$

Therefore we have the points (4, 256) and (2, 64). Therefore the average velocity is

$\frac{256-64}{4-2} = \frac{192}{2} = 96\frac{ft}{sec}$

Vocabulary

Slope

Slope is the steepness of a linear function. Slope denoted by the letter m is found by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ when we have the points $(x_1, y_1)(x_2, y_2)$

Tangent Line

A tangent line is a line that touches a function at exactly one point

Secant Line

A secant line is a straight line that joins two points in a function

Point Slope Form

Given the slope m of a line and the point (x_1, y_1) then the point slope form of a line is given by the formula $y - y_1 = m(x - x_1)$

Slope Intercept Form

The slope intercept form of a line is the equation $y = mx + b$ where **m** is the slope and **b** is the y-intercept.

Standard Form

The Standard form of a line is the equation $Ax + By = C$where A is a positive integer and B and C are integers.

General Form

The General form of a line is the equation $Ax + By + C = 0$where A is a positive integer and B and C are integers.

average rate of change

The average rate of change is the slope of the secant line given two points. If we have the points $(a, f(a))$ *and* $(b, f(b))$the formula of the average rate of change is $\frac{f(b) - f(a)}{b - a}$

In Summary

We have leaned about linear functions. We have learned how to find the slope of a linear function, how to write the different form of the lines including the point slope form, the slope intercept form, the standard form, the general form, a tangent line, a secant line, and how to find the average rate of change.

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2.5 Transformations of Functions

Here you will learn to identify the maximums and minimums in various graphs and be able to differentiate between global and relative extreme values.

TEKS

1. P.2.G

Lesson Objectives

In this section you will learn about:

1. Parent Functions.
2. How to Transform Functions (Graphing Variations).

Introduction

Now that we know what functions are and one type of function (linear functions) we can look at different other functions and how the graphs can be transformed. We will learn how to graph many different variations of the "common" functions by looking at the patterns of change.

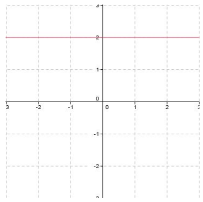
Vocabulary

parent function, horizontal translation, vertical translation, reflection, compression, stretch

1. **Parent Functions**

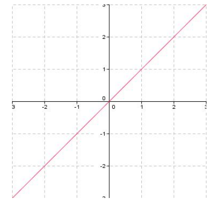
TABLE 2.13:

Constant Function



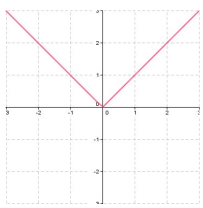
- $f(x) = c$
- Domain:$(-\infty, \infty)$
- Range:$[x]$
- Constant:$(-\infty, \infty)$
- Even Function

Linear Function



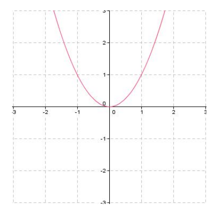
- $f(x) = x$
- Domain:$(-\infty, \infty)$
- Range:$(-\infty, \infty)$
- Increasing:$(-\infty, \infty)$
- Odd Function

Absolute Value Function



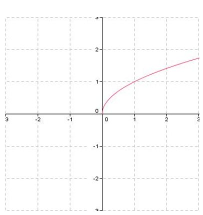
- $f(x) = |x|$
- Domain:$(-\infty, \infty)$
- Range:$[0, \infty)$
- Increasing:$(0, \infty)$
- Decreasing:$(-\infty, 0)$
- Even Function

Quadratic Function



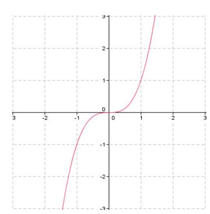
- $f(x) = x^2$
- Domain:$(-\infty, \infty)$
- Range:$[0, \infty)$
- Increasing:$(0, \infty)$
- Decreasing:$(-\infty, 0)$
- Even Function

Square Root Function



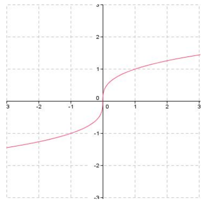
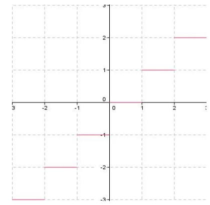
- $f(x) = \sqrt{x}$
- Domain:$[0, \infty)$
- Range:$[0, \infty)$
- Increasing:$[0, \infty)$
- Neither Odd nor Even Function

Cubic Function



- $f(x) = x^3$
- Domain:$(-\infty, \infty)$
- Range:$(-\infty, \infty)$
- Increasing:$(-\infty, \infty)$
- Odd Function

TABLE 2.13: (continued)

Cube Function	**Greatest Integer Function**
	
• $f(x)=\sqrt[3]{x}$ • Domain:$(-\infty,\infty)$ • Range:$(-\infty,\infty)$ • Increasing:$(-\infty,\infty)$ • Odd Function	• $f(x)=[x]$ • Domain: $(-\infty,\infty)$ • Range: all integers • Constant on $[k,k+1)$ for any integer k • Odd Function

1. **Transforming Functions (Graphing Variations)**

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The word translation means to shift or in geometry we call it a slide. It means to slide a geometric figure by adding or subtracting to the figure's endpoints depending on if you are sliding it up, down, left or right. We can do the same to functions, since functions can be represented as an infinite set of points.

Vertical Translation

TABLE 2.14:

Equation	**Words**
$y=f(x)+c$	raise the graph c units up
$y=f(x)-c$	lower the graph c units down

Horizontal Translation

TABLE 2.15:

Equation	**Words**
$y=f(x-c)$	shift right c units
$y=f(x+c)$	shift left c units

The word reflection in geometry we refer to as a "mirror" image. When we reflect there are two ways, across the x-axis or y-axis. For functions we say reflect with respect to the x-axis or y-axis.

Reflecting with Respect to the Y-Axis

TABLE 2.16:

Equation	Words
$y = f(-x)$	Reflect points across the y-axis

Reflecting with Respect to the X-Axis

TABLE 2.17:

Equation	Words
$y = -f(x)$	Reflect points across the x-axis

Stretching and Shrinking is also known as compressions and stretches. These are the act of making the graph "taller" and "narrower" or "shorter" and "wider".

Vertical Stretching/Shrinking

TABLE 2.18:

Equation	Words
$y = cf(x), c > 1$	Stretching vertically by c
$y = cf(x), 0 < c < 1$	Shrinking vertically by c

Horizontal Stretching/Shrinking

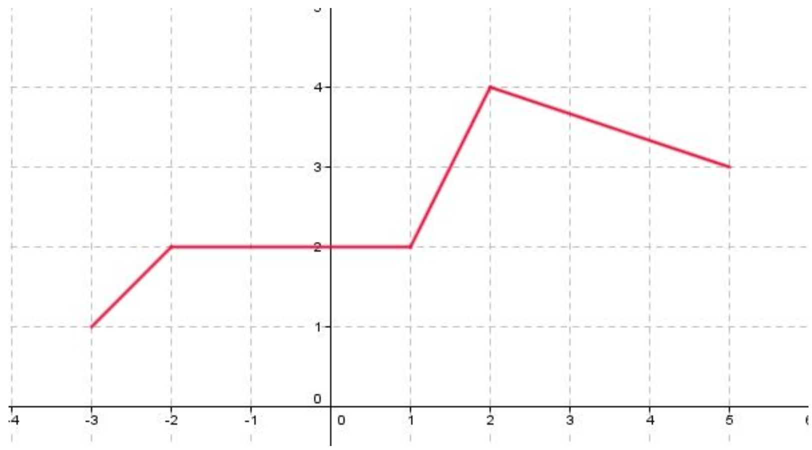
TABLE 2.19:

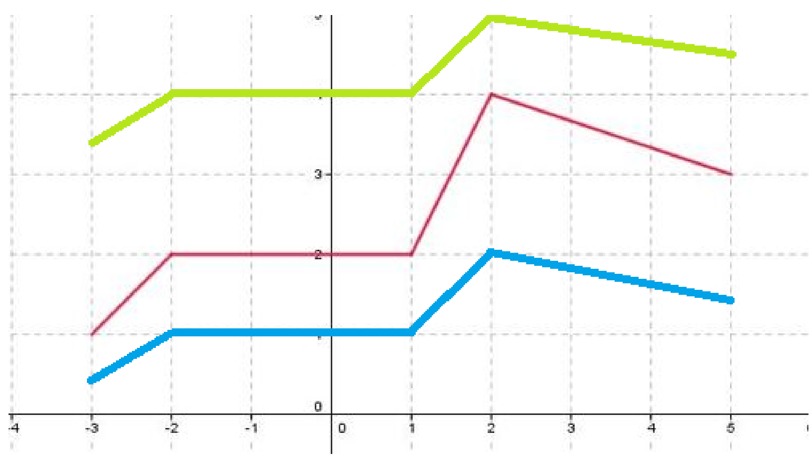
Equation	Words
$y = f(cx), c > 1$	Shrinking horizontally by $\frac{1}{c}$
$y = f(cx), 0 < c < 1$	Stretching horizontally by $\frac{1}{c}$

A suggested way to perform transformations is to use the sequence:

1. Stretch/Shrink
2. Reflections.
3. Horizontal Translation
4. Vertical Translation

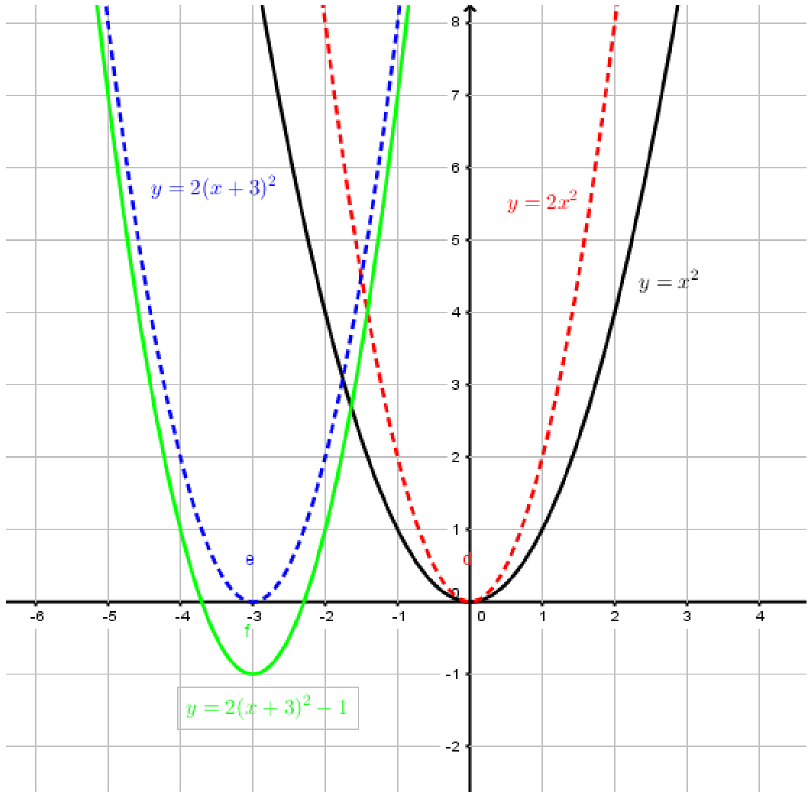
Example: Use the graph of $f(x)$ to graph $y = \frac{1}{2}(f(x)) + 3$.





The blue graph is vertically compressing the original red function by half first then green graph is when it is being translated up three units.

Example: Use the graph of $f(x) = x^2$ to graph $y = 2(x+3)^2 - 1$.



The parent function is the solid black graph.

Then red dotted graph is the vertical stretch by a factor of 2.

The blue dotted graph is the vertical stretch by factor 2 and the translation to the left 3 units.

Finally the green solid graph is the vertical stretch by factor 2, the translation of 3 units to the left and the translation down by 1 unit.

Vocabulary

Parent Functions

The simplest function of a family of functions for example $y=x, \ \ y=x^2, \ \ y=|x|, \ \ y=\sqrt{x}$are examples of parent functions.

Horizontal Translation

A translation left or right in a function. In a function f(x), the translation happens inside with x like **f(x-c)** and it translates **c** units opposite of the sign. If minus moves right, if plus moves left.

Vertical Translation

A translation up or down in a function. In a function f(x), the translation happens outside the x like **f(x)+d** and it translates **d** units accordingly to the sign. If plus moves up, if minus moves down.

Reflection

A reflection is a transformation in which a parent function reflects over the x or the y axis. If we have the function f(x) a vertical reflection (over x-axis) is denoted by -f(x) and a horizontal reflection(over y) is denoted by f(-x).

Stretch/Compression

A stretch/compression is a transformation that stretches and compresses a function f(x). A vertical stretch/compression is denoted by **af(x)** by a factor **a**. For a horizontal compression/stretch, it is denoted by f(bx) where the compression/stretch is by a factor of $\frac{1}{b}$

In Summary

We have leaned about the basic parent functions. We have learned about how to transform parent functions by performing vertical and horizontal translations, vertical and horizontal compressions and stretching and finally reflections over the x-axis and y-axis

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2.6 Algebra of Functions

TEKS

1. P.1.F

Lesson Objectives

In this section you will learn about:

1. Domains of Functions.
2. Adding and Subtracting Functions.
3. Multiplying and Dividing Functions.

Introduction

We can combine functions by using algebra of functions: addition, subtraction, multiplication, and division. We have to be be careful when using these operations to create a new function for the domain can change!

Vocabulary

domain, algebra of functions

When we are trying to find the domains of functions we run into problems when we encounter rational functions or square root (radical) functions therefore we need to take into account the following.

a. If the function is a polynomial function, the domain is all real numbers.
b. If the function is a rational function (fraction), the domain is all real numbers except the zeros of the denominator.
c. If the function is an even radical function like $y = \sqrt{x}$, then the inside of the function must be greater than or equal to zero,

1. **Domains of Functions**

The domain of a function is the largest set of all possible x values where the graph exists. We must exclude values that cause a division by zero or non-real numbers, like taking a square root of a negative.

Example: What is the domain of the following functions in interval notation?

A. $y = 3x^2 + 2x + 1$

Since this function does not have division by zero or the possibility of non real numbers, the domain is all real numbers $(-\infty, \infty)$

B. $y = \sqrt{3x-1}$

This function has the possibility of non real numbers because the square root can never be less than zero, therefore set the inside of the radical to greater than or equal to zero and solve as follows:

$$\begin{aligned} 3x - 1 &\geq 0 \\ 3x &\geq 1 \\ x &\geq \frac{1}{3} \end{aligned}$$

or $\left[\frac{1}{3}, \infty\right)$.

C.$y = \frac{x^2}{x+5}$

Since this is a rational function, then we need to check if the denominator can be zero by setting it equal to zero. Therefore set the denominator equal to zero and solve as follows:

$$\begin{aligned} x + 5 &\neq 0 \\ x &\neq -5 \end{aligned}$$

or $(-\infty, -5) \bigcup (-5, \infty)$

Which means all real numbers except -5.

NOTE: Rational functions will be discussed more in depth in chapter 3.

2. **Adding and Subtracting Functions**

Given that $f(x)$ and $g(x)$ are two functions. Then $(f+g)(x) = f(x) + g(x)$ and $(f-g)(x) = f(x) - g(x)$

To simplify, we combine like terms. Because we are combining two functions to make a new function we must state the new domain by checking each individual domain and the new domain will be the intersection of both domains.

Example: Find $(f+g)(x)$ and $(f-g)(x)$ and state the new domain for each.

A. $f(x) = 3x + 5; g(x) = x^2 + 2x - 4$

$(f+g)(x) = (3x+5) + (x^2 + 2x - 4)$
$= 3x + 5 + x^2 + 2x - 4$
$= x^2 + 5x + 1 \quad (f-g)(x) = (3x+5) - (x^2 + 2x - 4)$
$= 3x + 5 - x^2 - 2x + 4$
$= -x^2 + x + 9$

Domain of **f(x) =** $(-\infty, \infty)$

Domain of **g(x) =** $(-\infty, \infty)$

Therefore the domains of $(f+g)(x) = (-\infty, \infty)$ *and* $(f+g)(x) = (-\infty, \infty)$

The domain for both is all real numbers.

Example:

B. $f(x) = 2x^2 + 4; g(x) = \sqrt{x+1}$

$(f+g)(x) = (2x^2+4) + (\sqrt{x+1})$
$= 2x^2 + 4 + \sqrt{x+1}$ $(f-g)(x) = (2x^2+4) - (\sqrt{x+1})$
$= 2x^2 + 4 - \sqrt{x+1}$

Domain of **f(x) =** $(-\infty, \infty)$

Domain of **g(x) =** $x \geq -1$

Since the domain is not all real numbers for both f(x) and g(x) then we need to find the intersection of both of them. Both domains will intersect for all real numbers greater than or equal to -1.

Therefore the domain will be $[-1, \infty)$

NOTE: Think about it if we choose a number less than -1 the inside of the square root will be a negative number and the function will be undefined.

3. **Multiplying and Dividing Functions**

Given $f(x)$ and $g(x)$ are two functions, then $(fg)(x) = f(x)g(x)$ and $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Again, like adding and subtracting we are creating a new function so we must find and state the new domain.

Example:

A. $f(x) = 3x + 5; g(x) = x^2 + x - 12$

$(fg)(x) = (3x+5)(x^2+x-12)$
$= 3x^3 + 3x^2 - 36x + 5x^2 + 5x - 60$
$= 3x^3 + 8x^2 - 31x - 60$

The domain of **f(x) =** $(-\infty, \infty)$

The domain of **g(x) =** $(-\infty, \infty)$

Therefore the domain of the new function **(fg)(x) =**$(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{3x+5}{x^2+x-12}$

When you divide polynomials, finding the domain is a little different, you also need to take into account that in a fraction, the denominator cannot be zero

The domain of **f(x) =** $(-\infty, \infty)$

The domain of **g(x) =** $(-\infty, \infty)$

So the domain should be all real numbers, **NOT REALLY** since we have a rational function now we need to find out when the denominator is zero.

By using the quadratic formula or by factoring we can find out that the denominator is zero at x=-4 and x=3.

Finally the domain of $\left(\frac{f}{g}\right)(x) = (-\infty, -4] \bigcup (-4, 3) \bigcup (3, \infty)$which means all real numbers except -4 and -3.

Example:

B. $f(x) = x^2 - 5; g(x) = \sqrt{x-3}$

$(fg)(x) = (x^2-5)(\sqrt{x-3})$

The domain of **f(x) =** $(-\infty, \infty)$

The domain of **g(x) =** $[3,\infty)$

The domain of the new function **(fg)(x) =** $[3,\infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x^2-5}{\sqrt{x-3}}$

The domain of **f(x) =** $(-\infty,\infty)$

The domain of **g(x) =** $[3,\infty)$

The domain of the new function **(f/g)(x) =** $(3,\infty)$

In the final answer of this problem we had to exclude the 3 from the domain because it makes the denominator be zero.

Vocabulary

Domain

The set of values of the independent variable for which a function is defined

Algebra of functions

The sum, difference, product and quotient of functions.

In Summary

We have learned about the algebra of functions and their domains. We have learned that when we have radical functions and quotient/rational functions we have to be careful with the domain and we might have to exclude some values. Finally we learned how to multiply and divide functions.

Check for Understanding:

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2.7 Composition of Functions

TEKS

1. P.2.A
2. P.2.B
3. P.2.C

Lesson Objectives

In this section you will learn about:

1. Composition of Functions.
2. Evaluating Compositions.
3. Writing Functions as a Composition.

Introduction

In the previous section we learned about combining functions using algebraic operations; adding, subtracting, multiplying and dividing. We also learned that when we perform certain operations our domain changes. In this section we are going to be substituting all input values in one function with the equation of the other function.

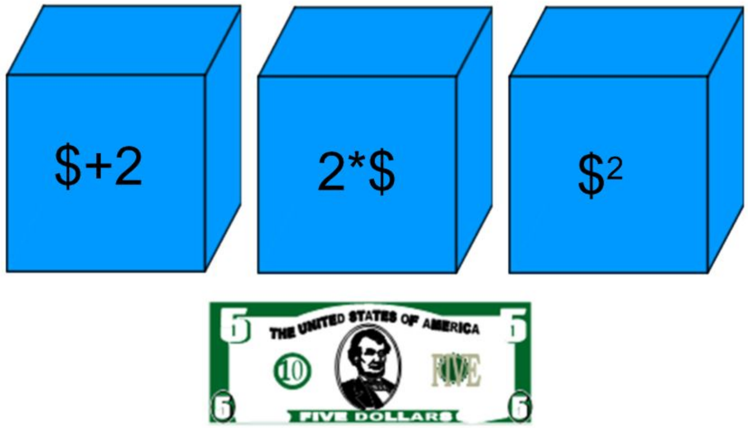
Vocabulary

composition of functions,$g \circ f$, $f \circ g$

1. **Composition of Functions**

lets say we have three boxes that if I put money inside it either adds two dollars, doubles it, or squares it. Once the box is used it can't be used again. I have $5.00, using only two boxes (one time each) what is the best sequence to get the most money?

Box A Box B Box C



Let us try different combinations:

If we put it in box A we obtain \$7 then in box B we obtain \$14.

If we put it in box A we obtain \$7 then in box C we obtain \$49.

If we put it in box B we obtain \$10 then in box A we obtain \$12.

If we put it in box B we obtain \$10 then in box C we obtain \$100.

If we put it in box C we obtain \$25 then in box A we obtain \$27.

If we put it in box C we obtain \$25 then in box B we obtain \$50.

This is a composition of functions.

We start with an input value or x and get a new value from one function. Then we take the new value and use it as an input for a second function to get a whole new value. In this case we would choose to put the money in box B then in box C obtaining the maximum amount possible.

Given $f(x)$ and $g(x)$ two functions then the composition of functions is defined by:

$(f \circ g)(x) = f(g(x))$ Where the range of g is used as the domain of f.

$(g \circ f)(x) = g(f(x))$ Where the range of f is used as the domain of g.

Obtaining the Domain of a composition of functions.

If we are trying to obtain the domain of the composition $(f \circ g)(x)$ then:

- - x has to be an element of the domain of g
  - g(x) has to be an element in the domain of f

In other words the x has to be in the domain of the inside function first and then g(x) has to be in the domain of the function f.

NOTE: We also need to take into account that if x is not in the domain of the inside function g, then it must not be in the domain of the composition (f o g)(x). and if g(x) is not in the domain of f then is not in the domain of (f o g)(x).

Example A

Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and state the domain of each.

$f(x) = 3x+2; g(x) = 2x+1$

$(f \circ g)(x) = f(g(x))$
$= f(2x+1)$
$= 3(2x+1)+2$
$= 6x+3+2$
$= 6x+5 \quad (g \circ f)(x) = g(f(x))$
$= g(3x+2)$
$= 2(3x+2)+1$
$= 6x+4+1$
$= 6x+5$

In this case the domain of (f(g))(x) and (g(f(x))(x) is all real numbers.

Example B

$f(x) = x+4; g(x) = 2x^2$

$(f \circ g)(x) = f(g(x))$
$= f(2x^2)$
$= (2x^2)+4$
$= 2x^2+4 \quad (g \circ f)(x) = g(f(x))$
$= g(x+4)$
$= 2(x+4)^2$
$= 2(x^2+8x+16)$
$= 2x^2+16x+32$

The domain is all real numbers for both compositions.

Example C

$f(x) = 2x+1; g(x) = \sqrt{x+1}$

$(f \circ g)(x) = f(g(x))$
$= f(\sqrt{x+1})$
$= 2(\sqrt{x+1})+1$
$= 2\sqrt{x+1}+1 \quad (g \circ f)(x) = g(f(x))$
$= g(2x+1)$
$= \sqrt{(2x+1)+1}$
$= \sqrt{2x+2}$

the domain is all real numbers greater than -1.

Example D

$f(x) = x-4; g(x) = x^2+2x+1$

$(f \circ g)(x) = f(g(x))$
$= f(x^2+2x+1)$
$= (x^2+2x+1)-4$
$= x^2+2x+1-4$
$= x^2+2x-3 \quad (g \circ f)(x) = g(f(x))$
$= g(x-4)$

$= (x-4)^2 + 2(x-4) + 1$
$= x^2 - 8x + 16 + 2x - 8 + 1$
$= x^2 - 6x + 9$

The domain for both functions is all real numbers.

Example E

$f(x) = \frac{2}{x+5}; \;\; g(x) = \frac{1}{x}$

Find the domain of $(f \circ g)(x)$

The domain of g(x) is all real numbers except zero. Therefore we need to exclude zero from the domain.

The domain of f(x) is all real numbers except -5. Therefore **g(x)**$\neq$ **-5** by setting solving **1/x = -5** and solving for **x, x** $\neq$ **-1/5**

Therefore the domain of the composition (f o g)(x) is all real numbers except **-1/5** and **0**.

2. **Evaluating Compositions**

Just like any other function we can substitute values of x with a number and find the result.

Example F:

Evaluate $(f \circ g)(2)$ and $(g \circ f)(-2)$.

A. $f(x) = 3x^2 + 2; g(x) = 4x^2 + 2x - 1$

$(f \circ g)(2) = f(g(2))$
$= f(4(2)^2 + 2(2) - 1)$
$= f(16 + 4 - 1)$
$= f(19)$
$= 3(19)^2 + 2$
$= 3(361) + 2$
$= 1085 \quad (g \circ f)(-2) = g(f(-2))$
$= g(3(-2)^2 + 2)$
$= g(3(4) + 2)$
$= g(12 + 2)$
$= g(14)$
$= 4(14)^2 + 2(14) - 1$
$= 4(196) + 28 - 1$
$= 811$

B **.**

$f(x) = x^2 + 4; g(x) = \sqrt{x+5}$

$(f \circ g)(2) = f(g(2))$
$= f(\sqrt{2+5})$
$= f(\sqrt{7})$
$= (\sqrt{7})^2 + 4$
$= 7 + 4$
$= 11$

$(f \circ g)(2) = 11$

$(g \circ f)(2) = g(f(2))$
$= g((2)^2 + 4)$
$= g(4+4)$
$= g(8)$
$= \sqrt{8+5}$
$= \sqrt{13}$

$(g \circ f)(2) = \sqrt{13}$

We can also evaluate using tables as well. Recall that in the definition of composition of functions the range of one function turns into the domain for the second function. In other words, we use the output of one function as the input of the other.

Example G: Use the tables to evaluate.

TABLE 2.20:

x	$f(x)$	x	$g(x)$
-2	5	-2	-2
0	6	0	3
1	4	1	5
2	2	2	1
3	1	3	0

A. $(f \circ g)(-2)$

$$(f \circ g)(-2) = f(g(-2)) = f(-2) = 5$$

B. $(g \circ f)(2)$

$$(g \circ f)(2) = g(f(2)) = g(2) = 1$$

C. $(f \circ g)(2)$

$$(f \circ g)(2) = f(g(2)) = f(1) = 4$$

D. $(g \circ f)(3)$

$$(g \circ f)(3) = g(f(3)) = g(1) = 5$$

E. $(f \circ f)(3)$

$$(f \circ f)(3) = f(f(3)) = f(1) = 4$$

3. Writing Functions as a Composition

In this final objective we will learn that there are many ways to write a function using a composition. In a sense it is like working backwards.

Given $h(x)$ is a function, we can re-write it as a composition of two functions f and g:

$h(x) = (f \circ g)(x) = f(g(x))$

Example H: Define two functions f and g such that $(f \circ g)(x) = h(x)$.

A. $h(x) = x^2 + 2x + 1$

$h(x) = x^2 + 2x + 1$
$= (x+1)(x+1)$
$= (x+1)^2$

$$f(x) = x^2$$
$$g(x) = x + 1$$

B. $h(x) = \sqrt{x+5}$

$$f(x) = \sqrt{x}$$
$$g(x) = x + 5$$

C. $h(x) = \frac{1}{x-2}$

$$f(x) = \frac{1}{x}$$
$$g(x) = x - 2$$

Vocabulary

Composition of functions

The composition of the function f with the function g is denoted by $f \circ g$ and is defined by the equation $f(g(x))$and it is read as f of g of x

In Summary

We have learned about what composition of functions are and their characteristics. We have learned how to perform a composition, how to evaluate a composition and how to rewrite a function as a composition of two functions.

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2.8 Inverse Functions

TEKS

1. P.2.E
2. P.2.F

Lesson Objectives

In this section you will learn about:

1. Definition of an inverse.
2. Horizontal line test.
3. Verifying Inverses.
4. Finding Inverses.
5. Graphing Inverses.

Introduction

We have learned that we can take two functions and substitute them into one and other. If we do this composition $(f \circ g)(x)$ and $(g \circ f)(x)$, simplify and end up with just x then the two functions are inverses of each other, along with some other special properties.

Vocabulary

inverse, horizontal line test

The use of inverse functions is in our every day life without you probably knowing it. Every time you send a text or a picture on your phone or computer, the phone converts your message into a code by performing a function and the receiving phone must use the inverse function you applied to put the message back together. This process is the same with pictures. So what is an inverse and how it is found?

1. What is an inverse?

- An inverse is a relation in which the ordered pair (x,y) is changed to the ordered pair (y,x) and it is denoted by f^{-1}

- In other words, we need to switch the x and the y coordinates of all the points in the relation.
- Graphically, an inverse is a reflection over the y =x line.
- If we have the function f(x) then with domain and range, then the range becomes the domain of the inverse and the domain becomes the range of the inverse.
- The inverse of a function is also a function if and only if the function is a one to one function. If it is not one to one function, then we can restrict the domain to find an inverse function that is a function.
- We can use the horizontal line test to determine if the function is one to one.

Example A.

Given the relation f = $\{(1,5),(2,10),(3,15),(4,20)\}$

a) Find the domain and range of f

b) Find the inverse

c) Find the domain and range of the inverse.

a) The domain of f = {1, 2,3, 4}, the range of f ={5, 10, 15, 20}

b) The inverse of f = $\{(5,1),(10,2),(15,3),(20,4)\}$

c) The domain of the inverse = {5, 10, 15, 20} and the range {1, 2, 3, 4}

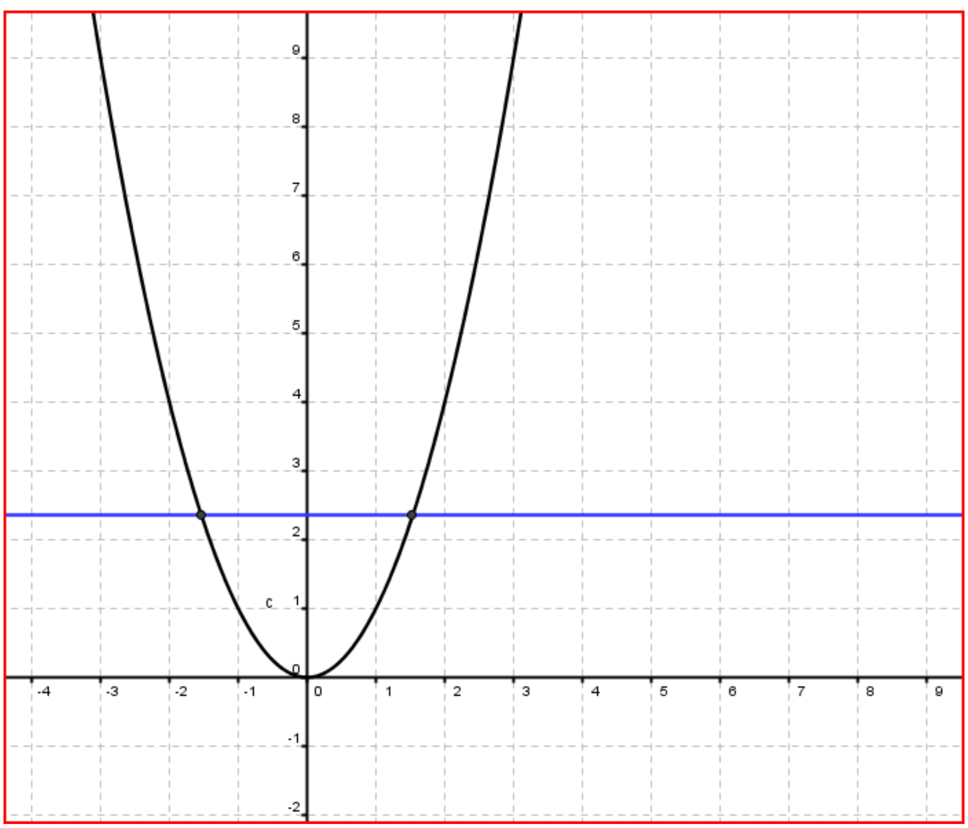
As you can see the domain and range of the function switch around on the inverse function.

2.Horizontal Line Test

The horizontal line test works just like the vertical line test. Sometimes when we find an inverse of a function, the inverse is strictly a relation, not a function. There is a way to tell if a function will have an inverse that is also a function without having to graph the inverse, it is the horizontal line test.

Example B.

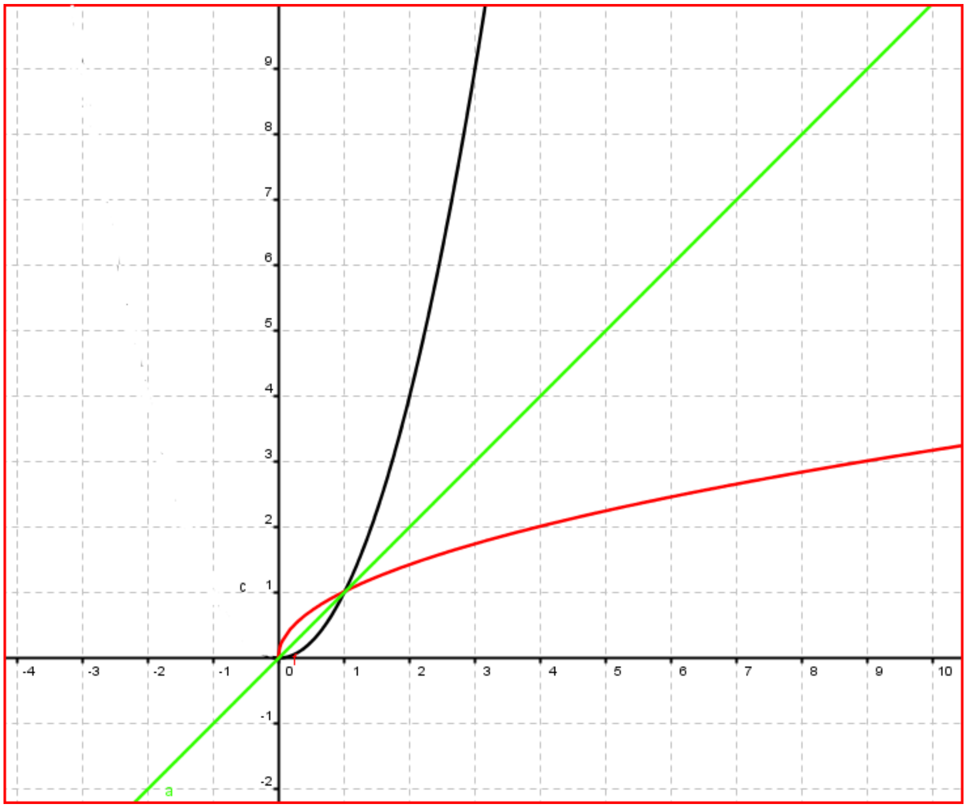
Lets graph the function $y = x^2$and perform the horizontal line test.



Obviously the line touches more than once so it is not a one to one function. Since the domain is all real numbers we need to restrict it to $[0,\infty)$

Now it is a one to one function and we can have an inverse function.

The following is the graph with the restricted domain and the graph of its inverse.



3. Verifying Inverses

Given $f(x)$ and $g(x)$ two functions, if $(f\circ g)(x)=x$ **AND** $(g\circ f)(x)=x$ then $g(x)$ is the inverse of $f(x)$, we can then say $g(x)=f^{-1}(x)$. The domain and range of f switch to give us the function g.

Example C: Verify that f and g are inverses.

A. $f(x)=2x+1; g(x)\frac{x-1}{2}$

$$
\begin{aligned}
&(f\circ g)(x)=f(g(x))\\
&=f\left(\tfrac{x-1}{2}\right)\\
&=2\left(\tfrac{x-1}{2}\right)+1\\
&=x-1+1\\
&=x \quad (g\circ f)(x)=g(f(x))\\
&=g(2x+1)\\
&=\tfrac{(2x+1)-1}{2}\\
&=\tfrac{2x+1-1}{2}\\
&=\tfrac{2x}{2}\\
&=x
\end{aligned}
$$

Therefore since (f o g)(x) = x = (g o f)(x) then f and g are inverses

Note: Both compositions must be equal to x for the functions to be inverses.

B. $f(x)=3x+2; g(x)\frac{x}{3}-2$

$(f \circ g)(x) = f(g(x))$
$= f\left(\frac{x}{3} - 2\right)$
$= 3\left(\frac{x}{3} - 2\right) + 2$
$= \frac{3x}{3} - 6 + 2$
$= x - 4$

Since (f o g)(x) $\neq$ x automatically f(x) and g(x) **are not** inverses and we don't have to check the other composition.

4. Finding Inverses

When we perform the composition of functions to verify if two functions are inverses we are replacing the domain of one function with the range of the other, meaning the domain and range switch. The domain or independent variable *x* then switches with the range or dependent variable *y*.

Steps to Find an Inverse:

1. Write the equation in the form $y = f(x)$.
2. Switch the *x* and *y* (switch the domain and range)
3. Solve the equation for *y*
4. Write the equation in the form $f^{-1}(x) = y$

To check if your inverse is correct you can always use composition of functions to verify!

Example: Find the inverse.

A. $f(x) = 5x + 2$

$$\begin{aligned} y &= 5x + 2 \\ x &= 5y + 2 \\ x - 2 &= 5y \\ \frac{x-2}{5} &= y \\ f^{-1}(x) &= \frac{x-2}{5} \end{aligned}$$

B. $f(x) = \frac{x}{2} + 3$

$$\begin{aligned} y &= \frac{x}{2} + 3 \\ x &= \frac{y}{2} + 3 \\ x - 3 &= \frac{y}{2} \\ 2(x-3) &= y \\ f^{-1}(x) &= 2x - 6 \end{aligned}$$

C. $f(x) = \frac{15}{2x+1}$

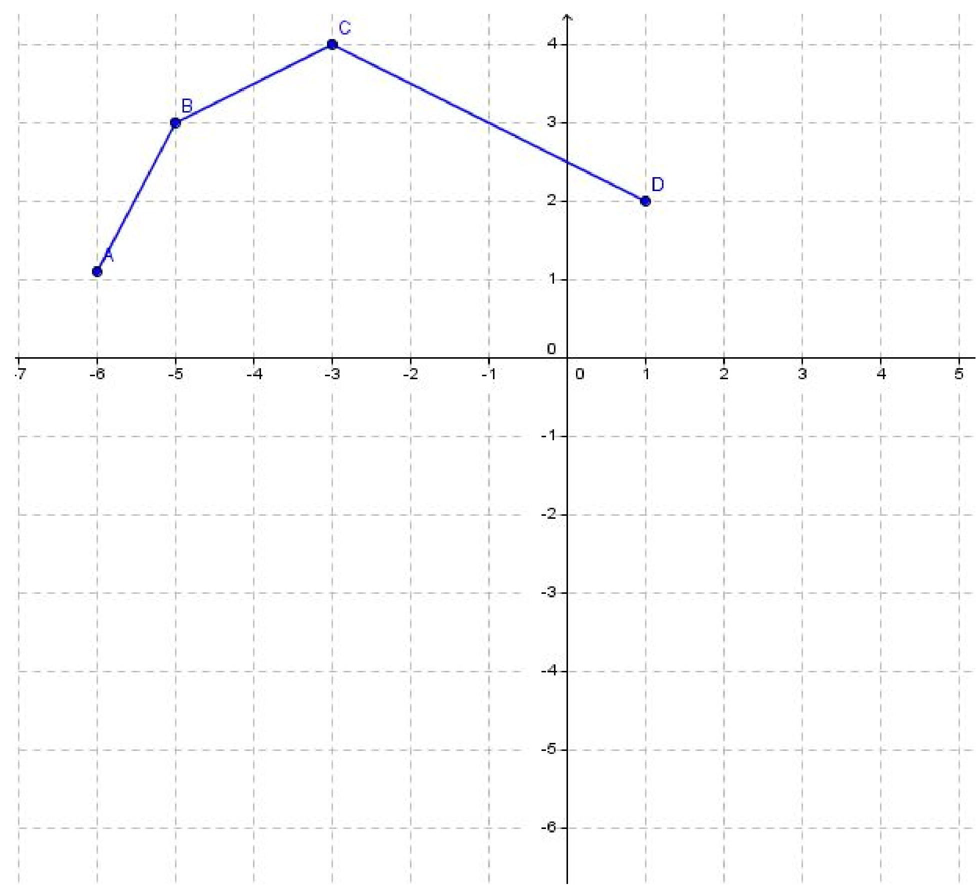
$$\begin{aligned}
y &= \frac{15}{2x+1} \\
x &= \frac{15}{2y+1} \\
x(2y+1) &= 15 \\
2xy+x &= 15 \\
2xy &= 15-x \\
y &= \frac{15-x}{2x} \\
f^{-1}(x) &= \frac{15-x}{2x}
\end{aligned}$$

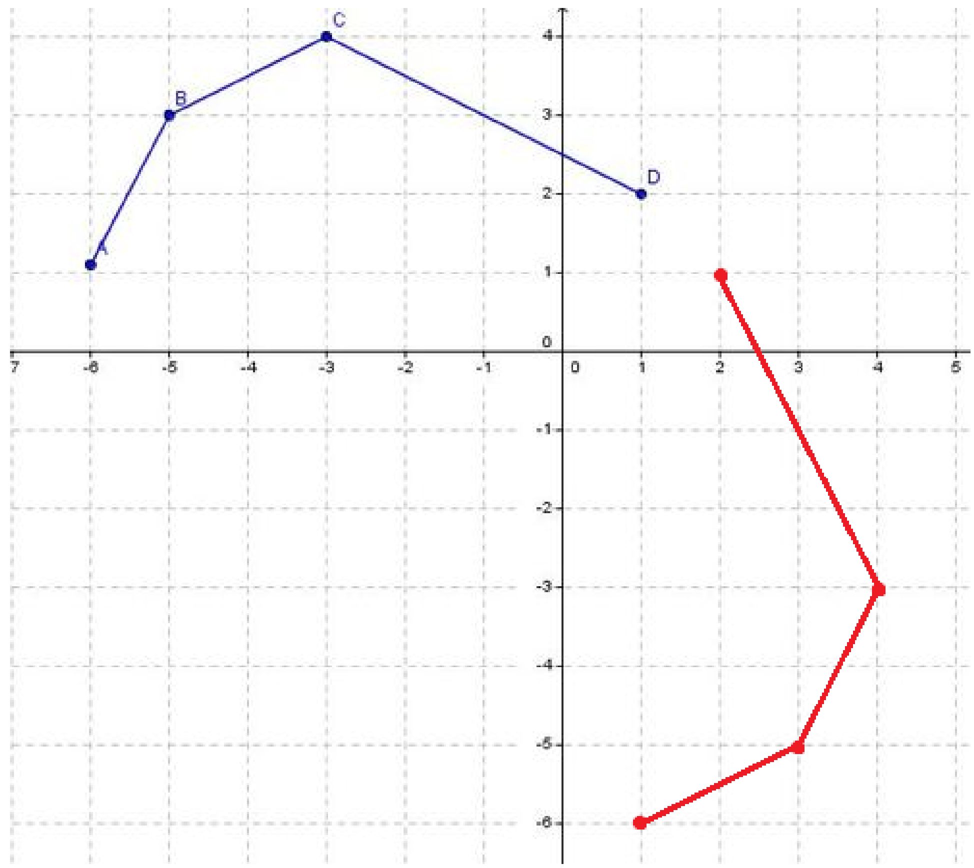
5. Graphing Inverses

Since the domain and range switch in the function rules, we can switch the x and y values in the ordered pairs of the coordinates of the points of any relation to get an inverse relation.

Graphically an inverse can be seen as a reflection across the line $y = x$.

Example: Graph the inverse of the relation given the graph.





The red graph is the inverse of the blue function. Notice that the red graph is a reflection of the blue graph over the line y = x and is not a function because the original function is not a one to one function.

Vocabulary

Horizontal Line Test

Is a test that is used to verify if a function is a one-to-one function.

Inverse Function

In math, an inverse function is a function that reverses another function. If the function f(x) is a mapping from $x \rightarrow y$, then $f^{-1}(x)$ would be a mapping from $y \rightarrow x$..

In Summary

We have learned about the definition of an inverse function and how the horizontal line test is used to determine if the function's inverse is also a function. We have learned how verify if two functions are inverses of each other, how to find inverses and how to graph inverses given a graph.

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Key features of functions were explored through the use of ten basic function families. Transforming functions and finding inverses of functions were also considered.

CHAPTER 3 Polynomials and Rational Functions

Chapter Outline

Here you will deepen you knowledge about polynomials from your work in algebra. You will review factoring and learn more advanced factoring techniques. You will learn how to divide polynomials and how polynomials and rational expressions are related. You will then focus on rational functions and learn about their unique features including three different types of asymptotes. Finally, you will put it all together and learn how to graph rational functions by hand.

3.1 Complex numbers

TEKS

1. P.1.F
2. P.2.F
3. P.2.I

Lesson Objectives

In this lesson you will learn about:
1. The definition of a complex number and what a complex conjugate is.
2. Adding and Subtracting complex numbers.
3. Multiplying Complex Numbers.
4. Dividing Complex Numbers.
5. Performing operations that involve square roots of negative numbers.
6. Solving quadratic equations with complex imaginary solutions.

Introduction

When solving quadratic equations, we sometimes don't have real solutions for example the function $f(x) = x^2 + 1$ never crosses the x-axis so it has no real solutions, but it has imaginary/complex number solutions. In this section you will learn what a complex number is and to perform operations with complex numbers.

Vocabulary

Imaginary unit i, complex number, complex conjugate.

The imaginary unit *i*

The **imaginary unit i** is defined as $i = \sqrt{-1}$ where $i^2 = -1$.

Complex Numbers and Imaginary Numbers

The set of all numbers in the form $a + bi$ is called the **set of complex numbers**.

In this set of numbers the a part is called the real part, the bi is called the imaginary part where $b \neq 0$.

Complex Conjugate of a complex Number

If you have a complex number in the form $a+bi$ then the number $a-bi$ is called the **complex conjugate** of the complex number.

NOTE: The only part that changes is the imaginary part, many students do the mistake of also changing the real part.

We usually use the imaginary number i to express the square roots of negative numbers. For example:

$$\sqrt{-36} = \sqrt{-1}\sqrt{36} = \pm 6i$$

In the next couple of examples, you will learn how to add, subtract, multiply and divide imaginary numbers.

Adding/Subtracting imaginary numbers

To add and subtract imaginary numbers, you have to add/subtract **real part with real part** and **imaginary part with imaginary part.**

Example A.

Adding and subtracting complex numbers

Add and subtract the following two imaginary numbers $(5-2i)$ **and** $(3-5i)$

Solution

$(5-2i)+(3-5i) = (5+3)+(-2i+-5i) = 8-7i$

$(5-2i)-(3-5i) = (5-3)+(-2i--5i) = 2+3i$

Adding and subtracting is not too complicated, you just need to be careful when you subtract, don't forget to distribute the subtraction to both parts of the complex number.

Multiplication is very similar as multiplying binomials, we are going to require the help of our friend the **FOIL** method.

Example B

Multiply $7i(2-i)$

To perform this multiplication we apply the distributive property in this case

$(7i)(2)+(7i)(-i) = 14i-7i^2$

using our knowledge that $i^2 = -1$

then we obtain $14i+7$ therefore our answer is $7+14i$

Note: Make sure you write it in complex form format $a+bi$

Example C

Multiplying complex numbers

Multiply $(-4+12i)$ **and** $(3-7i)$

Solution

By using the distributive property and FOIL method we obtain the following

$(-4+12i)(3-7i) = (-4)(3)+(-4)(-7i)+(12i)(3)+(12i)(-7i) = 12+28i+36i-84i^2$

recall that $i^2 = -1$ therefore the -84 becomes +84 and by combining like terms we obtain

$96 + 64i$ which is our answer.

Example D

Multiplying complex conjugates

Multiply $(-5-3i)(-5+3i)$

Solution

By performing FOIL we obtain:

$$\begin{aligned}(-5)(-5)+(-5)(3i)+(-3i)(-5)+(-3i)(3i)\\=25-15i+15i-9i^2\\=25+9\\=34\end{aligned}$$

Notice that the imaginary numbers disappeared leaving us with just a real number.

Lets analyze what happens when you multiply a complex number and its complex conjugate in general

$$\begin{aligned}(a+bi)(a-bi)\\=(a)(a)+(a)(-bi)+(bi)(a)+(bi)(-bi)\\=a^2-abi+abi-b^2i^2\\=a^2+b^2\end{aligned}$$

Using this fact in example 3, we would obtain $(-5)^2+(-3)^2 = 25+9 = 34$.

Knowing this fact will help us a lot in the division of complex numbers.

Dividing Complex Numbers

Recall that a complex number is a number in the form $a+bi$ therefore when we divide complex numbers our answer should be in the same format, to accomplish the division we need to multiply by the complex conjugate of the denominator so that the denominator becomes a real number.

Strategy to divide complex numbers.

1. Identify the complex number in the denominator.
2. Find its complex conjugate.
3. Multiply by a value of 1 which will be $\frac{a-bi}{a-bi} where \ \ a-bi$ its the complex conjugate.
4. Use the complex conjugate multiplication fact to get the denominator.
5. FOIL the numerator and simplify.
6. Write your solution in complex number form.

Example E

Dividing Complex Numbers

Divide the following $\frac{7-3i}{3+4i}$

The complex number in the denominator is $3+4i$ and its complex conjugate is $3-4i$ therefore we nee to multiply by $\frac{3-4i}{3-4i}$as follows

$\frac{(7-3i)}{(3+4i)} \cdot \frac{(3-4i)}{(3-4i)}$ This process will cancel the imaginary numbers from the denominator we will obtain

$= \frac{(7)(3)+(7)(-4i)+(-3i)(3)+(-3i)(-4i)}{(3)^2+(4)^2}$

$= \frac{21-28i-9i+12i^2}{25}$

$= \frac{9-37i}{25}$

$= \frac{9}{25} - \frac{37}{25}i$

This is the solution and you have to perform this process every time you are dividing complex numbers.

Powers of *i*

When we have i to a power we get the following pattern.

$$\begin{aligned} i& \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}$$

So if we want i^5 for example the value will be i or $i^{11} = -i$ if the numbers are small enough its easy to just count but what if we want i^{1333} ? It would take us a lot of time to count, luckily we have a method to compute the value fast.

Since there is four values in the pattern, we divide the exponent by the 4 and find the remainder.

If the remainder is **1**, the value is i,

if the remainder is **2**, the value is -1,

if the remainder is **3**, the value is $-i$,

if the remainder is **0**, the value is 1.

For example i^8 then we obtain $8 \div 4 = 2$ *remainder* 0 therefore the value will be 1.

The remainder gives us the value, the 2 represents how many times you went through the pattern.

Example F

Powers of *i*

Evaluate i^{1333}

By doing long division we obtain $1333 \div 4 = 333$ *remainder* 1

Therefore our value will be i.

Example G

Performing operations with negative square roots $\sqrt{-36} \cdot \sqrt{-4}$

First we need to change the numbers as $i\sqrt{36} \cdot i\sqrt{4}$ then performing the operation we obtain $6i \cdot 2i = 12i^2 = -12$

Example 8

Simplify the following expression $\frac{-15+\sqrt{-75}}{10}$

First we need to rewrite the numerator as $-15+i\sqrt{25}\sqrt{3}$

thus we have the following $\frac{-15+5\sqrt{3}i}{10}$ and simplifying we obtain

$\frac{-3}{2}+\frac{\sqrt{3}}{2}i$

Example H

Solving quadratic equations with complex solutions.

Solve the quadratic equation $x^2-6x+10=0$.

We can solve all quadratic equations using the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ in this case **a=1**, **b=-6** and **c= 10** therefore we have

$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(10)}}{2(1)}$

$x=\frac{6\pm\sqrt{-4}}{2}$

$x=\frac{6\pm 2i}{2}$

$x=3+i, x=3-i$

We obtained an imaginary number solution because we inside of the square root was negative.

Quadratic functions will be covered more in depth in the next section.

Vocabulary

Imaginary unit i
: The imaginary unit i is defined as $i=\sqrt{-1}$ *where* $i^2=-1$

Complex Number
: A number in the form $a+bi$ where a is the real part of the number and b is called the imaginary part where $b\neq 0$

Complex Conjugate
: The complex conjugate of a number $a+bi$ is the complex number $a-bi$

In summary

We learned what a complex number is and what is its complex conjugate. We learned how to add, subtract, multiply and divide complex numbers. Also we learned how to find the powers of **i** to a very large power. We also learned how to perform operations with negative square roots and how to solve a quadratic equation using the quadratic formula.

Practice

Perform the following operations in complex numbers.

1. $(12+9i)+(5-2i)$

2. $(-3-6i)+(7+2i)$

3. $(4-5i)-(8+3i)$

4. $(-4-5i)-(-11-3i)$

5.$-12i(4-3i)$

6. $3i(5+2i)$

7. $(5+3i)(8+6i)$

8. $(-9+i)(2-5i)$

9. $(5+2i)(5-2i)$

10. $(11-7i)(11+7i)$

11. $\frac{5-2i}{7+3i}$

12. $\frac{7+4i}{5-9i}$

13. Find $i^{13}, i^{527}, i^{1028}, i^{3322}$

14. Solve $0 = x^2 + 5x + 15$

15. Solve $-2x^2 + 3x - 20 = 0$

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3.2 Quadratic Functions

TEKS

1. P.1.F
2. P.2.G
3. P.2.I

Lesson Objectives

In this section you will learn about:
1. Identifying the characteristics of a parabola including the vertex and the way it opens.
2. Finding minimums and maximums of parabolas.
3. Graphing parabolas.
4. Identifying the domain and range of parabolas.
5. Solving problems involving quadratic functions.

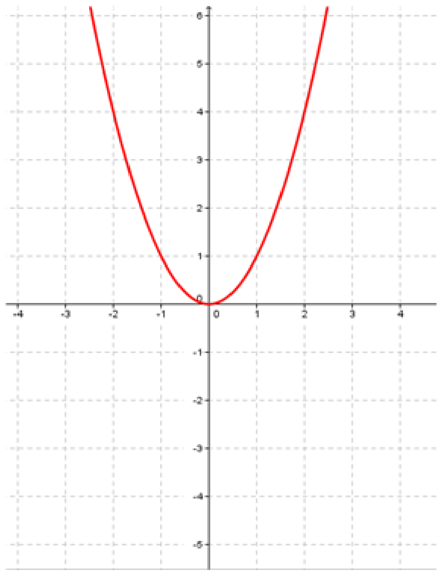
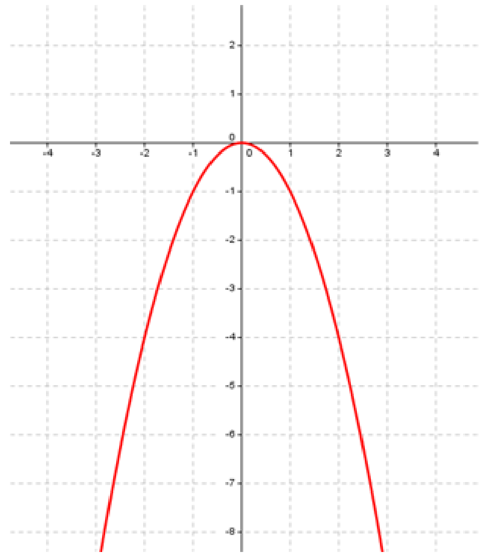
Introduction

Quadratic functions are one of the most natural events in real life, every time you throw an object into the air, it creates a parabolic curve which is a quadratic function. In this section you will learn about quadratic functions and how to solve them.

Vocabulary

Parabola, Quadratic equation, axis of symmetry, vertex, quadratic equation,

From Algebra 1 and Algebra two we know that the graph of a quadratic function is called a **parabola**. The graphs of Parabolas are u shaped graphs. The following are graphs of a parabola opening up and a parabola opening down.

When the coefficient of the x^2 term is positive, the parabola opens upwards, when the coefficient of x^2 term is negative it opens downward as in the figure above.

Polynomial form of Quadratic Function

The polynomial form of a quadratic function is the function$f(x) = ax^2 + bx + c$ where a,b, c, are real numbers.

From this form we can obtain some information.

The axis of symmetry is given by the equation $x = \frac{-b}{2a}$.

The vertex of the parabola will be the point $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. This just means to find the x value and replace it in the function to find the value of y. It looks very complicated but is not you will see in the examples.

Vertex Form of a Quadratic Function

The vertex form of a quadratic function is the function $f(x) = a(x-h)^2 + k,\ \ where\ \ a \neq 0$.

The point (h,k) is the vertex of the parabola.

The axis of symmetry will be $x = h$.

If $a > 0$ then the parabola opens upward.

If $a < 0$ then the parabola opens downward.

NOTE: If the parabola opens upward it has a minimum point and if the parabola opens downward then it has a maximum point.

Finding the x and y intercepts of the parabola.

To find the y intercept of the parabola in both forms you need to let $x = 0$

To find the x-intercepts if any of the parabola in both forms you need to let $f(x) = 0$ and solve for x. We can obtain these by factoring if its obvious or by doing the quadratic formula.

The Domain and Range of a Parabola

The domain of a parabola is the set of all real numbers $(-\infty, \infty)$

The range depends on the **y-coordinate** of the vertex and the value of **a**

Example A.

Quadratic Function in Polynomial Form

Given the parabola in polynomial form $f(x) = 3x^2 + 18x + 17$

a)Find the axis of symmetry,

b)Find the vertex

c) Find the y intercept and the x-intercepts(if any)

d) Graph the parabola

e) Write parabola in standard form

f) State the domain and range

Solution

a) The axis of symmetry is found by $x = \frac{-b}{2a} = \frac{-18}{2(3)}$ $therefore \;\; x = -3$

b) The vertex is found by $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ therefore since $x = -3, \;\; then \;\; f(-3) = 3(-3)^2 + 18(-3) + 17 = -10$.

The coordinate of the vertex is $(-3, -10)$

c) The y intercept is found by $f(0) = 3(0)^2 + 18(0) + 17 = 17$

The coordinate of the y-intercept is $(0, 17)$

The x intercepts are found by using the quadratic formula

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-(18) \pm \sqrt{(18)^2 - 4(3)(17)}}{2(3)} \\ &= \frac{-18 \pm \sqrt{120}}{6} = \frac{-18 \pm 10.95}{6} \\ therefore \;\; x &= -1.175, x = -4.825 \end{aligned}$$

The coordinates of the x intercepts will be $(-4.825, 0) \;\; and \;\; (-1.175, 0)$

d) To graph the parabola, graph the vertex, the y-intercept and the x-intercepts if any.

e) The standard form of the parabola will be $f(x) = 3(x+3)^2 + 10$

Since $a > 0$ the parabola opens upward, the dotted line is the axis of symmetry.

f) Finally Domain is $(-\infty, \infty)$ and the Range is $[-10, \infty)$

Example B.

Given the quadratic function $f(x) = -2x^2 + 10x$

a)Find the axis of symmetry,

b)Find the vertex

c) Find the y intercept and the x-intercepts(if any)

d) Graph the parabola

e) Write parabola in standard form

f) State the domain and range

Solution

a) The axis of symmetry is $x = \frac{-b}{2a} = \frac{-10}{2(-2)} = 2.5$

b) The vertex is $f(2.5) = -2(2.5)^2 + 10(2.5) = 12.5$

The coordinate is $(2.5, 12.5)$.

c) The y intercept by letting $x = 0$, *its* $(0,0)$ This is also one of the x intercepts the other intercept int his case we can obtain it by factoring the quadratic factors as $-2x(x-5)$ by making both factors equal to zero, we obtain $x = 0$ *and* $x = 5$ therefore the x intercepts are $(0,0)$ *and* $(5,0)$.

d) The graph of the quadratic is

e) The equation of the parabola in standard form is $f(x) = -2(x-2.5)^2 + 12.5$

f) Since $a < 0$ then it has a maximum point therefore the Domain is $(-\infty, \infty)$ and the Range is $(-\infty, 12.5]$

Example C.

Given the quadratic function in standard form $f(x) = \frac{1}{2}(x-3)^2 - 2$

a)Find the axis of symmetry,

b)Find the vertex

c) Find the y intercept and the x-intercepts(if any)

d) Graph the parabola

e) State the domain and range

Solution

a) The axis of symmetry is $x = 3$

NOTE: When it is in standard form, it is the opposite of what you see in the equation.

b) The vertex is $(-3, -2)$

c) The y intercept you let $x = 0$ therefore $\frac{1}{2}((0)-3)^2 - 2 = 2.5$ The coordinate is $(0, 2.5)$

The x intercept you let $f(x) = 0$ Therefore $0 = \frac{1}{2}(x-3)^2 - 2$ and solve for x.

$$\begin{aligned} 2 &= \frac{1}{2}(x-3)^2 \\ 4 &= x - 3^2 \\ \sqrt{4} &= \sqrt{(x-3)^2} \\ \pm 2 &= (x-3) \\ 3 \pm 2 &= x \end{aligned}$$

Therefore $x = 1$, $x = 5$ and the coordinates are $(1,0)$ *and* $(5,0)$

d) The graph of the quadratic is

e) Since $a > 0$ the Domain is $(-\infty, \infty)$ and the Range is $[-2, \infty)$.

The position for a free falling object near the surface of the earth is given by the function

$s(t) = -16t^2 + v_o t + s_o$

where v_o is the initial velocity in ft/sec, s_o is the initial height in ft. and the height is in feet.

Example E.

Applying the position formula

A football is punted into the air from a height of 3ft with an initial velocity of 75 ft/sec. Find the maximum height of the ball.

solution

The position will be given by the formula $s(t) = -16t^2 + 75t + 3$

The maximum height of the ball is given by the vertex therefore $t = \frac{-75}{2(-16)} = 2.34 sec$,

and the maximum height will be $s(2.34) = -16(2.34)^2 + 75(2.34) + 3 = 90.89 ft$

The maximum height of the ball is **90.89 ft** at **2.34 seconds**.

Here is the graph of the height of the football with respect to time.

Vocabulary

Parabola

The graph of a quadratic equation

Axis of Symmetry

Vertical line that divides a parabola into two equal parts.

Vertex

The highest or lowest point in a parabola given by $(\frac{-b}{2a}, f(\frac{-b}{2a}))$

In Summary

We have learned how to find the standard form of a quadratic equation. We also learned to find the axis of symmetry, the vertex, the domain and range, and when a parabola opens upward or downward. We learned how to graph a quadratic equation. Finally we applied the procedures of a quadratic function to a real life situation.

Practice

For the following quadratic equations find the following:

a) the axis of symmetry and vertex of the parabola.

b) x-intercept and y-intercept

c) Write the equation in vertex form

1. $f(x) = x^2 + x - 20$
2. $f(x) = 4x^2 + 9x + 20$
3. $f(x) = x^2 - 4x - 30$
4. $f(x) = -4x^2 + 13x - 3$
5. $f(x) = -x^2 + 169$

For the following, set the functions equal to zero and solve the quadratic equation using the quadratic formula. Then graph the quadratics.

6. Number 1.
7. Number 2.
8. Number 3.
9. Number 4.
10. Number 5

11. Explain the similarities between the x intercepts and the solutions to the quadratics when they are equal to zero.

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3.3 Characteristics of Polynomials

Here you will review factoring techniques from Algebra 1 and 2 in preparation for more advanced factoring techniques. The review will include factoring out a greatest common factor, factoring into binomials and the difference of squares.

TEKS

1. P.2.I
2. P.2.J

Lesson Objectives

In this section you will learn about:
1. Definition of a polynomial.
2. How to identify a polynomial.
3. Determine characteristics and end behavior of polynomials.

Introduction

In the previous two sections you learned about complex numbers and quadratic equations, but there are more than quadratic functions. These functions will be called polynomials. In this section you will learn about some of its characteristics and the graphs of polynomials including its end behavior.

Vocabulary

Polynomial, degree of polynomial, leading coefficient

What is a polynomial? A polynomial is a function in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ were $\boldsymbol{n}$ is a non-negative integer and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are any real numbers with $a^n \neq 0$.

- $\boldsymbol{n}$ is called the degree of the polynomial

- a_n is called the leading coefficient which is the coefficient of the highest power of the polynomial.
- A polynomial of n degree has at most $n-1$ turning points.

The graphs of polynomials are smooth and continuous, they don't have any sharp turns, holes or breaks. Hint: In an algebraic equation, you can find which of the functions are polynomials by using the following,

1. All the powers of x have to be positive whole numbers.
2. The variable x cannot be in the denominator, this makes the exponent negative.
3. $f(x) = c$ is a polynomial function of degree zero. It can be written as $f(x) = cx^0$

Example A

Polynomial functions VS non-polynomial functions In this example, the functions 1, 2, 3, 4 they are all polynomials, all the exponents of x are positive integers and the square root on number 3 is only for the number not for x. From 5 to 8 they are not polynomials for the following reasons.

- Number 5 has a negative exponent.
- Number 6 the square root is affecting x to the power of x becomes 1/2.
- Number 7 x is on the bottom of a fraction.
- Number 8 the exponent is a fraction.

Example B Graphs of Polynomials and Non-Polynomials The three graphs above are the graphs of polynomial functions. They are smooth continuous curves. They do not have any sharp turns, holes or breaks. The two graphs above are not the graphs of polynomials because they are not continuous, they have breaks.

Example C

Is the graph of $f(x) = |x|$ a polynomial?

The answer is no because it contains a sharp turn.

CHARACTERISTICS OF POLYNOMIALS AND END BEHAVIOR

As we discussed before Polynomial Functions are smooth curves, but as x increases or decreases without bound, the graph of the polynomial eventually rises or falls depending on the degree of the polynomial and the leading coefficient. **Even Degree Polynomials**

- If the degree of the polynomial is even, and the leading coefficient is positive, then it rises to the left and rises to the right.
- If the degree of the polynomial is even, and the leading coefficient is negative, then it falls to the left and falls to the right.

Example D. Graphs of even power functions

$f(x) = x^2 - x - 5$ **,** $f(x) = x^4 - x^2 - 2$**,** $f(x) = x^6 - x^4 - x^2 - 2$

The leading coefficient in the three cases above is positive 1, since the leading coefficient is positive and the degree of the polynomial is even, then the polynomial rises to the left and rises to the right. The first graph is of degree 2 and it has 1 turning point. The second graph is of degree 4 and it has 3 turning points. The third graph is of degree 6 and it has 3 turning points. Remember a polynomial has at most $n-1$ turning points, but it does not mean that it will have $n-1$turning points.

Example E. Graphs of Even degree functions with negative leading coefficient

$f(x) = \frac{-1}{2}x^2 + 5$ **,** $f(x) = -x^4 + 2x^3 + 5$ **,** $f(x) = -2x^6 + x^4 + x^2 + 3$

Since the three polynomials are of even degree, and the first has a leading coefficient of $\frac{-1}{2}$, the second has a leading coefficient of -1 and the third has a leading coefficient of -2, then all three polynomials fall to the left and fall to the right. $\frac{-1}{2}$

Odd Degree Polynomials

- If the degree of the polynomial is odd, and the leading coefficient is positive, then it falls to the left and rises to the right.
- If the degree of the polynomial is odd, and the leading coefficient is negative, then it rises to the left and falls to the right.

Example F. Odd degree polynomials with positive leading coefficient $f(x)=x^3$ **,** $f(x)=3x^3-3x$**,** $f(x)=x^5-x^3-x$

Since the polynomials are of odd degree and the leading coefficient of all three of them is positive, then the graph falls to the left and rises to the right.

Example I
Odd degree polynomial with negative leading coefficient $f(x)=-x^3$**,** $f(x)=-3x^3+3x$**,** $f(x)=-x^5+x^3+x$

Since the polynomials are of odd degree and the leading coefficient of all three of them is negative, then the graph rises to the left and falls to the right.

Table for determining the end behavior of a polynomial.

Example J
Use the table and your knowledge about polynomials to determine the end behavior of the following polynomials.

1. $f(x)=-5x^2+4x-5$
2. $g(x)=2x^7-6x^5+x^4-2x^3+x^2-1$
3. $h(x)=-2x^3+5x+1$
4. $k(x)=3x^4+5x^3-6$

solution:

$f(x)$ **is of even degree and negative leading coefficient therefore it falls to the left and falls to the right.**

$g(x)$ **is of odd degree and positive leading coefficient therefore it falls to the left and rises to the right.**

$h(x)$ **is of odd degree and negative leading coefficient therefore it rises to the left and falls to the right.**

$k(x)$ **is of even degree and positive leading coefficient therefore it rises to the left and rises to the right.**

Vocabulary

A ***polynomial*** is a mathematical expression that is often represented as a sum of terms or a product of factors.

The **degree of a polynomial** is the highest exponent.

The **leading coefficient** is the coefficient of the term with the largest exponent of a polynomial.
In summary: If degree is even and the leading coefficient is even then it rises to left and rises to the right. If degree is even and leading coefficient is negative, it falls to the left and falls to the right. If degree is odd and leading coefficient is positive then it falls to the left and rises to the right. If degree is odd and the leading coefficient is negative, then it rises to the left and falls to the right. The number of turning points of a polynomial is at most $n-1$ where **n** is the degree of the polynomial.

PRACTICE

1. Explain when a function is a polynomial function. Give examples.
2. Explain when a function is not a polynomial function. Give examples.

Find the degree of the following polynomial functions identify the leading coefficient and their end behavior.

3. $f(x) = -5x^4 + 6x^3 - 2x^2 + x - 1$
4. $f(x) = 2x^5 - 9x^3 + 6x^2 + 3$
5. $f(x) = 3x^2 + 9x^6 - 2x^3 + x - 1$
6. $f(x) = -6x^5 - 7x^3 + 11x^2 + 9x + 10$
7. True or False the function $f(x) = -15x^4 + x^{\frac{1}{2}} + 9$ is a polynomial.
8. True or False. All parent functions are polynomials.
9. Create a function that is a polynomial.
10. Create a function that is not a polynomial

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3.4 Pascals Triangle and Binomial Expansion

Here you will explore patterns with binomial and polynomial expansion and find out how to get coefficients using Pascal's Triangle.

TEKS

1. P.1.E
2. P.1.F
3. P.5.F

Lesson Objectives

In this section you will learn:
1. To expand binomials by using pascal's triangle.
2. To factor a polynomial by recognizing the coefficients.

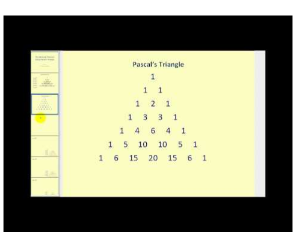
Introduction

The expression $(2x+3)^5$ would take a while to multiply out. Is there a pattern you can use?
How about expanding $(2x+3)^{10}$?

Vocabulary

Binomial Expansion, Pascal's Triangle

Watch This



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http://www.youtube.com/watch?v=NLQmQGA4a3M James Sousa: The Binomial Theorem Using Pascal's Triangle

Pascal was a French mathematician in the 17^{th} century, but the triangle now named Pascal's Triangle was studied long before Pascal used it. The pattern was used around the 10^{th} century in Persia, India and China as well as many other places.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

The primary purpose for using this triangle is to introduce how to expand binomials.

$$
\begin{aligned}
&(x+y)^0 = 1 \\
&(x+y)^1 = x+y \\
&(x+y)^2 = x^2+2y+y^2 \\
&(x+y)^3 = x^3+3x^2y+3xy^2+y^3
\end{aligned}
$$

Notice that the coefficients for the x and y terms on the right hand side line up exactly with the numbers from Pascal's triangle. This means that given $(x+y)^n$ for any power n you can write out the expansion using the coefficients from the triangle. When you study how to count with combinations then you will be able to calculate the value of any coefficient without writing out the whole triangle.

There are many patterns in the triangle. Here are just a few.

1

1 1

1 2 1

1 3 3 1

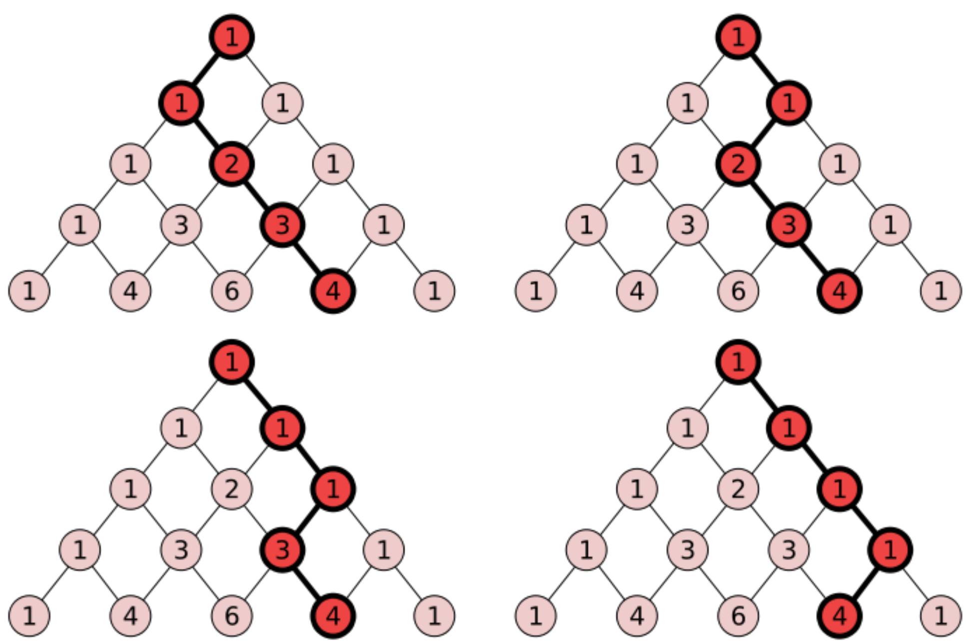
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

⋮

1. Notice the way each number is created by summing the two numbers above on the left and right hand side.
2. As you go further down the triangle the values in a row approach a bell curve. This is closely related to the normal distribution in statistics.
3. For any row that has a second term that is prime, all the numbers besides 1 in that row are divisible by that prime number.
4. In the game Plinko where an object is dropped through a triangular array of pegs, the probability (which corresponds proportionally to the values in the triangle) of landing towards the center is greater than landing towards the edge. This is because every number in the triangle indicates the number of ways a falling object can get to that space through the preceding numbers.



Example A

Expand the following binomial using Pascal's Triangle: $(3x-2)^4$

Solution: The coefficients will be 1, 4, 6, 4, 1; however, since there are already coefficients with the x and the constant term you must be particularly careful.

$1\cdot(3x)^4+4\cdot(3x)^3\cdot(-2)+6\cdot(3x)^2\cdot(-2)^2+4\cdot(3x)\cdot(-2)^3+1\cdot(-2)^4$

Then it is only a matter of multiplying out and keeping track of negative signs.

$81x^4-216x^3+216x^2-96x+16$

Example B

Expand the following trinomial: $(x+y+z)^4$

Solution: Unfortunately, Pascal's triangle does not apply to trinomials. Instead of thinking of a two dimensional triangle, you would need to calculate a three dimensional pyramid which is called Pascal's Pyramid. The sum of all the terms below is your answer.

$$\begin{aligned}&1x^4+4x^3z+6x^2z^2+4xz^3+1z^4\\&4x^3y+12x^2yz+12xyz^2+4yz^3\\&6x^2y^2+12xy^2z+6y^2z^2\\&4xy^3+4y^3z\\&1y^4\end{aligned}$$

Notice how many patterns exist in the coefficients of this layer of the pyramid.

Example C

Expand the following binomial: $\left(\frac{1}{2}x-3\right)^5$

Solution: You know that the coefficients will be 1, 5, 10, 10, 5, 1.

$$1\left(\frac{1}{2}x\right)^5+5\left(\frac{1}{2}x\right)^4(-3)+10\left(\frac{1}{2}x\right)^3(-3)^2+10\left(\frac{1}{2}x\right)^2(-3)^3+5\left(\frac{1}{2}x\right)(-3)^4+1\cdot(-3)^5$$

$$=\frac{x^5}{32}-\frac{15x^4}{16}+\frac{90x^3}{8}-\frac{270x^2}{4}+\frac{405x}{2}-243$$

Remember to simplify fractions.

$$=\frac{x^5}{32}-\frac{15x^4}{16}+\frac{45x^3}{4}-\frac{135x^2}{2}+\frac{405x}{2}-243$$

Concept Problem Revisited

Pascal's triangle allows you to identify that the coefficients of $(2x+3)^5$ will be 1, 5, 10, 10, 5, 1 like in Example C. By carefully substituting, the expansion will be:

$$1 \cdot (2x)^5 + 5 \cdot (2x)^4 \cdot 3 + 10 \cdot (2x)^3 \cdot 3^2 + 10 \cdot (2x^2) \cdot 3^3 + 5(2x)^1 \cdot 3^4 + 3^5$$

Simplifying is a matter of arithmetic, but most of the work is done thanks to the patterns of Pascal's Triangle.

Vocabulary

A ***binomial expansion*** is a polynomial that can be factored as the power of a binomial.

Pascal's Triangle is a triangular array of numbers that describes the coefficients in a binomial expansion.

In Summary

We learned a faster way to expand a binomial by using the patterns of Pascal's triangle.

Guided Practice

1. Factor the following polynomial by recognizing the coefficients.

$x^4 + 4x^3 + 6x^2 + 4x + 1$

2. Factor the following polynomial by recognizing the coefficients.

$8x^3 - 12x^2 + 6x - 1$

3. Expand the following binomial using Pascal's Triangle.

$(A - B)^6$

Answers:

1. $(x+1)^4$

2. Notice that the first term of the binomial must be $2x$, the last term must be -1 and the power must be 3. Now all that remains is to check.

$(2x-1)^3 = (2x)^3 + 3(2x)^2 \cdot (-1) + 3(2x)^1(-1)^2 + (-1)^3 = 8x^3 - 12x^2 + 6x - 1$

3. $(A-B)^6 = A^6 - 6A^5B + 15A^4B^2 - 20A^3B^3 + 15A^2B^4 - 6AB^5 + B^6$

Practice

Factor the following polynomials by recognizing the coefficients.

1. $x^2 + 2xy + y^2$
2. $x^3 + 3x^2 + 3x + 1$
3. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$
4. $27x^3 - 27x^2 + 9x - 1$
5. $x^3 + 12x^2 + 48x + 64$

Expand the following binomials using Pascal's Triangle.

6. $(2x-3)^3$
7. $(3x+4)^4$
8. $(x-y)^7$

9. $(a+b)^{10}$
10. $(2x+5)^5$
11. $(4x-1)^4$
12. $(5x+2)^3$
13. $(x+y)^6$
14. $(3x+2y)^3$
15. $(5x-2y)^4$

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3.5 Graphs of Polynomials, Zeros and Their Multiplicities

TEKS

1. P.1.F
2. P.2.F
3. P.2.G
4. P.2.I
5. P.2.J

Lesson Objectives

In this section you will learn about:
1. Identifying the zeros of a polynomial.
2. Finding the y intercept of a polynomial.
3. Identifying the multiplicities of the zeros of a polynomial.
4. Factoring polynomial by using the zeros of a graph.
5. Graphing a polynomial given the polynomial in linear factored form.

Introduction

On section 3.1, we learned about the basic characteristics of polynomials, now we will look at them more in depth.

Vocabulary

Zeros, multiplicity, x-intercepts , y-intercept

When you have a polynomial of **even degree**, there is a possibility that it may not have any real zeros, for example the polynomial $f(x) = x^2 + 1$ does not have any real zeros because it never touches or crosses the x-axis. We know that as x approaches $-\infty$ or ∞ it rises to the left and rises to the right.

If we have a polynomial of **odd degree**, we know that it falls to the left and rises to the right or vice versa depending on the leading coefficient, therefore it has **at least one x-intercept.**

What is a zero of a polynomial?

A zero of a polynomial is when the y coordinate of the polynomial is equal to zero. That is $f(x) = 0$

What is the multiplicity of a zero?

The multiplicity of a zero is how many times that zero repeats.

What is the degree of a polynomial if it is in linear factored form?

The degree of a polynomial in linear factored form will be the sum of the exponents of the linear factors.

Linear Factorization Theorem

Let a_n be any real number where $a_n \neq 0$ and $n \geq 1$, then a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ can be written as a product of linear factors $f(x) = a_n (x - c_1)(x - c_2) \ldots (x - c_n)$ where $c_1, c_2, \ldots, c_n$ are complex numbers (possibly real and not necessarily distinct).

Properties of Polynomials and its roots.

1. A polynomial of n^{th} degree, can have at most n roots
2. If is a root of a polynomial equation with real coefficients then its complex conjugate is also a root. Imaginary roots if they exist occur in conjugate pairs.

NOTE: If a polynomial is in linear factored form, then the degree of the polynomial is the sum of the exponents of the factors

Example A

Graph the polynomial $f(x) = x^2 + x - 6$ **find the zeros and the y intercept.**

By using the factoring algorithms in the previous sections, we can see that the polynomial in factored form is $f(x) = (x+3)(x-2)$ and setting both factors equal to zero it yields that $x = -3$ and $x = 2$.

The y-intercept is found by letting $x = 0$ therefore $f(0) = (0)^2 + (0) - 6 = -6$

This polynomial is of degree 2 **.**

From this example we learn the following.

1. The y-intercept is found by letting all **x's = 0**
2. The linear factors of the polynomial will be the x-intercepts of the graph with the opposite signs. The x-intercepts in the graph were $x = -3$ and $x = 2$ and the factors will were $(x+3)$ and $(x-2)$ notice that they have the opposite signs.
3. The multiplicity of both zeros is 1 which is the exponents of the linear factors.

The multiplicities of the zeros also tell us some information about how the function behaves near the zeros and at the zero.

- If the multiplicity of the zero is even, then the graph at that zero touches the x-axis and turns around (does not cross the x-axis) and the graph looks like a parabola or a flat parabola depending on the multiplicity.
- If the multiplicity is 1, then it crosses the x-axis at that particular zero. It crosses the x axis like a line.
- If the multiplicity an odd number positive integer, then it crosses the x-axis at that particular zero. The graph crosses like the shape of a cubic function.

Example B

Graph the function $f(x) = (x+5)^3 (x-5)(x-8)^2 (x+9)$ **then find the degree of the polynomial, the zeros and their multiplicities and the y intercept.**

Since the polynomial is in linear factored form, then the zeros are

-5 of multiplicity 3, so it will cross the x-axis like a cubic function

5 of multiplicity 1, so ti will cross the x-axis like a line

8 of multiplicity 2, so it will not cross the x-axis, it will touch and come back

-9 of multiplicity 1, so it will cross like x-axis like a line

The degree of the polynomial will be the sum of the multiplicities so this polynomial is of degree 7.

The y-intercept will be -360,000 and its found by $f(x) = (0+5)^3 (0-5)(0-8)^2 (0+9)$

To graph it:

- plot the zeros from left to right,
- since its an odd degree and the leading coefficient is positive, then it will start at the bottom left and end at the top right.
- the leftmost zero is -9 and multiplicity 1 it crosses like a line going up, the zero -5, crosses like a cubic function going down, the zero 5 crosses like a line going up and the zero -8 will touch and go back up.

Notice: that the scale on the graph is very big on the y axis, if you graph it on the standard window you will not see anything.

Example C

Graph the function $f(x) = -2(x+3)^2 (x-1)(x+7)^3$ **then find the degree, the zeros and their multiplicities, the y-intercept**.

The degree of the polynomial will be degree 6, since the leading coefficient is negative, then it will start on the bottom left and end on the bottom right, meaning that it will fall to the left and fall to the right.

The zeros and the multiplicities are the following.

-3 with multiplicity 2 so it will not cross the x axis

6 with multiplicity 1 so it will cross the x axis like a line

-7 with multiplicity 3 so it will cross the x axis like a cubic function

The y intercept will be $(0, 6174)$ and it is found by letting $x = 0$.

Notice the shape of the function near the zeros and at the zeros.

Example D

Given the graphs of the polynomials, write the possible equations of the polynomials in factored form and polynomial form, determine the degree, the zeros and multiplicities.

Since the graph rises to the left and rises to the right the degree must be an even number. The zeros of the polynomial are -3 of multiplicity 3, 0 of multiplicity 1, and 3 of multiplicity 2. By adding the multiplicities this polynomial is of degree 6.

The factored form of the polynomial would be $f(x) = x(x+2)^3(x-3)^2$ in polynomial form the equation would be $f(x) = x^6 - 15x^4 - 10x^3 + 60x^2 + 72x$

Since the graph rises to the left and falls to the right the degree must be an odd number. The zeros of the polynomial are -5 of multiplicity 1, 0 of multiplicity 1, and 5 of multiplicity 1. By adding the multiplicities this polynomial is of degree 3, but since it rises to the left and falls to the right then it must have an negative leading coefficient.

The factored form of the polynomial would be $f(x) = -x(x+5)(x-5)$ and the polynomial form would be $f(x) = -x^3 + 25x$

Example E

Having a polynomial go through a specific point

Write the equation of a polynomial such that the zeros of the polynomial are -5, 0, 5 all of multiplicity one passing through the point (3,90)

To have the polynomial pass through the given point we need to write the polynomial in factored form as $f(x) = a_n x(x+5)(x-5)$ and substitute $x = 3$ and $f(3) = 90$ and we need to solve for a_n

$90 = a_n(3)((3)+5)((3)-5)$

$90 = -48a_n$

$\frac{-15}{8} = a_n$

Finally rewrite the factored form as $f(x) = \frac{-15}{8}x(x+5)(x-5)$ or polynomial form as $f(x) = \frac{-15}{8}x^3 + \frac{375}{8}x$

In the following graph you can see that the curve passes through the point (3,90). Compare the graph to example 4 graph 2

Having a polynomial pass though a given point will be discussed more in depth in section 3.6

Vocabulary

Zeros
The zero's of a polynomial function is when the value of the y coordinate is zero, $f(x) = 0$.

Multiplicity of a zero
The multiplicity of a zero is how many times the zero repeats.

X-intercepts
Are the zeros of the polynomial, that is when the graph of the polynomial crosses the x-axis.

Y-intercept
It is where the graph crosses the y axis, it is the point (0,y) obtained by letting x =0 it is found by finding $f(0)$

In Summary

We learned how to identify the zeros of a polynomial with their respective multiplicity from a graph.

We also learned to find the y intercept of a polynomial and the end behavior of the polynomials by identifying the leading coefficient of the polynomial and if the polynomial is of even degree or odd degree.

We learned to graph a polynomial given the zeros and their multiplicities.

We learned how to obtain a possible polynomial given the graph of the polynomial and make it pass through a given point.

Practice Problems

Identify the zeros and their multiplicity of the following polynomials and also the degree of the polynomial.

1. $f(x) = (x+3)^2(x-3)(x+7)^3$

2. $f(x) = (x+5)^2(x-4)^3(x+10)^3$

3. $f(x) = (x+\frac{1}{2})^2(x-4)(x-9)^4$

4. $f(x) = x^2 + x - 12$

5. $f(x) = x^2 + 8x + 15$

6. $f(x) = x^3 - 7x^2 - 5x + 75$

Sketch a graph of and identify the end behavior.

7. Number 1

8. Number 2

9. Number 3

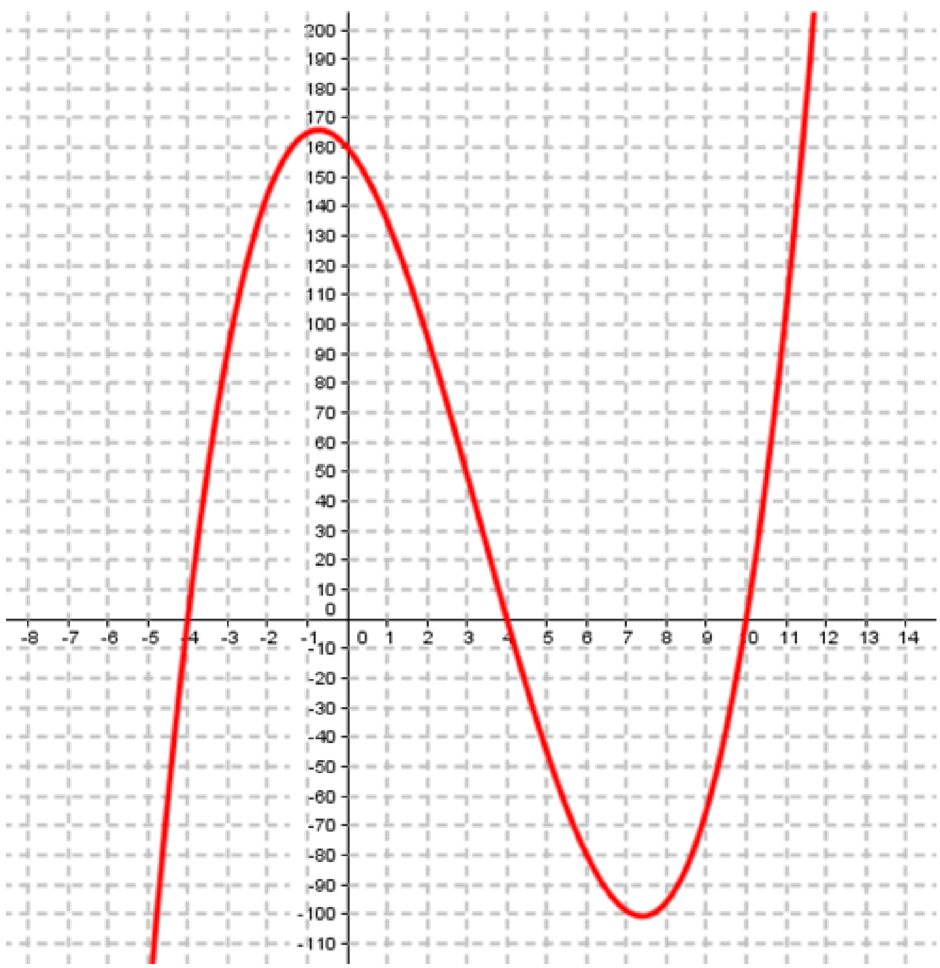
10. Number 4

11. Number 5

12. Number 6

For the next three graphs obtain the equation in factored form

13.



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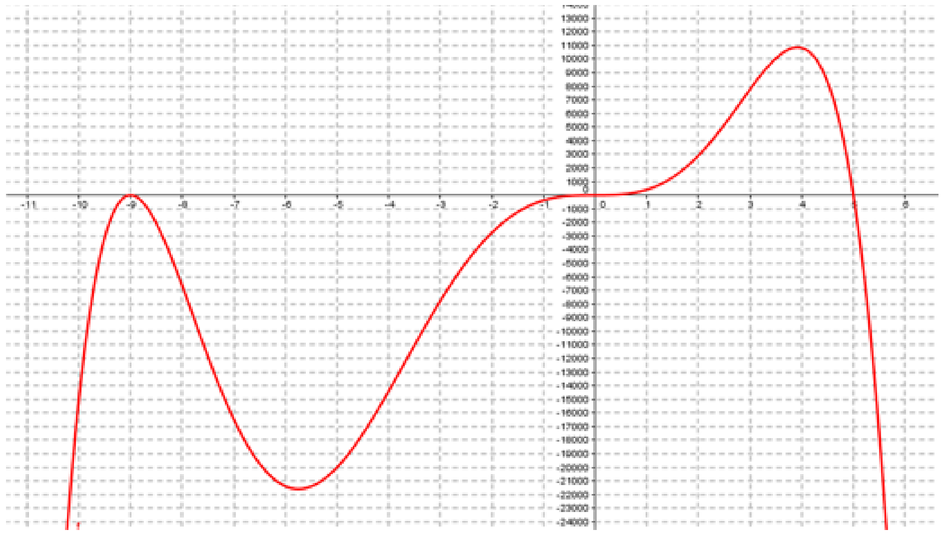

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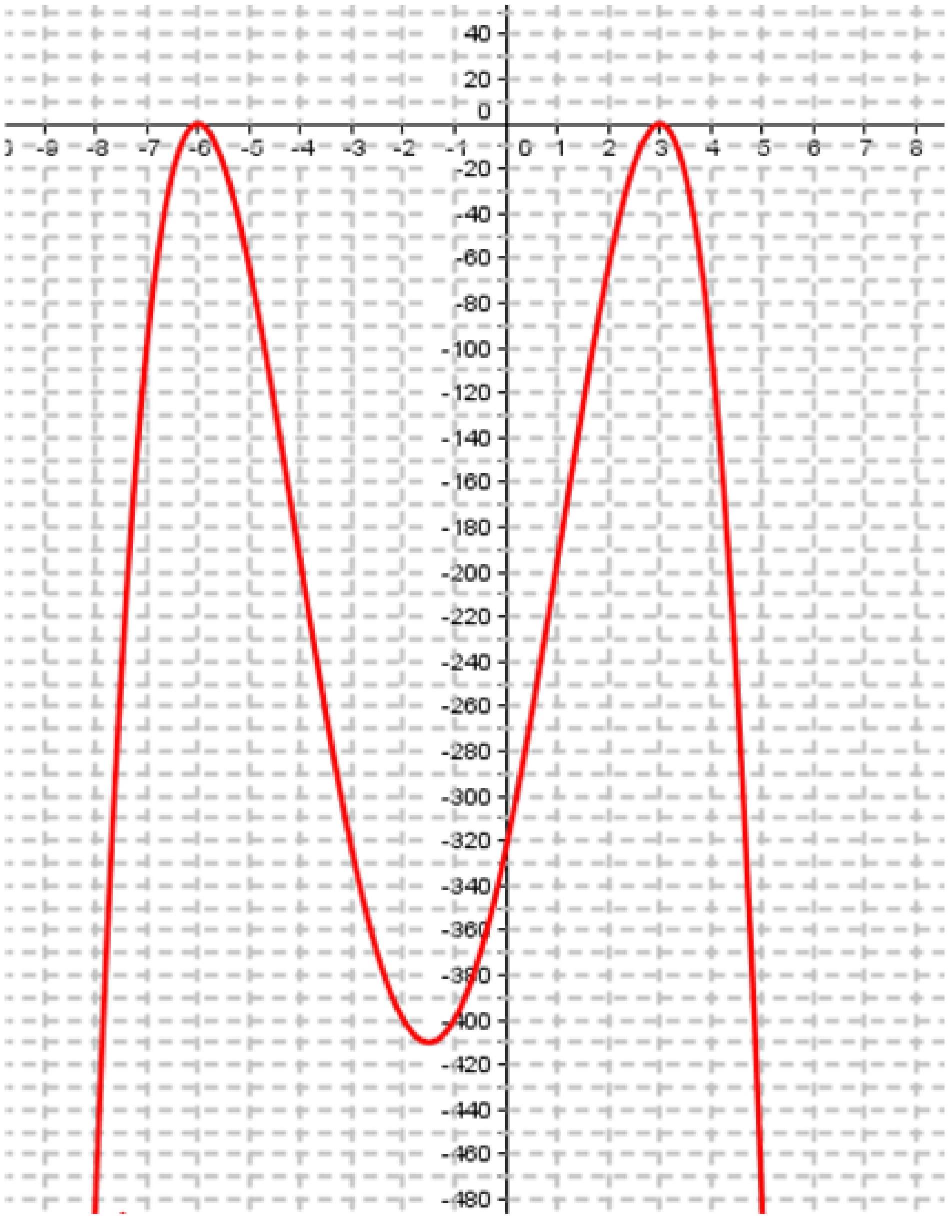

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3.6 Factoring Review (Optional)

Here you will be exposed to a variety of factoring techniques for special situations. Additionally, you will see alternatives to trial and error for factoring.

TEKS

1. P.1.B
2. P.1.C
3. P.5.F

Lesson Objectives

In this section you will learn about:
1. How to factor a trinomial of degree 2.
2. How to factor a difference of squares.
3. How to factor sums and differences of cubes.

Introduction

In chapter 1 we learned different techniques about how to factor polynomials, in case you forgot here is a review of factoring polynomials.

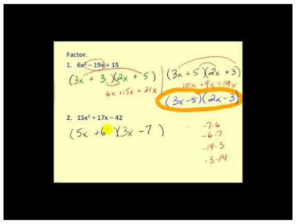
Vocabulary

Difference of squares, sum and difference of cubes, quadratic equation, factors,

The difference of perfect squares can be generalized as a factoring technique. By extension, any difference between terms that are raised to an even power like $a^6 - b^6$ can be factored using the difference of perfect squares technique. This is because even powers can always be written as perfect squares: $a^6 - b^6 = (a^3)^2 - (b^3)^2$.

What about the sum or difference of terms with matching odd powers? How can those be factored?

Watch This



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http://www.youtube.com/watch?v=55wm2c1xkp0 James Sousa: Factoring: Trinomials using Trial and Error and Grouping

Factoring a trinomial of the form ax^2+bx+c is much more difficult when $a \neq 1$. In Examples A and B, you will see how the following expression can be factored using educated guessing and checking and the quadratic formula. Additionally, you will see an algorithm (a step by step procedure) for factoring these types of polynomials without guessing. The proof of the algorithm is beyond the scope of this book, but is a reliable technique for getting a handle on tricky factoring questions of the form:

$6x^2-13x-28$

When you compare the computational difficulty of the three methods, you will see that the algorithm described in Example A is the most efficient.

A second type of advanced factoring technique is factoring by grouping. Suppose you start with an expression already in factored form:

$(4x+y)(3x+z) = 12x^2+4xz+3xy+yz$

Usually when you multiply the factored form of a polynomial, two terms can be combined because they are like terms. In this case, there are no like terms that can be combined. In Example C, you will see how to factor by grouping.

The last method of advanced factoring is the sum or difference of terms with matching odd powers. The pattern is:

$a^3+b^3 = (a+b)(a^2-ab+b^2)$

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

This method is shown in the guided practice and the pattern is fully explored in the exercises.

Example A

Factor the following expression: $6x^2-13x-28$

Solution: While this trinomial can be factored by using the quadratic formula or by guessing and checking, it can also be factored using a factoring algorithm. Here, you will learn how this algorithm works.

$6x^2-13x-28$

First, multiply the first coefficient with the last coefficient:

$x^2-13x-168$

Second, factor as you normally would with $a=1$:

$(x-21)(x+8)$

Third, divide the second half of each binomial by the coefficient that was multiplied originally:

$\left(x-\frac{21}{6}\right)\left(x+\frac{8}{6}\right)$

Fourth, simplify each fraction completely:

$\left(x-\frac{7}{2}\right)\left(x+\frac{4}{3}\right)$

Lastly, move the denominator of each fraction to become the coefficient of x:

$(2x-7)(3x+4)$

Note that this is a procedural algorithm that has not been proved in this text. It does work and can be a great time saving tool.

Example B

Factor the following expression using two methods different from the method used in Example A: $6x^2-13x-28$

Solution: The educated guess and check method can be time consuming, but since there are a finite number of possibilities, it is still possible to check them all. The 6 can be factored into the following four pairs:

1, 6

2, 3

-1, -6

-2, -3

The -28 can be factored into the following twelve pairs:

1, -28 or -28, 1

-1, 28 or 28, -1

2, -14 or -14, 2

-2, 14 or 14, -2

4, -7 or -7, 4

-4, -7 or -7, -4

The correctly factored expression will need a pair from the top list and a pair from the bottom list. This is 48 possible combinations to try.

If you try the first pair from each list and multiply out you will see that the first and the last coefficients are correct but the b coefficient does not.

$(1x+1)(6x-28) = 6x-28x+6x-28$

A systematic approach to every one of the 48 possible combinations is the best way to avoid missing the correct pair. In this case it is:

$(2x-7)(3x+4) = 6x^2+8x-21x-28 = 6x^2-13x-28$

This method can be extremely long and rely heavily on good guessing which is why the algorithm in Example A is provided and preferable.

An alternative method is using the quadratic formula as a clue even though this is not an equation set equal to zero.

$6x^2-13x-28$

$a=6, b=-13, c=-28$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{13\pm\sqrt{169-4\cdot 6\cdot -28}}{2\cdot 6} = \frac{13\pm 29}{12} = \frac{42}{12}\ or - \frac{16}{12} = \frac{7}{2}\ or\ -\frac{4}{3}$

This means that when set equal to zero, this expression is equivalent to

$\left(x-\frac{7}{2}\right)\left(x+\frac{4}{3}\right)=0$

Multiplying by 2 and multiplying by 3 only changes the left hand side of the equation because the right hand side will remain 0. This has the effect of shifting the coefficient from the denominator of the fraction to be in front of the x just like in Example A.

$6x^2 - 13x - 28 = (2x-7)(3x+4)$

Example C

Factor the following expression using grouping: $12x^2 + 4xz + 3xy + yz$

Solution: Notice that the first two terms are divisible by both 4 and x and the last two terms are divisible by y. First, factor out these common factors and then notice that there emerges a second layer of common factors. The binomial $(3x+z)$ is now common to both terms and can be factored out just as before.

$$\begin{aligned} 12x^2 + 4xz + 3xy + yz &= 4x(3x+z) + y(3x+z) \\ &= (3x+z)(4x+y) \end{aligned}$$

Concept Problem Revisited

The sum or difference of terms with matching odd powers can be factored in a precise pattern because when multiplied out, all intermediate terms cancel each other out.

$$a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$$

When a is distributed: $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4$

When b is distributed: $+a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5$

Notice all the inside terms cancel: $a^5 + b^5$

Vocabulary

To Factor
: It is to rewrite a polynomial expression given as sum of terms into a product of factors.

Difference ofSquares
: $(a^2 - b^2) = (a+b)(a-b)$

Difference of Cubes
: $(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$

Sum of Cubes
: Definition$(a^3 + b^3) = (a+b)(a^2 - ab + b^2)$

In Summary

You have learned how to factor different kinds of polynomials including difference of squares, difference of cubes and sums of cubes.

Guided Practice

1. Show how $a^3 - b^3$ factors by checking the result given in the guidance section.

2. Show how $a^3 + b^3$ factors by checking the result given in the guidance section.

3. Factor the following expression without using the quadratic formula or trial and error:

$8x^2+30x+27$

Answers:

1. Factoring,

$$\begin{aligned}a^3-b^3 &= (a-b)(a^2+ab+b^2)\\ &= a^3+a^2b+ab^2-a^2b-ab^2-b^3\\ &= a^3-b^3\end{aligned}$$

2. Factoring,

$$\begin{aligned}a^3+b^3 &= (a+b)(a^2-ab+b^2)\\ &= a^3-a^2b+ab^2+ba^2-ab^2+b^3\\ &= a^3+b^3\end{aligned}$$

3. Use the algorithm described in Example A.

$$\begin{aligned}8x^2+30x+27 &\to x^2+30x+216\\ &\to (x+12)(x+18)\\ &\to \left(x+\frac{12}{8}\right)\left(x+\frac{18}{8}\right)\\ &\to \left(x+\frac{3}{2}\right)\left(x+\frac{9}{4}\right)\\ &\to (2x+3)(4x+9)\end{aligned}$$

Practice

Factor each expression completely.

1. $2x^2-5x-12$
2. $12x^2+5x-3$
3. $10x^2+13x-3$
4. $18x^2+9x-2$
5. $6x^2+7x+2$
6. $8x^2+34x+35$
7. $5x^2+23x+12$

8. $12x^2 - 11x + 2$

Expand the following expressions. What do you notice?

9. $(a+b)(a^8 - a^7b + a^6b^2 - a^5b^3 + a^4b^4 - a^3b^5 + a^2b^6 - ab^7 + b^8)$

10. $(a-b)(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6)$

11. Describe in words the pattern of the signs for factoring the difference of two terms with matching odd powers.

12. Describe in words the pattern of the signs for factoring the sum of two terms with matching odd powers.

Factor each expression completely.

13. $27x^3 - 64$

14. $x^5 - y^5$

15. $32a^5 - b^5$

16. $32x^5 + y^5$

17. $8x^3 + 27$

18. $2x^2 + 2xy + x + y$

19. $8x^3 + 12x^2 + 2x + 3$

20. $3x^2 + 3xy - 4x - 4y$

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3.7 Dividing Polynomials

Here you will learn how to perform long division with polynomials. You will see how synthetic division abbreviates this process. In addition to mastering this procedure, you will see the how the remainder root theorem and the rational root theorem operate.

TEKS

1. P.1.F
2. P.2.J
3. P.2.K

Lesson Objectives

In this section you will learn about:
1. How to identify possible zeros of a polynomial.
2. How to divide polynomials by Long Division.
3. How to divide polynomials by Synthetic Division.
4. How to factor polynomials using Long Division or Synthetic Division.

Introduction

In section 3.5 you learned about obtaining the zeros from a graph, and how to graph a polynomial using the zeros and the degree, in this section you will learn about how to divide two polynomials and you will learn when is a polynomial a factor of the other polynomial.

Vocabulary

Long Division, Synthetic Division, Quotient, Remainder, Divisor, Dividend

While you may be experienced in factoring, there will always be polynomials that do not readily factor using basic or advanced techniques. How can you identify the roots of these polynomials?

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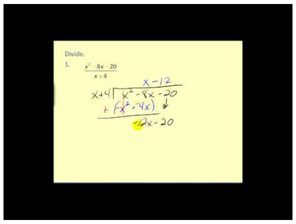

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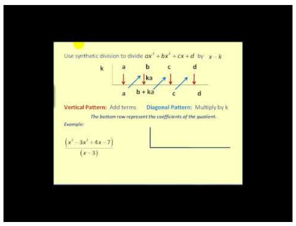
http://www.youtube.com/watch?v=brpNxPAkv1c James Sousa: Dividing Polynomials-Long Division



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http://www.youtube.com/watch?v=5dBAdzl2Mns James Sousa: Dividing Polynomials-Synthetic Division

There are numerous theorems that point out relationships between polynomials and their factors. For example there is a theorem that a polynomial of degree n must have exactly n solutions/factors that may or may not be real numbers. The **Rational Root Theorem** and the **Remainder Theorem** are two theorems that are particularly useful starting places when manipulating polynomials.

RATIONAL ZERO THEOREM AND THE REMAINDER THEOREM

- The **Rational Zero Theorem** states that in a polynomial, every rational solution can be written as a reduced fraction $\left(x = \frac{p}{q}\right)$, where p is an integer factor of the constant term and q is an integer factor of the leading coefficient. Example 1 shows how all the possible rational solutions can be listed using the Rational Root Theorem.
- The **Remainder Theorem** states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x-a)$ is equal to $f(a)$. The Remainder Theorem is only useful after you have performed polynomial long division because you are usually never given the divisor and the remainder to start. The main purpose of the Remainder Theorem in this setting is a means of double checking your application of polynomial long division. Example 2 shows how the Remainder Theorem is used.

THE DIVISION ALGORITHM

If $f(x)$ and $d(x)$ are polynomials with $d(x) \neq 0$ and the degree of $d(x)$ is less than or equal to the degree of $f(x)$ then there is unique polynomials $q(x)$ and $r(x)$ shuch that

$f(x) = d(x) \cdot q(x) + r(x)$

the remainder $r(x)$ equal 0 or it is of degree less than the degree of $d(x)$. If $r(x) = 0$ then $d(x)$ divides evenly into $f(x)$ and that means that $d(x)$ and $q(x)$ are factors of $f(x)$.

Polynomial long division is identical to regular long division. Synthetic division is a condensed version of regular long division where only the coefficients are kept track of. In Example 2 polynomial long division is used and in Example 3 synthetic long division is used.

Example A

Identify all possible rational solutions of the following polynomial using the Rational Root Theorem.

$12x^{18} - 91x^{17} + x^{16} + \cdots + 2x^2 - 14x + 5 = 0$

Solution: The integer factors of 5 are 1, 5. The integer factors of 12 are 1, 2, 3, 4, 6 and 12. Since pairs of factors could both be negative, remember to include $\pm$.

$\pm\frac{p}{q}=\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6},\frac{1}{12},\frac{5}{1},\frac{5}{2},\frac{5}{3},\frac{5}{4},\frac{5}{6},\frac{5}{12}$

While narrowing the possible solutions down to 24 possible rational answers may not seem like a big improvement, it surely is. This is especially true considering there are only a handful of integer solutions. If this question required you to find a solution, then the Rational Root Theorem would give you a great starting place.

Example B

Use Polynomial Long Division to divide

$\frac{x^3+2x^2-5x+7}{x-3}$

Solution: First note that it is clear that 3 is not a root of the polynomial because of the Rational Root Theorem and so there will definitely be a remainder. Start a polynomial long division question by writing the problem like a long division problem with regular numbers:

$x-3\overline{)x^3+2x^2-5x+7}$

Just like with regular numbers ask yourself "how many times does x go into x^3?" which in this case is x^2.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-5x+7} \end{array}$$

Now multiply the x^2 by $x-3$ and copy below. Remember to subtract the entire quantity.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-5x+7} \\ \underline{-(x^3-3x^2)} \end{array}$$

Combine the rows, bring down the next number and repeat.

$$\begin{array}{r} x^2\ +\ 5x\ +\ 10 \\ x-3\overline{)x^3+2x^2-5x+7} \\ \underline{-(x^3-3x^2)} \\ 5x^2-5x \\ \underline{-(5x^2-15x)} \\ 10x+7 \\ \underline{-(10x-30)} \\ 37 \end{array}$$

The number 37 is the remainder. There are two things to think about at this point. First, interpret in an equation:

$\frac{x^3+2x^2-5x+7}{x-3}=(x^2+5x+10)+\frac{37}{x-3}$

Second, check your result with the Remainder Theorem which states that the original function evaluated at 3 must be 37. Notice the notation indicating to substitute 3 in for x.

$(x^3+2x^2-5x+7)|_{x=3}=3^3+2\cdot 3^2-5\cdot 3+7=27+18-15+7=37$

Example C

Use Synthetic Division to divide the same rational expression as the previous example.

Solution: Synthetic division is mostly used when the leading coefficients of the numerator and denominator are equal to 1 and the divisor is a first degree binomial.

$\frac{x^3+2x^2-5x+7}{x-3}$

Instead of continually writing and rewriting the x symbols, synthetic division relies on an ordered spacing.

$+3\overline{)1\ \ 2\ -5\ \ 7}$

Notice how only the coefficients for the denominator are used and the divisor includes a positive three rather than a negative three. The first coefficient is brought down and then multiplied by the three to produce the value which goes beneath the 2.

$$\begin{array}{r|rrrr} +3) & 1 & 2 & -5 & 7 \\ & & \downarrow 3 & & \\ \hline & & 1 & & \end{array}$$

Next the new column is added. $2+3=5$, which goes beneath the 2^{nd} column. Now, multiply $5 \cdot +3 = 15$, which goes underneath the -5 in the 3^{rd} column. And the process repeats...

$$\begin{array}{r|rrrr} +3) & 1 & 2 & -5 & 7 \\ & \downarrow & 3 & 15 & 30 \\ \hline & 1 & 5 & 10 & 37 \end{array}$$

The last number, 37, is the remainder. The three other numbers represent the quadratic that is identical to the solution to Example 2.

$1x^2+5x+10$

Example D

Use synthetic division to divide $\frac{x^3-8}{x-2}$

Since the numerator is missing powers, then we need to replace it with 0's. We need to write the numerator as x^3+0x^2+0x+8

Now using synthetic division by using the coefficients

$$\begin{array}{r|rrrr} +2) & 1 & 0 & 0 & -8 \\ & \downarrow & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 0 \end{array}$$

finally

it would factor as $(x-2)(x^2+2x+4)$

Concept Problem Revisited

Identifying roots of polynomials by hand can be tricky business. The best way to identify roots is to use the rational root theorem to quickly identify likely candidates for solutions and then use synthetic or polynomial long division to quickly and effectively test them to see if their remainders are truly zero.

Guided Practice

1. Divide the following polynomials.

$\frac{x^3+2x^2-4x+8}{x-2}$

2. Completely factor the following polynomial.

$x^4+6x^3+3x^2-26x-24$

3. Divide the following polynomials.

$\frac{3x^5-2x^2+10x-5}{x-1}$

Answers:

1. $\frac{x^3+2x^2-4x+8}{x-2}=x^2+4x+4+\frac{16}{x-2}$

2. Notice that possible roots are $\pm 1,2,3,4,6,8,24$. Of these 14 possibilities, four will yield a remainder of zero. When you find one, repeat the process.

$$\begin{aligned}&x^4+6x^3+3x^2-26x-24\\&=(x+1)(x^3+5x^2-2x-4)\\&=(x+1)(x-2)(x^2+7x+12)\\&=(x+1)(x-2)(x+3)(x+4)\end{aligned}$$

3. $\frac{3x^5-2x^2+10x-5}{x-1}=3x^4+3x^3+3x^2+x+11+\frac{6}{x-1}$

Vocabulary

Polynomial long division is a procedure with rules identical to regular long division. The only difference is the dividend and divisor are polynomials.

Synthetic division is an abbreviated version of polynomial long division where only coefficients are used.

In Summary

We have learned how to do long division of polynomials and synthetic division of polynomials when possible. We have also learned that when a polynomial is missing a power, we have to replace it with a zero.

Practice

Identify all possible rational solutions of the following polynomials using the Rational Root Theorem.

1. $15x^{14}-12x^{13}+x^{12}+\cdots+2x^2-5x+5=0$
2. $18x^{11}+42x^{10}+x^9+\cdots+x^2-3x+7=0$
3. $12x^{16}+11x^{15}+3x^{14}+\cdots+6x^2-2x+11=0$
4. $14x^7-7x^6+x^5+\cdots+x^2+6x+3=0$

5. $9x^9 - 10x^8 + 3x^7 + \cdots + 4x^2 - 2x + 2 = 0$

Completely factor the following polynomials.

6. $2x^4 - x^3 - 21x^2 - 26x - 8$

7. $x^4 + 7x^3 + 5x^2 - 31x - 30$

8. $x^4 + 3x^3 - 8x^2 - 12x + 16$

9. $4x^4 + 19x^3 - 48x^2 - 117x - 54$

10. $2x^4 + 17x^3 - 8x^2 - 173x + 210$

Divide the following polynomials.

11. $\frac{x^4+7x^3+5x^2-31x-30}{x+4}$

12. $\frac{x^4+7x^3+5x^2-31x-30}{x+2}$

13. $\frac{x^4+3x^3-8x^2-12x+16}{x+3}$

14. $\frac{2x^4-x^3-21x^2-26x-8}{x^3-x^2-10x-8}$

15. $\frac{x^4+8x^3+3x^2-32x-28}{x^3+10x^2+23x+14}$

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$$x^4 + 6x^3 + 3x^2 - 26x - 24$$

3.8 Zeros of Polynomials

TEKS

1. P.1.F
2. P.2.I
3. P.2.J

Lesson Objectives

In this section you will learn about:
1. Using the Rational Zero Theorem to find possible zeros.
2. Solving polynomial equations.
3. Using the Factor Theorem to find polynomials with given zeros.

Introduction

In section 3-5 we learned how to factor a Polynomial by using the graphs of polynomials, now we will learn how to completely factor a polynomial algebraically.

Vocabulary

Rational Zero Theorem, Remainder Theorem, Factor theorem, Intermediate Value Theorem

The Following theorems will be useful in finding the zeros of a polynomial.

- The **Rational Zero Theorem** states that in a polynomial, every rational solution can be written as a reduced fraction $\left(x = \frac{p}{q}\right)$, where p is an integer factor of the constant term and q is an integer factor of the leading coefficient. In words, this means the following: $Possible.rational.zeros = \pm\frac{Factors.of.the.constant.term}{Factors.of.the.leading.coefficient}$
- The **Remainder Theorem** states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x-a)$ is equal to $f(a)$. The Remainder Theorem is only useful after you have performed polynomial long division because you are usually never given the divisor and the remainder to start. The main purpose of the Remainder Theorem in this setting is a means of double checking your application of polynomial long division.
- The **Factor Theorem** states that if $f(c) = 0$, then $(x-c)$ is a factor of $f(x)$. And if $(x-c)$ is a factor of $f(x)$, then $f(c) = 0$.

- The **Intermediate Value Theorem** states that If $f(x)$ is a polynomial with real coefficients and if $f(a)$ and $f(b)$ have opposite signs (one is positive and one is negative), then there is at least one value of c between a and b for which $f(c) = 0$. Equivalently if the equation $f(x) = 0$ has at least one real zero between a and b.
- Properties of Roots of Polynomial Equations

1. 1. If a polynomial is of n^{th} degree, then counting all roots (complex and real) the polynomial will have $n$roots
   2. If it has a complex root $a+bi$ then the complex conjugate $a-bi$ will also be a root. They occur in conjugate pairs.

Example A.

List all the possible rational zeros of $f(x) = 2x^2 - 4x - 30$**. By using the Rational Zero Theorem find an actual zero of the polynomial, then factor the polynomial.**

The possible rational zeros are $\pm\frac{1,2,3,5,6,10,15}{1,2}$ and simplifying we have $\pm 1, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3, 5, 6, \frac{15}{2}, 10, 15$ therefore since it is a degree two, two of those zeros might be a zero of the polynomial.

By using the **Factor Theorem,** lets check if **3** is a zero of the polynomial. $f(3) = 2(3)^2 - 4(3) - 30 = -24$ Therefore this is not a zero of the polynomial.

Now lets check 6 $f(6) = 2(6)^2 - 4(6) - 30 = 18$

Since $f(3)$is negative and $f(6)$ is positive (have opposite signs) and applying the **Intermediate Value Theorem** then that means that a number in between 3 and 6 must be a zero of the polynomial. From the options we have from the rational zero theorem, then 5 can be a zero.

Lets check 5 $f(5) = 2(5)^2 - 4(5) - 30 = 0$.

By using synthetic division and we can factor the polynomial as $f(x) = (x-5)(2x-6)$**.** The second factor obviously factors as $2(x-3)$ so the complete factored polynomial will be $f(x) = 2(x-5)(x+3)$

Therefore the two zeros of the polynomial will be -3 and 5 and by letting $x = 0$ the y-int is -30.

Look at the following graph.

This is a tedious process if you try to check all the possible zeros, if you have access to a graphing utility choose the zeros to factor from the graph.

Example B

Find the zeros of a the following polynomial function$f(x) = 4x^2 - 4x - 35$ **and factor the polynomial.**

list possible rational zeros $\pm\frac{1,5,7}{1,2,4} or \pm 1, 5, 7, \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{1}{4}, \frac{5}{4}, \frac{7}{4}$

We can try to do the same process as Example 1, but if we check the graph, it looks like the zeros can be $\frac{7}{2} and \frac{-5}{2}$. By using the remainder and factor theorems they are the factors so the polynomial can be factored as $f(x) = (x-\frac{7}{2})(x+\frac{5}{2})$ and finally by using one of the factoring algorithms $f(x) = (2x-7)(2x+5)$

Example C

Finding all roots of a polynomial including complex roots

Find all the roots of the following polynomial $f(x) = x^3 - 3x^2 + x - 3$ **and factor completely the polynomial**

The possible rational zeros are $\pm\frac{1,3}{1}$ which equals$\pm 1, \pm 3$.

By taking the zero 3, and performing synthetic division on the polynomial.

we obtain a remainder of zero. Therefore the polynomial factors as $f(x) = (x-3)\left(x^2+1\right)$ look at the following graph.

This polynomial only has one real zero, the second factor is a quadratic polynomial so we can find the other two zeros by using the quadratic formula as follows. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where a =1 b= 0 c=1. Therefore by substituting we obtain $x = \frac{-0 \pm \sqrt{0^2-4(1)(1)}}{2(1)}$ = $x = \frac{\pm\sqrt{-4}}{2} = \pm\frac{2i}{2} = \pm i$

Finally we have the zeros $3, -i, i$ so the polynomial factors as $f(x) = (x-3)(x-(i))(x-(-i))$

Note: every time a polynomial has complex zeros, they come in pairs, the zero and its complex conjugate.

Note 2: A polynomial of n^{th} degree will have n solutions they might be all real or a mixture between real and non-real(complex).

Example D

Solve the polynomial equation $x^4 - x^3 - 4x^2 - 2x - 12 = 0$

First, start by finding the possible rational zeros $\pm\frac{1,2,3,4,6,12}{1}$

Then find one that is a factor by applying the remainder theorem, the intermediate value theorem and the factor theorem. In this case they will be **3** and **-2**. These are two of the four solutions. We can also obtain them from a graph.

Then use synthetic division or long division to factor the polynomial.

If the remaining polynomial is a quadratic that cannot be factored use the quadratic formula to find the remaining solutions. In the final factor a= 1, b= 0 and c= 2 then by the quadratic formula

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-0 \pm \sqrt{0^2-4(1)(2)}}{2} = \pm\frac{\sqrt{-8}}{2} = \frac{\pm 2i\sqrt{2}}{2} = \pm i\sqrt{2}$ therefore the solutions are $3, -2, i\sqrt{2}, -i\sqrt{2}$

Example E

Find a polynomial given that the zeros are -3 and 1-2i

Since one of the zeros is a complex number then its complex conjugate is also a zero, therefore we have the zeros -3, 1-2i, 1+2i The polynomial in factored form will be

$f(x) = (x+3)(x-(1-2i))(x-(1+2i))$

$f(x) = (x-3)\left(x^2-2x+5\right)$

$f(x) = x^3 + x^2 - x + 15$

This is the graph of the polynomial

As you can see the only real zero will be -3.

Example F

Find a polynomial given that the zeros are 5, 7- 3i of degree 4.

Since it has complex numbers, the zeros are 5, 7-3i, 7+3i, but those are only three, therefore the 5 has to be of multiplicity 2, therefore the polynomial in linear factored form will be

$$\begin{aligned} f(x) &= (x-5)(x-5)(x-(7-3i))(x-(7+3i)) \\ &= \left(x^2-10x+25\right)\left(x^2-14x+58\right) \\ &= x^4 - 24x^3 + 223x^2 - 930x + 1450 \end{aligned}$$

this is the graph of the polynomial.

The y intercept will be 1450 by letting x = 0 and as you can see the zero only real zero is 5 of degree 2.

Example G.

Find a polynomial with zeros -9, -2, 5 passing through the point (2, 50).

First write the polynomial in factored form as $f(x) = a_n(x+9)(x+2)(x-5)$ if we let $a_n = 1$ then the polynomial will pass through the point **(2,-132)** therefore we need to find the correct a_n by replacing $x = 2$ and $f(x) = 50$ therefore

$$50 = a_n((2)+9)((2)+2)((2)-5)$$

$$50 = -132a_n$$

$$\frac{50}{-132} = a_n = \frac{-25}{66}$$

The correct polynomial is $f(x) = \frac{-25}{66}(x+9)(x+2)(x-5)$ and will pass through (2,50) here is the graph

The leading coefficient is negative so it will rise to left and fall to the right since the degree of the polynomial will be degree 3.

Vocabulary

Remainder Theorem

The **Remainder Theorem** states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x-a)$ is equal to $f(a)$. The Remainder Theorem is only useful after you have performed polynomial long division because you are usually never given the divisor and the remainder to start. The main purpose of the Remainder Theorem in this setting is a means of double checking your application of polynomial long division.Definition

Rational Zero Theorem

Rational Zero Theorem states that in a polynomial, every rational solution can be written as a reduced fraction$\left(x = \frac{p}{q}\right)$, where p is an integer factor of the constant term and q is an integer factor of the leading coefficient. In words, this means $Possible.rational.zeros = \pm\frac{Factors.of.the.constant.term}{Factors.of.the.leading.coefficient}$

Factor Theorem

Factor Theorem states that if $f(c) = 0$, then $(x-c)$is a factor of$f(x)$. And if $(x-c)$is a factor of$f(x)$, then$f(c) = 0$

Intermediate Value Theorem

Intermediate Value Theorem states that If $f(x)$is a polynomial with real coefficients and if $f(a)$and $f(b)$have opposite signs (one is positive and one is negative), then there is at least one value of c between a and b for which$f(c) = 0$. Equivalently if the equation$f(x) = 0$ has at least one real zero between a and b

In Summary

We learned how to list the possible rational zeros of a polynomial and to find a zero of the polynomial by using the remainder theorem, the factor theorem and the intermediate value theorem. We learned how to solve polynomials and find all the roots. We also learned to find polynomials that have complex roots/zeros Finally we learned how to make a polynomial pass through a given point.

Practice

Use the rational zero theorem to list all the possible rational zeros of the following polynomials.

1. $f(x) = x^2 + 14x + 48$

2.$f(x) = x^2 - 144$

3.$f(x) = 2x^3 + 23x^2 + 80x + 75$

4. $f(x) = 4x^3 + 5x^2 - 36x - 45$

Use synthetic division to divide the polynomials and state the remainder and tell if it is a factor of the polynomial.

5. polynomial $x^5 + 4x^4 - 3x^2 + 2x + 3 \;\; by \;\; x - 3$

6. polynomial $x^5 + x^3 - 2 \;\; by \;\; x - 1$

7. polynomial $x^5 - 2x^4 - x^3 + 3x^2 - x + 1 \;\; by \;\; x - 2$

Find an *nth* degree polynomial given the zeros of the polynomial.

8. $degree \;\; 3; \;\; zeros = 5, -3, \;\; 7$

9. $degree \;\; 3; \;\; zeros = 2, 5 + 3i$

10. $degree \;\; 4; \;\; zeros = 1, 5$

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3.9 Rational Expressions

Here you will add, subtract, multiply and divide rational expressions in order to help you solve and graph rational expressions in the future.

TEKS

1. P.1.F
2. P.2.K
3. P.2.L
4. P.2.M

Lesson Objectives

In this section you will learn about:
1. What is a rational expression.
2. To add subtract and divide rational expressions.
3. To simplify rational expressions.

Introduction

Simplifying fractions are simple when they are numerical values, but how about when the fractions are algebraic expressions? In this section you will learn how to simplify rational expressions.

Vocabulary

Rational Expression

A rational expression is a ratio just like a fraction. Instead of a ratio between numbers, a rational expression is a ratio between two expressions. One driving question to ask is:

Are the rules for simplifying and operating on rational expressions are the same as the rules for simplifying and operating on fractions?

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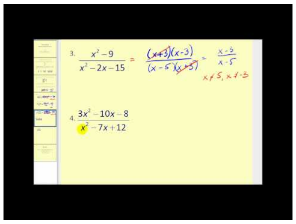

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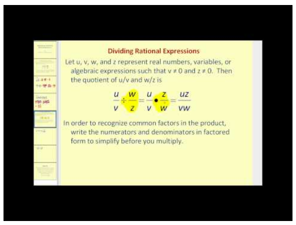

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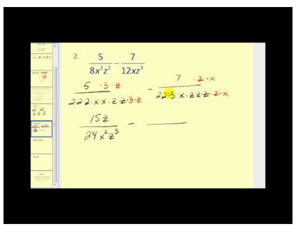

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URL: http://www.ck12.org/flx/render/embeddedobject/60714

http://www.youtube.com/watch?v=mJ6qOMno4-g James Sousa: Adding and Subtracting Rational Expressions

Guidance

A rational expression is a ratio of two polynomial expressions. When simplifying or operating on rational expressions, it is vital that each polynomial be fully factored. Once all expressions are factored, identical factors in the numerator and denominator may be canceled. The reason they can be "canceled" is that any expression divided by itself is equal to 1. An identical expression in the numerator and denominator is just an expression being divided by itself, and so equals 1.

- To multiply rational expressions, you should write the product of all the numerator factors over the product of all the denominator factors and then cancel identical factors.
- To divide rational expressions, you should rewrite the division problem as a multiplication problem. Multiply the first rational expression by the reciprocal of the second rational expression. Follow the steps above for multiplying.

To add or subtract rational expressions, it is essential to first find a common denominator. While any common denominator will work, using the least common denominator is a means of keeping the number of additional factors under control. Look at each rational expression you are working with and identify your desired common denominator. Multiply each expression by an appropriate clever form of 1 and then you should have your common denominator.

In both multiplication and division problems answers are most commonly left entirely factored to demonstrate everything has been reduced appropriately. In addition and subtraction problems the numerator must be multiplied, combined and then re-factored. Example B shows you how to finish an addition and subtraction problem appropriately.

Example A

Simplify the following rational expression.

$$\frac{x^2+7x+12}{x^2+4x+3}\cdot\frac{x^2+9x+8}{2x^2-128}\div\frac{x+4}{x-8}\cdot\frac{14}{1}$$

Solution: First factor everything. Second, turn division into multiplication (only one term). Third, cancel appropriately which will leave the answer.

$$\begin{aligned}
&=\frac{(x+3)(x+4)}{(x+3)(x+1)}\cdot\frac{(x+8)(x+1)}{2(x+8)(x-8)}\cdot\frac{(x-8)}{(x+4)}\cdot\frac{14}{1}\\
&=\frac{\cancel{(x+3)(x+4)}}{\cancel{(x+3)(x+1)}}\cdot\frac{\cancel{(x+8)(x+1)}}{2\cancel{(x+8)(x-8)}}\cdot\frac{\cancel{(x-8)}}{\cancel{(x+4)}}\cdot\frac{14}{1}\\
&=\frac{14}{2}\\
&=7
\end{aligned}$$

In this example, the strike through is shown. You should use this technique to match up factors in the numerator and the denominator when simplifying.

Example B

Combine the following rational expressions.

$$\frac{x^2-9}{x^4-81}-\frac{4x}{x^2+9}\div\frac{x-3}{2}$$

Solution:

$$\begin{aligned}
&\frac{\cancel{(x+3)(x-3)}}{(x^2+9)\cancel{(x+3)(x-3)}}-\frac{4x}{(x^2+9)}\cdot\frac{2}{(x-3)}\\
&=\frac{1}{(x^2+9)}-\frac{8x}{(x^2+9)(x-3)}\\
&=\frac{(x-3)}{(x^2+9)(x-3)}-\frac{8x}{(x^2+9)(x-3)}\\
&=\frac{(-7x-3)}{(x^2+9)(x-3)}
\end{aligned}$$

The numerator cannot factor at this point, so in this example there is not a factor that cancels at the end. Remember that the sum of perfect squares does not factor.

Example C

Combine the following rational expressions.

$$\frac{1}{x^2+5x+6}-\frac{1}{x^2-4}+\frac{(x-7)(x+5)+5}{(x+2)(x-2)(x+3)(x-4)}$$

Solution: First factor everything and decide on a common denominator. While the numerators do not really need to be factored, it is sometimes helpful in simplifying individual expressions before combining them. Note that the numerator of the expression on the right hand seems factored but it really is not. Since the 5 is not connected to the rest of the numerator through multiplication, that part of the expression needs to be multiplied out and like terms need to be combined.

$$\begin{aligned}
&=\frac{1}{(x+2)(x+3)}-\frac{1}{(x+2)(x-2)}+\frac{x^2-2x-35+5}{(x+2)(x-2)(x+3)(x-4)}\\
&=\frac{1}{(x+2)(x+3)}-\frac{1}{(x+2)(x-2)}+\frac{x^2-2x-30}{(x+2)(x-2)(x+3)(x-4)}
\end{aligned}$$

Note that the right expression has 4 factors in the denominator while each of the left expressions have two that match and two that are missing from those four factors. This tells you what you need to multiply each expression by in order to have the denominators match up.

$$=\frac{(x-2)(x-4)}{(x+2)(x-2)(x+3)(x-4)}-\frac{(x+3)(x-4)}{(x+2)(x-2)(x+3)(x-4)}+\frac{x^2-2x-30}{(x+2)(x-2)(x+3)(x-4)}$$

Now since the rational expressions have a common denominator, the numerators may be multiplied out and combined. Sometimes instead of rewriting an expression repeatedly in mathematics you can use an abbreviation. In this case, you can replace the denominator with the letter D and then replace it at the end.

$$\begin{aligned}
&=\frac{(x-2)(x-4)-(x+3)(x-4)+x^2-2x-30}{D}\\
&=\frac{x^2-6x+8-[x^2-x-12]+x^2-2x-30}{D}
\end{aligned}$$

Notice how it is extremely important to use brackets to indicate that the subtraction applies to all the terms of the middle expression not just x^2. This is one of the most common mistakes.

$$\begin{aligned}
&=\frac{x^2-6x+8-x^2+x+12+x^2-2x-30}{D}\\
&=\frac{x^2-8x-10}{D}
\end{aligned}$$

After the numerator has been entirely simplified try to factor the remaining expression and see if anything cancels with the denominator which you now need to replace.

$$\begin{aligned}
&= \frac{\cancel{(x+2)}(x-5)}{\cancel{(x+2)}(x-2)(x+3)(x-4)} \\
&= \frac{(x-5)}{(x-2)(x+3)(x-4)}
\end{aligned}$$

Concept Problem Revisited

Rational expressions are an extension of fractions and the operations of simplifying, adding, subtracting, multiplying and dividing work in exactly the same way.

Vocabulary

A ***rational expression*** is a ratio of two polynomial expressions.

In Summary

We learned how to add subtract divide and simplify rational expressions.

Guided Practice

1. Simplify the following expression.

$\frac{\frac{1}{x+1}-\frac{1}{x+2}}{\frac{1}{x-2}+\frac{1}{x+1}}$

2. Subtract the following rational expressions.

$\frac{x-2}{x+3}-\frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)}$

Answers:

1. The expression itself does not look like a rational expression, but it can be rewritten so it is more recognizable. Also working with fractions within fractions is an important skill.

$$\begin{aligned}
&= \left(\frac{1}{x+1}-\frac{1}{x+2}\right)\div\left(\frac{1}{x-2}+\frac{1}{x+1}\right) \\
&= \left[\frac{(x+2)}{(x+1)(x+2)}-\frac{(x+1)}{(x+1)(x+2)}\right]\div\left[\frac{(x+1)}{(x+1)(x-2)}-\frac{(x-2)}{(x+1)(x-2)}\right] \\
&= \left[\frac{1}{(x+1)(x+2)}\right]\div\left[\frac{3}{(x+1)(x-2)}\right] \\
&= \frac{1}{(x+1)(x+2)}\cdot\frac{(x+1)(x-2)}{3} \\
&= \frac{(x-2)}{3(x+2)}
\end{aligned}$$

2. Being able to factor effectively is of paramount importance.

$$\begin{aligned}
&= \frac{x-2}{x+3} - \frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)} \\
&= \frac{(x-2)}{(x+3)} - \frac{x^2(x-3)+8(x-3)}{2(x+2)(x^2-9)} \\
&= \frac{(x-2)}{(x+3)} - \frac{(x-3)(x^2+8)}{2(x+2)(x+3)(x-3)}
\end{aligned}$$

Before subtracting, simplify where possible so you don't contribute to unnecessarily complicated denominators.

$$= \frac{(x-2)}{(x+3)} - \frac{x^2+8}{2(x+2)(x+3)}$$

The left expression lacks $2(x+2)$, so multiply both its numerator and denominator by $2(x+2)$.

$$\begin{aligned}
&= \frac{2(x+2)(x-2)}{2(x+2)(x+3)} - \frac{(x^2+8)}{2(x+2)(x+3)} \\
&= \frac{2(x^2-4)-x^2-8}{2(x+2)(x+3)} \\
&= \frac{x^2-16}{2(x+2)(x+3)}
\end{aligned}$$

Practice

Perform the indicated operation and simplify as much as possible.

1. $\frac{x^2+5x+4}{x^2+4x+3} \cdot \frac{5x^2+15x}{x+4}$
2. $\frac{x^2-4}{x^2+4x+4} \cdot \frac{7}{x-2}$
3. $\frac{4x^2-12x}{5x+10} \cdot \frac{x+2}{x} \div \frac{x-3}{1}$
4. $\frac{4x^3-4x}{x} \div \frac{2x-2}{x-4}$
5. $\frac{2x^3+8x}{x+1} \div \frac{x}{2x^2-2}$
6. $\frac{3x-1}{x^2+2x-15} - \frac{2}{x+5}$
7. $\frac{x^2-8x+7}{x^2-4x-21} \cdot \frac{x^2-9}{1-x^2}$
8. $\frac{2}{x+7} + \frac{1}{x-7}$
9. $\frac{6}{x-7} - \frac{6}{x+7}$
10. $\frac{3x+35}{x^2-25} + \frac{2}{x+5}$
11. $\frac{2x+20}{x^2-4x-12} + \frac{2}{x+2}$
12. $\frac{2}{x+6} - \frac{x-9}{x^2-3x-18}$
13. $-\frac{5x+30}{x^2+11x+30} + \frac{2}{x+5}$
14. $\frac{x+3}{x+2} + \frac{x^3+4x^2+5x+20}{2x^4+2x^2-40}$

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3.10 Rational Functions

Here you will extend your knowledge of linear and quadratic equations to rational equations in general. You will gain insight as to what extraneous solutions are and how to identify them.

TEKS

1. P.1.F
2. P.2.K
3. P.2.L
4. P.2.M
5. P.2.N

Lesson Objectives

In this lesson you will learn about:
1. How to graph rational functions.
2. How to find the domains of rational functions.
3. How to identify the vertical asymptotes, horizontal asymptotes, holes and slant asymptotes.

Introduction

The graphs of rational functions are not like the graphs of polynomials, they can contain discontinuities like vertical and horizontal asymptotes, holes or slant asymptotes. In this section you will learn about rational functions and their graphs.

Vocabulary

Rational Function, Vertical Asymptote, Horizontal Asymptote, Slant Asymptote, Hole, Domain, Range, Zero's, Rational Function, Degree of Polynomial, Multiplicity of zeros, X-intercept, Y-intercept.

A **RATIONAL FUNCTION** is a function that its the quotient of two polynomial functions we can express the function as $f(x) = \frac{p(x)}{q(x)}$ where $q(x) \neq 0$.

Remember in a ratio(fraction) the denominator cannot equal 0.

GRAPHING RATIONAL FUNCTIONS TECHNIQUES

Step 1. Find the zeros of both the numerator and the denominator including their multiplicities.

Step 2. If the Numerator and Denominator have no zeros in common then those zeros represent the following.

a) The **domain** of the function will be the set of all real numbers with exception of all the zeros of the denominator.

b) The **zeros from the Numerator** are the **x-intercepts**.(recall a rational function is zero when the numerator is zero).

c) The **zeros from the denominator** are the **Vertical Asymptotes.**

d) To find the **y-intercept** let all x's on the function equal 0 and simplify

Step 3. If there is zeros in common between the numerator and the denominator, we need to check multiplicities.

a) If the multiplicity of the common zero is greater or equal on the numerator then that zero is a **hole,** factor both polynomials and simplify, then plug in the x coordinate of the hole to find the y-coordinate of the hole.

b) If the multiplicity of the common zero is greater on the denominator then that zero is a **Vertical Asymptote.**

Step 4. Get coordinates of points between the vertical asymptotes if there are more than one and after the asymptotes.

HORIZONTAL ASYMPTOTES

A rational function has a horizontal asymptote in two cases. Let $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq \mathbf{0}$.

a) If the **degree of q(x) >degree of p(x)** then the Horizontal Asymptote is the line $y = 0$

b) If the **degree of q(x) = degree of p(x)** then the Horizontal Asymptote is the line $y = \frac{a}{b}$ where a and b are the leading coefficients of p and q.

c) If the **degree of p(x) >degree of q(x)** then there is no Horizontal Asymptote, instead we obtain a slant or oblique asymptote that is found by long division or synthetic division if possible.

Example A.

Graphing function $f(x) = \frac{1}{x}$

zeros of the numerator: none **degree of numerator:** 0

zeros of the denominator: 0, **degree of denominator:** 1

Domain: $all real numbers x \neq \mathbf{0}$, **Holes:** *none*,

V.A.: $x = 0$, **H.A.:** $y = 0$, **S.A.:** *none*,

x-int: *none*, **y-int:** *none*,

Example B.

Graphing the function $f(x) = \frac{(x+1)(x-1)}{(x+5)(x-6)}$ **which is the same as** $f(x) = \frac{x^2-1}{x^2-x-30}$

zeros of the numerator: 1, -1 **degree of numerator:** 2

zeros of the denominator: -5, 6 **degree of denominator:** 2

Since there are no zeros in common between the numerator and the denominator and the degree of the denominator is bigger then we know the following.

The y intercept is obtained by letting $x = 0$ so we have $\frac{0^2-1}{0^2-0-30} = \frac{1}{30}$

Domain: all real numbers x $\neq$ -5, 6 **Holes:** *none* **,**

V.A.: x = 0, x = 2, x = 6, **H.A.:** y = 1, **S.A.:** *none* **,**

x-int: (-1,0),(1,0), **y-int:** $(0, \frac{1}{30})$

NOTE: When we distribute both polynomials the leading coefficients are 1 on the numerator and 1 on the denominator.

Example C.

Graphing the function $f(x) = \frac{x^2-9}{x^2-2x-15}$

by factoring both the numerator and the denominator it factors to

$f(x) = \frac{(x+3)(x-3)}{(x+3)(x-5)}$

and simplifying gives us $f(x) = \frac{(x-3)}{(x-5)}$ then we have the following

zeros of the numerator: -3, 3 **degree of numerator:** 2

zeros of the denominator: -3, 5 **degree of denominator:** 2

Since there are zeros that repeat in the numerator and denominator then we need to look at their multiplicities. The multiplicity of -3 in numerator is one and in the denominator is one, therefore **at x=-3** we have a **hole** in the function.

Domain: all real numbers x $\neq$ -3, 5, **Holes:** x = -3 coordinate (-3, 3/4),

V.A.: x = 5 , **H.A.:** y = 1, **S.A.:** *none*,

x-int: (3,0), **y-int:** (0,3/5)

NOTE: to find the coordinate of the hole you have to input the x-coordinate on the simplified function.

Example D.

Graphing the function $f(x) = \frac{8x^2-72}{2x^2-4x-30}$

by factoring both the numerator and the denominator it factors to

$f(x) = \frac{8(x+3)(x-3)}{2(x+3)(x-5)}$

and simplifying gives us $f(x) = \frac{4(x-3)}{(x-5)}$ then we have the following

zeros of the numerator: -3, 3 **degree of numerator:** 2

zeros of the denominator: -3, 5 **degree of denominator:** 2

Since there are zeros that repeat in the numerator and denominator then we need to look at their multiplicities.

The multiplicity of -3 in numerator is one and in the denominator is one, therefore **at x=-3** we have a hole in the function.

Also since the **degrees of the numerator and denominator are the same** then we have a **H.A.** $y = \frac{8}{2}$ which is **y = 4**

Domain: all real numbers x $\neq$ -3, 5, **Holes:** x = -3 coordinate (-3, 3),

V.A.: x = 5 , **H.A.:** y = 4, **S.A.:** *none*,

x-int: (3,0), **y-int:** (0,12/5) or (0, 2.4)

NOTE: to find the coordinate of the hole you have to input the x-coordinate on the simplified function.

SLANT ASYMPTOTES

SLANT ASYMPTOTES appear when the dividend (the numerator) polynomial is of one degree higher than the degree of the divisor (the denominator). By performing long division, then the quotient will be the equation of the slant asymptote. If possible use synthetic division to find the equation of the slant asymptote.

Example E.

Graph the function $f(x) = \frac{x^2+1}{x}$

zeros of the numerator: **Degree of numerator:** 2

zeros of the denominator: 0 **Degree of denominator:** 1

Since the degree of the numerator is bigger than the degree of the denominator there is no Horizontal Asymptote, instead we obtain a Slant Asymptote

Domain: all real numbers x $\neq$ 0, **Holes:** *none*,

V.A.: x = 0 , **H.A.:** , **S.A.:** y = x,

x-int: *none* , **y-int:** *none*

Example F

Graph the function $f(x) = \frac{(x-2)(x+3)(x-1)}{x^2-1}$

begin by factoring the denominator into $(x+1)(x-1)$ form there we can see the following

zeros of the numerator: -3, 1, 2 **Degree of numerator:** 3

zeros of the denominator: -1,1 **Degree of denominator:** 2

The simplified function will be $f(x) = \frac{(x-2)(x+3)}{x+1}$

both the numerator and the denominator have a zero in common in this case 1, the multiplicity of 1 is one on numerator and one on the denominator therefore it will be a **hole at x = 1**

The **y intercept** let **x = 0** so we obtain -6/1 = **-6**

The degree of the numerator is one more than the degree of the denominator therefore we will have a slant asymptote.

Domain: all real numbers x $\neq$ -1, 1, **Holes:** x = 1 coordinate (1,-2)

V.A.: x = -1 , **H.A.:** none, **S.A.:** y = x,

x-int: (-3, 0),(2, 0), **y-int:** (0, -6)

Example G

Graph $f(x) = \frac{2x^2-x-3}{x-3}$

by factoring the numerator we obtain the function $f(x) = \frac{(2x-3)(x+1)}{x-3}$

zeros of the numerator: -1, 3/2 **Degree of numerator:** 2

zeros of the denominator: 3 **Degree of denominator:** 1

Since the degree of the numerator is bigger by one, we have a slant asymptote

Domain: all real numbers x $\neq$ 1, 3/2, **Holes:** *none* ,

V.A.: x = 3 , **H.A.:** , **S.A.:** y = 2x + 5,

x-int: (-1,0), (3/2, 0), **y-int:** (0, 1)

Example I Graph $f(x) = \frac{(x-3)^2(x-5)(x+7)}{(x-3)(x+1)}$

zeros of the numerator: -7, 3, 5 **Degree of numerator:** 4

zeros of the denominator: -1, 3 **Degree of denominator:** 2

If you notice the numerator of this function is more than one degree bigger than the degree of the denominator. When this happens then the slant asymptote is not a line anymore, in this case the asymptote will be a parabola.

The simplified form of the function is $f(x) = \frac{(x-3)(x-5)(x+7)}{x+1}$

and the rational function will be $f(x) = \frac{x^3-x^2-41x+105}{x+1}$

Domain: all real numbers x $\neq$ -1, 3, **Holes:** x = 3 coordinate (3, 0)

V.A.: x = -1 , **H.A.:** , **S.A.:** y = x^2 - 2x - 39,

x-int: (-7, 0),(5, 0), **y-int:** (0, 105)

As you can see besides vertical, horizontal and slant asymptotes, we also obtain other asymptotes which could be parabolic or a greater power depending on the degree of the top and bottom polynomials of the rational function.

Vocabulary

Rational Function
: Is a function that is the quotient of two polynomial functions, we can express the function as $f(x) = \frac{p(x)}{q(x)}$ *where* $q(x) \neq 0$

Vertical Asymptote
: The line **x = a** of the graph of a rational function is a vertical asymptote if as **x** is approaching **a** from the left and from the right, the value of **f(x)** increases or decreases without bound. In a rational function, they are the zero's of the denominator if they are not holes.

Horizontal Asymptote
: The line **y = b** is a horizontal asymptote of the graph of a rational function if **f(x)** approaches **b** as **x** increases without bound.

Slant Asymptote
: The line in form **y = mx + b** obtained by dividing the rational function. We obtain a slant asymptote when the degree of the numerator is one more than the degree of the denominator.

Hole
: A break in the graph of a rational function that occurs when both the numerator and the denominator have

factors in common and the multiplicity of that factor in the numerator is greater than or equal to the multiplicity in the denominator.

Domain
The set of values for which the rational function is defined.

Degree of Polynomial
The highest degree of a polynomial.

Zeros
The x-intercepts of the polynomials that make the rational function.

Multiplicity of zeros
How many times a factor/zeros repeat.

In Summary

We have learned what a rational function is and its characteristics. We have learned to find the domain of a rational function, to find the vertical asymptotes, the horizontal asymptotes, the holes(if any), the slant asymptote(if any), the x-intercepts, the y-intercept. We have also learned that there is more than slant asymptotes, they could be parabolic or a greater power.

PRACTICE PROBLEMS

For the following rational functions do the following.

a) Find the zeros of the numerator and the denominator.

b) Identify the Domain

c) Identify the Vertical Asymptotes.

d) Horizontal Asymptotes

e) Holes

f) Slant Asymptotes (if any)

g) x-intercepts

h) y- intercept

i) graph the rational function.

1. $f(x) = \frac{5x+20}{x^2+x-12}$

2. $f(x) = \frac{(x-2)(x+3)(x-1)}{x^2-1}$

3. $f(x) = \frac{(x-1)(x-2)(x-3)}{x(x-4)^2}$

4. $f(x) = \frac{x^2+3x}{x^3+4x^2+4x}$

5. $f(x) = \frac{2x^2-x-3}{x-3}$

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3.11 Rational Equations

TEKS

1. P.1.F
2. P.5.K
3. P.5.L

Lesson Objectives

In this section you will learn about:
1. Solving Rational Equations.

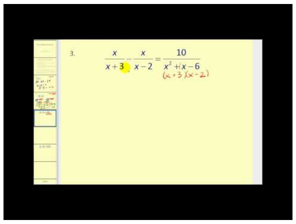
Introduction

The techniques for solving rational equations are extensions of techniques you already know. Recall that when there are fractions in an equation you can multiply through by the denominator to clear the fraction. The same technique helps turn rational expressions into polynomials that you already know how to solve. When you multiply by a constant there is no problem, but when you multiply through by a value that varies and could possibly be zero interesting things happen.
Since every equation is trivially true when both sides are multiplied by zero, how do you account for this when solving rational equations?
In this section you will learn about solving rational equations.

Vocabulary

Extraneous solution, Rational Equation, Rational Expression.

Watch This



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http://www.youtube.com/watch?v=MMORnvhR4wA James Sousa: Solving Rational Equations

Guidance

The first step in solving rational equations is to transform the equation into a polynomial equation. This is accomplished by clearing the fraction which means multiplying the entire equation by the common denominator of all the rational expressions. Then you should solve using what you already know. The last thing to check once you have the solutions is that they do not make the denominators of any part of the equation equal to zero when substituted back into the original equation. If so, that solution is called **extraneous** and is a "fake" solution that was introduced when both sides of the equation were multiplied by a number that happened to be zero.

Example A

Solve the following rational equation.

$x-\frac{5}{x+3}=12$

Solution: Multiply all parts of the equation by $(x+3)$, the common denominator.

$$\begin{aligned}
x(x+3)-5&=12(x+3)\\
x^2+3x-5-12x-36&=0\\
x^2-9x-41&=0\\
x&=\frac{-(-9)\pm\sqrt{(-9)^2-4\cdot 1\cdot(-41)}}{2\cdot 1}\\
x&=\frac{9\pm 7\sqrt{5}}{2}
\end{aligned}$$

The only potential extraneous solution would have been -3, so both answers are possible.

Example B

Solve the following rational equation

$\frac{3x}{x+4}-\frac{1}{x+2}=-\frac{2}{x^2+6x+8}$

Solution: Multiply each part of the equation by the common denominator of $x^2+6x+8=(x+2)(x+4)$.

$$\begin{aligned}
(x+2)(x+4)\left[\frac{3x}{x+4}-\frac{1}{x+2}\right] &= \left[\frac{-2}{(x+2)(x+4)}\right](x+2)(x+4)\\
3x(x+2)-(x+4) &= -2\\
3x^2+6x-x-2 &= 0\\
3x^2+5x-2 &= 0\\
(3x-1)(x+2) &= 0\\
x &= \frac{1}{3},-2
\end{aligned}$$

Note that -2 is an extraneous solution. The only actual solution is $x=\frac{1}{3}$.

Example C

Solve the following rational equation for y.

$x=2+\frac{1}{2+\frac{1}{y+1}}$

Solution: This question can be done multiple ways. You can use the clearing fractions technique twice.

$$\begin{aligned}
\left(2+\frac{1}{y+1}\right)x &= \left[2+\frac{1}{2+\frac{1}{y+1}}\right]\left(2+\frac{1}{y+1}\right)\\
2x+\frac{x}{y+1} &= 2\left(2+\frac{1}{y+1}\right)+1\\
2x+\frac{x}{y+1} &= 4+\frac{2}{y+1}+1\\
(y+1)\left[2x+\frac{x}{y+1}\right] &= \left[5+\frac{2}{y+1}\right](y+1)\\
2x(y+1)+x &= 5(y+1)+2\\
2xy+2x+x &= 5y+5+2
\end{aligned}$$

Now just get the y variable to one side of the equation and everything else to the other side.

$$\begin{aligned}
2xy-5y &= -3x+7\\
y(2x-5) &= -3x+7\\
y &= \frac{-3x+7}{2x-5}
\end{aligned}$$

Concept Problem Revisited

In order to deal with extra solutions introduced when both sides of an equation are multiplied by a variable, you must check each solution to see if it makes the denominator of any fraction in the original equation zero. If it does, it is called an extraneous solution.

Vocabulary

An ***extraneous solution*** is a "fake" solution to a rational equation that is introduced when both sides of an equation are multiplied through by zero.

A ***rational equation*** is an equation with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

In Summary

In this lesson we learned different techniques on solving rational equations. We also learned about extraneous solutions that sometimes appear when solving rational expressions and how to check them.

Guided Practice

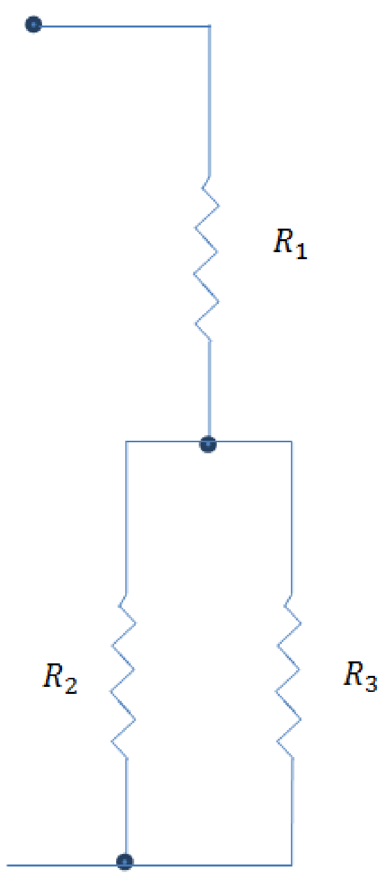
1. Solve the following rational equation.

$\frac{3x}{x-5}+4=x$

2. In electrical circuits, resistance can be solved for using rational expressions. This is an electric circuit diagram with three resistors. The first resistor R_1 is run in series to the other two resistors R_2 and R_3 which are run in parallel. If the total resistance R is 100 ohms and R_1 and R_3 are each 22 ohms, what is the resistance of R_2?

The equation of value is:

$$R=R_1+\frac{R_2R_3}{R_2+R_3}$$



3. Solve the following rational equation.

$$\frac{x+2}{x} - \frac{3}{x+2} = \frac{6}{x^2+2x}$$

Answers:

1. $\frac{3x}{x-5} + 4 = x$

$$\begin{aligned} 3x + 4x - 20 &= x^2 - 5x \\ 0 &= x^2 - 12x + 20 \\ 0 &= (x-2)(x-10) \\ x &= 2, 10 \end{aligned}$$

Neither solution is extraneous.

2. $R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$

$$\begin{aligned} 100 &= 22 + \frac{x \cdot 22}{x+22} \\ 78(x+22) &= 22x \\ 78x + 1716 &= 22x \\ 56x &= -1716 \\ x &= -30.65 \end{aligned}$$

A follow up question would be to ask whether or not ohms can be negative which is beyond the scope of this text.

3. $\frac{x+2}{x} - \frac{3}{x+2} = \frac{6}{x^2+2x}$

$$\begin{aligned} (x+2)(x+2) - 3x &= 6 \\ x^2 + 4x + 4 - 3x - 6 &= 0 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ x &= -2, 1 \end{aligned}$$

Note that -2 is an extraneous solution.

Practice

Solve the following rational equations. Identify any extraneous solutions.

1. $\frac{2x-4}{x} = \frac{16}{x}$

2. $\frac{4}{x+1} - \frac{x}{x+1} = 2$
3. $\frac{5}{x+3} + \frac{2}{x-3} = 1$
4. $\frac{3}{x-4} - \frac{5}{x+4} = 6$
5. $\frac{x}{x+1} - \frac{6}{x+2} = 4$
6. $\frac{x}{x-4} - \frac{4}{x-4} = 8$
7. $\frac{4x}{x-2} + 3 = 1$
8. $\frac{-2x}{x+1} + 6 = -x$
9. $\frac{1}{x+2} + 1 = -2x$
10. $\frac{-6x-3}{x+1} - 3 = -4x$
11. $\frac{x+3}{x} - \frac{3}{x+3} = \frac{6}{x^2+3x}$
12. $\frac{x-4}{x} - \frac{2}{x-4} = \frac{8}{x^2-4x}$
13. $\frac{x+6}{x} - \frac{2}{x+6} = \frac{12}{x^2+6x}$
14. $\frac{x+5}{x} - \frac{3}{x+5} = \frac{15}{x^2+5x}$
15. Explain what it means for a solution to be extraneous.

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3.12 Polynomial and Rational Inequalities

TEKS

1. P.1.F
2. P.5.K
3. P.5.L

Lesson Objectives

In this section you will learn about:
1. What a Polynomial Inequality is.
2. How to solve Polynomial Inequalities.
3. What a Rational Inequality is.
4. How to solve Rational Inequalities.

Introduction

In the previous sections of this chapter we have learned how to solve polynomial functions/equations and how to work with rational functions, but what about polynomial or rational inequalities? Do we use the same procedures? You will find out in this section.

Vocabulary

Polynomial Inequality, Rational Inequality, boundary points

Solving an inequality is very similar than solving an equation the only difference is that when you are solving an equation you are solving for a specific number(s) that make the equation true, while the inequalities you still need to solve the inequality like if it was an equation, but those solutions will be your **boundary points**, then you will have more solutions to the left or right that will make the inequality true.

Here are some steps we can use to solve the polynomial inequalities.

1. Make the inequality in the form $f(x) < 0, or f(x) > 0$
2. Solve it like if it where $f(x) = 0$. These are your boundary points.

3. Divide the number line in intervals using the boundary points.
4. Test a point in each interval to see if they satisfy the inequality, if they do the interval is part of the solution.
5. Write your solution set.

NOTE: This procedure also works if the inequalities are $\leq, \geq$ the only difference is that the boundary points will be part of your solution.

Example A. (Review)

Write the solutions to the following inequalities and graph the solution set on a number line.

a) $x \geq 5$ **b)** $x < -3$ **c)** $-7 \leq x < 10$

Solution

a) The solutions is $[5, \infty)$

b) the solution is $(-\infty, -3)$

c) the solution is $[-7, 10)$

Recall

For an inequality if it is $\leq, \geq$ they should be closed brackets [a,b] and on a graph they should be closed points.

For an inequality if it is a $<, >$ they should be parenthesis (a ,b) and on a graph they should be open points.

Finally if it is on infinity at any side, the infinity side has to have a parenthesis.

Here is the solutions for a, b, and c in a number line.

Example B.

Solve and graph the solution set on a number line of $x^2 + 4x \leq 5$

Solution

We write the polynomial as $x^2 + 4x - 5 \leq 0$

From there, we make the polynomial equal to zero and solve it $x^2 + 4x - 5 = 0$

This polynomial factors as $(x-1)(x+5) = 0$ making each factor equal to zero will give us the solutions $x = 1, x = -5$

Using these points as our boundary points, the test a point before in the middle and after, in this case we will test $x = -6, x = 0, x = 2$

testing $x = -6$

$(-6)^2 + 4(-6) - 5 \leq 0$ **is false** because the inequality $7 \leq 0$ its false

testing $x = 0$

$(0)^2 + 4(0) - 5 \leq 0$ **is true** because the inequality $-5 \leq 0$ its true and

testing $x = 3$

$(3)^2 + 4(3) - 5 \leq 0$ **is false** because the inequality $16 \leq 0$ its false

Therefore our solutions will be the intervals where the values are true.

Finally our solution set will be $[-5,1]$.with brackets on both ends because it was a $\leq$ inequality.

Here is the graphical solution on a number line

Example C.

Solve and graph the solution set on a number line of $x^2+4x>5$**.**

Notice, this example is the same as example 2 with a different inequality, therefore we will use the same procedure and obtain the same boundary points, if we test the points as example 2, we will obtain the following

testing $x=-6$

$(-6)^2+4(-6)-5>0$ **is true** because the inequality $7>0$ its true

testing $x=0$

$(0)^2+4(0)-5>0$ **is false** because the inequality $-5>0$ it is false

testing $x=2$

$(3)^2+4(3)-5>0$ **is true** because the inequality $16>0$ it is true,

The solutions then are the intervals where the values of the tested point are true in this case they will be $(-\infty,-5)\bigcup(1,\infty)$

here is the solution on a number line

Example D.

Solve and graph the solution set of $x^3-2x^2-29x+30>50$

First we need to write the polynomial as $x^3-2x^2-29x-20>0$ by subtracting 50 from both sides and solve as if it were equal to zero.

Then we need to find the boundary points. By using the rational zero theorem, the possible rational zeros are $\pm\frac{1,2,4,5,10,20}{1}$ and choosing -4 will be a zero of the polynomial. By using synthetic division we obtain

therefore the polynomial factors as $(x+4)\left(x^2-6x-5\right)>0$**.** Therefore $x=-4$ is a boundary point and using the quadratic formula for the quadratic factor we obtain $x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-5)}}{2(1)}$ so then by simplifying we obtain $x=3\pm\sqrt{14}$.

Therefore the boundary points are $x=-4,x=3-\sqrt{14},x=3+\sqrt{14}$

We will test the values $x=-5,x=-2,x=0,x=7$ (one before, one in between each boundary point, one after) to see which of them satisfy our inequality.

testing $x=-5$

$(-5)^3-2(-5)^2-29(-5)-20>0$ **is false** because the inequality $-50>0$ its false.

testing $x=-2$

$(-2)^3-2(-2)^2-29(-2)-20>0$ **is true** because the inequality $22>0$ it is true$22>0$

testing $x=0$

$(0)^3-2(0)^2-29(0)-20>0$ **is false** because the inequality $-20>0$ it is false$-20>0$

testing $x=7$

$(7)^3-2(7)^2-29(7)-20>0$ **is true** because the inequality $22>0$ it is true

Therefore our solutions are $\left(-4,3-\sqrt{14}\right)\bigcup\left(3+\sqrt{14},\infty\right)$

The number line solution is

Solving Rational Inequalities

The difference between a polynomial inequality and a rational inequality is that you need to find the zeros of both the numerator and the denominator to find the asymptotes and all the zeros will become your boundary points. You also need to find the common denominator and simplify to solve the rational inequalities.

Here are some steps we can use to solve rational inequalities.

1. Find the x-intercepts, y-intercepts and asymptotes as if you were graphing a rational function.
2. Express the rational function as $f(x) < 0$ or $f(x) > 0$ and locate the zeros of the numerator and denominator, these will be your boundary points.
3. Divide the number line using your boundary points.
4. Choose a value in the interval to test if they satisfy the inequality, if they do they are part of the solution.
5. Write the solution set.

NOTE: You cannot multiply both sides by the denominator and solve it like a polynomial.

Example E.

Solve and graph the solution set of $\frac{2x+6}{x-2} \geq 0$

Since this is a rational inequality, from the previous section we know the following, by setting $2x+6=0$ then $x=3$ this is an x-intercept, by setting $x-2=0$ then $x=2$ this is a vertical asymptote.

Since the numerator and the denominator have the same degree, then the horizontal asymptote is $y=2$ obtained by dividing the leading coefficients and the y-intercept is **(0,-3)** by letting **x = 0**.

Here is the graph.

Our boundary points come from the zeros of the numerator and the denominator, in this case they were $x=-3, x=2$ and the number line is broken into three intervals from $(-\infty,-3),(-3,2),(2,\infty)$. Therefore check a value on each interval and see if its true, in this case we will check $x=-4, x=0, x=4$.

Checking $x=-4$

$\frac{2(-4)+6}{(-4)-2}=\frac{1}{3}$ therefore **its true** because the inequality $\frac{1}{3} \geq 0$ its true.

Checking $x=0$

$\frac{2(0)+6}{(0)-2}=-3$ therefore **its false** because the inequality $-3 \geq 0$ its not true.

Checking $x=4$

we obtain $\frac{2(4)+6}{(4)-2}=7$ therefore it **is true** because the inequality $7 \geq 0$ its true.

Finally the solutions to this rational inequality will be $(-\infty,-3] \bigcup (2,\infty)$ it excludes 2 because its an asymptote and its undefined.

Here is the solution on a number line.

Example F.

Solve and graph the solution set of $\frac{x+5}{x-6} > 2$**.**

Since this is a rational inequality, and performing the rational function procedures for graphing, we can find out the following: **x-int = -5**, **vertical asymptote** is **x = 6**, the **y-int = (0, -5/6)** and the horizontal asymptote **y = 1**.

Here is the graph of the rational function $\frac{x+5}{x-6}$

The steps to solving the inequality is to make it >0 by subtracting two from both sides thus obtaining the inequality

$\frac{x+5}{x-6} - 2 > 0$ To simplify this expression we need to find the common denominator by multiplying -2 by $\frac{x-6}{x-6}$

Therefore we have

$\frac{x+5}{x-6} - 2\left(\frac{x-6}{x-6}\right) > 0$ and simplifying we obtain $\frac{x+5-2(x-6)}{x-6} > 0$ thus we obtain

$\frac{-x+17}{x-6} > 0$ so by setting both the numerator and the denominator equal to zero we obtain $x = 17, x = 6$ and these become our boundary points, therefore the number line gets broken into three intervals, $(-\infty, 6), (6, 17), (17, \infty)$ thus by checking a point in each interval, we will check $x = 0, x = 7, x = 18$.

checking $x = 0$

$\frac{-(0)+17}{(0)-6} = \frac{17}{-6}$ then this value **is false** because the inequality $\frac{-17}{6} > 0$ is not true

checking $x = 7$

$\frac{-(7)+17}{(7)-6} = \frac{10}{1} = 10$ then this value **is true** because the inequality $10 > 0$ its true

checking $x = 18$

$\frac{-(18)+17}{(18)-6} = \frac{-1}{12}$ then this value **is false** because the inequality $\frac{-1}{12} > 0$ is false.

Therefore the solution set to this rational inequality is $(-6, 17)$ because it was a $>$ inequality. We can see this is true on the graph above.

Here is the solution set on a number line

Example G

Application

An object is thrown up from the ground level at a velocity of 90 ft/sec. In which time interval will the object have a height greater than or equal to 100 ft.

Solution

Recall from section 3.2 that the position function of a free falling object near the surface of the earth is given by

$s(t) = -16t^2 + v_o t + s_o$ where v_o is the initial velocity in ft/sec and s_o is the initial height in ft. and position is given in ft.

Therefore we need to solve the inequality $-16t^2 + 90t \geq 100$

By rewriting it as $-16t^2 + 90t - 100 \geq 0$ we can use the quadratic formula with **a=-16**, **b=90** and **c =-100** to obtain the solutions.

$t = \frac{-(90) \pm \sqrt{(90)^2 - (4)(-16)(-100)}}{2(-16)} = \frac{-90 \pm \sqrt{1700}}{-32} = \frac{-90 \pm 41.30}{-32}$ therefore obtaining $t = 1.52sec, t = 4.10sec$

therefore the solution set is $[1.52, 4.10]$ therefore the object has a height greater than or equal to 100 ft. between 1.52 seconds and 4.10 seconds.

Here is the graph that shows the height of the object with respect to time. The red line show a height greater than 100 ft.

Vocabulary

A **polynomial inequality** is an inequality that can be put into one of the following forms $f(x) < 0, f(x) > 0, f(x) \leq 0, f(x) \geq 0$,.

A **rational inequality** is an inequality that can be put into one of the forms like the polynomial inequality where $f(x)$ is a rational expression.

In Summary

We learned how to solve a polynomial inequality by making it equal to zero, solving for zero and determining the intervals where the inequality is satisfied. Similarly the rational inequalities were solved, but we also needed to find the zeros of the denominator.

Practice

Solve the following polynomial inequalities, state the solution in interval notation and graph the solution on a number line.

1. $x^2+2x-48 \leq 0$
2. $x^2-25 \geq 0$
3. $12x^2+7x > 10$
4. $x^3+8x^2-9x-72 < 0$
5. $-4x^2-144 \geq 0$

Solve the following rational inequalities, state the solution in interval notation and graph the solution on a number line.

6. $\frac{x-4}{x-1} \geq 0$
7. $\frac{(x+5)(x-2)}{x+1} \leq 0$
8. $\frac{x+4}{2x-1} < 3$
9. $\frac{x}{x+5} > 5$
10. $\frac{x^2+6x-27}{x^3+4x^2-x-4} > 0$

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Polynomials and rational functions were explored. Special features of rational functions such as holes and asymptotes were introduced. Graphing rational functions by hand, which utilized work from many concepts, concluded the chapter.

3.13 References

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CHAPTER 4 Exponential and Logarithmic Functions

Chapter Outline

Exponential growth has been called one of the most powerful forces in the universe. You may already know the basic rules of exponents. Here you will explore their relationship with logarithms and their application to different types of growth and decay.

4.1 Exponential Functions

TEKS

1. P.2.G
2. P.2.F
3. P.2.I
4. P.2.J

Lesson Objectives

In this section you will learn about:

1. How to evaluate exponential functions (including base e).
2. Graphing exponential functions (including base e).
3. Interest Formulas (n compoundings and compounding continuously).

Introduction

You learned about exponents in Algebra 2, including properties and simplifying exponential expressions. We will now learn about how to evaluate, graph and apply exponential equations to real life situations, not just talking about simplifying expressions.

Vocabulary

base, exponent, exponential function, natural base (Euler's constant), principal, compound interest, continuous compound

1. **Exponential Functions and Evaluating:**

Definition of an Exponential Function:

$f(x) = b^x$or $y = b^x$

Where b is the base, and x is the exponent and $b > 0$ and $b \neq 0$. The first condition that $b > 0$ is important because if the value of the base is negative the relation is no longer a function due to the fact that the sign of the value of the dependent value will alternate. The second condition that $b \neq 0$ is just as important because if the value of the base is 1, then the function is constant, no matter what the value of x is, the result will always be 1.

Example: Evaluate the exponential function for the given values.

$f(x) = 3(4.3)^x$

$f(0), f(1), f(2)$

Solution:

$f(0) = 3(4.3)^0$
$= 3(1)$
$= 3 \quad f(1) = 3(4.3)^1$
$= 3(4.3)$
$= 12.9 \quad f(2) = 3(4.3)^2$
$= 3(18.9)$
$= 55.47$

Definition of the Natural Base "e" (Euler's Constant):

e is an irrational number known as the natural base or Euler's Constant.

$f(x) = e^x$ is the natural exponential function where $e \approx 2.718281827$

Example: Evaluate the exponential function for the given values.

$f(x) = 1.38e^{4x}$

$f(0), f(1), f(2)$

Solution:

$f(0) = 1.38e^{4(0)}$
$= 1.38(1)$
$= 1.38 \quad f(1) = 1.38e^{4(1)}$
$= 1.38(54.598)$
$= 75.35 \quad f(2) = 1.38e^{4(2)}$
$= 1.38(2980.96)$
$= 4113.72$

2. **Graphing Exponential Functions and Variations:**

To graph an exponential function we can use the point plotting technique learned in Chapter 2.

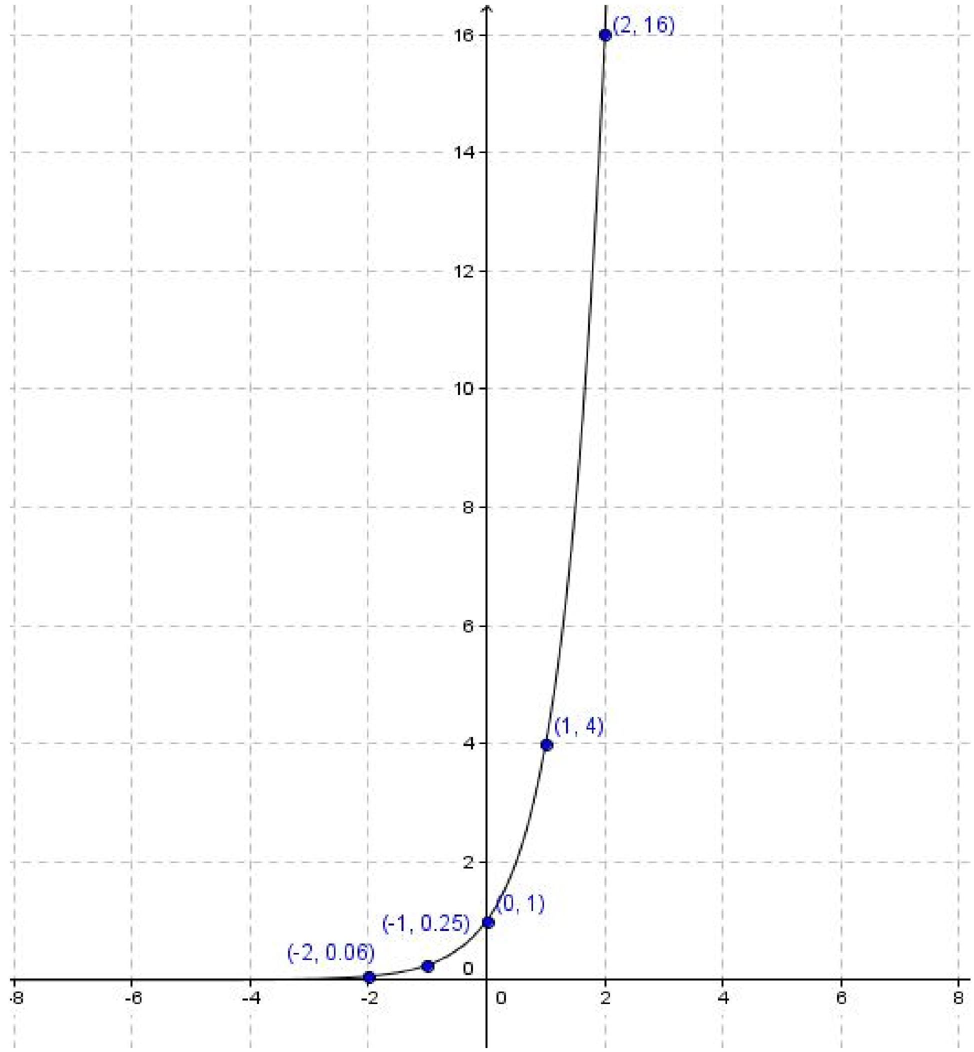
Example: Graph.

A. $f(x) = 4^x$

TABLE 4.1:

x	$f(x) = 4^x$	$(x, f(x))$
-2	$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$	$\left(-2, \frac{1}{16}\right)$
-1	$4^{-1} = \frac{1}{4^1} = \frac{1}{4}$	$\left(-1, \frac{1}{4}\right)$
0	$4^0 = 1$	$(0, 1)$
1	$4^1 = 4$	$(1, 4)$
2	$4^2 = 16$	$(2, 16)$

Then we plot the points to get the graph:

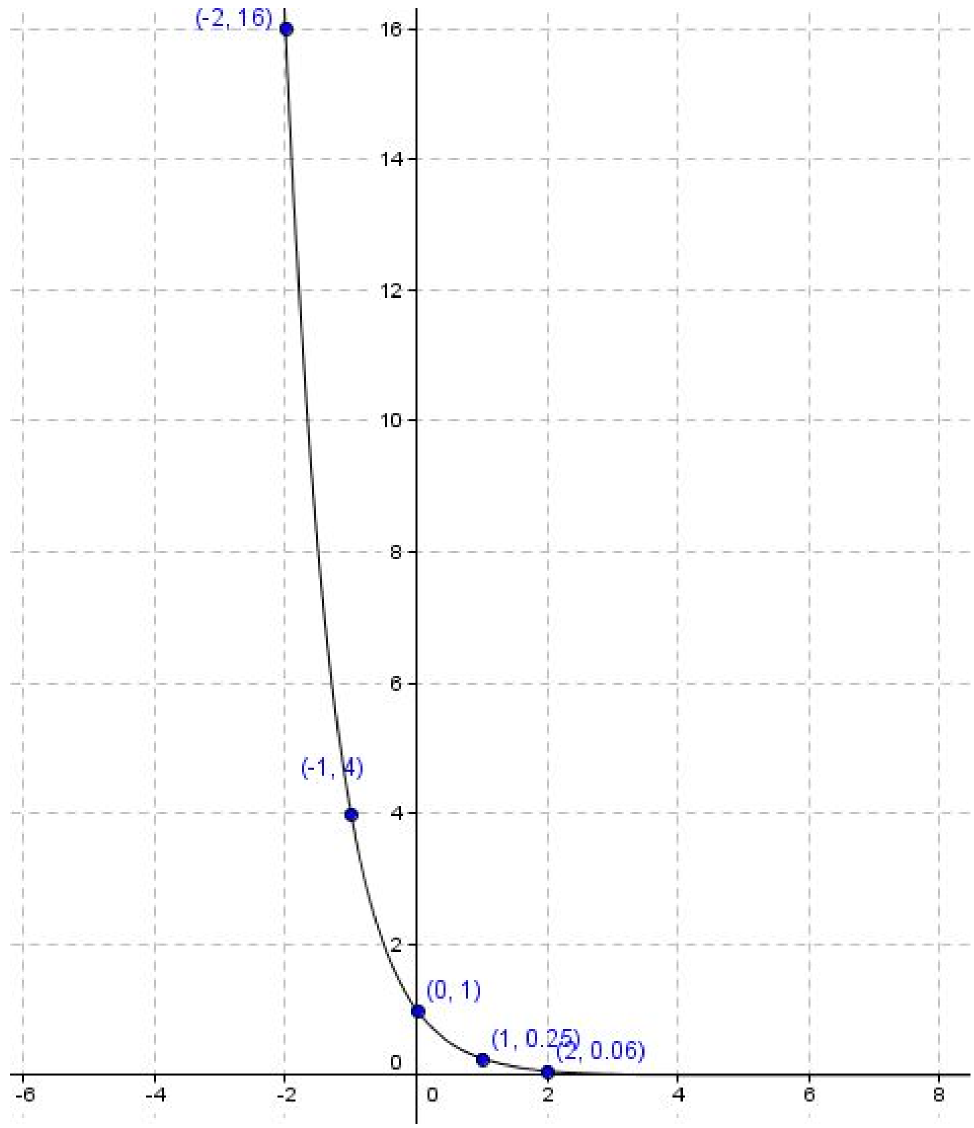


B. $f(x) = \left(\frac{1}{4}\right)^x$

TABLE 4.2:

x	$f(x) = \left(\frac{1}{4}\right)^x$	$(x, f(x))$
-2	$\left(\frac{1}{4}\right)^{-2} = 4^2 = 16$	$(-2, 16)$
-1	$\left(\frac{1}{4}\right)^{-1} = 4^{-1} = 4$	$(-1, 4)$
0	$\left(\frac{1}{4}\right)^0 = 1$	$(0, 1)$
1	$\left(\frac{1}{4}\right)^1 = \frac{1}{4^1} = \frac{1}{4}$	$\left(1, \frac{1}{4}\right)$
2	$\left(\frac{1}{4}\right)^2 = \frac{1}{4^2} = \frac{1}{16}$	$\left(2, \frac{1}{16}\right)$

Then we plot the points to get the graph:



Note some characteristics of the exponential function graph:

1. The domain of the exponential function is $(-\infty, \infty)$ and the range is $(0, \infty)$
2. Functions $f(x) = b^x$ pass through $(0, 1)$ because $f(x) = b^0$ no matter what value of b.
3. If $b > 1$ it is increasing, the bigger the value of b the steeper the graph.
4. If $0 < 1$ the graph is decreasing and the smaller the value of b the steeper the graph.
5. $f(x) = b^x$ is one to one, meaning that $f(x)$ is a function AND its inverse $f(x)^{-1}$ is also a function.
6. The graph $f(x) = b^x$ approaches 0 but never reaches 0.

Variations of $f(x) = b^x$:

TABLE 4.3:

Vertical Translation	$g(x) = b^x + c$	If $c > 0$ the graph is translated up If $c < 0$ the graph is translated down
Horizontal Translation	$g(x) = b^{x+c}$	If $c > 0$ the graph is translated to the left If $c < 0$ the graph is translated to the right

TABLE 4.3: (continued)

Reflections	$g(x) = -b^x$	Reflects across the x-axis
	$g(x) = b^{-x}$	Reflects across the y-axis
Vertical Stretch/Compress	$g(x) = cb^x$	If $c > 1$ stretches the graph
		If $0 < 1$ compresses the graph
Horizontal Stretch/Compress	$g(x) = b^{cx}$	If $c > 1$ compresses the graph
		If $0 < 1$ stretches the graph

3. **Interest Formulas:**

Compound Interest is interest computed on your original investment as well as any accumulated interest. In Algebra II you learned about the interest formula

$A = P(1+r)$, where A is the accumulated amount, P is the principal or original amount, and r is the rate as a decimal. This equation shows the accumulated amount over just ONE year, if we want to calculate for t years at once a year we would be multiplying by $(1+r)$ for each year, giving us $A = P(1+r)^t$.

What about those interest questions that compound more than once a year? For example semiannually (two times a year), quarterly (three times a year), or monthly (twelve times a year)...

We use the variable n represents the number of compoundings per year where we would have to divide the rate by how many times per year and multiply the year time the number of times per year giving us a new formula $A = P\left(1+\frac{r}{n}\right)^{nt}$.

What about those interest rates that say they are compounded continuously? When a interest rate is said to be compounded continuously it means the number of compounding periods per year increase infinitely giving us the formula $A = Pe^{rt}$.

Example: You decide to invest \$4,000 for 5 years and you have a choice between two types of accounts. One pays 6% per year compounded monthly and the other pays 6.35% per year compounded continuously. Which is the better investment?

First lets look at the first type of account, 6% compounded monthly for five years. Compounded monthly means that $n = 12$, $r = 0.06$, and $t = 5$. We will use the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ to find out what the final value of this account is:

$$\begin{aligned}
A &= P\left(1+\frac{r}{n}\right)^{nt} \\
A &= 4000\left(1+\frac{0.06}{12}\right)^{12(5)} \\
&= 4000(1.005)^{60} \\
&= 4000(1.34885) \\
&= \$5395.40
\end{aligned}$$

For the second type of account, 6.35% compounded continuously for five years means that $r = 0.0635$ and $t = 5$. We will use the formula $A = Pe^{rt}$ to find out what the final value of this account:

$$\begin{aligned}
A &= Pe^{rt} \\
A &= 4000e^{0.0635(5)} \\
&= 4000(1.37369) \\
&= \$5494.76
\end{aligned}$$

So out of the two accounts the one that is a better investment is the second , for it pays out more than the first.

Vocabulary

Base

The number or variable being raised to a power. For example in the equation $y = a \cdot b^x$, b is the base.

Exponent

The number or variable to which the base is raised to. For example in the equation $y = a \cdot b^x$, x is the exponent

Exponential Function

A mathematical function of the form $y = a \cdot b^x$where a is a constant, b is the base and x is the exponent.

Natural Base e

The natural base "e" is the number approximately 2.718281

Principal

The total amount of money that is borrowed or invested, not including any dividends or interest.

Compound Interest

Is the interest added to a principal of a deposit or loan so that the added interest also earns interest from then on. The formula for compound interest it $A = P(1 + \frac{r}{n})^{nt}$where A is the ending amount, P is the principal, r is the interest rate, n is the number of compounding periods and t is the time in years.

Continuous Compound

The process of earning interest on top of interest on a continuous or instantaneous basis. The formula for compound interest is $A = Pe^{rt}$where A is the ending amount, P is the principal, r is the interest rate and t is the time in years.

In Summary

We have learned how to evaluate exponential functions including base e. We have also learned to graph exponential functions and the characteristics of the exponential functions. We also learned about the transformations of exponential functions and to work with the compound interest formula and the continuous compound formula.

Check for Understanding:

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4.2 Logarithmic Functions

TEKS

1. P.2.G
2. P.2.I
3. P.2.J

Lesson Objectives

In this section you will learn about:

1. What a logarithmic function is.
2. How to convert from exponential to logarithmic form.
3. How to evaluate logarithms.
4. Basic logarithmic properties.
5. Graphing and domain and range.
6. Natural logarithms.

Introduction

In the previous section we learned about the exponential functions. One of the characteristics of an exponential function is that the inverse does not exist but what is the inverse?

Vocabulary

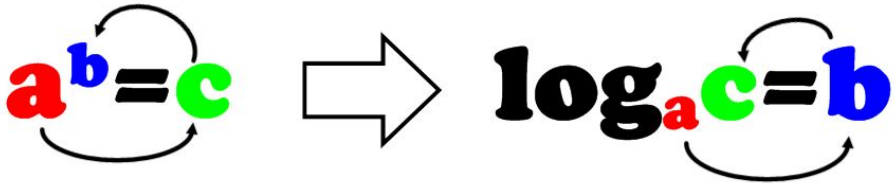
logarithm, common logarithm, natural logarithm

1. **Definition of Logarithmic Function:**

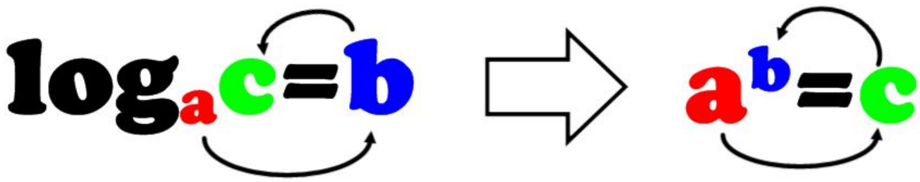
   When $x > 0$ and $b \neq 1$ we have that $y = log_b x$ is the same as $b^y = x$ where the first is called the logarithmic form and the second is the exponential form.

2. **Change between Exponential and Logarithmic Forms:**

   We can think of converting as a circle, we always start at the base and go in the direction of the equals sign.



Similarly:



Example: Convert between the two forms.

A. $7^x = 5$

$log_7 5 = x$

B. $3^a = 27$

$log_3 27 = a$

C. $log_4 x = 16$

$4^{16} = x$

D. $log_3 9 = x$

$3^x = 9$

3. Evaluate Logarithms:

When we are evaluating logarithms we ask ourselves, "what exponent or power will i need to raise b to to get x?"

Example: Evaluate.

A. $log_3 9$

$3^? = 9$

$$3^0 = 1$$
$$3^1 = 3$$
$$3^2 = 9$$

Therefore $log_3 9 = 2$

B. $log_4 \frac{1}{16}$

$4^? = \frac{1}{16}$

$$4^0 = 1$$
$$4^{-1} = \frac{1}{4}$$
$$4^{-2} = \frac{1}{16}$$

Therefore $log_4 \frac{1}{16} = -2$

C. $log_{36} 6$

$36^? = 6$

$$36^0 = 1$$
$$36^1 = 36$$
$$36^{\frac{1}{2}} = 6$$

Therefore $log_{36} 6 = \frac{1}{2}$

4. Basic Logarithmic Properties:

TABLE 4.4:

$log_b b = 1$	because $b^1 = b$ no matter what the value of b is.
$log_b 1 = 0$	because $b^0 = 1$ as long as $b \neq 0$
$log_b b^x = x$	because a logarithm is the inverse of an exponential, if the bases match they inverse each other leaving x.
$b^{log_b x} = x$	again, because a logarithm is the inverse of an exponential, if the bases match they inverse each other leaving only x.

Example: Evaluate using basic logarithmic properties.

A. $log_8 8$

Since $log_b b = 1$, and in this case $b = 8$, so $log_8 8 = 1$.

B. $log_{13} 1$

Since $log_b 1 = 0$, and in this case $b = 13$, so $log_{13} 1 = 0$.

C. $log_3 3^{5x}$

Since $log_b b^x = x$, and in this case $b = 3$, so $log_3 3^{5x} = 5x$

D. $7^{log_7 4}$

Since $b^{log_b x} = x$, and in this specific case $b = 7$ and $x = 4$ so $7^{log_7 4} = 4$

5. Graphing and Domain and Range:

Note the Characteristics of Logarithmic Functions:

1. The domain of $f(x) = log_b x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

2. The graphs of all functions in the form $f(x) = log_b x$ pass through the point $(1,0)$
3. If $b > 1$ then the graph of $f(x) = log_b x$ is increasing.
4. If $0 < 1$ then the graph of $f(x) = log_b x$ is decreasing.
5. The graph of $f(x) = log_b x$ approaches the y-axis but never reaches or touches it.

Variations of $f(x) = log_b x$:

TABLE 4.5:

Vertical Translation	$g(x) = log_b x + c$	If $c > 0$ translates up If $c < 0$ translates down
Horizontal Translation	$g(x) = log_b(x+c)$	If $c > 0$ translates to the left If $c < 0$ translates to the right
Reflection	$g(x) = -log_b x$ $g(x) = log_b(-x)$	Reflects about the x-axis Reflects about the y-axis
Vertical Stretch/Compress	$g(x) = clog_b x$	If $c > 1$ it stretches If $0 < 1$ it compresses
Horizontal Stretch/Compress	$g(x) = log_b(cx)$	If $c > 1$ it compresses If $0 < 1$ it stretches

Example

Given the function $f(x) = log_b(x-4)$state its domain and range.

The domain of the parent logarithmic function is $(0,\infty)$and the range is $(-\infty,\infty)$ then since the function f(x) is a horizontal translation of 4 units to the right therefore:

Domain of f(x) = $(4,\infty)$

Range of f(x) = $(-\infty,\infty)$

6. Natural Logarithms:

A logarithmic function with base e is called a Natural Logarithm. The natural logarithmic function $f(x) = log_e x$ is expressed $f(x) = lnx$.

The natural logarithmic function takes on the same basic properties as a logarithmic function, as notated in sub-section 4 of this section.

Vocabulary

Logarithm

The logarithm of a number y with respect to a base b is the exponent we have to raise b to get y. For example in the expression $b^x = y$ the logarithmic form is $log_b y = x$

Common logarithm

The common logarithm is the logarithm of base 10. It is denoted by Log it means it has base 10.

Natural logarithm

The natural logarithm is the logarithm of base e. It is denoted by Ln it means it has base e.

In Summary

We have learned what a logarithmic function is and to be able to change between logarithmic form and exponential form. We have also learned how to evaluate logarithmic expressions and the basic properties of logarithms. Finally we learned about transformations of logarithms and the natural logarithm ln(x).

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4.3 Properties of Logarithms

TEKS

1. P.1.E
2. P.1.F

Lesson Objectives

In this section you will learn about:

1. The Product and Quotient Rules.
2. The Power Rule.
3. Expanding and Condensing Logarithmic Expressions.
4. Change of Base Formula.

Introduction

In the previous section you learned about logarithmic functions and how they are the inverse of exponential functions. We can create logarithmic expressions using variables just like with exponential expressions. Since a logarithm is essentially an exponent, it follows some similar rules that exponents follow. In this section we will explore how properties of exponents and the logarithmic rules are similar and use them to simplify or expand and condense logarithmic expressions.

Vocabulary

<no new vocabulary>

1. **The Product and Quotient Rules**

For properties of exponents we know the following rules for multiplying and dividing exponential expressions with like bases:

$a^m \times a^n = a^{m+n}$ and $\frac{a^m}{a^n} = a^{m-n}$

so if a logarithm is an exponent then we have:

Product Rule: $log_x(a \times b) = log_x a + log_x b$

Quotient Rule: $log_x \left(\frac{a}{b}\right) = log_x a - log_x b$

Example: Use the product or quotient rules to evaluate.

A. $log_3 18 - log_3 2$

This is a subtraction of two logs with the same base, therefore we are going to use the quotient rule for logarithms that states $log_x \frac{a}{b} = log_x a - log_x b$.

$$\begin{aligned} log_3 18 - log_3 2 &= log_3 \frac{18}{2} \\ &= log_3 9 \\ &= 2 \end{aligned}$$

B. $log_2 4 + log_2 2$

This is an addition of two logs with the same base, therefore we are going to use the product rule for logarithms that states $log_x(a \times b) = log_x a + log_x b$.

$$\begin{aligned} log_2 4 + log_2 2 &= log_2(4 \times 2) \\ &= log_2 8 \\ &= 3 \end{aligned}$$

2. **The Power Rule**

For properties of exponents we know:

$(a^m)^n = a^{m \times n}$

Naturally for logarithms we then have that:

$log_x a^n = n \times log_x a$

Example: Use the power rule to evaluate.

A. $log_3 3^x$

Since $log_x a^n = n \times log_x a$ then:

$$\begin{aligned} log_3 3^x &= x \times log_3 3 \\ &= x \times 1 \\ &= x \end{aligned}$$

B. $log 100^{2x}$

In the same fashion as above:

$$\begin{aligned} log 100^{2x} &= 2x \times log 100 \\ &= 2x \times 2 \\ &= 4x \end{aligned}$$

3. **Expanding and Condensing Logarithmic Expressions**

We can combine the properties/rules to either condense or expand logarithmic expressions to simplify. By simplifying we either can condense our logarithmic expression in terms of one logarithm, or expand to where each term only has a log, a base and a value to evaluate.

Example: Simplify by expanding completely.

A. $log_b\left(x^3\sqrt[3]{y}\right)$

$$\begin{aligned} log_b\left(x^3\sqrt[3]{y}\right) &= log_b x^3 + log_3\sqrt[3]{y} \\ &= 3log_b x + log_b y^{\frac{1}{3}} \\ &= 3log_b x + \left(\frac{1}{3}\right) \times log_b y \end{aligned}$$

B. $log_6\left(\frac{\sqrt{x}}{26y^2}\right)$

$$\begin{aligned} log_b\left(\frac{\sqrt{x}}{26y^2}\right) &= log_b\sqrt{x} - log_b 26y^2 \\ &= log_b x^{\frac{1}{2}} - \left(log_b 26 + log_b y^2\right) \\ &= \left(\frac{1}{2}\right) \times log_b x - log_b 26 - 2 \times log_b y \end{aligned}$$

Example: Simplify by condensing and writing as a single logarithm.

A.$log_2 2 + log_2 32$

$$\begin{aligned} log_2 2 + log_2 32 &= log_2(2 \times 32) \\ &= log_2 64 \\ &= 6 \end{aligned}$$

B. $log(4x-3) - logx$

$$log(4x-3) - logx = log\left(\frac{4x-3}{x}\right)$$

4. **Change of Base Formula**

Given any logarithm of base x, we can change the base to base 10 or base y by doing the following:

$log_x M = \frac{logM}{logx}$ or $log_x M = \frac{log_y M}{log_y x}$

Example

Use the change of base formula to find the value of $\log_5 625 = x$

Solution

By changing the expression to exponential form we have $5^x = 625$ therefore the solution is **x = 4** but we want to use the change of base formula as follows.

$$\begin{aligned}
\log_5 625 &= \frac{log625}{log5} \\
&= \frac{2.795880017}{.6989700043} \\
&= 4
\end{aligned}$$

NOTE: Some calculators can only do base 10 logarithms therefore you need to know how to convert from any base into base 10 by using the change of base formula **.**

Vocabulary

No new vocabulary

In Summary

In this lesson we have learned about the product rule, the quotient rule and the power rule of logarithms. We have learned about condensing and expanding logarithmic expressions and to use the change of base formula.

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4.4 Exponential and Logarithmic Equations

Here you will apply the new algebraic techniques associated with logs to solve equations.

TEKS

1. P.5.G
2. P.5.H
3. P.5.I

Lesson Objectives

In this section you will learn about:

1. Solving exponential equations using like bases
2. Solving exponential equations using logarithms
3. Solving logarithmic equations
4. Applying methods of solving to real world situations

Introduction

In the previous sections we learned how to manipulate logarithmic expressions, how to evaluate and the special relationship between logarithms and exponentials. We can now use all these concepts to solve problems involving exponential and logarithmic equations.

Vocabulary

logarithmic equation, exponential equation, half-life

1. **Solving exponential equations using like bases**

   Some numbers can be expressed as a known exponential, for example 9 can be written as the exponential 3^2. This is useful because if we want to solve an exponential equation and notice that the two expressions can be written with like bases then it makes it much easier to solve because the exponents become equal.

   Example: Solve.

A. $5^{2x-16} = 125$

$5^{2x-16} = 125$
$5^{2x-16} = 5^3$
then, $2x-1 = 3$
$2x = 4$
$x = 2$

B. $27^{x+4} = 9^x$

$27^{x+4} = 9^x$
$\left(3^3\right)^{x+4} = \left(3^2\right)^x$
then, $3\left(x+4\right) = 2x$
$3x+12 = 2x$
$x+12 = 0$
$x = -12$

2. **Solving exponential equations using logarithms**

The next logical question to ask is, "What happens if the exponential cannot be written with like bases?" The solution is easy. We learned that the inverse of an exponential is a logarithm, therefore we can use a logarithm to essentially isolate the variable. We can use either a common log or a natural log to do this.

Steps for Solving with Logarithms

1. Isolate the exponential to be solved.
2. Take the log, or natural log of both sides of the equation.
3. Simplify using the properties of logarithms.
4. Solve for the variable.

Example: Solve.

A. $2^x = 15$

$$\begin{aligned}2^x &= 15\\ log2^x &= log15\\ x \times log2 &= log15\\ x &= \frac{log15}{log2}\\ x &= 3.907\end{aligned}$$

B. $10^x = 84000$

$$\begin{aligned}10^x &= 84000\\ log10^x &= log84000\\ x \times log10 &= log84000\\ x &= \frac{log84000}{log10}\\ x &= 4.924\end{aligned}$$

C. $30e^{0.3x} - 2 = 327$

$$
\begin{aligned}
30e^{0.3x} - 2 &= 327 \\
30e^{0.3x} &= 329 \\
e^{0.3x} &= \frac{329}{30} \\
lne^{0.3x} &= ln\frac{329}{30} \\
0.3x &= ln\frac{329}{30} \\
x &= \frac{ln\frac{329}{30}}{0.3} \\
x &= 7.983
\end{aligned}
$$

D. $5^{x+2} = 4^{2x-1}$

$$
\begin{aligned}
5^{x+2} &= 4^{2x-1} \\
ln5^{x+2} &= ln4^{2x-1} \\
(x+2)\,ln5 &= (2x-1)\,ln4 \\
xln5 + 2ln5 &= 2xln4 - ln4 \\
xln5 - 2xln4 &= -ln4 - 2ln5 \\
x(ln5 - 2ln4) &= -ln4 - 2ln5 \\
x &= \frac{-ln4 - 2ln5}{ln5 - 2ln4} \\
x &= 3.959
\end{aligned}
$$

3. **Solving Logarithmic Equations**

A logarithmic equation is an equation that has a variable in a logarithmic expression. We can use some properties we know for logarithms to isolate the variable and solve. Once we solve we have to make sure to check that our answers exist.

Example: Solve.

A. $log_9(x+4) = 3$

$$
\begin{aligned}
log_9(x+4) &= 3 \\
9^{log_9(x+4)} &= 9^3 \\
x+4 &= 729 \\
x &= 725
\end{aligned}
$$

B. $2ln(5x) = 4$

$$
\begin{aligned}
2ln(5x) &= 4 \\
ln(5x) &= \frac{4}{2} \\
ln(5x) &= 2 \\
e^{ln(5x)} &= e^2 \\
5x &= e^2 \\
x &= \frac{e^2}{5} \\
x &= 1.478
\end{aligned}
$$

C. $log_3 x + log_3(x+7) = 2$

$$
\begin{aligned}
log_3 x + log_3(x+7) &= 2 \\
log_3\left(x(x+7)\right) &= 2 \\
3^{log_3(x^2+7x)} &= 3^2 \\
x^2+7x &= 9 \\
x^2+7x-9 &= 0 \\
x = \frac{-7 \pm \sqrt{49+36}}{2} & \\
x = \frac{-7 \pm \sqrt{85}}{2} &
\end{aligned}
$$

D. $ln(x+2) - ln(4x+4) = ln\frac{1}{x}$

$$
\begin{aligned}
ln(x+2) - ln(4x+4) &= ln\frac{1}{x} \\
ln\left(\frac{x+2}{4x+4}\right) &= ln\frac{1}{x} \\
e^{ln\left(\frac{x+2}{4x+4}\right)} &= e^{ln\frac{1}{x}} \\
\frac{x+2}{4x+4} &= \frac{1}{x} \\
\frac{x+2}{2(x+2)} &= \frac{1}{x} \\
\frac{1}{2} &= \frac{1}{x} \\
x &= 2
\end{aligned}
$$

4. **Applications**

The half-life of a substance is the time required for half of a given sample to degenerate. We use carbon-14 to date fossils and other artifacts. Carbon-14 decays exponentially with a half-life of 5715 years. We use the exponential growth model:

$A = A_o e^{kt}$

A_o is the original amount, k is the rate of growth or decay and t is the time.

Example: Carbon-14 Dating

A. Use the fact that carbon-14 has a half-life of 5715 years to write an equation for A.

B. In 1957 an explorer found an artifact in a cave. Analysis indicated that the artifact contained 78% of its original carbon-14. Estimate the age.

Solution:

Part A.

Since we want half of the original amount, then we proceed as follows:

$$\begin{aligned}
\frac{1}{2}A_o &= A_o e^{kt} && \textit{divide both sides by } A_o \\
\frac{1}{2} &= e^{k \cdot 5715} && \textit{take ln of both sides} \\
\ln\frac{1}{2} &= \ln e^{5715k} && \textit{the } ln(e) \textit{ cancels} \\
\ln\frac{1}{2} &= 5715k && \textit{divide both sides by } 5715 \\
\frac{\ln\frac{1}{2}}{5715} &= k \\
-.000121285596 &= k
\end{aligned}$$

Therefore our equation is $A = A_o e^{-.000121285596t}$

Part B.

Since the artifact contains 78% of the original carbon-14 we proceed as follows:

$$\begin{aligned}
.78A_o &= A_o e^{-.000121285596t} \\
.78 &= e^{-.000121285596t} \\
\ln(.78) &= \ln e^{-.000121285596t} \\
\ln(.78) &= -.000121285596t \\
\frac{\ln(.78)}{-.000121285596} &= t \\
2048.56 years &= t
\end{aligned}$$

Vocabulary

Exponential Equation

An equation that involves exponents that are variables.

Logarithmic Equation

An equation that involves logarithms.

Half-Life

The amount of time it takes any substance to become half of its original amount.

In Summary

In this section we have learned about solving exponential equations using like bases and using logarithms. We also learned about solving logarithmic equations and about solving half life problems.

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Exponential functions demonstrate applications of geometric growth and decay in the real world. After practicing with the rules and procedures to gain fluency with exponents and scientific notation, you transferred your knowledge to logarithms and their properties. Lastly, you explored how a new type of function, the logistic function, improves on exponential growth models for real world application.

CHAPTER 5 Basic Triangle Trigonometry

Chapter Outline

Trigonometry is the study of triangles and the relationships between their sides, angles and areas. Using what you know from Geometry like the Pythagorean theorem and the formula for the area of a triangle, you will be able to derive more powerful formulas and learn to apply them appropriately in different situations.

5.1 Angles in Radians and Degrees

Here you will learn how to translate between different ways of measuring angles.

TEKS

1. P.4.C
2. P.4.D

Lesson Objectives

In this lesson you will learn about:
1. The definition of a radian.
2. How to convert from degrees to radians.
3. How to convert from radians to degrees.

Introduction

Most people are familiar with measuring angles in degrees. It is easy to picture angles like 30°, 45°or 90°and the fact that 360°makes up an entire circle. Over 2000 years ago the Babylonians used a base 60 number system and divided up a circle into 360 equal parts. This became the standard and it is how most people think of angles today. However, there are many units with which to measure angles. For example, the gradian was invented along with the metric system and it divides a circle into 400 equal parts. The sizes of these different units are very arbitrary.
A radian is a unit of measuring angles that is based on the properties of circles. This makes it more meaningful than gradians or degrees. How many radians make up a circle?

Vocabulary

degrees, radians.

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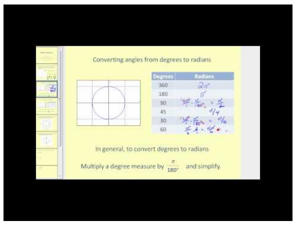
A radian is a unit of measuring angles that is based on the properties of circles. This makes it more meaningful than gradians or degrees. How many radians make up a circle?

Watch This



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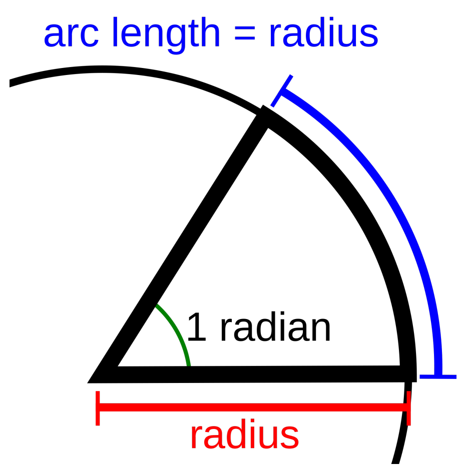
http://www.youtube.com/watch?v=nAJqXtzwpXQ James Sousa: Radian Measure

Guidance

A radian is defined to be the central angle where the subtended arc length is the same length as the radius.



Another way to think about radians is through the circumference of a circle. The circumference of a circle with radius r is $2\pi r$. Just over six radii (exactly 2π radii) would stretch around any circle.

To define a radian in terms of degrees, equate a circle measured in degrees to a circle measured in radians.

$360\ degrees = 2\pi\ radians$, so $\frac{180}{\pi}\ degrees = 1\ radian$

Alternatively; $360\ degrees = 2\pi\ radians$, so $1\ degree = \frac{\pi}{180}\ radians$

The conversion factor to convert degrees to radians is: $\frac{\pi}{180^\circ}$

The conversion factor to convert radians to degrees is: $\frac{180^\circ}{\pi}$

If an angle has no units, it is assumed to be in radians.

Example 1

Convert 150° into radians.

Solution: $150^\circ \cdot \frac{\pi}{180^\circ} = \frac{15\pi}{18} = \frac{5\pi}{6}\ radians$

Make sure the degree units cancel.

Example 2

Convert $\frac{\pi}{6}$ radians into degrees.

Solution: $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{6} = 30^\circ$

Often the π's will cancel.

Example 3

Convert $(6\pi)^\circ$ into radians.

Solution: Don't be fooled just because this has π. This number is about 19°

$(6\pi)^\circ \cdot \frac{\pi}{180^\circ} = \frac{6\pi^2}{180} = \frac{\pi^2}{3}$

It is very unusual to ever have a π^2 term, but it can happen.

Concept Problem Revisited

Exactly 2π radians describe a circular arc. This is because 2π radiuses wrap around the circumference of any circle.

Guided Practice

1. Convert $\frac{5\pi}{6}$ into degrees.

2. Convert 210° into radians.

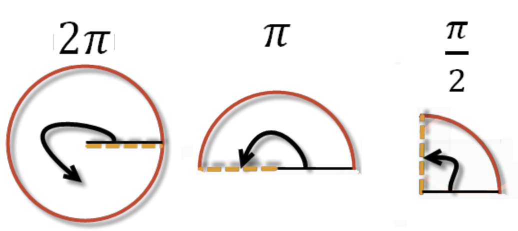
3. Draw a $\frac{\pi}{2}$ angle by first drawing a 2π angle, halving it and halving the result.

Answers:

1. $\frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{5 \cdot 30^\circ}{1} = 150^\circ$

2. $210^\circ \cdot \frac{\pi}{180^\circ} = \frac{7 \cdot 30 \cdot \pi}{6 \cdot 30} = \frac{7\pi}{6}$

3. $\frac{\pi}{2} = 90^\circ$



Vocabulary

A ***radian***

Is defined to be the central angle where the subtended arc length is the same length as the radius.

A ***subtended arc***

Is the part of the circle in between the two rays that make the central angle

A **Degree**

Is a unit of measurement of an angle. Its one three hundred and sixtieth of a circle.

In Summary

We have learned the definition of a radian. We have also learned that how to convert units from radians to degrees and from degrees to radians. Finally we have learned that a complete circle is 2π radians.

Practice

Find the radian measure of each angle.

1. 120°
2. 300°
3. 90°
4. 330°
5. 270°
6. 45°
7. $(5\pi)^\circ$

Find the degree measure of each angle.

8. $\frac{7\pi}{6}$
9. $\frac{5\pi}{4}$
10. $\frac{3\pi}{2}$
11. $\frac{5\pi}{3}$
12. π
13. $\frac{\pi}{6}$
14. 3
15. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.

5.2 Circular Motion and Dimensional Analysis

Here you'll review converting between linear and angular speeds using radians and circumference.

TEKS

1. P.4.D

Lesson Objectives

In this lesson you will learn about:
1. Dimensional Analysis.
2. Circular Motion.
3. Linear Speed.
4. Angular Speed.

Introduction

Converting between units is essential for mathematics and science in general. Radians are very powerful because they provide a link between linear and angular speed. One radian is an angle that always corresponds to an arc length of one radius. This will allow you to convert the rate at which you pedal a bike to the actual speed you can travel.

$$1\ revolution = 2\pi r$$

The gear near the pedals on a bike has a radius of 5 inches and spins once every second. It is connected by a chain to a second gear that has a 3 inch radius. If the second wheel is connected to a tire with a 17 inch radius, how fast is the bike moving in miles per hour?

Vocabulary

Angular Speed, Linear Speed, Dimensional Analysis

Converting between units is essential for mathematics and science in general. Radians are very powerful because they provide a link between linear and angular speed. One radian is an angle that always corresponds to an arc length of one radius. This will allow you to convert the rate at which you pedal a bike to the actual speed you can travel.

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The gear near the pedals on a bike has a radius of 5 inches and spins once every second. It is connected by a chain to a second gear that has a 3 inch radius. If the second wheel is connected to a tire with a 17 inch radius, how fast is the bike moving in miles per hour?

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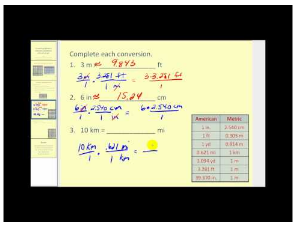

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URL: http://www.ck12.org/flx/render/embeddedobject/58061

http://www.youtube.com/watch?v=sn8Y7qpYLCY James Sousa: American and Metric Conversions

Guidance

Dimensional analysis just means converting from one unit to another. Sometimes it must be done in several steps in which case it is best to write the original amount on the left and then multiply it by all the different required conversions. To convert 3 miles to inches you write:

$\frac{3 \ mile}{1} \cdot \frac{5280 \ feet}{1 \ mile} \cdot \frac{12 \ inches}{1 \ foot} = \frac{3 \cdot 5280 \cdot 12 \ inches}{1} = 190080 \ inches$

Notice how miles and feet/foot cancel leaving the desired unit of inches. Also notice each conversion factor is the same distance on the numerator and denominator, just written with different units.

Circular motion refers to the fact that on a spinning wheel points close to the center of the wheel actually travel very slowly and points near the edge of the wheel actually travel much quicker. While the two points have the same angular speed, their linear speed is very different.

Example 1

Suppose Summit High School has a circular track with two lanes for running. The interior lane is 30 meters from the center of the circle and the lane towards the exterior is 32 meters from the center of the circle. If two people run 4 laps together, how much further does the person on the outside go?

Solution: Calculate the distance each person ran separately using 1 lap to be 1 circumference and find the difference.

$$\frac{4 \ laps}{1} \cdot \frac{2\pi \cdot 30 \ meters}{1 \ lap} \approx 754 \ meters$$
$$\frac{4 \ laps}{1} \cdot \frac{2\pi \cdot 32 \ meters}{1 \ lap} \approx 804 \ meters$$

The person running on the outside of the track ran about 50 more meters.

Example 2

Andres races on a bicycle with tires that have a 17 inch radius. When he is traveling at a speed of 30 feet per second, how fast are the wheels spinning in revolutions per minute?

Solution: Look for ways to convert feet to revolutions and seconds to minutes.

$\frac{30\ feet}{1\ second} \cdot \frac{60\ seconds}{1\ minute} \cdot \frac{12\ inches}{1\ foot} \cdot \frac{1\ revolution}{2\pi \cdot 17\ inches} = \frac{30 \cdot 60 \cdot 12\ revolutions}{2\pi \cdot 17\ minute} \approx 202.2 \frac{rev}{min}$

Example 3

When a car travels at 60 miles per hour, how fast are the tires spinning if they have 30 inch diameters?

$$\frac{60\ miles}{1\ hour} \cdot \frac{5280\ feet}{1\ mile} \cdot \frac{12\ inches}{1\ foot} \cdot \frac{1\ revolution}{2\pi \cdot 15\ inches} \cdot \frac{1\ hour}{60\ minute}$$
$$= \frac{60 \cdot 5280 \cdot 12\ revolutions}{2\pi \cdot 15 \cdot 60\ minute} \approx 672.3 \frac{rev}{min}$$

Concept Problem Revisited

The gear near the pedals on the bike has radius 5 inches and spins once every second. It is connected by a chain to a second gear that has a 3 inch radius. If the second wheel is connected to a tire with a 17 inch radius, how fast is the bike moving in miles per hour?

A bike has pedals that rotate a gear at a circular speed. The gear translates this speed to a linear speed on the chain. The chain then moves a second gear, which is a conversion to angular speed for the rear tire. This tire then converts the angular speed back to linear speed which is how fast you are moving. Instead of doing all these calculations in one step, it is easier to do each conversion in small pieces.

First convert the original gear into the linear speed of the chain.

$\frac{1\ revolution}{1\ second} \cdot \frac{2\pi \cdot 5\ inches}{1\ revolution} = 10\pi \frac{in}{sec}$

Then convert the speed of the chain into angular speed of the back gear which is the same as the angular speed of the rear tire.

$\frac{10\pi\ inches}{1\ second} \cdot \frac{1\ revolution}{2\pi \cdot 3\ inches} = \frac{10}{6} \frac{rev}{sec}$

Lastly convert the angular speed of the rear tire to the linear speed of the tire in miles per hour.

$$\frac{10\ rev}{6\ sec} \cdot \frac{2\pi \cdot 17\ in}{1\ rev} \cdot \frac{1\ ft}{12\ in} \cdot \frac{1\ mile}{5280\ ft} \cdot \frac{60\ sec}{1\ min} \cdot \frac{60\ min}{1\ hour}$$
$$= \frac{10 \cdot 2 \cdot \pi \cdot 17 \cdot 60 \cdot 60}{6 \cdot 12 \cdot 5280} \frac{miles}{hour}$$
$$\approx 10.1 \frac{miles}{hour}$$

Guided Practice

1. Does a tire with radius 4 inches need to spin twice as fast as a tire with radius 8 inches to keep up?

2. An engine spins a wheel with radius 4 inches at 1200 rpm. How fast is this wheel spinning in miles per hour?

3. Mike rides a bike with tires that have a radius of 15 inches. How many revolutions must Mike make to ride a mile?

Answers:

1. If the small wheel spins at 2 revolutions per minute, then the linear speed is:

$$\frac{2\ rev}{1\ min} \cdot \frac{2\pi \cdot 4\ in}{1\ rev} = 16\pi \frac{in}{min}$$

If the large wheel spins at 1 revolution per minute, then the linear speed is:

$$\frac{1\ rev}{1\ min} \cdot \frac{2\pi \cdot 8\ in}{1\ rev} = 16\pi \frac{in}{min}$$

Yes, the small wheel does need to spin at twice the rate of the larger wheel to keep up.

2. $\frac{1200\ rev}{1\ min} \cdot \frac{60\ min}{1\ hour} \cdot \frac{2\pi \cdot 4\ in}{1\ rev} \cdot \frac{1\ ft}{12\ in} \cdot \frac{1\ mi}{5280\ ft} \approx 28.6\ mph$

3. $\frac{1\ rev}{2\pi \cdot 15\ in} \cdot \frac{12\ in}{1\ ft} \cdot \frac{5280\ ft}{1\ mi} \approx 672.3 \frac{rev}{mile}$

Vocabulary

Angular speed is the ratio of revolutions that occur per unit of time.

Linear speed is the ratio of distance per unit of time.

Dimensional analysis means converting from one unit to another

In Summary

We learned how to perform dimensional analysis. We learned to change into convert from angular speed to linear speed and from linear speed to angular speed.

Practice

For 1-10, use the given values in each row to find the unknown value (x) in the specified units in the row.

TABLE 5.1:

Problem Number	Radius	Angular Speed	Linear Speed
1.	$5\ inches$	$60\ rpm$	$x \frac{in}{min}$
2.	$x\ feet$	$20\ rpm$	$2 \frac{in}{sec}$
3.	$15\ cm$	$x\ rpm$	$12 \frac{cm}{sec}$
4.	$x\ feet$	$40\ rpm$	$8 \frac{ft}{sec}$
5.	$12\ inches$	$32\ rpm$	$x \frac{in}{sec}$
6.	$8\ cm$	$x\ rpm$	$12 \frac{cm}{min}$
7.	$18\ feet$	$4\ rpm$	$x \frac{mi}{hr}$
8.	$x\ feet$	$800\ rpm$	$60 \frac{mi}{hr}$
9.	$15\ in$	$x\ rpm$	$60 \frac{mi}{hr}$
10.	$2\ in$	$x\ rpm$	$13 \frac{in}{sec}$

11. An engine spins a wheel with radius 5 inches at 800 rpm. How fast is this wheel spinning in miles per hour?

12. A bike has tires with a radius of 10 inches. How many revolutions must the tire make to ride a mile?

13. An engine spins a wheel with radius 6 inches at 600 rpm. How fast is this wheel spinning in inches per second?

14. Bob has a car with tires that have a 15 inch radius. When he is traveling at a speed of 30 miles per hour, how fast are the wheels spinning in revolutions per minute?

15. A circular track has two lanes. The interior lane is 25 feet from the center of the circle and the lane towards the exterior is 30 feet from the center of the circle. If you jog 6 laps, how much further will you jog in the exterior lane as opposed to the interior lane?

5.3 Special Right Triangles

Here you will review properties of 30-60-90 and 45-45-90 right triangles.

TEKS

1. P.4.E

Lesson Objectives

In this lesson you will learn about:
1. Rules of special right triangles.
2. Solving for the lengths of sides of special right triangles.
3. Rationalizing values.

Introduction

The Pythagorean Theorem is great for finding the third side of a right triangle when you already know two other sides. There are some triangles like 30-60-90 and 45-45-90 triangles that are so common that it is useful to know the side ratios without doing the Pythagorean Theorem each time. Using these patterns also allows you to totally solve for the missing sides of these special triangles when you only know one side length.
Given a 45-45-90 right triangle with sides 6 inches, 6 inches and x inches, what is the value of x?

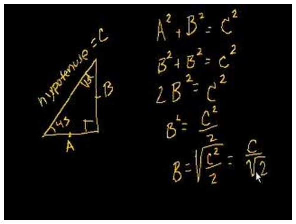
Vocabulary

Pythagorean Theorem, Pythagorean triples, right triangle, special right triangles, corresponding angles and sides.

Watch This

MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/58104

http://www.youtube.com/watch?v=681WWiNxMIQ Khan Academy: Intro to 30 60 90 Triangles Geometry



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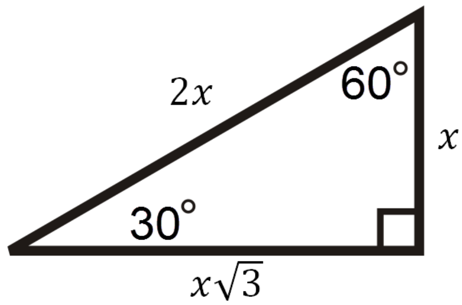
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/58106

http://www.youtube.com/watch?v=tSHitjFIjd8 Khan Academy: 45 45 90 Triangles

Guidance

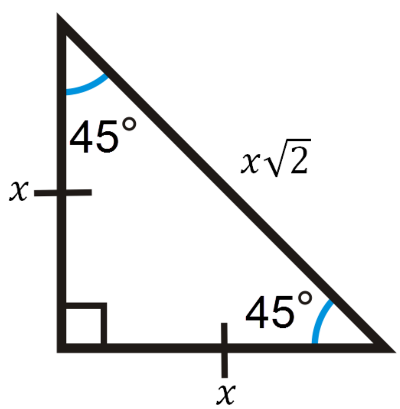
A 30-60-90 right triangle has side ratios $x, x\sqrt{3}, 2x$.



Confirm with Pythagorean Theorem:

$$\begin{aligned} x^2 + \left(x\sqrt{3}\right)^2 &= (2x)^2 \\ x^2 + 3x^2 &= 4x^2 \\ 4x^2 &= 4x^2 \end{aligned}$$

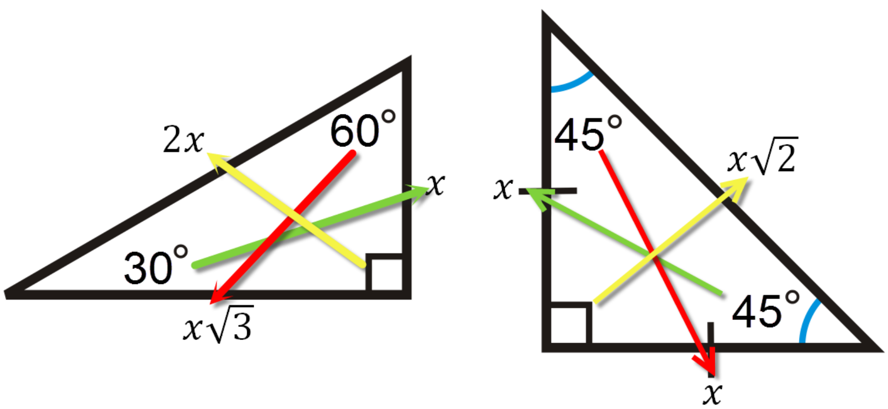
A 45-45-90 right triangle has side ratios $x, x, x\sqrt{2}$.



Confirm with Pythagorean Theorem:

$$x^2 + x^2 = \left(x\sqrt{2}\right)^2$$
$$2x^2 = 2x^2$$

Note that the order of the side ratios $x, x\sqrt{3}, 2x$ and $x, x, x\sqrt{2}$ is important because each side ratio has a corresponding angle. In all triangles, the smallest sides correspond to smallest angles and largest sides always correspond to the largest angles.



Pythagorean number triples are special right triangles with integer sides. While the angles are not integers, the side ratios are very useful to know because they show up everywhere. Knowing these number triples also saves a lot of time from doing the Pythagorean Theorem repeatedly. Here are some examples of Pythagorean number triples:

- 3, 4, 5
- 5, 12, 13
- 7, 24, 25
- 8, 15, 17
- 9, 40, 41

More Pythagorean number triples can be found by scaling any other Pythagorean number triple. For example:

$3, 4, 5 \rightarrow 6, 8, 10$ (scaled by a factor of 2)

Even more Pythagorean number triples can be found by taking any odd integer like 11, squaring it to get 121, halving the result to get 60.5. The original number 11 and the two numbers that are 0.5 above and below (60 and 61) will always be a Pythagorean number triple.

$11^2 + 60^2 = 61^2$

Example 1

A right triangle has two sides that are 3 inches. What is the length of the third side?

Solution: Since it is a right triangle and it has two sides of equal length then it must be a 45-45-90 right triangle. The third side is $3\sqrt{2}$ inches.

Example 2

A 30-60-90 right triangle has hypotenuse of length 10. What are the lengths of the other two sides?

Solution: The hypotenuse is the side opposite 90. Sometimes it is helpful to draw a picture or make a table.

TABLE 5.2:

30	60	90
x	$x\sqrt{3}$	$2x$
		10

From the table you can write very small subsequent equations to solve for the missing sides.

$$\begin{aligned} 10 &= 2x \\ x &= 5 \\ x\sqrt{3} &= 5\sqrt{3} \end{aligned}$$

Example 3

A 30-60-90 right triangle has a side length of 18 inches corresponding to 60 degrees. What are the lengths of the other two sides?

Solution: Make a table with the side ratios and the information given, then write equations and solve for the missing side lengths.

TABLE 5.3:

30	60	90
x	$x\sqrt{3}$	$2x$
	18	

$$\begin{aligned} 18 &= x\sqrt{3} \\ \frac{18}{\sqrt{3}} &= x \\ x &= \frac{18}{\sqrt{3}} = \frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3} \end{aligned}$$

Note that you need to rationalize denominators.

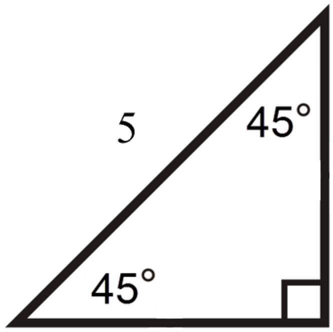
Concept Problem Revisited

If you can recognize the pattern for 45-45-90 right triangles, a right triangle with legs 6 inches and 6 inches has a hypotenuse that is $6\sqrt{2}$ inches. $x = 6\sqrt{2}$.

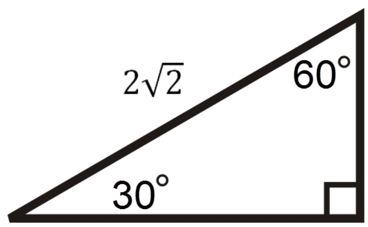
Guided Practice

Using your knowledge of special right triangle ratios, solve for the missing sides of the following right triangles.

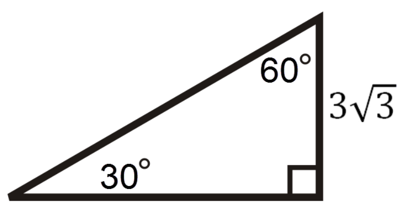
1.



2.



3.



Answers:

1. The other sides are each $\frac{5\sqrt{2}}{2}$.

TABLE 5.4:

45	45	90
x	x	$x\sqrt{2}$
		5

$$x\sqrt{2} = 5$$
$$x = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

2. The other sides are $\sqrt{2}$ and $\sqrt{6}$.

TABLE 5.5:

30	60	90
x	$x\sqrt{3}$	$2x$
		$2\sqrt{2}$

$$2x = 2\sqrt{2}$$
$$x = \sqrt{2}$$
$$x\sqrt{3} = \sqrt{2} \cdot \sqrt{3} = \sqrt{6}$$

3. The other sides are 9 and $6\sqrt{3}$.

TABLE 5.6:

30	60	90
x	$x\sqrt{3}$	$2x$
$3\sqrt{3}$		

$$x = 3\sqrt{3}$$
$$2x = 6\sqrt{3}$$
$$x\sqrt{3} = 3\sqrt{3} \cdot \sqrt{3} = 9$$

Vocabulary

Corresponding angles and sides are angles and sides that are on opposite sides of each other in a triangle. Capital letters like A, B, C are often used for the angles in a triangle and the lower case letters a, b, c are used for their corresponding sides (angle A corresponds to side a etc).

Right Triangle

It is a triangle in which the measure of one of its angles is 90° .

Pythagorean Theorem

A theorem attributed to the famous mathematician Pythagoras that states that the sum of the squares of the legs of a right triangle equals the square of the hypotenuse. The famous formula $a^2+b^2=c^2 a^2+b^2=c^2$

Pythagorean number triples

Are 3 whole numbers that satisfy the Pythagorean Theorem.

A ***45-45-90 triangle***

Is a special right triangle with angles of $45^\circ, 45^\circ$, and 90°.

A ***30-60-90 triangle***

Is a special right triangle with angles of $30^\circ, 60^\circ$, and 90°.

In Summary

We have learned how to find the lengths of the sides of special right triangles given one of the sides.

Practice

For 1-4, find the missing sides of the 45-45-90 triangle based on the information given in each row.

TABLE 5.7:

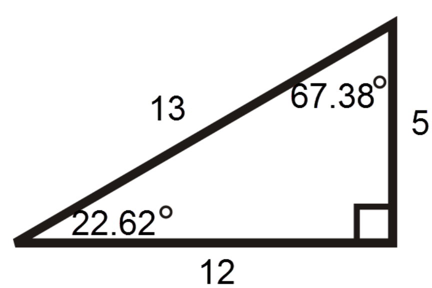
Problem Number	Side Opposite 45°	Side Opposite 45°	Side Opposite 90°
1.	3		
2.		7.2	
3.			16
4.	$5\sqrt{2}$		

For 5-8, find the missing sides of the 30-60-90 triangle based on the information given in each row.

TABLE 5.8:

Problem Number	Side Opposite 30°	Side Opposite 60°	Side Opposite 90°
5.	$3\sqrt{2}$		
6.		4	
7.			15
8.			$12\sqrt{3}$

Use the picture below for 9-11.



9. Which angle corresponds to the side that is 12 units?

10. Which side corresponds to the right angle?

11. Which angle corresponds to the side that is 5 units?

12. A right triangle has an angle of $\frac{\pi}{6}$ radians and a hypotenuse of 20 inches. What are the lengths of the other two sides of the triangle?

13. A triangle has two angles that measure $\frac{\pi}{4}$ radians. The longest side is 3 inches long. What are the lengths of the other two sides?

For 14-19, verify the Pythagorean Number Triple using the Pythagorean Theorem.

14. 3, 4, 5

15. 5, 12, 13

16. 7, 24, 25

17. 8, 15, 17

18. 9, 40, 41

19. 6, 8, 10

20. Find another Pythagorean Number Triple using the method explained for finding "11, 60, 61".

5.4 Right Triangle Trigonometry

Here you will learn the six right triangle ratios and how to use them to completely solve for the missing sides and angles of any right triangle.

TEKS

1. P.4.E

Lesson Objectives

In this lesson you will learn about:

1. What is trigonometry?
2. The definition of the six trigonometric ratios.
3. Solving a right triangle using sine cosine and tangent.

Introduction

Trigonometry is the study of triangles. If you know the angles of a triangle and one side length, you can use the properties of similar triangles and proportions to completely solve for the missing sides.

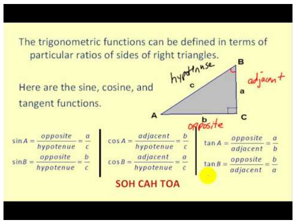
Vocabulary

Sine, Cosine, Tangent, SOH CAH TOA, theta θ, alpha α, beta β

Trigonometry is the study of triangles. If you know the angles of a triangle and one side length, you can use the properties of similar triangles and proportions to completely solve for the missing sides.

Imagine trying to measure the height of a flag pole. It would be very difficult to measure vertically because it could be several stories tall. Instead walk 10 feet away and notice that the flag pole makes a 65 degree angle with your feet. Using this information, what is the height of the flag pole?

Watch This



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http://www.youtube.com/watch?v=Ujyl_zQw2zE James Sousa: Introduction to Trigonometric Functions Using Triangles

Guidance

The six trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant. *Opp* stands for the side opposite of the angle θ, *hyp* stands for hypotenuse and *adj* stands for side adjacent to the angle θ.

$\sin\theta = \frac{opp}{hyp}$

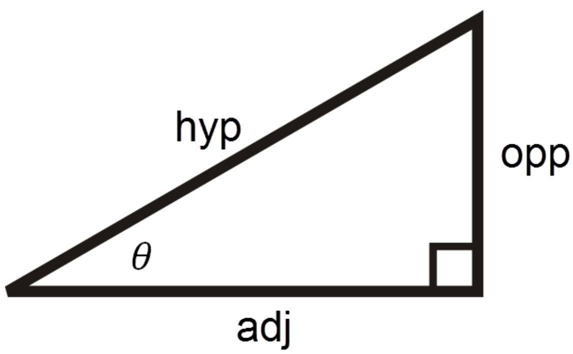
$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj}$

$\cot\theta = \frac{adj}{opp}$

$\sec\theta = \frac{hyp}{adj}$

$\csc\theta = \frac{hyp}{opp}$



The reason why these trigonometric functions exist is because two triangles with the same interior angles will have side lengths that are always proportional. Trigonometric functions are used by identifying two known pieces of information on a triangle and one unknown, setting up and solving for the unknown. Calculators are important because the operations of sin, cos and tan are already programmed in. The other three (cot, sec and csc) are not usually in calculators because there is a reciprocal relationship between them and tan, cos and sec.

$\sin\theta = \frac{opp}{hyp} = \frac{1}{\csc\theta}$

$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sec\theta}$

$\tan\theta = \frac{opp}{adj} = \frac{1}{\cot\theta}$

In order to remember the ratios, you can use the phrase **SOH, CAH, TOA**

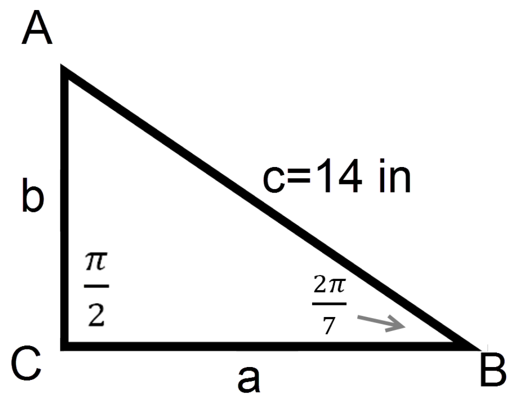
Also remember that the sum of all the angles in a triangle is 180 degrees.

Keep in mind that your calculator can be in degree mode or radian mode. Be sure you can toggle back and forth so that you are always in the appropriate units for each problem.

Note: The images throughout this concept are not drawn to scale.

Example 1

Solve for side b.



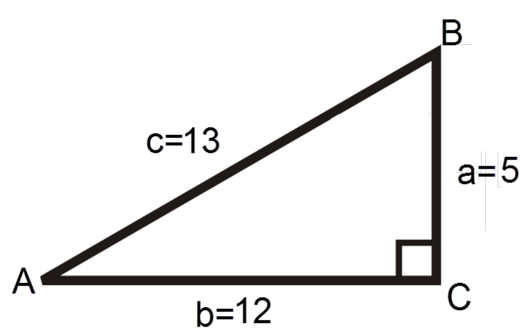
Solution:

$$\sin\left(\frac{2\pi}{7}\right) = \frac{b}{14}$$
$$b = 14 \cdot \sin\left(\frac{2\pi}{7}\right) \approx 10.9\ in$$

Note: Make sure the calculator is on radian mode

Example 2

Solve for angle A.



Solution: This problem can be solved using sin, cos or tan because the opposite, adjacent and hypotenuse lengths are all given.

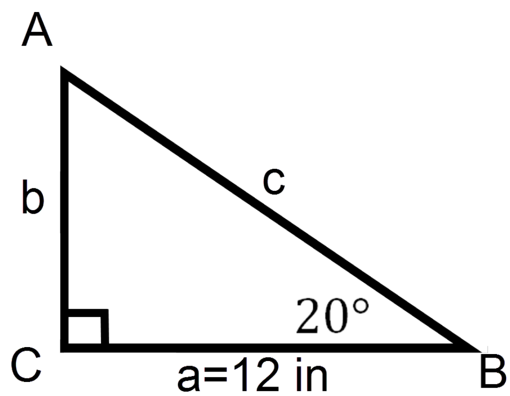
The argument of a sin function is always an angle. The arcsin or $\sin^{-1}\theta$ function on the calculator on the other hand has an argument that is a side ratio. It is useful for finding angles that have that side ratio.

$$\sin A = \frac{5}{13}$$
$$A = \sin^{-1}\left(\frac{5}{13}\right) \approx 0.39\ radian \approx 22.6^\circ$$

Note: If you have the calculator in degree mode, you automatically get the degrees, you don't have to convert. Usually in a triangle, the angle measurement is given in degrees.

Example 3

Given a right triangle with $a = 12\ in$, $m\angle B = 20^\circ$, and $m\angle C = 90^\circ$, find the length of the hypotenuse.



Solution:

$$\cos 20^\circ = \frac{12}{c}$$
$$c = \frac{12}{\cos 20^\circ} \approx 12.77\ in$$

Note: Make sure the calculator is in degree mode since the angle is 20 degrees.

In the previous three examples, we have the three possible cases we might have when using the trigonometric ratios. If we have the variable in the top, we multiply the denominator to the other side. If we the variable on the bottom of the fraction, we switch the variable and the trigonometric expression. If we are looking for the angle, we perform the inverse trigonometric function as follows.

Case 1. The variable is on the top.

If we have an equation like:

$$\sin 30 = \frac{y}{5}\ \ then$$
$$5 \cdot \sin 30 = y$$

Case 2. The variable is on the bottom.

If we have an equation like:

$$\cos(25) = \frac{10}{x} \quad then$$
$$x = \frac{10}{\cos(25)}$$

Note: In this case a very common mistake is that students switch the variable x with the number 25, you have to switch x with the whole expression cos(25)

Case 3. The variable is the angle.

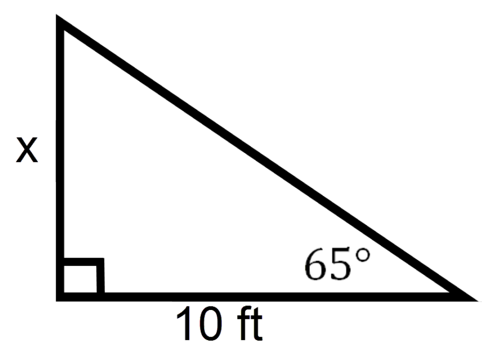
If we have an equation like:

$$\tan\theta = \frac{7}{10} \quad then$$
$$\theta = tan^{-1}\left(\frac{7}{10}\right)$$

Concept Problem Revisited

Instead walk 10 feet away and notice that the flag pole makes a 65° angle with your feet.

If you walk 10 feet from the base of a flagpole and assume that the flagpole makes a 90° angle with the ground.



$$\tan 65^\circ = \frac{x}{10}$$
$$x = 10\tan 65^\circ \approx 30.8 \ ft$$

Guided Practice

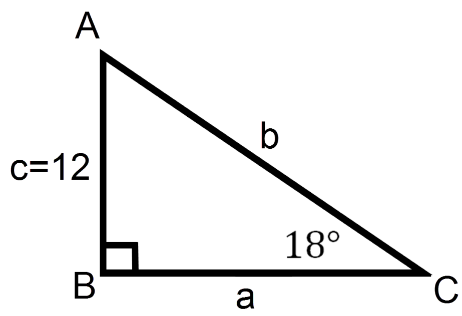
1. Given $\triangle ABC$ where B is a right angle, $m\angle C = 18^\circ$, and $c = 12$. What is a?

2. Given $\triangle XYZ$ where Z is a right angle, $m\angle X = 1 \ radian$, and $x = 3$. What is z?

3. Given $\triangle MNO$ where O is a right angle, $m = 12$, and $n = 14$. What is the measure of angle M?

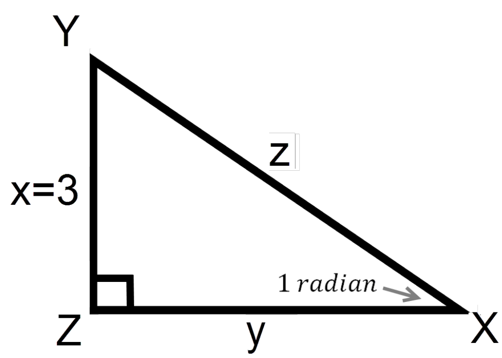
Answers:

1. Drawing out this triangle, it looks like:



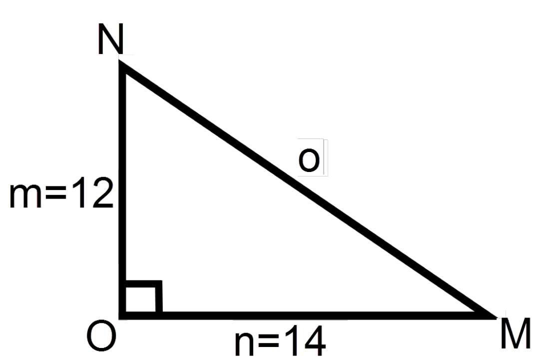
$$
\begin{aligned}
\tan 18^\circ &= \frac{12}{a} \\
a &= \frac{12}{\tan 18^\circ} \approx 36.9
\end{aligned}
$$

2. Drawing out the triangle, it looks like:



$$
\begin{aligned}
\sin 1 &= \frac{3}{z} \\
z &= \frac{3}{\sin 1} \approx 3.6
\end{aligned}
$$

3. Drawing out the triangle, it looks like:



$$\tan M = \frac{12}{14}$$
$$M = \tan^{-1}\left(\frac{12}{14}\right) \approx 0.7\ radian \approx 40.6^\circ$$

Vocabulary

The ***six trigonometric ratios***

Are universal proportions that are always true of similar triangles (triangles with congruent corresponding angles). In right triangles the six trigonometric ratios are sine, cosine, tangent, cosecant, secant, cotangent (sin, cos, tan, csc, sec, cot)

θ ***(theta)***, α **(alpha)**, β **(beta)**

Are Greek letters used in mathematics to stand for an unknown angle.

In Summary

We have learned about the six trigonometric ratios that can be used in right triangles. We also learned how to use sine, cosine, tangent to solve for the missing sides of a right triangle or the angle of a right triangle.

Practice

For 1-15, information about the sides and/or angles of right triangle *ABC* is given. Completely solve the triangle (find all missing sides and angles) to 1 decimal place.

TABLE 5.9:

Problem Number	A	B	C	a	b	c
1.	90°				4	7
2.	90°		37°	18		
3.		90°	15°		32	
4.			90°	6		11
5.	90°	12°		19		
6.		90°			17	10
7.	90°	10°			2	
8.	4°	90°		0.3		
9.	$\frac{\pi}{2}$ radian		1 radian			15
10.		$\frac{\pi}{2}$ radian		12	15	
11.			$\frac{\pi}{2}$ radian		9	14
12.	$\frac{\pi}{4}$ radian	$\frac{\pi}{4}$ radian			5	
13.	$\frac{\pi}{2}$ radian			26	13	
14.		$\frac{\pi}{2}$ radian			19	16
15.			$\frac{\pi}{2}$ radian	10		$10\sqrt{2}$

5.5 The Unit Circle

TEKS

1. P.1.F

Lesson Objectives

In this lesson you will learn about:
1. Creating the unit circle using right triangle trigonometry.
2. Evaluating the six trigonometric functions of the special angles of the unit circle.
3. Finding a Reference angle and a Co-terminal angle

Introduction

The formula for the circumference of a circle is $C = 2\pi \cdot r$ well, what does this formula have to do with the unit circle? In this section you will learn to create the unit circle and how the circumference formula plays an important part in it.

Vocabulary

Unit circle, reference angle, co-terminal angle.

If we have a circle of radius one, then the circumference around the circle is

$$amp;C = 2\pi \cdot r$$

Therefore a complete circle is 2π *radians* which is 360 degrees

We can divide the circle in half so half a circle is

π *radians* which is which is 180°

If we divide π *radians* in half we obtain $\frac{\pi}{2}$ *radians* which is 90°

If we divide π *radians* into four parts we have $\frac{\pi}{4}$ *radians* which will be 45°

If we divide π *radians* into six parts, we obtain $\frac{\pi}{6}$ *radians* which is 30°

If we start creating the circle using the previous information and reducing the fractions we will obtain the following unit circle

The coordinates of the circle come from using the special right triangle trigonometry for the $30^\circ, 60^\circ, 90^\circ$ if the hypotenuse is 1 then the short side will be $\frac{1}{2}$ the the long side will be $\frac{\sqrt{3}}{2}$.

This information will fit the 30° $or \frac{\pi}{6}$ angle giving us the coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

By flipping the coordinates this information also fits 60° or $\frac{\pi}{3}$ angle so the coordinates will be $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

For the 45° or $\frac{\pi}{4}$ angle if the hypotenuse is 1 then the length of the legs will be $\frac{1}{\sqrt{2}}$ and by rationalizing the value will be $\frac{\sqrt{2}}{2}$ therefore the coordinate at 45° or $\frac{\pi}{4}$ will be $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

The coordinates at 0 is $(1,0)$, at $\frac{\pi}{2}$ is $(0,1)$, at π is $(-1,0)$, and at $\frac{3\pi}{2}$ is $(0,-1)$.

These previous four angles are called **Quadrantal Angles**.

Finally for the other three quadrants, we just map the 1st quadrant coordinates across the y or x axis to fill the complete circle.

FIGURE 5.1

Now that we have the complete circle we can define the trigonometric functions with respect to the unit circle.

In the unit circle, the radius of the circle is 1. Therefore we have the following definitions of the trigonometric functions in terms of the unit circle.

$$\begin{array}{ccc} sin\ \theta = y & cos\ \theta = x & tan\ \theta = \frac{y}{x} \\ csc\ \theta = \frac{1}{y} & sec\ \theta = \frac{1}{x} & cot\ \theta = \frac{x}{y} \end{array}$$

From the previous definition we can conclude that $\sin\theta$ is the **y-coordinate** and that $\cos\theta$ is the **x-coordinate.** Also we obtain two identities you will be using in the following chapters.

$$\begin{array}{ccc} csc\ \theta = \frac{1}{\sin\theta} & and & \sec\theta = \frac{1}{\cos\theta} \\ \tan\theta = \frac{\sin\theta}{\cos\theta} & and & \cot\theta = \frac{\cos\theta}{\sin\theta} \end{array}$$

Example 1.

Find the six trigonometric ratios of the angle $\theta = \frac{\pi}{6}$

Solution.

By looking at the unit circle, at the angle $\theta = \frac{\pi}{6}$ the coordinates are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ therefore

$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \csc\frac{\pi}{6} = \frac{1}{\frac{1}{2}} = 2$$

Example 2

Find the six trigonometric ratios of the angle $\theta = \frac{5\pi}{3}$

Solution

By looking at the unit circle at the angle $\theta = \frac{5\pi}{3}$ the coordinates are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ therefore

$$\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} \qquad \csc\frac{5\pi}{3} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

Example 3

Find the six trigonometric ratios of the angle $\theta = 120^\circ$

Solution

By looking at the unit circle at the angle $\theta = 120^\circ$the coordinates are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ therefore

$$\sin 120^\circ = \frac{\sqrt{3}}{2} \qquad \csc 120^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Example 4

Find the six trigonometric ratios of the angle $\theta = 225^\circ$

Solution

By looking at the unit circle at the angle $\theta = 225^\circ$ the coordinates are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ therefore

$$\sin\ 225^\circ = -\frac{\sqrt{2}}{2} \qquad \csc\ 225^\circ = -\frac{2}{\sqrt{2}} = -\frac{2\sqrt{2}}{2} = -\sqrt{2}$$

Reference Angles

Reference angles are the angles made by the terminal side of the angle and the x-axis. For example if we have 210° then the reference angle will be 30°. Depending on what quadrant the terminal side of the angle is there is different forms to find the reference angle. Here is a reference table of how to find the reference angle in each quadrant.

FIGURE 5.2

Let θ be the angle in the unit circle and α be the reference angle.

TABLE 5.10:

Quadrant	Reference angle in degrees	Reference angle in radians
I	$\alpha = \theta$	$\alpha = \theta$
II	$\alpha = 180^\circ - \theta$	$\alpha = \pi - \theta$
III	$\alpha = \theta - 180^\circ$	$\alpha = \theta - \pi$
IV	$\alpha = 360^\circ - \theta$	$\alpha = 2\pi - \theta$

Note: The quadrantal angles have no reference angles.

Example 5

Find the reference angle of 125°.

Solution

Since the angle lies on the second quadrant the reference angle will be

$$\alpha = 180^\circ - 125^\circ = 55^\circ$$

Example 6.

Find the reference angle of $\frac{11\pi}{6}$

Solution

Since the angle lies on the third quadrant the reference angle will be

$$\alpha = 2\pi - \frac{11\pi}{6}$$
$$\alpha = \frac{12\pi}{6} - \frac{11\pi}{6}$$
$$\alpha = \frac{\pi}{6}$$

Example 7

Find the reference angle of $\frac{5\pi}{4}$

Solution

Since the angle lies on the third quadrant the reference angle will be

$$\alpha = \frac{5\pi}{4} - \pi$$
$$\alpha = \frac{5\pi}{4} - \frac{4\pi}{4}$$
$$\alpha = \frac{\pi}{4}$$

Co-terminal Angles.

Co-terminal Angles are angles that start on the same initial side and end on the same terminal side.

To find Co-terminal angles is very simple, you have to go around one whole circle either in a clockwise direction or a counter clockwise direction.

FIGURE 5.3

Let α be the co-terminal angle

TABLE 5.11:

In Degrees	In Radians
$\alpha = \theta \pm 360^\circ$	$\alpha = \theta \pm 2\pi$

Example 8

Find a co-terminal angle in a counterclockwise direction and in a clockwise direction for the angle $\theta = 50^\circ$

Solution

The co-terminal angles are

$$\alpha = 50^\circ + 360^\circ = 410^\circ$$
$$\alpha = 50^\circ - 360^\circ = 310^\circ$$

Example 9

Find a co-terminal angle in a counterclockwise direction and in a clockwise direction for the angle $\theta = \frac{2\pi}{3}$

Solution

The co-terminal angles are

$$\alpha = \frac{2\pi}{3} + 2\pi \qquad \alpha = \frac{2\pi}{3} - 2\pi$$

Vocabulary

Unit Circle

It is the circle of radius one that is used to define the six trigonometric functions.

Quadrantal Angles

Are the angles that lie in the x or y axis

Reference angle

It is the angle made by the terminal side of the angle and the x-axis

Co-terminal angle

They are two angles that start on the same initial side and end on the same terminal side.

In Summary

We have learned what the unit circle is and how to create it. We have learned to obtain the values of the six trigonometric functions using the unit circle. We also learned what Quadrantal angles are. We also learned to find the reference angle of an angle and to find the co-terminal angles.

Practice

Evaluate the 6 trigonometric functions of

Find the reference angle of

Find the co-terminal angles of

5.6 Solving Simple Trig equations

TEKS

1. P.2.I
2. P.4.A

Lesson Objectives

In this section you will learn about.

1. Solving equations in the form $\sin\theta = C$
2. Solving equations in the form $A\sin\theta + C = D$
3. Solving equations in the form $A\sin(B\theta) + C = D$
4. State all possible solutions for the equations.

Introduction

Solving trigonometric equations is very similar to solving equations, the difference is you need to take the inverse trig function to cancel the trigonometric function and you need to take into account the unit circle. In this section you will learn to solve simple trigonometric equations. Later in the chapter you will learn about solving more complicated equations.

Vocabulary

Trigonometric Equation.

When you are solving an equation you are actually performing either the additive inverse or the multiplicative inverse to solve for a variable. For example if we have the equation $ax + b = c$ and we are solving for x we perform the following process

$$= c - b \qquad =$$

So when we subtract b we are doing the additive inverse to cancel b and when we divide by a we are doing the multiplicative inverse of a therefore we do the same for the trigonometric equations.

In the following examples we will solve the equations in radian units.

Example 1.

Solve the equation $\sin\theta = \frac{1}{2}$

Solution

What this equation is actually asking is at what angle θ is the value of the function $\sin$ equal to $\frac{1}{2}$. If its one of the values in the unit circle then we can actually get the answer from the circle itself. Remember that the y-coordinate represents the value of sin so by checking the unit circle when the value of the y-coordinate is $\frac{1}{2}$ then $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ The problem is what if the value in not one of the values on the unit circle, then we need to actually perform the process as follows:

$$= \frac{1}{2}\sin^{-1}(\sin\theta)$$

But that is only one solution how do we obtain the other solution? If we are solving a trigonometric equation involving *sin* then to obtain the second solution can use a little trick we do $\pi - answer$ therefore our second solution will be

$$= \quad \pi - solution\theta_2$$

$[-2\pi, 2\pi]$Since the period of the sine function is 2π therefore all possible solutions to this trigonometric equation will be:

$$= \frac{\pi}{6} \pm 2\pi \cdot k\theta_2$$

Where k represents the number of cycles around the circle.

Example 2.

Solve the equation $\cos\theta = \frac{1}{2}$

Solution

We know that the cosine function is represented by the x-coordinate of the unit circle. By looking in the unit circle and using the same analogy as example 1, we can see that the value of $\cos\theta = \frac{1}{2}$at $\frac{\pi}{3} and \frac{5\pi}{3}$ but instead of doing $\frac{5\pi}{3}$ we can do $\frac{-\pi}{3}$ so our solution from the unit circle will be $\theta = \pm\frac{\pi}{3}$

Algebraically we do the following process:

$$= \frac{1}{2}\cos^{-1}(\cos\theta)$$

Remember that the period of the cosine function is also 2π

Therefore all possible solutions to the equation is

$$\theta = \pm\frac{\pi}{3} \pm 2\pi \cdot k$$

Where k represents the number of turns/cycles around the circle.

Example 3

Solve the equation $\tan\theta = 1$

Solution

We know from a previous section that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ therefore $\tan\theta$ will be equal to 1 if $\sin\theta = \cos\theta$ that means that in the unit circle both the x-coordinate and the y-coordinate have to be the same therefore from the unit circle this will happen when $\theta = \frac{\pi}{4}$

Algebraically we will do the following process

$$= 1\tan^{-1}(\tan\theta)$$

Since the period of tangent is π then all possible solutions are given by

$$\theta = \frac{\pi}{4} \pm \pi \cdot k$$

Example 4

Solve the equation $3\sin\theta - 1 = 0$ and state all possible solutions.

Solution

First we need to add 1 to both sides, divide by 3 and take the inverse sin function obtaining the equation as follows

$$= 03\sin\theta$$

$$=$$

Therefore since its a sine function we obtain the following solutions.

$$= .3398 \pm 2\pi \cdot k\theta_2$$

Example 6

Solve the following equation $5\cos(2\theta) + 3 = 0$

Solution

This equation looks a little bit different, it has a number next to the θ which means it is being horizontally compressed or stretched by a factor of $\frac{1}{B}$, therefore at the end we need to do one more step. So first we are going to subtract 3 to both sides and divide by 5 obtaining the following equation.

$$\cos 2\theta = \frac{-3}{5}$$

Therefore when we take the inverse cosine of this function we are actually solving for 2θ therefore we will obtain the solutions:

$$= \cos^{-1}\left(\frac{-3}{5}\right) 2\theta$$

Since we have to solve for theta then we have to divide both sides by 2 therefore our final solutions are:

$$\theta = \pm 1.107 \pm \pi \cdot k$$

Example 7

Solve the equation

$-3\tan\left(\frac{1}{2}\theta\right) = 15$

Solution

First we need to divide both sides by -3 therefore we obtain:

$$\tan\left(\frac{1}{2}\theta\right) = -5$$

Then by performing the inverse tangent function on both sides and the period of tangent being π we will obtain

$$= \tan^{-1}(-5)\,\frac{1}{2}\theta$$

Finally we need to multiply both sides by 2 therefore our final solution is

$$= -2.7468 \pm 2\pi \cdot k$$

Example 8

Solve the equation $5\sin(3\theta) - 4 = 0$

Solution

First we need to add 4 to both sides and divide by 5 therefore we obtain the equation:

$$\sin(3\theta) = \frac{4}{5}$$

Then by performing the inverse sine function to both sides we will obtain the equation:

$$= \sin^{-1}\left(\frac{4}{5}\right)3\theta$$

But since its a sine function we need to obtain the second answer by subtracting your solution from π therefore our two solutions will be:

$$= .927 \pm 2\pi \cdot k3\theta_2$$

Finally we need to divide both sides of the solutions by 3 therefore our final answers will be:

$$= .309 \pm \frac{2\pi}{3} \cdot k\theta_2$$

Vocabulary

Trigonometric Equation

An equation that involves sine, cosine or tangent.

In Summary

We have learned how to solve simple trigonometric equations involving sine, cosine and tangent. We also learned to state all solutions to the functions.

Practice

##################################

5.7 Law of Cosines

Here you will solve non-right triangles with the Law of Cosines.

TEKS

1. P.4.H

Lesson Objectives

In this lesson you will learn about:

1. The Law of Cosines.
2. How to solve for a side with the Law of Cosines.
3. How to solve for an angle using the Law of Cosines.

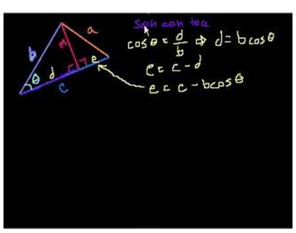
Introduction

The Law of Cosines is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle. Suppose you have a triangle with sides 11, 12 and 13. What is the measure of the angle opposite the 11?

Vocabulary

Law of cosines

Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58142

http://www.youtube.com/watch?v=pGaDcOMdw48 Khan Academy: Law of Cosines

The Law of Cosines is:

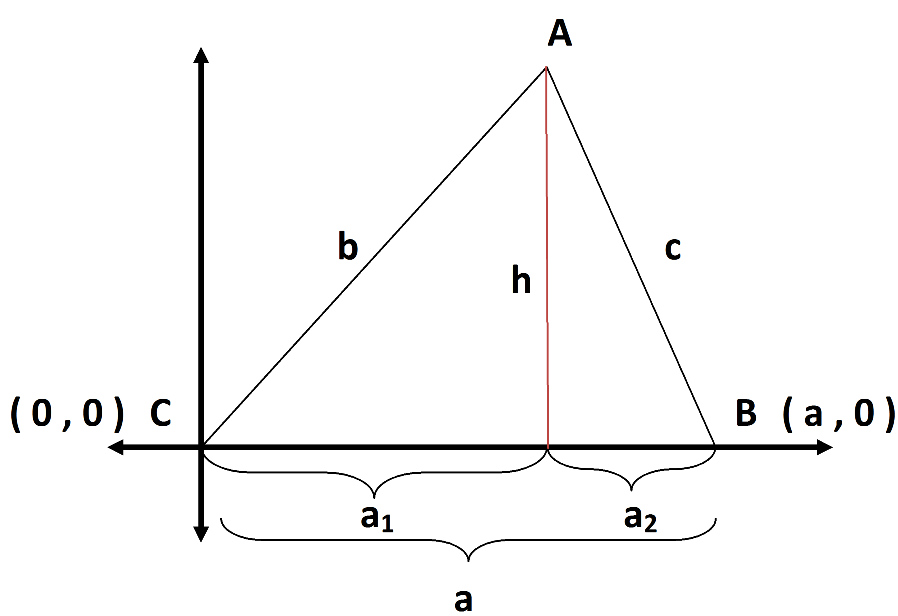
$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

There exists 3 different versions of the formula. The previous formula is the easiest to remember because at the beginning it looks like Pythagorean Theorem. The 3 versions of the formula are:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

You also need to remember that the capital A, B, C represent the angles in the triangle.

It is important to understand the proof:



You know four facts from the picture:

$a = a_1 + a_2 \qquad (1)$

$b^2 = a_1^2 + h^2 \qquad (2)$

$c^2 = a_2^2 + h^2 \qquad (3)$

$\cos C = \frac{a_1}{b} \qquad (4)$

Once you verify for yourself that you agree with each of these facts, check algebraically that these next two facts must be true.

$a_2 = a - a_1 \qquad (5,\text{ from } 1)$

$a_1 = b \cdot \cos C \qquad (6,\text{ from } 4)$

Now the Law of Cosines is ready to be proved using substitution, FOIL, more substitution and rewriting to get the order of terms right.

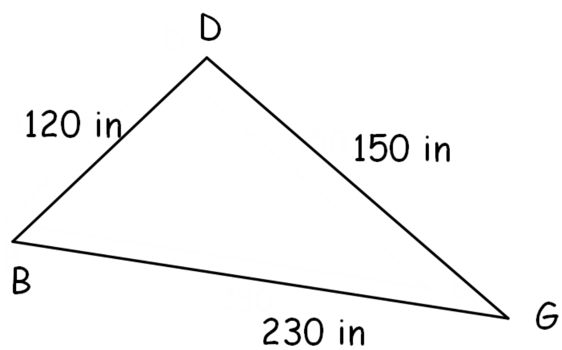
$$
\begin{aligned}
c^2 &= a_2^2 + h^2 && \text{(3 again)}\\
c^2 &= (a-a_1)^2 + h^2 && \text{(substitute using 5)}\\
c^2 &= a^2 - 2a\cdot a_1 + a_1^2 + h^2 && \text{(FOIL)}\\
c^2 &= a^2 - 2a\cdot b\cdot \cos C + a_1^2 + h^2 && \text{(substitute using 6)}\\
c^2 &= a^2 - 2a\cdot b\cdot \cos C + b^2 && \text{(substitute using 2)}\\
c^2 &= a^2 + b^2 - 2ab\cdot \cos C && \text{(rearrange terms)}
\end{aligned}
$$

There are only two types of problems in which it is appropriate to use the Law of Cosines.

1. The first is when you are given all three sides of a triangle and asked to find an unknown angle. This is called **SSS** like in geometry.
2. The second situation where you will use the Law of Cosines is when you are given two sides and the included angle and you need to find the third side. This is called **SAS**.

Example 1

Determine the measure of angle D.



Solution: It is necessary to set up the Law of Cosines equation very carefully with D corresponding to the opposite side of 230. The letters are not ABC like in the proof, but those letters can always be changed to match the problem as long as the angle in the cosine corresponds to the side used in the left side of the equation.

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cdot \cos C\\
230^2 &= 120^2 + 150^2 - 2\cdot 120\cdot 150\cdot \cos D\\
230^2 - 120^2 - 150^2 &= -2\cdot 120\cdot 150\cdot \cos D\\
\frac{230^2 - 120^2 - 150^2}{-2\cdot 120\cdot 150} &= \cos D\\
D &= \cos^{-1}\left(\frac{230^2 - 120^2 - 150^2}{-2\cdot 120\cdot 150}\right) \approx 116.4^\circ \approx 2.03 \textit{ radians}
\end{aligned}
$$

Another method to find the measure of the angle is a **3 number method.** The first number is the left side of the equation in this case c^2, the second number would be the sum of $a^2 + b^2$ the third number will be the product of $-2ab$.

If you use this method, this is the process to find the angle.

First number **minus** the second number, **divided** by the third number.

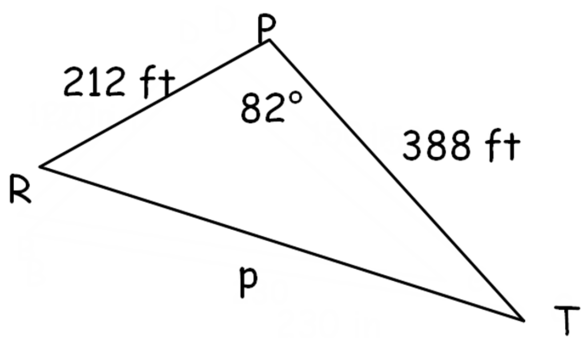
Once you get the decimal you will have to take the $\cos^{-1}$ of the decimal.

Here is how the previous problem would look by using this method.

$$c^2 = \qquad a^2+b^2-2ab\cdot cosC230^2 =$$

Example 2

Determine the length of side p.



Solution:

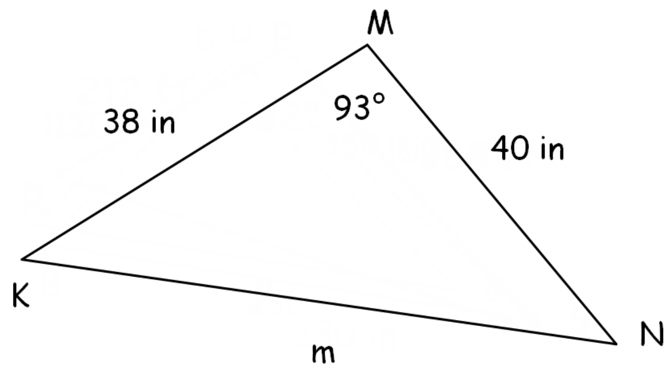
$$\begin{aligned}
c^2 &= a^2+b^2-2ab\cdot\cos C\\
p^2 &= 212^2+388^2-2\cdot 212\cdot 388\cdot\cos 82^\circ\\
p^2 &\approx 194192.02\ldots\\
p &\approx 440.7
\end{aligned}$$

Note: If you are solving for length of a side of a triangle using the law of cosines, all you have to do is the square root of the whole expression

$$p = \sqrt{212^2+388^2-2(212)(388)\cos 82} \approx 440.7$$

Example 3

Determine the degree measure of angle N.



Solution: This problem must be done in two parts. First apply the Law of Cosines to determine the length of side m. This is a SAS situation like Example B. Once you have all three sides you will be in the SSS situation like in Example 1 and can apply the Law of Cosines again to find the unknown angle N.

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cdot\cos C \\ m^2 &= 38^2+40^2-2\cdot 38\cdot 40\cdot\cos 93^\circ \\ m^2 &\approx 3203.1\ldots \\ m &\approx 56.59\ldots \end{aligned}$$

Now that you have all three sides you can apply the Law of Cosines again to find the unknown angle N. Remember to match angle N with the corresponding side length of 38 inches. It is also best to store m into your calculator and use the unrounded number in your future calculations.

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cdot\cos C \\ 38^2 &= 40^2+(56.59\ldots)^2-2\cdot 40\cdot(56.59\ldots)\cdot\cos N \\ 38^2-40^2-(56.59\ldots)^2 &= -2\cdot 40\cdot(56.59\ldots)\cdot\cos N \\ \frac{38^2-40^2-(56.59\ldots)^2}{-2\cdot 40\cdot(56.59\ldots)} &= \cos N \\ N &= \cos^{-1}\left(\frac{38^2-40^2-(56.59\ldots)^2}{-2\cdot 40\cdot(56.59\ldots)}\right)\approx 42.1^\circ \end{aligned}$$

With the 3 number method

$$c^2 = \qquad a^2+b^2-2ab\cdot cosC38^2 =$$

Concept Problem Revisited

A triangle that has sides 11, 12 and 13 is not going to be a right triangle. In order to solve for the missing angle you need to use the Law of Cosines because this is a SSS situation.

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\
11^2 &= 12^2 + (13)^2 - 2 \cdot 12 \cdot 13 \cdot \cos C \\
C &= \cos^{-1}\left(\frac{11^2 - 12^2 - 13^2}{-2 \cdot 12 \cdot 13}\right) \approx 52.02\ldots^\circ
\end{aligned}
$$

With the 3 step method

$$c^2 = \qquad a^2 + b^2 - 2ab \cdot cosC11^2 =$$

Vocabulary

The **Law of Cosines** is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle.

SSS refers to Side, Side, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that all three sides are known in the problem.

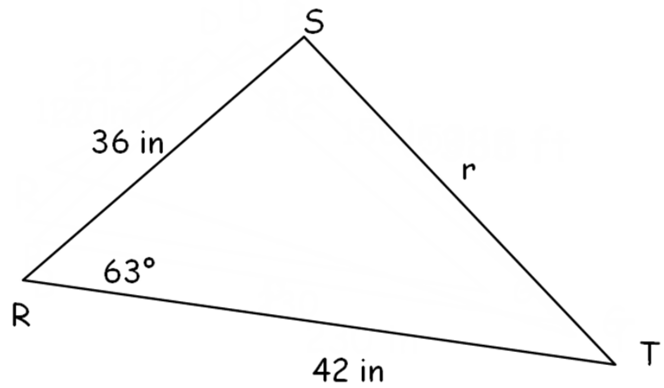
SAS refers to Side, Angle, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that the known quantities of a triangle are two sides and the included angle.

Included angle is the angle between two sides.

In Summary

We learned the formula called the law of cosines. We learned that the law of cosines can be used when in any triangle where we have **SAS** to find the length of the third side or when we have **SSS** to find the measure of all of the angles.

Guided Practice



1. Determine the length of side r.

2. Determine the measure of angle T in degrees.

3. Determine the measure of angle S in radians.

Answers:

1. $r^2 = 36^2 + 42^2 - 2 \cdot 36 \cdot 42 \cdot \cos 63$

$r = 41.07\ldots$

2. $36^2 = (41.07\ldots)^2 + 42^2 - 2 \cdot (41.07\ldots) \cdot 42 \cdot \cos T$

$T \approx 51.34\ldots°$

3. You could repeat the process from the previous question, or use the knowledge that the three angles in a triangle add up to 180.

$$63 + 51.34\ldots + S = 180$$
$$S \approx 65.65° \cdot \frac{\pi}{180°} \approx 1.145\ldots radians$$

Practice

For all problems, find angles in degrees rounded to one decimal place.

In $\Delta ABC, a = 12, b = 15$, and $c = 20$.

1. Find the measure of angle A.

2. Find the measure of angle B.

3. Find the measure of angle C.

4. Find the measure of angle C in a different way.

In $\Delta DEF, d = 20, e = 10$, and $f = 16$.

5. Find the measure of angle D.

6. Find the measure of angle E.

7. Find the measure of angle F.

In $\Delta GHI, g = 19, \angle H = 55°$, and $i = 12$.

8. Find the length of h.

9. Find the measure of angle G.

10. Find the measure of angle I.

11. Explain why the Law of Cosines is connected to the Pythagorean Theorem.

12. What are the two types of problems where you might use the Law of Cosines?

Determine whether or not each triangle is possible.

13. $a = 5, b = 6, c = 15$

14. $a = 1, b = 5, c = 4$

15. $a = 5, b = 6, c = 10$

5.8 Law of Sines

Here you will further explore solving non-right triangles in cases where a corresponding side and angle are given using the Law of Sines.

TEKS

1. P.4.G

Lesson Objectives

In this lesson you will learn about:
1. The law of sines.
2. The ambiguous case.
3. Finding the area of a non right triangle.

Introduction

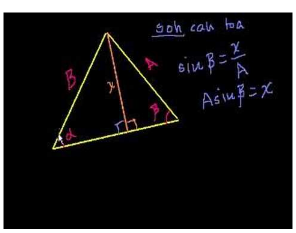
When given a right triangle, you can use basic trigonometry to solve for missing information. When given SSS or SAS, you can use the Law of Cosines to solve for the missing information. But what happens when you are given two sides of a triangle and an angle that is not included? There are many ways to show two triangles are congruent, but SSA is not one of them. Why not?

Vocabulary

Law of sines, ambiguous

Watch This

http://www.youtube.com/watch?v=APNkWrD-U1k Khan Academy: Proof: Law of Sines



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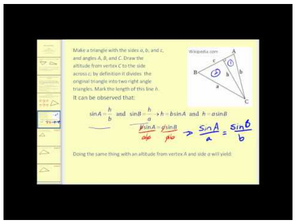
URL: http://www.ck12.org/flx/render/embeddedobject/58144

http://www.youtube.com/watch?v=dxYVBbSXYkA James Sousa: The Law of Sines: The Basics



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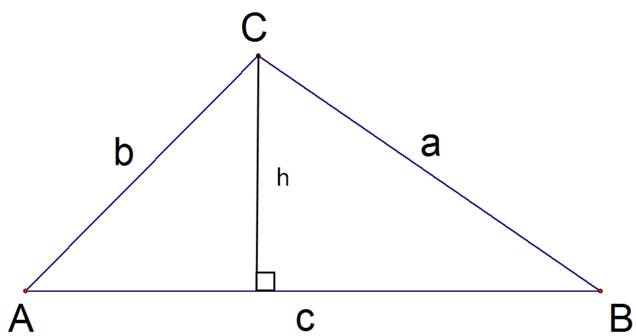
Guidance

When given two sides and an angle that is not included between the two sides, you can use the Law of Sines. The Law of Sines states that in every triangle the ratio of each side to the sine of its corresponding angle is always the same. Essentially, it clarifies the general concept that opposite the largest angle is always the longest side.

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Here is a proof of the Law of Sines:



Looking at the right triangle formed on the left:

$$\sin A = \frac{h}{b}$$
$$h = b \sin A$$

Looking at the right triangle formed on the right:

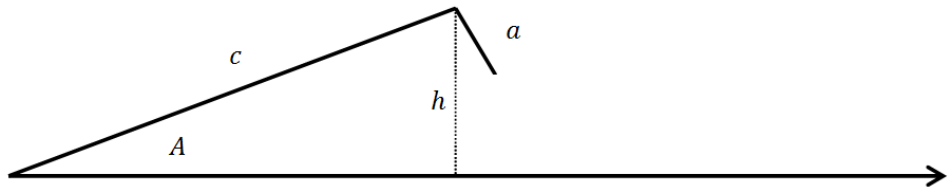
$$\sin B = \frac{h}{a}$$
$$h = a \sin B$$

Equating the heights which must be identical:

$$a \sin B = b \sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The best way to use the Law of Sines is to draw an extremely consistent picture each and every time even if that means redrawing and relabeling a picture. The reason why the consistency is important is because sometimes given SSA information defines zero, one or even two possible triangles.

Always draw the given angle in the bottom left with the two given sides above.



In this image side a is deliberately too short, but in most problems you will not know this. You will need to compare a to the height.

$$\sin A = \frac{h}{c}$$
$$h = c\ \sin A$$

Case 1: $a < h$

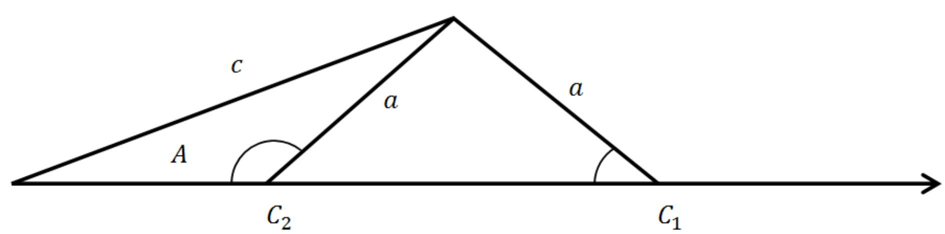
Simply put, side a is not long enough to reach the opposite side and the triangle is impossible.

Case 2: $a = h$

Side a just barely reaches the opposite side forming a 90° angle.

Case 3: $h < a < c$

In this case side a can swing toward the interior of the triangle or the exterior of the triangle- there are two possible triangles. This is called the ambiguous case because the given information does not uniquely identify one triangle. To solve for both triangles, use the Law of Sines to solve for angle C_1 first and then use the supplement to determine C_2.



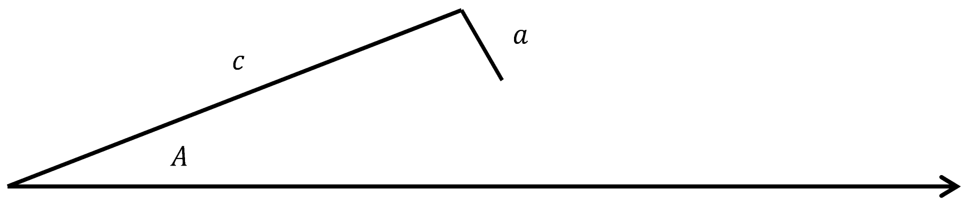
Case 4: $c \leq a$

In this case, side a can only swing towards the exterior of the triangle, only producing C_1.

NOTE: If there is a possibility that the angle you are solving for with the law of sines is obtuse, try not to solve for that angle. If you solve for the smallest angles first, you won't have any problems. If possible use the law of cosines to solve for the angles.

Example 1

$\angle A = 40^\circ, c = 13$, and $a = 2$. If possible, find $\angle C$.



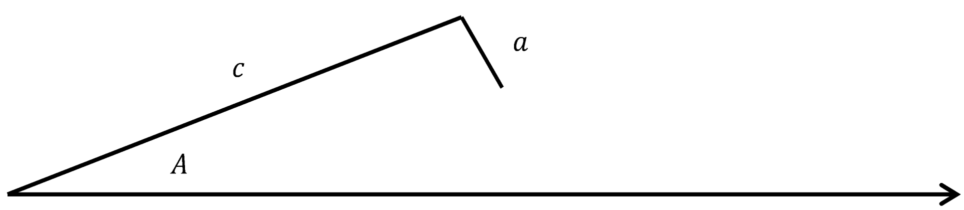
Solution:

$$\sin 40^\circ = \frac{h}{13}$$
$$h = 13 \sin 40^\circ \approx 8.356$$

Because $a < h$ $(2 < 8.356)$, this information does not form a proper triangle.

Example 2

$\angle A = 17^\circ, c = 14$, and $a = 4.0932\ldots$ If possible, find $\angle C$.



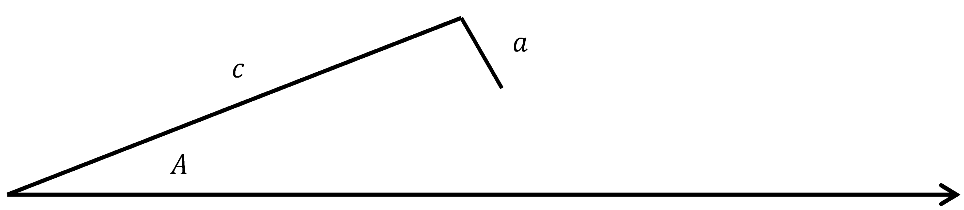
Solution:

$$\sin 17^\circ = \frac{h}{14}$$
$$h = 14 \ \sin 17^\circ \approx 4.0932\ldots$$

Since $a = h$, this information forms exactly one triangle and angle C must be 90°.

Example 3

$\angle A = 22^\circ, c = 11$ and $a = 9$. If possible, find $\angle C$.



Solution:

$$\sin 22^\circ = \frac{h}{11}$$
$$h = 11\ \sin 22^\circ \approx 4.12\ldots$$

Since $h < a < c$, there must be two possible angles for angle C.

Apply the Law of Sines:

$$\frac{9}{\sin 22^\circ} = \frac{11}{\sin C_1}$$
$$9 \sin C_1 = 11 \sin 22^\circ$$
$$\sin C_1 = \frac{11 \sin 22^\circ}{9}$$
$$C_1 = \sin^{-1}\left(\frac{11 \sin 22^\circ}{9}\right) \approx 27.24\ldots^\circ$$
$$C_2 = 180 - C_1 = 152.75\ldots^\circ$$

When we have a right triangle, it is easy to find the area of a triangle. Recall that the area of a triangle is given by the formula $A_\triangle = \frac{1}{2}b \cdot\ h$

where we take the base and the height to be the legs of the right triangle, but what happens when the triangle is not a right triangle? Here is how to obtain the formula look at the following figure

FIGURE 5.4

We have the triangle ABC with corresponding sides a,b,c. The base of our triangle will be **side a**, to find the height, we need to look at angle θ (**angle C**) then to find the height, then we would have to look at the highlighted blue triangle and find the height using right triangle trigonometry.

$$\sin C = \qquad \frac{h}{b} b \cdot \sin C =$$

Therefore the height of the triangle will be $h = b \cdot \sin C$

Finally if we want the area of any triangle if we have SAS information, then we can use the formula

$$A_\triangle = \frac{1}{2} a \cdot b \cdot \sin C$$

Therefore if we have SAS in a triangle the new area formula of a triangle will be:

$$A_\triangle = \frac{1}{2} a \cdot b \cdot sin\, C$$

where a, b are the sides and C is the included angle. If you use A,B,C as your variables, then this formula can be written in three forms.

$$A_\triangle = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$
$$A_\triangle = \frac{1}{2} \cdot b \cdot c \cdot \sin A$$
$$A_\triangle = \frac{1}{2} \cdot a \cdot c \cdot \sin B$$

Example 4.

Find the area of the following triangle.

FIGURE 5.5

Solution

Since we in this triangle we have **SAS**, we don't need to find the length of the third side or any other angles to find the area. In this case $a = 3, b = 5, \angle C = 120^\circ$.

Therefore the area would be as follows:

$$= \frac{1}{2} \cdot a \cdot b \cdot sin(C) A_\triangle$$

Concept Problem Revisited

SSA is not a method from Geometry that shows two triangles are congruent because it does not always define a unique triangle.

Guided Practice

1. Given ΔABC where $A = 10^\circ, b = 10, a = 11$, find $\angle B$.

2. Given ΔABC where $A = 12^\circ, B = 50^\circ, a = 14$ find b.

3. Given ΔABC where $A = 70^\circ, b = 8, a = 3$, find $\angle B$ if possible.

Answers:

1. $\frac{10}{\sin B} = \frac{11}{\sin 10^\circ}$

$B = \sin^{-1}\left(\frac{10 \sin 10^\circ}{11}\right) \approx 9.08\ldots^\circ$

2. $\frac{14}{\sin 12^\circ} = \frac{b}{\sin 50^\circ}$

$b = \frac{14 \sin 50^\circ}{\sin 12^\circ} \approx 51.58\ldots$

3. $\sin 70^\circ = \frac{h}{8}$

$h = 8 \sin 70^\circ \approx 7.51\ldots$

Because $a < h$, this triangle is impossible.

Vocabulary

Ambiguous

means that the given information may not uniquely identify one triangle.

In Summary

We learned what the law of sines is. With the law of sines, we learned that it might give us a problem if we are solving for the biggest side of the triangle. Also, we learned how to find the area of non right triangle by using a new area formula that involves the trigonometric function $\sin\theta$.

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Practice

For 1-3, draw a picture of the triangle and state how many triangles could be formed with the given values.

1. $A = 30^\circ, a = 13, b = 15$

2. $A = 22^\circ, a = 21, b = 12$

3. $A = 42^\circ, a = 36, b = 37$

For 4-7, find all possible measures of $\angle B$ (if any exist) for each of the following triangle values.

4. $A = 86^\circ, a = 15, b = 11$

5. $A = 30^\circ, a = 24, b = 43$

6. $A = 48^\circ, a = 34, b = 39$

7. $A = 80^\circ, a = 22, b = 20$

For 8-12, find the length of b for each of the following triangle values.

8. $A = 94^\circ, a = 31, B = 34^\circ$

9. $A = 112^\circ, a = 12, B = 15^\circ$

10. $A = 78^\circ, a = 20, B = 16^\circ$

11. $A = 54^\circ, a = 15, B = 112^\circ$

12. $A = 39^\circ, a = 9, B = 98^\circ$

13. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce two triangles?

14. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce no triangles?

15. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce one triangle?

For 16-18, Find the area of the triangle.

16. $B = 130^\circ, a = 9, c = 14$

17. $A = 94^\circ, b = 31, c = 34$

18. $C = 69^\circ, a = 10, b = 7$

For 19 and 20 solve the triangles (find all sides and all angles)

19. $A = 67^\circ, B = 27^\circ, a = 15$

20. $a = 7, b = 13, c = 17$

5.9 Applications of Basic Triangle Trigonometry

Here you will apply your knowledge of trigonometry and problem solving in context.

TEKS

1. P.4.F
2. P.4.G
3. P.4.H

Lesson Objectives

In this lesson you will learn about:
1. Solving real world situations using triangle trigonometry.
2. Solving problems involving bearings.

Introduction

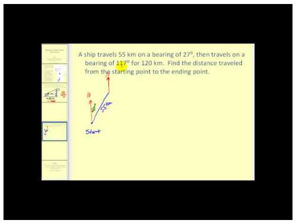
Deciding when to use SOH, CAH, TOA, Law of Cosines or the Law of Sines is not always obvious. Sometimes more than one approach will work and sometimes correct computations can still lead to incorrect results. This is because correct interpretation is still essential.

Vocabulary

Angle of elevation, Angle of depression, bearing

If you use both the Law of Cosines and the Law of Sines on a triangle with sides 4, 7, 10 you end up with conflicting answers. Why?

Watch This



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URL: http://www.ck12.org/flx/render/embeddedobject/58154

http://www.youtube.com/watch?v=-QOEcnuGQwo James Sousa: Solving Right Triangles-Part 2 Applications

Guidance

When applying trigonometry, it is important to have a clear toolbox of mathematical techniques to use. Some of the techniques may be review like the fact that all three angles in a triangle sum to be 180°, other techniques may be new like the Law of Cosines. There also may be some properties that are true and make sense but have never been formally taught.

Toolbox:

- The three angles in a triangle sum to be 180°.
- There are 360° in a circle and this can help us interpret negative angles as positive angles.
- The Pythagorean Theorem states that for legs a, b and hypotenuse c in a right triangle, $a^2+b^2=c^2$.
- The Triangle Inequality Theorem states that for any triangle, the sum of any two of the sides must be greater than the third side.
- The Law of Cosines: $c^2=a^2+b^2-2ab\cos C$
- The Law of Sines: $\frac{a}{\sin A}=\frac{b}{\sin B}$ or $\frac{\sin A}{a}=\frac{\sin B}{b}$ (Be careful for the ambiguous case)
- SOH CAH TOA is a mnemonic device to help you remember the three original trig functions:

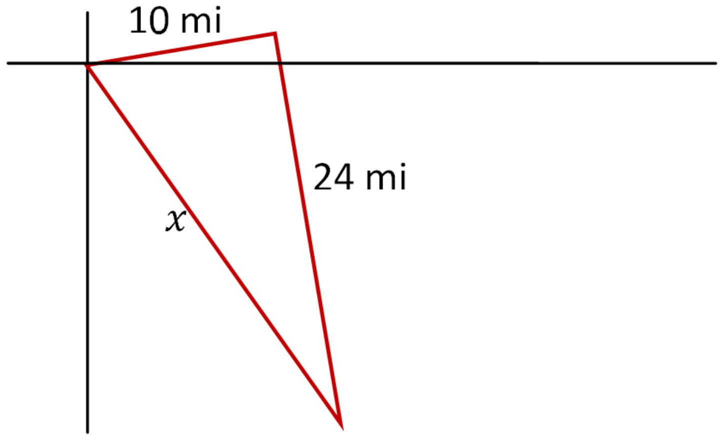
$$\sin\theta=\frac{opp}{hyp} \quad \cos\theta=\frac{adj}{hyp} \quad \tan\theta=\frac{opp}{adj}$$

- 30-60-90 right triangles have side ratios $x, x\sqrt{3}, 2x$
- 45-45-90 right triangles have side ratios $x, x, x\sqrt{2}$
- Pythagorean number triples are exceedingly common and should always be recognized in right triangle problems. Examples of triples are 3, 4, 5 and 5, 12, 13.

Example 1

Bearing is how direction is measured at sea. North is 0°, East is 90°, South is 180° and West is 270°. A ship travels 10 miles at a bearing of 88° and then turns 90° to the right to avoid an iceberg for 24 miles. How far is the ship from its original position?

Solution: First draw a clear sketch.



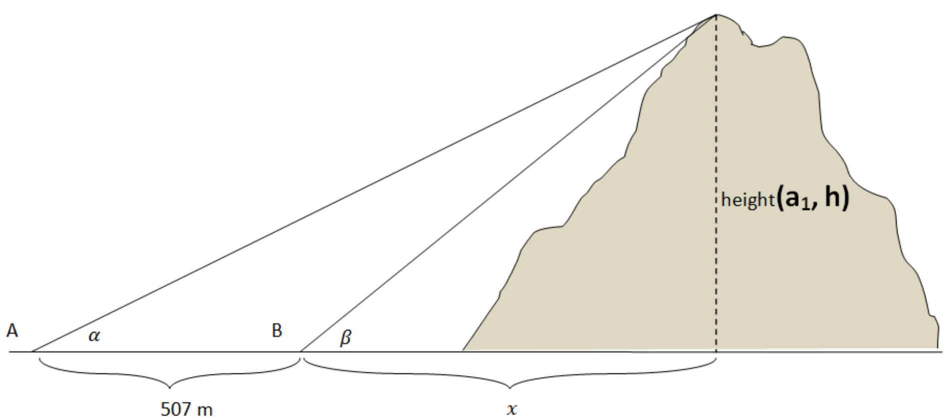
Next, recognize the right triangle with legs 10 and 24. This is a multiple of the 5, 12, 13 Pythagorean number triple and so the distance x must be 26 miles.

Example 2

A surveying crew is given the job of verifying the height of a cliff. From point A, they measure an angle of elevation to the top of the cliff to be $\alpha = 21.567^\circ$. They move 507 meters closer to the cliff and find that the angle to the top of the cliff is now $\beta = 25.683^\circ$. How tall is the cliff?

Note that α *is just the Greek letter alpha and in this case it stands for the number* 21.567°. β *is the Greek letter beta and it stands for the number* 25.683°.

Solution: First, sketch the image and label what you know.



Next, because the height is measured at a right angle with the ground, set up two equations. Remember that α and β are just numbers, not variables.

$$\tan\alpha = \frac{h}{507+x}$$
$$\tan\beta = \frac{h}{x}$$

Both of these equations can be solved for h and then set equal to each other to find x.

$$\begin{aligned}
h = \tan\alpha(507+x) &= x\tan\beta \\
507\tan\alpha + x\tan\alpha &= x\tan\beta \\
507\tan\alpha &= x\tan\beta - x\tan\alpha \\
507\tan\alpha &= x(\tan\beta - \tan\alpha) \\
x = \frac{507\tan\alpha}{\tan\beta - \tan\alpha} &= \frac{507\tan 21.567^\circ}{\tan 25.683^\circ - \tan 21.567^\circ} \approx 228.7\ meters
\end{aligned}$$

Since the problem asked for the height, you need to substitute x back and solve for h.

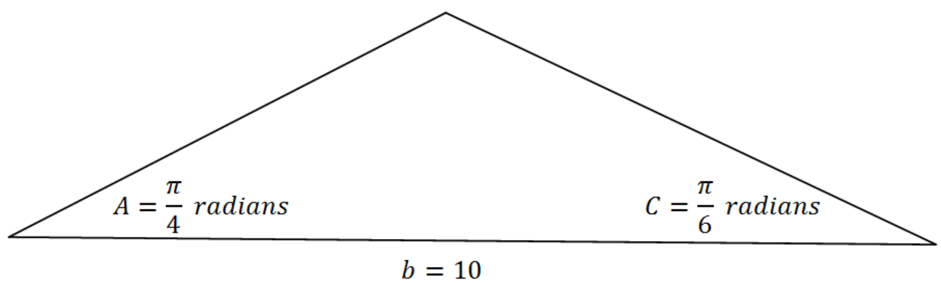
$h = x\tan\beta = 228.7\tan 25.683^\circ \approx 109.99\ meters$

Example 3

Given a triangle with SSS or SAS you know to use the Law of Cosines. In triangles where there are corresponding angles and sides like AAS or SSA it makes sense to use the Law of Sines. What about ASA?

Given ΔABC with $A = \frac{\pi}{4}\ radians, C = \frac{\pi}{6}\ radians$ and $b = 10\ in$ what is a?

Solution: First, draw a picture.



The sum of the angles in a triangle is 180°. Since this problem is in radians you either need to convert this rule to radians, or convert the picture to degrees.

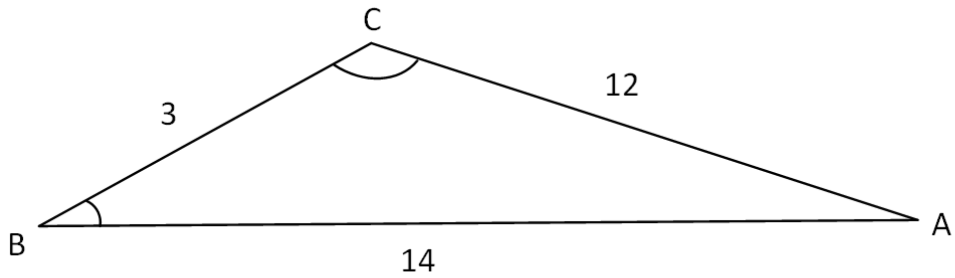
$$\begin{aligned}
A &= \frac{\pi}{4} \cdot \frac{180^\circ}{\pi} = 45^\circ \\
C &= \frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ
\end{aligned}$$

The missing angle must be $\angle B = 105^\circ$. Now you can use the Law of Sines to solve for a.

$$\begin{aligned}
\frac{\sin 105^\circ}{10} &= \frac{\sin 45^\circ}{a} \\
a &= \frac{10\sin 45^\circ}{\sin 105^\circ} \approx 7.32\ in
\end{aligned}$$

Concept Problem Revisited

Sometimes when using the Law of Sines you can get answers that do not match the Law of Cosines. Both answers can be correct computationally, but the Law of Sines may involve interpretation when the triangle is obtuse. The Law of Cosines does not require this interpretation step.



First, use Law of Cosines to find $\angle B$:

$$12^2 = 3^2 + 14^2 - 2 \cdot 3 \cdot 14 \cdot \cos B$$
$$\angle B = \cos^{-1}\left(\frac{12^2 - 3^2 - 14^2}{-2 \cdot 3 \cdot 14}\right) \approx 43.43\ldots^\circ$$

Then, use Law of Sines to find $\angle C$. Use the unrounded value for B even though a rounded value is shown.

$$\frac{\sin 43.43^\circ}{12} = \frac{\sin C}{14}$$
$$\frac{14 \sin 43.43^\circ}{12} = \sin C$$
$$\angle C = \sin^{-1}\left(\frac{14 \sin 43.43^\circ}{12}\right) \approx 53.3^\circ$$

Use the Law of Cosines to double check $\angle C$.

$$14^2 = 3^2 + 12^2 - 2 \cdot 3 \cdot 12 \cdot \cos C$$
$$C = \cos^{-1}\left(\frac{14^2 - 3^2 - 12^2}{-2 \cdot 3 \cdot 12}\right) \approx 126.7^\circ$$

Notice that the last two answers do not match, but they are supplementary. This is because this triangle is obtuse and the $\sin^{-1}\left(\frac{opp}{hyp}\right)$ function is restricted to only producing acute angles.

Guided Practice

1. The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{10}$. The lighthouse is 200 feet tall. Find the distance from the base of the lighthouse to the boat.

2. From the third story of a building (50 feet) David observes a car moving towards the building driving on the streets below. If the angle of depression of the car changes from 21° to 45° while he watches, how far did the car travel?

3. If a boat travels 4 miles SW and then 2 miles NNW, how far away is it from its starting point?

Answers:

1. When you draw a picture, you see that the given angle $\frac{\pi}{10}$ is not directly inside the triangle between the lighthouse, the boat and the base of the lighthouse. It is complementary to the angle you need.

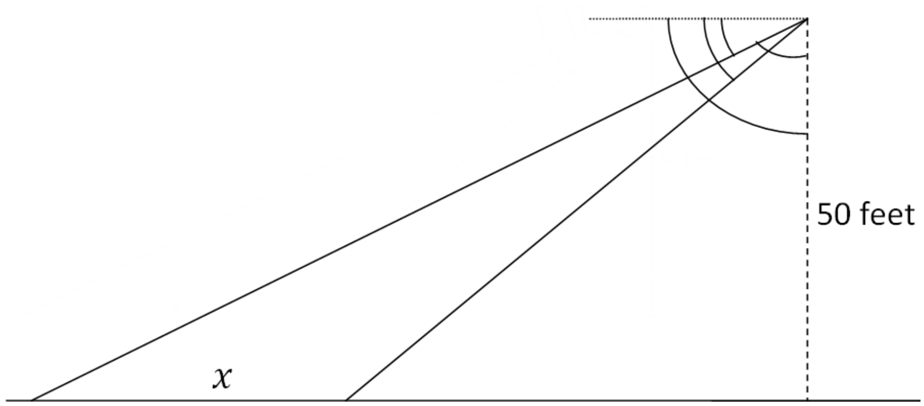
$$\frac{\pi}{10}+\theta=\frac{\pi}{2}$$
$$\theta=\frac{2\pi}{5}$$

Now that you have the angle, use tangent to solve for x.

$$\tan\frac{2\pi}{5}=\frac{x}{200}$$
$$x=200\tan\frac{2\pi}{5}\approx 615.5\ldots ft$$

Alternatively, you could have noticed that $\frac{\pi}{10}$ is alternate interior angles with the angle of elevation of the lighthouse from the boat's perspective. This would yield the same distance for x.

2. Draw a very careful picture:



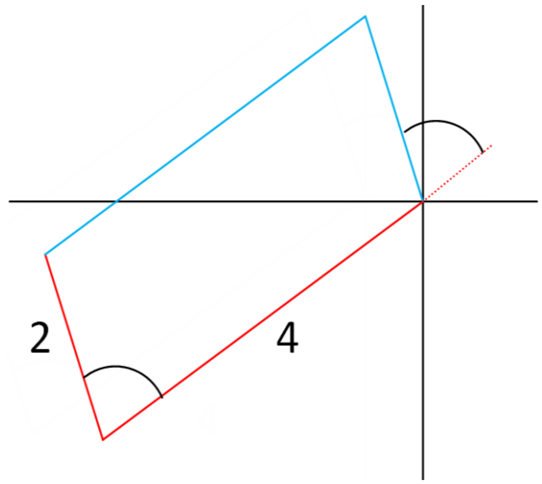
In the upper right corner of the picture there are four important angles that are marked with angles. The measures of these angles from the outside in are $90°, 45°, 21°, 69°$. There is a 45-45-90 right triangle on the right, so the base must also be 50. Therefore you can set up and solve an equation for x.

$$\tan 69°=\frac{x+50}{50}$$
$$x=50\tan 69°-50\approx 80.25\ldots ft$$

The hardest part of this problem is drawing a picture and working with the angles.

3. 4 miles SW and then 2 miles NNW

Translate SW and NNW into degrees bearing. SW is a bearing of 225° and NNW is a bearing of 315°. Draw a picture in two steps. Draw the original 4 miles traveled and draw the second 2 miles traveled from the origin. Then translate the second leg of the trip so it follows the first leg. This way you end up with a parallelogram, which has interior angles that are easier to calculate.



The angle between the two red line segments is 67.5° which can be seen if the red line is extended past the origin. The shorter diagonal of the parallelogram is the required unknown information.

$$x^2 = 4^2 + 2^2 - 2 \cdot 4 \cdot 2 \cdot \cos 67.5^\circ$$
$$x \approx 3.7 \text{ } miles$$

Vocabulary

Angle of elevation

is the angle at which you view an object above the horizon/horizontal.

Angle of depression

is the angle at which you view an object below the horizon/horizontal. This can be thought of negative angles of elevation.

Bearing

is how direction is measured at sea. North is 0°, East is 90°, South is 180° and West is 270°.

Greek letters *alpha* and *beta* (α, β)

are often used as placeholders for known angles. Unknown angles are often referred to as θ ***(theta).***

ASA

refers to the situation from geometry when there are two known angles in a triangle and one known side that is between the known angles.

In Summary

We have been reminded about some of the facts about right triangles like Pythagorean Theorem and about the sum of the angles in a triangle being 180°, also about the rules for special right triangles. We have learned to apply the law of sines, law of cosines, or right triangle trigonometry to solve real life problems. We have also learned about bearings, how they are measured from North and to solve bearing problems.

Practice

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{6}$. The lighthouse is 150 feet tall. You want to find the distance from the base of the lighthouse to the boat.

1. Draw a picture of this situation.
2. What methods or techniques will you use?
3. Solve the problem.

From the third story of a building (60 feet) Jeff observes a car moving towards the building driving on the streets below. The angle of depression of the car changes from 34° to 62° while he watches. You want to know how far the car traveled.

4. Draw a picture of this situation.
5. What methods or techniques will you use?
6. Solve the problem.

A boat travels 6 miles NW and then 2 miles SW. You want to know how far away the boat is from its starting point.

7. Draw a picture of this situation.
8. What methods or techniques will you use?
9. Solve the problem.

You want to figure out the height of a building. From point A, you measure an angle of elevation to the top of the building to be $\alpha = 10^\circ$. You move 50 feet closer to the building to point B and find that the angle to the top of the building is now $\beta = 60^\circ$.

10. Draw a picture of this situation.
11. What methods or techniques will you use?
12. Solve the problem.
13. Given ΔABC with $A = 40^\circ, C = 65^\circ$ and $b = 8\ in$, what is a?
14. Given ΔABC with $A = \frac{\pi}{3}\ radians, C = \frac{\pi}{8}\ radians$ and $b = 12\ in$ what is a?
15. Given ΔABC with $A = \frac{\pi}{6}\ radians, C = \frac{\pi}{4}\ radians$ and $b = 20\ in$ what is a?

The relationship between the sides and angles of all kinds of triangles were explored though the use of the six basic trigonometric functions.

5.10 References

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Chapter 6 Graphs of Trigonometric Functions

Chapter Outline

6.1 Graphs of Sine, Cosine, Tangent and Vertical stretches

TEKS

Lesson Objectives

In this section you will learn about:

1. The graph of sine and its characteristics.
2. The graph of cosine and its characteristics.
3. The graph of tangent and its characteristics.
4. The vertical stretch and compress of the 3 graphs.

Introduction

Situations in real life can be modeled by trigonometric functions like sine and cosine. For example the average temperature in a certain city can be approximated form year to year by a sine or cosine function. Another example is the alternating current we use in our homes. Throughout this section and this chapter you will learn about the graphs of the trigonometric functions.

Vocabulary

Trigonometric Graphs, Amplitude, Period, Domain, Range.

THE GRAPH OF $f(\theta) = \sin\theta$

The graph of $f(\theta) = \sin\theta$ has an **Amplitude** of 1 and a **Period** of 2π. Some important values of the function are:

At 0 the value is 0.

At $\frac{\pi}{2}$the value is 1.

At π the value is 0.

At $\frac{3\pi}{2}$the value is -1.

At 2π the value is 0.

Here is the graph of the parent function $f(\theta) = \sin\theta$ in the interval $[0, 2\pi]$

FIGURE 6.1

The **Domain** is $(-\infty,\infty)$ and the **Range** is $[-1,1]$

THE GRAPH OF $f(\theta) = \cos\theta$

The graph of $f(\theta) = \cos\theta$ has an **Amplitude** of 1 and a **Period** of 2π. Some important values of the function are:

At 0 the value is 1.

At $\frac{\pi}{2}$the value is 0.

At π the value is -1.

At $\frac{3\pi}{2}$the value is 0.

At 2π the value is 1.

Here is the graph of the parent function $f(\theta) = \cos\theta$ in the interval $[0,2\pi]$

FIGURE 6.2

The **Domain** is $(-\infty,\infty)$ and the **Range** is $[-1,1]$.

THE GRAPH OF $f(\theta) = \tan\theta$

The graph of $f(\theta) = \tan\theta$ has **NO Amplitude** and a **Period** of π. Some important values of the function are:

At $-\frac{\pi}{2}$ the value is *undefined*.

At $-\frac{\pi}{4}$ the value is -1.

At 0 the value is 0.

At $\frac{\pi}{4}$ the value is 1.

At $\frac{\pi}{2}$ the value is *undefined*.

Here is the graph of the function $f(\theta) = \tan\theta$ in the interval $[-\pi,\pi]$.

FIGURE 6.3

The red lines are the asymptotes.

The **Domain** is $\left\{\theta \mid \theta \neq \frac{\pi}{2}n\right\}$ and the **Range** is $(-\infty,\infty)$

On the tangent function the domain is all real numbers except $\frac{\pi}{2}$and its multiples.

VERTICAL STRETCH AND COMPRESS

The general form of a trigonometric function will be in the form $f(\theta) = A\ sin(B\theta + C) + D$.

We are just going to analyze what the value of **A** does to the graph of the trigonometric function.

- If the $|A| > 1$ then the graph stretches vertically **.**
- If $0 < 1$ then the graph will compress vertically
- If $A < 0$ then the graph will reflect across the x-axis.

Example 1.

Graph the parent function $f(\theta) = \sin\theta$ and the transformation $f(\theta) = 3\sin\theta$ on the interval $[-2\pi,2\pi]$ and find the amplitude, period, domain and range.

FIGURE 6.4

The red graph is the graph of the transformed function. The only points that do not change are the zeros, every other point gets multiplied by 3 which is the Amplitude. The following is the information about the transformed function.

Amplitude = 3, **Period =** 2π, **Domain=** $(-\infty,\infty)$, **Range =** $[-3,3]$

Example 2.

Graph the parent function $f(\theta) = \cos\theta$ and graph the transformation $f(\theta) = \frac{1}{2}\cos\theta$ on the interval $[-2\pi, 2\pi]$ and find the amplitude, period, domain and range.

FIGURE 6.5

The blue graph is the graph of the transformed function. The only points that do not change are the zeros, every other point gets multiplied by .5 which is the Amplitude. The following is the information about the transformed function.

Amplitude = $\frac{1}{2}$, **Period =** 2π, **Domain =** $(-\infty,\infty)$, **Range =** $\left[-\frac{1}{2},\frac{1}{2}\right]$.

Example 3.

Graph the parent function $f(\theta) = \tan\theta$ and graph the transformation $f(\theta) = 3\tan\theta$ on the interval $[-\pi, \pi]$ and find the period, domain and range.

FIGURE 6.6

The purple graph is the graph of the transformed function. The asymptotes did not change.

NO amplitude. **Period =** π, **Domain =** $\left\{\theta \mid \theta \neq \frac{\pi}{2}n\right\}$, **Range =** $(-\infty,\infty)$

Example 4

Graph the parent function $f(\theta) = \cos\theta$ and graph the transformation$f(\theta) = -2\cos\theta$ and identify the Amplitude, period, domain and range.

FIGURE 6.7

The blue graph is the graph of the transformed function.

Since A is negative, then the graph reflects over the x axis and vertically stretches by 2.

Amplitude = 2, **Period =** 2π, **Domain =** $(-\infty,\infty)$, **Range =** $[-2,2]$

Vocabulary

Trigonometric Graphs

The graphs of sine, cosine, tangent, cosecant, secant and cotangent.

Amplitude

The height from the mean or rest value of the function to its maximum or minimum. It's half the distance between the highest and lowest point.

Period

The distance required for a function to complete one full cycle.

Domain

The set of independent values for which the function is defined.

Range

The set of dependent values obtained form the domain values.

In Summary

We have learned about the graphs of sine, cosine and tangent and about their characteristics like the amplitude and the period. We have also learned about the vertical stretch transformation and to find the Domain and Range of the transformed function.

Practice

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6.2 Graphs of Cosecant, Secant and Cotangent and Vertical Stretches

TEKS

Lesson Objectives

In this lesson you will learn about:
1. The graph of cosecant and its characteristics.
2. The graph of secant and its characteristics.
3. The graph of cotangent and its characteristics.

Introduction

$\csc\theta, \sec\theta, \cot\theta$ are the reciprocals of $\sin\theta, \cos\theta, \tan\theta$. In this section you will learn how to graph $\csc\theta, \sec\theta, \cot\theta$ with vertical stretch or compress.

Vocabulary

reciprocal trigonometric functions, $\csc\theta, \sec\theta, \cot\theta$

In **section 5.5**, we mentioned the following identities.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\sin\theta}{\cos\theta}$$

By looking at the identities, if we want to graph $f(\theta) = \csc\theta = \frac{1}{\sin\theta}$ then the value of $\sin\theta \neq 0$ because it will be undefined.

By looking at the identities, if we want to graph $f(\theta) = \sec\theta = \frac{1}{\cos\theta}$ then the value of $\cos\theta \neq 0$ because it will be undefined.

By looking at the identities, if we want to graph $f(\theta) = \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$then the value of $\sin\theta \neq 0$ because it will be undefined.

To graph $f(\theta) = \csc\theta \ \ or \ \ f(\theta) = \sec\theta$ you can use the following steps.

1. First graph the reciprocal function either sin x or cos x.
2. Then take away by graphing an asymptote the values where the function is zero.
3. Then flip the graph around and that will be your graph.

THE GRAPH OF $f(\theta) = \csc\theta$

FIGURE 6.8

IMAGE NOT AVAILABLE

The green graph is the graph of cosecant, the red dotted graph is the graph of sine, it was used to help graph the cosecant function. The period of cosecant function is 2π

Domain: $\{\theta \mid \theta \neq 0 \pm n \cdot \pi\}$ (all real numbers except the multiples of π)

Range: $(-\infty, -1] \bigcup [1, \infty)$

THE GRAPH OF $f(\theta) = \sec\theta$

FIGURE 6.9

IMAGE NOT AVAILABLE

The orange graph is the graph of the secant fucntion, the blue dotted line is the graph of cosine, it was used to help graph the secant fucntion. The period of secant function is 2π.

Domain: $\left\{\theta \mid \theta \neq \frac{\pi}{2} \pm n \cdot \pi\right\}$ (All real numbers except $\frac{\pi}{2}$ plus or minus the multiples of π)

Range: $(-\infty, -1] \bigcup [1, \infty)$

THE GRAPH OF $f(\theta) = \cot\theta$

FIGURE 6.10

IMAGE NOT AVAILABLE

The graph of cotangent is the purple graph it differs from the tangent graph by being shifted $\frac{\pi}{2}$ units to the right and reflected over the x axis. The period of the cotangent function is π

Domain: $\{\theta | \theta \neq n\pi\}$ (all real numbers except the multiples of π)

Range: (∞, ∞)

To vertically stretch and compress the cosecant and secant functions we graph first the stretched/compressed reciprocal function, take away the zeros and then flip the function around.

Example 1.

Graph the function $f(\theta) = 4\csc\theta$ on the interval $[-2\pi, 2\pi]$

To achieve this first graph $f(\theta) = 4\sin\theta$ then take away all the points where the graph is zero and draw vertical asymptotes, then flip the graph.

FIGURE 6.11

IMAGE NOT AVAILABLE

The green graph is the graph of $f(\theta) = 4\csc\theta$

Domain: $\{\theta | \theta \neq 0 \pm n \cdot \pi\}$ (all real numbers except all the multiples of π)

Range: $(-\infty,-4]\bigcup[4,\infty)$

Period: 2π

Example 2.

Graph the function $f(\theta)=2\sec\theta$ on the interval $[-2\pi,2\pi]$

To achieve this first graph $f(\theta)=2\cos\theta$ then take away all the points where the graph is zero and draw vertical asymptotes, then flip the graph.

FIGURE 6.12

The orange graph is the graph of $f(\theta)=2\sec\theta$

Domain: $\left\{\theta|\theta\neq\frac{\pi}{2}\pm n\cdot\pi\right\}$ (all real numbers except $\frac{\pi}{2}$ plus or minus the multiples of π)

Range: $(-\infty,-2]\bigcup[2,\infty)$

Period: 2π

Example 3.

Graph the function $f(\theta)=\frac{1}{2}\cot\theta$ on the interval $[-\pi,\pi]$

To achieve this, we know that the values of the function $f(\theta)=\cot\theta$ are $f(\frac{\pi}{4})=1$, $f(\frac{\pi}{2})=0$ and $f(\frac{3\pi}{4})=-1$

By dividing these values by $\frac{1}{2}$ we obtain $f(\frac{\pi}{4})=\frac{1}{2},f(\frac{\pi}{2})=0,f(\frac{3\pi}{4})=-\frac{1}{2}$

FIGURE 6.13

The black graph is the graph of $f(\theta)=\cot\theta$ and the purple graph is the graph of $f(\theta)=\frac{1}{2}\cot\theta$. As you can see the purple graph is vertically compressed by $\frac{1}{2}$

Domain: $\{\theta|\theta\neq n\pi\}$ (all real numbers except all the multiples of π)

Range: $(-\infty,\infty)$

Period: π

Vocabulary

Reciprocal Trigonometric Functions

They are the functions cosecant, secant, and cotangent and they are the reciprocals of sine, cosine, and tangent respectively.

Cosecant function

The cosecant function is defined by $\csc\theta=\frac{1}{\sin\theta}$

Secant function

The secant function is defined by $\sec\theta=\frac{1}{\cos\theta}$

Cotangent function

The cotangent function is defined by $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

In Summary

We learned what the graphs of cosecant, secant and cotangent are. We learned about their domain and range and how to graph them when they are vertically stretched or compressed.

Practice

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6.3 Transformations of Trigonometric Functions Horizontal Stretch and Compression

TEKS

Lesson Objectives

In this lesson you will learn about:
1. Transforming trigonometric functions by horizontal stretching and compressions.

Introduction

It would be nice if in real life everything would be whole numbers, or if the trigonometric equations would always repeat every 2π. In this section you will learn about transforming the trigonometric functions by compressions.

Vocabulary

Period, horizontal stretch and compress.

As mentioned in the previous chapter, the form of the trigonometric functions is $f(\theta) = A\sin(B\theta + C) + D$, in this section we will analyze what the value of **B** does to the graph of the trigonometric functions.

The period of the functions $\sin\theta$ *and* $\cos\theta$ is 2π and the period of the tangent function is π, this is because the value of the variable B is 1, but what happens if the value of $B \neq 1$.

If the value of $B \neq 1$ then the period of the functions is found by the following formulas.

For sine and cosine: **Period =** $\frac{2\pi}{B}$

For tangent : **Period =** $\frac{\pi}{B}$

Note: The value of **B** tells us how many sine or cosine waves fit in 2π and for tangent in π. Also the wave will compress or stretch by a factor of $\frac{1}{B}$

Example 1

Graph the function $f(\theta) = \sin 2\theta$ in the interval $[0, 2\pi]$.

If you notice the value of **B = 2**. This means that there will be two complete waves in a distance of 2π. Therefore the new period will be $period = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.

Here is the graph of $f(\theta) = \sin 2\theta$

FIGURE 6.14

The black curve is the graph of the parent sine function while the red graph is the graph of the transformed function. Notice that there is exactly two complete red waves in the interval $[0,2\pi]$

To obtain the coordinates of the new points on a table you take the points of the parent function and divide the x-coordinate by 2. The following table shows the coordinates of one complete cycle for both functions.

TABLE 6.1:

$f(\theta)=\sin\theta$	$(0,0)$	$\left(\frac{\pi}{2},1\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-1\right)$	$(2\pi,0)$
$f(\theta)=\sin 2\theta$	$(0,0)$	$\left(\frac{\pi}{4},1\right)$	$\left(\frac{\pi}{2},0\right)$	$\left(\frac{3\pi}{4},-1\right)$	$(\pi,0)$

Domain: $(-\infty,\infty)$

Range: $[-1,1]$

Example 2

Graph the function $f(\theta)=\cos 6\theta$ in the interval $[0,2\pi]$.

If you notice the value of **B = 6**. This means that there will be **six** complete waves in a distance of 2π .Therefore the new period will be $period=\frac{2\pi}{B}=\frac{2\pi}{6}=\frac{\pi}{3}$.

FIGURE 6.15

The black curve is the graph of the parent cosine function while the blue graph is the graph of the transformed function. Notice that there is exactly six complete waves on the interval $[0,2\pi]$

To obtain the coordinates of the new points on a table you take the points of the parent function and divide the x-coordinate by 6. The following table shows the coordinates of one complete cycle for both functions.

TABLE 6.2:

$f(\theta)=\cos\theta$	$(0,1)$	$\left(\frac{\pi}{2},0\right)$	$(\pi,-1)$	$\left(\frac{3\pi}{2},0\right)$	$(2\pi,1)$
$f(\theta)=\cos 6\theta$	$(0,1)$	$\left(\frac{\pi}{12},0\right)$	$\left(\frac{\pi}{6},-1\right)$	$\left(\frac{\pi}{4},0\right)$	$\left(\frac{\pi}{3},1\right)$

Domain: $(-\infty,\infty)$

Range: $[-1,1]$

Example 3

Graph the function $f(\theta)=\tan\frac{1}{2}\theta$ in the interval $[-\pi,\pi]$.

If you notice the value of B = $\frac{1}{2}$. This means that there will be **half** complete waves in a distance of π .Therefore the new period will be $period=\frac{\pi}{B}=\frac{\pi}{\frac{1}{2}}=2\pi$.

FIGURE 6.16

The black graph is the graph of the parent tangent function while the purple graph is the graph of the transformed tangent function.

Notice that the purple graph stretched horizontally that is because the value of B was between zero and one. The blue dotted lines are the new asymptotes.

To obtain the coordinates of the new points from a table you need to take the points from the parent tangent function and multiply the x-coordinate by two. The following table shows the coordinates of one complete cycle.

TABLE 6.3:

$f(\theta) = \tan\theta$	$\left(-\frac{\pi}{2}, undefined\right)$	$\left(-\frac{\pi}{4}, -1\right)$	$(0,0)$	$\left(\frac{\pi}{4}, 1\right)$	$\left(\frac{\pi}{2}, undefined\right)$
$f(\theta) = \tan\frac{1}{2}\theta$	$(-\pi, undefined)$	$\left(-\frac{\pi}{2}, -1\right)$	$(0,0)$	$\left(\frac{\pi}{2}, 1\right)$	$(\pi, undefined)$

Domain: $\{\theta \mid \theta \neq \pi \pm 2\pi\}$

Range:$(-\infty, \infty)$

Example 4

Graph the function $f(\theta) = \sin\frac{1}{2}\theta$ in the interval $[0, 4\pi]$.

If you notice the value of **B =**$\frac{1}{2}$. This means that there will be **half** complete waves in a distance of 2π .Therefore the new period will be $period = \frac{2\pi}{B} = \frac{2\pi}{\frac{1}{2}} = 4\pi$.

FIGURE 6.17

The black curve is the graph of the parent sine function while the red graph is the graph of the transformed sine function. Notice in the interval $[0, 2\pi]$ there is only half a wave.

To obtain the coordinates of the new function in a table you take the coordinates of the parent sine function and multiply the x-coordinate by 2.

TABLE 6.4:

$f(\theta) = \sin\theta$	$(0,0)$	$\left(\frac{\pi}{2}, 1\right)$	$(\pi, 0)$	$\left(\frac{3\pi}{2}, -1\right)$	$(2\pi, 0)$
$f(\theta) = \sin\frac{1}{2}\theta$	$(0,0)$	$(\pi, 1)$	$(2\pi, 0)$	$(3\pi, -1)$	$(4\pi, 0)$

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Vocabulary

Period

It is the distance required for a function to complete one complete cycle.

Horizontal Stretch/Compress

A horizontal stretch or compress occurs when the coefficient of θ , the value of $B \neq 1$

In Summary

In this section we have learned how to compress or stretch a trigonometric function horizontally. We have learned that the value of B tells us how many waves in 2π for sine and cosine and for tangent in π

Practice

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6.4 Transformations 2: Vertical Shift, Horizontal Shift (Phase Shift)

TEKS

Lesson Objectives

In this lesson you will learn about:
1. The vertical shift of trigonometric functions.
2. The horizontal (phase shift) of trigonometric functions.

Introduction

As we have seen in the previous sections, the graphs of trigonometric functions can be vertically stretched or compressed, horizontally stretched or compressed, in this section we will learn about horizontal and vertical translations of the trigonometric functions.

Vocabulary

Vertical shift, Horizontal(phase shift)

As we have mentioned before the general form of a trigonometric function is $f(\theta) = A\sin(B\theta + C) + D$. In this chapter we will learn about the values of **C** and **D** and how it affects the graph of the trigonometric functions.

The easiest one to understand is the value of **D**. This value is the vertical translation and the graph of the function.

Vertical Translation

- If $D \geq 0$ then the graph shifts up **D** units.
- If $D \leq 0$ then the graph shifts down **D** units.

Example 1

Graph the function $f(\theta) = \cos\theta + 3$ in the interval $[0, 2\pi]$

FIGURE 6.18

The black graph is the graph is the graph of the parent cosine function while the blue graph is the graph of the transformed function.

To obtain the coordinates of the new function, we need to get the coordinates of the parent function and add 3 to the y coordinate.

TABLE 6.5:

$f(\theta) = \cos\theta$	$(0,1)$	$\left(\frac{\pi}{2},0\right)$	$(\pi,-1)$	$\left(\frac{3\pi}{2},0\right)$	$(2\pi,1)$
$f(\theta) = \cos\theta+3$	$(0,4)$	$\left(\frac{\pi}{2},3\right)$	$(\pi,2)$	$\left(\frac{3\pi}{2},3\right)$	$(2\pi,4)$

Domain: $(-\infty,\infty)$

Range: $(2,4)$

Example 2

Graph the function $f(\theta) = \sin\theta - 2$ in the interval $[0,2\pi]$

FIGURE 6.19

The black graph is the graph is the graph of the parent sine function while the red graph is the graph of the transformed function.

To obtain the coordinates of the new function, we need to get the coordinates of the parent function and subtract two to the y coordinate.

TABLE 6.6:

$f(\theta) = \sin\theta$	$(0,0)$	$\left(\frac{\pi}{2},1\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-1\right)$	$(2\pi,0)$
$f(\theta) = \sin\theta-2$	$(0,-2)$	$\left(\frac{\pi}{2},-1\right)$	$(\pi,-2)$	$\left(\frac{3\pi}{2},-3\right)$	$(2\pi,-2)$

Domain: $(-\infty,\infty)$

Range: $[-3,1]$

Horizontal Shift (phase shift)

The horizontal shift it a bit more complicated to understand. We need to take into account the value of **B**.

- The graph of the curve shifts opposite to the sign of the variable **C**,
- If the value of $B = 1$ then the horizontal shift will be **C** units.
- If the value of $B \neq 1$ then the horizontal shift will be $\frac{C}{B}$ units and the period will change to $\frac{2\pi}{B}$.

Note: If there is no parenthesis next to the trigonometric function, then there will be no horizontal shift.

Example 3

Graph the function $f(x) = \sin\left(\theta - \frac{\pi}{2}\right)$ in the interval $[0,2\pi]$

Since the value of $B = 1$ there will be no horizontal compression and the sign of **C** is **negative** it will shift to the right $\frac{\pi}{2}$ units.

FIGURE 6.20

The black graph is the graph is the graph of the parent sine function while the red graph is the graph of the transformed function.

To obtain the coordinates of the new function, we need to get the coordinates of the parent function and add $\frac{\pi}{2}$ to the x coordinate.

TABLE 6.7:

$f(\theta) = \sin\theta$	$(0,0)$	$\left(\frac{\pi}{2},1\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-1\right)$	$(2\pi,0)$
$f(\theta) = \sin\left(\theta - \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2},0\right)$	$(\pi,1)$	$\left(\frac{3\pi}{2},0\right)$	$(2\pi,-1)$	$\left(\frac{5\pi}{2},0\right)$

Domain: $(-\infty,\infty)$

Range: $[-1,1]$

Example 4

Graph the function $f(\theta) = \sin(6\theta+\pi)$ in the interval $\left[-\frac{\pi}{2},2\pi\right]$

Since the value of $B \neq 1$ then there will be a horizontal compression, and since the sign of **C** is **positive** then the function will shift to the left $\frac{\pi}{6}$ units and the period will be $\frac{2\pi}{6} = \frac{\pi}{3}$.

FIGURE 6.21

The black graph is the graph is the graph of the parent sine function while the red graph is the graph of the transformed function. The point A represents the point (0,0) after the transformation.

To obtain the coordinates of the new function, we need to get the coordinates of the parent function and divide the x-coordinate by 6 then subtract $\frac{\pi}{6}$units to the x-coordinate.

TABLE 6.8:

$f(x) = \sin\theta$	$(0,0)$	$\left(\frac{\pi}{2},1\right)$	$(\pi,0)$	$\left(\frac{3\pi}{2},-1\right)$	$(2\pi,0)$
$f(\theta) = \sin(6\theta+\pi)$	$\left(-\frac{\pi}{6},0\right)$	$\left(-\frac{\pi}{12},1\right)$	$(0,0)$	$\left(\frac{\pi}{12},-1\right)$	$\left(\frac{\pi}{6},0\right)$

Vocabulary

Horizontal Shift (Phase Shift)

The shift to the left or to the right from the y-axis of a trigonometric function.

In Summary

We have learned that the value of **D** shifts the graph of the trigonometric function up or down from the x-axis. Also we have learned that the value of **C** shifts the graph left or right opposite to its sign and if the value of $B \neq 1$ then the amount of shift will be $\frac{C}{B}$.

Practice

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6.5 Tranformations 3: Stretching and Shifting (Putting Them Together)

TEKS

Introduction

In this lesson you will learn about:
1. Graphing trigonometric functions with stretches and shifting together.

Introduction

Graphing trigonometric functions that include both vertical and horizontal compressions or stretching and horizontal and vertical shifting can be a difficult task. In this section you will learn some techniques to graph these kinds of transformations.

Vocabulary

Amplitude, Period, Vertical shift, Horizontal shift

It is easier to graph functions that either only have compressions or stretches or shifts, when dealing with trigonometric functions that have multiple transformations, then we should find all the information about the transformations. These are the Amplitude, Period, Horizontal Shift and Vertical Shift.

To graph multiple transformations follow the following steps.

- Obtain all the information from the equation like the Amplitude, Period, Horizontal Shift and Vertical Shift.
- Perform the Horizontal and Vertical shifts at the same time. (this means you are moving the origin to that new point.
- Use that new point as your origin, then stretch the graph by the vertical stretch and use the period to graph one cycle of the graph.

Here is how to find the information given the function in the form

$$f(\theta) = A\sin(B\theta + C) + D$$

Amplitude = $|A|$, **Period =** $\frac{2\pi}{B}$, **Horizontal shift =** $\frac{C}{B}$, **Vertical shift =** D

Example 1.

Graph the function $f(\theta) = -4\sin(3\theta - \pi) + 5$

First lets find all the important information.

Amplitude = $|-4| = 2$, **Period =** $\frac{2\pi}{3}$, **Horizontal shift =** $\frac{\pi}{3}$ right, **Vertical shift =** $5 \ \ up$,

First start by shifting the origin point to point A by performing the horizontal and vertical shifts and use this as your new origin.

Then use the period and the value of A to amplify and compress and graph one complete cycle. Since A is negative it starts opening down from point A.

Then continue the pattern. Then graph the rest of the cycles as shown in the graph.

FIGURE 6.22

The black graph is the graph of the sine function and the red graph is the graph of the transformed function.

Domain: $(-\infty, \infty)$

Range: $[1, 9]$

Example 2

Graph the function $f(\theta) = 3\cos(2\theta + 2\pi) - 4$

Find the important information:

Amplitude = $|3| = 3$, **Period =** $\frac{2\pi}{2} = \pi$, **Horizontal shift =** $\frac{2\pi}{2} = \pi \ \ left$, **Vertical shift =** $4 \ \ down$

Start by performing the translations first so we shift the origin units to the left and four down. It is represented by point A in the graph.

Then use the Amplitude and the Period to graph one cycle with respect to point A and continue pattern.

FIGURE 6.23

The black graph is the graph of the parent cosine function while the blue graph is the graph of the transformed function.

Domain: $(-\infty, \infty)$

Range: $[-7, -1]$

Example 3

Graph the function $f(\theta) = 3\sec(2\theta + 2\pi) - 4$

This example is the same as example 2, but we are using the secant function instead. So to graph this we will need to graph the cosine function exactly as in example 2, then we need to take out all the points that intersect the line that represents the new x-axis and reflect the graph as follows.

FIGURE 6.24

The black curve is the parent cosine function, the blue dotted curve is the graph of the transformed cosine function and the blue dotted lines are the lines of the asymptotes and the orange curve is the graph of the transformed secant function.

Domain: $\left\{\theta \mid \theta \neq \frac{\pi}{4} \pm \frac{\pi}{2} \cdot n\right\}$

Range: $(-\infty, -7] \bigcup [-1, \infty)$

Example 4

Graph the function $f(\theta) = \frac{1}{2} \tan\left(2\theta - \frac{\pi}{3}\right) + 4$

Find the important information first

Amplitude = N/A, **Period =** $\frac{\pi}{2}$, **Horizontal shift =** $\frac{\pi}{6}$ *right*, **Vertical shift =** 4 *up* ,

FIGURE 6.25

The black curve is the graph of the parent tangent function, the purple curve is the graph of the transformed tangent function and the blue dotted lines are the asymptotes of the transformed function.

Domain = $\left\{\theta \mid \theta \neq \frac{\pi}{4} \pm \frac{\pi}{2} \cdot n\right\}$

Range = $(-\infty, \infty)$

Vocabulary

Amplitude

Is the height of the mean or rest value of the function to its maximum or minimum

Period

The distance required for a function to complete one full cycle.

Horizontal Shift

A left or right translation of a function sometimes called the phase shift.

Vertical Shift

A translation up or down of a function.

In Summary

We have learned about graphing trigonometric functions that include horizontal compressions or stretches, vertical compressions or stretches, horizontal shifts and vertical shifts.

Practice

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6.6 Graphs of arcsin, arccos, and arctan (Inverse Trigonometric Functions)

TEKS

P.2.H

Lesson Objectives

In this section you will learn about:
1. Inverse trigonometric functions
2. The graph of arcsin, arccos, arctan ($sin^{-1}\theta$, $cos^{-1}\theta$, $tan^{-1}\theta$)
3. Find the values of composite trigonometric functions.

Introduction

In chapter 5 we learned about solving simple trigonometric equations by using the inverse functions with the calculator or with the unit circle, but how do the graphs of the inverse trigonometric functions look? Or why does the law of sines has an ambiguous case? In this section you will learn about the graphs of inverse trigonometric functions and the properties of the inverse trigonometric functions.

Vocabulary

arcsin, arccos, arctan, domain, range, composite trigonometric functions.

From earlier in our study of Pre-calculus we know that the inverse of a function is obtained by switching x's and y's and solving for y, also we learned that the inverse of a function is a reflection over the line y = x so lets analyze what happens to the inverse of the function $y = \sin\theta$ by being reflected over the line y = x where x is represented by θ

FIGURE 6.26

The blue graph is the graph of $y = \sin\theta$ while the red graph is the reflected graph over the line y = x, obviously the red graph is not a function because it is not a one to one function and it doesn't pass the vertical line test, therefore we need to restrict the domain.

The Graph of $y = \sin^{-1}\theta$

The blue graph is the graph of the $y = \sin\theta$ on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ while the red graph represents the graph of $y = \sin^{-1}\theta$ obtained by performing the following process:

$= \sin\theta\theta$ $= s$

By restricting the domain of the function $y = \sin\theta$ from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$the function was one to one and will has an inverse that is a function as shown in the following graphs.

FIGURE 6.27

For the graph of $y = \sin\theta$ then with restricted domain

Domain = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range = $[-1, 1]$

For the graph of $y = \sin^{-1}\theta$ then

Domain = $[-1, 1]$

Range =$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Since the domain of the inverse sine function is $[-1, 1]$ when you are solving for the angle the number you are taking the inverse of must be between -1 and 1. That is why if you take the inverse of a number that is not between -1 and 1 you will get an error message because its not part of the domain. The answer will be between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or [-1.57,1.57] radians which in degrees will be [-90,90] therefore it does not distinguish for an obtuse angle creating the ambiguity in the law of sines.

The Graph of $y = \cos^{-1}\theta$

Similarly as with the sine function we will also need to restrict the domain for the cosine function. For the cosine function we will restrict the domain to $[0, \pi]$ leaving us with a range of $[-1, 1]$

FIGURE 6.28

Similarly as with the sine function the function $y = \cos^{-1}\theta$ was obtained with the following process

$= \cos\theta\theta$ $= c$

For the graph of $y = \cos\theta$ with restricted domain

Domain = $[0, \pi]$

Range = $[-1, 1]$

For the graph of $y = \cos^{-1}\theta$

Domain = $[-1, 1]$

Range = $[0, \pi]$

Since we have a range of $[0, \pi]$ then the inverse cosine function distinguishes between $[0, \pi]$ which is in degrees between [0,180] therefore in the law of cosines we do not get the ambiguity.

The Graph of $y = \tan^{-1}\theta$

For the tangent function and its inverse we will be looking at the cycle with Domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and having a range of $[-\infty, \infty]$. By reflecting the function over the line y = x then we obtain the inverse tangent function. We switch θ and y and solve for y similar like the sine example.

FIGURE 6.29

The slanted dotted line is the line $y = x$ and the vertical dotted lines are the asymptotes of the tangent function at $x = \pm\frac{\pi}{2}$ and the horizontal dotted lines are the asymptotes of the inverse function at $y = \pm\frac{\pi}{2}$

Similarly as the inverse functions of sine and cosine, the $y = \tan^{-1}\theta$ function was obtained with the following process

$$= \tan\theta\theta \qquad = t$$

For $y = \tan\theta$ with restricted domain

Domain = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range = $(-\infty, \infty)$

For $y = \tan^{-1}\theta$

Domain = $(-\infty, \infty)$

Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Since the range of the tangent function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then our values will be between $[-90, 90]$ degrees.

PROPERTIES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

For the sine function and its inverse

$$\sin\left(\sin^{-1}\theta\right) = \theta \quad in\ the\ interval\ \ [-1,1]$$
$$\sin^{-1}(\sin\theta) = \theta \quad in\ the\ interval\ \ \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

For the cosine function and its inverse

$$\cos\left(\cos^{-1}\theta\right) = \theta \quad in\ the\ interval\ \ [-1,1]$$
$$\cos^{-1}(\cos\theta) = \theta \quad in\ the\ interval\ \ [0,\pi]$$

For the tangent function and its inverse

$$\tan\left(\tan^{-1}\theta\right) = \theta \quad in\ the\ interval\ \ [-\infty,\infty]$$
$$\tan^{-1}(\tan\theta) = \theta \quad in\ the\ interval\ \ \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Example 1

Find the exact values of

$$a)\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad b)\ \cos^{-1}\left(-\frac{1}{2}\right) \qquad c)\ \tan^{-1}\left(\sqrt{3}\right)$$

Solution

Lets look at the part a.

When you see a trigonometric expression involving an inverse what is actually asking is at what angle is the value of the function that value. You need to translate it as follows.

$$= \theta\frac{\sqrt{3}}{2} \qquad = \mathrm{si}$$

So at what angle is the value of the sine function equal to $\frac{\sqrt{3}}{2}$ therefore the value of the expression will be at $\frac{\pi}{3}$ radians or 60^o

Part b

Similarly like part a, the expression is asking at what angle is the value of the cosine function $\frac{1}{2}$ therefore we will solve it as follows

$$= \theta - \frac{1}{2}$$

$= c$

Therefore at what angle is the value of the cosine function $-\frac{1}{2}$? The solution is at $\frac{2\pi}{3}$ *or* 120^o

Part c

Similarly like part a and b we need to translate the equation as follows.

$$= \theta\sqrt{3}$$

$= ta$

Therefore at what angle is the value of the tangent function $\sqrt{3}$? The answer is at $\frac{\pi}{3}$ *or* 60^o because by recalling that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ that is the angle where the value of $\sin\theta = \frac{\sqrt{3}}{2}$ *and* $\cos\theta = \frac{1}{2}$

Evaluating Composite Trigonometric Expressions

A composite trigonometric expression is when you have two trigonometric functions in on a single term for example $\cos(tan^{-1}\theta)$ is a composite expression.

To evaluate composite trigonometric expressions you need to evaluate the inside expression first, sometimes you will need to create a right triangle if we have mixed expressions to evaluate it. First we will solve some with the inverse properties then we will do it with a right triangle.

Example 2

Evaluate

$a) \sin^{-1}\left(\sin\frac{5\pi}{2}\right)$ $\quad b)\ \cos\left(cos^{-1} - 1.7\right)$ $\quad c)\ \cos\left(\cos^{-1}0.8\right)$

Solution

For part a

The inverse property only applies for every x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ therefore since $\frac{5\pi}{2}$is not on the domain, we need to evaluate the inside, therefore $\sin\left(\frac{5\pi}{2}\right) = 1$

Now we have the expression $\sin^{-1}(1)$ so at what value is the $\sin\theta = 1$ therefore our values is $\frac{\pi}{2}$.

For part b

The inverse property only applies for every x in the interval $[-1, 1]$ therefore since -1.7 is not in this interval the value of $\cos^{-1}(-1.7)$ is undefined so the value of the whole expression is undefined.

For part c

The inverse property only applies for every x in the interval $[-1,1]$ therefore since 0.8 is in this interval, the value of the expression is 0.8

To evaluate composite expressions with different trigonometric functions we need to create a right triangle, find all three sides of the right triangle and then evaluate the expression as in the following example.

Example 3

Find the value of $\cos\left(\sin^{-1}\frac{3}{5}\right)$

Solution

To obtain the value of the expression we need to solve the inside by creating a right triangle. So the inside expression is asking us at what angle is the value of the sine function $\frac{3}{5}$

Recall that $\sin\theta = \frac{opposite}{hypotenuse}$ therefore by creating the right triangle and finding the third side using Pythagorean theorem we obtain that the third side is 4. The triangle will be in the first quadrant because both are positive as shown in the following figure.

FIGURE 6.30

Now this right triangle represents your inside function therefore what is the value of $\cos(\triangle)$ and the value is $\frac{4}{5}$ since $\cos(\theta) = \frac{adjacent}{hypotenuse}$

Therefore $\cos\left(\sin^{-1}\frac{3}{5}\right) = \frac{4}{5}$

NOTE: When we are doing composite trigonometric functions remember that the hypotenuse is always positive.

Example 4.

Find the value of the expression $\sin\left(\tan^{-1}\left(-\frac{5}{12}\right)\right)$

Solution

First we need to solve the inside of the expression, since the $\tan^{-1}$function has a domain of$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$and knowing that $\tan\theta = \frac{opposite}{adjacent}$ then the angle will be on the fourth quadrant because the opposite will be -5 and the adjacent will be 12. Therefore by using Pythagorean theorem we conclude that the length of the third side will be 13 as shown in the next figure.

FIGURE 6.31

Now this triangle represents the inside of your expression therefore $\sin(\triangle) = \frac{-5}{13}$since the definition of $\sin\theta = \frac{opposite}{hypotenuse}$.

Therefore the value of the expression $\sin\left(\tan^{-1}\left(-\frac{5}{12}\right)\right) = \frac{-5}{13}$.

Example 5

Find the value of the expression $\sec\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)$

Solution

Similarly as the previous examples, this triangle will be in the second quadrant because the definition of $\cos\theta =$

$\frac{adjacent}{hypotenuse}$ therefore the value of the adjacent side will be -1 and the hypotenuse will be three. By performing Pythagorean theorem we obtain that the length of the third side will be $\sqrt{8} = 2\sqrt{2} \approx 2.82$ as shown in the next figure.

FIGURE 6.32

Now this triangle represents the inside of the expression therefore $\sec(\triangle) = \frac{hypotenuse}{adjacent} = \frac{3}{-1} = -\frac{3}{1} = -3$.

Therefore the value of the expression $\sec\left(\cos^{-1}\left(-\frac{1}{3}\right)\right) = -3$

Vocabulary

arcsin, arccos, arctan

They are the inverse trigonometric functions represented by $\sin^{-1}\theta, \ \cos^{-1}\theta, \ \tan^{-1}\theta$

Composite Trigonometric Expression

A mathematical expression involving two or more trigonometric functions in a composition.

In summary

We have learned about the inverse trigonometric functions and its properties. We have also learned how to evaluate inverse trigonometric functions and how to evaluate composite inverse functions.

Practice

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6.7 Equations of Trigonometric Graphs

TEKS

Lesson Objectives

In this lesson you will learn about:
1. Identifying the values of A, B, C, and D from a graph.
2. To write a trigonometric equation from a graph.

Introduction

As we mentioned before real life situations can be modeled by a trigonometric function, but what if we just have the graph and not the equation itself? In this section you will learn how to write a trigonometric equation given the graph.

Vocabulary

Amplitude, Period, Horizontal shift, Vertical Shift

To obtain the equation of a sine and cosine function we need to identify the Amplitude, Period, Horizontal shift and Vertical shift from the graph itself. Once you have identified these, you will need to relate them as A,B,C,D and then write the equation.

Follow these procedures

$A = \frac{max - min}{2}$

$B = \frac{2\pi}{Period}$ for sine and cosine and $B = \frac{\pi}{period}$ for tangent.

$D = max - amplitude$ Shift the x-axis(θ-axis) this much up or down.

$\frac{C}{B}$ is the value of the horizontal shift and it depends if the function is sine, cosine.

- For $\sin\theta$ its the distance from the y-axis to a point on the new axis.
- For $\cos\theta$ its the distance from the y-axis to a high or low point.

Then write the formula with B outside the parenthesis first and then distribute it inside. Write it as $f(\theta) = A\sin B(\theta + \frac{C}{B}) + D$ then when you distribute the value of B you will obtain $f(\theta) = A\sin(B\theta + C) + D$

Example 1

Given the following graph, write a sine equation and a cosine equation that matches the graph.

FIGURE 6.33

To find the value of $A = \frac{12-2}{2} = \frac{10}{2} = 5$

The period is from high point to high point will be $\frac{2\pi}{3}$therefore the value of $B = \frac{2\pi}{\frac{2\pi}{3}} = \frac{6\pi}{2\pi} = 3$

The vertical shift will be the value of $D = max \quad - \quad amplitude = 12 - 5 = 7 \ \ up$

Since the new axis represented by the dotted line goes through the curve at the y-axis then the value of $\sin\theta$ the value of $\frac{C}{B} = 0$.

For the function $\cos\theta$ we will choose the distance from the y-axis to the top right point therefore the value of $\frac{C}{B} = \frac{\pi}{6} \ \ right$

Finally both equations will be as follows:

$$amp; f(\theta) = 5\sin 3(\theta + 0) + 7 = 5\sin 3\theta + 7$$

Note: There are many different equations that can be a solution for example the following are also solutions

$$amp; f(\theta) = -5\sin 3(\theta - \frac{\pi}{3}) + 7 = -5\sin(3\theta - \pi) + 7$$

The negatives on the amplitude are because in sine after the shifts the graph is going down and for cosine because it is a low point.

Example 2.

Given the following graph write a sine and cosine equation that matches the following graph

FIGURE 6.34

The value of $A = \frac{2--6}{2} = \frac{8}{2} = 4$

The period will be from high point to high point will be π therefore the value of $B = \frac{2\pi}{\pi} = 2$

The vertical shift will $D = 2 - 4 = -2$

The horizontal shift for a sine equation $\frac{C}{B} = \frac{\pi}{12}$ $left$

The horizontal shift for a cos equation $\frac{C}{B} = \frac{\pi}{6}$ *right*

Finally both equations will be as follows

$$amp; f(\theta) = -4\sin 2(\theta + \frac{\pi}{12}) - 2 = -4\sin(2\theta + \frac{\pi}{6}) - 2$$

By now you should notice that the values of A, B, D are the same, what changes is the vertical shift.

Example 3

Given the following graph write a sine and cosine equation that matches the graph.

FIGURE 6.35

The value of $A = \frac{3-2}{2} = \frac{1}{2}$

The period will be from high point to high point which is $\frac{\pi}{3}$ therefore the value of $B = \frac{2\pi}{\frac{\pi}{3}} = \frac{6\pi}{\pi} = 6$

The vertical shift will be $D = 3 - \frac{1}{2} = \frac{5}{2} = 2.5$

The horizontal shift for a sine equation $\frac{C}{B} = \frac{\pi}{12}$ $left$

The horizontal shift for a cos equation $\frac{C}{B} = 0$ (already at high or low point)

Finally both equation will be as follows

$$amp; f(\theta) = \frac{1}{2}\sin 6(\theta + \frac{\pi}{12}) + 2.5 = \frac{1}{2}\sin(6\theta + \frac{\pi}{2}) + 2.5$$

Example 4

Given the following graph write a sine and cosine equation that matches the graph.

FIGURE 6.36

The value of $A = \frac{10--4}{2} = \frac{14}{2} = 7$

The period will be from high point to high point which is 3π therefore the value of $\frac{2\pi}{3\pi} = \frac{2}{3}$

The vertical shift will be$D = 10-7 = 3$

The horizontal shift for a sine equation will be $\frac{\pi}{4}\ \ right$

The vertical shift for a cosine equation will be $\pi\ \ right$

Finally both equations will be as follows

$$amp; f(\theta) = 7\sin\frac{2}{3}(\theta - \frac{\pi}{4}) + 3 = 7\sin(\frac{2}{3}\theta - \frac{\pi}{6}) + 3$$

Obtaining an equation for tangent

For tangent, it is a little bit different, the vertical stretch is not called an amplitude, to find the value of A you need to find the half of the distance between the center point and the asymptote and see what you need to multiply by to get the value.

To find the period you need to find the distance from asymptote to asymptote and the value of $B = \frac{\pi}{period}$.

The horizontal translation is the move from the middle point to the left or to the right its the value of C.

The vertical translation is the mover from the middle point up or down the value of D.

Example 5

Given the following graph, write a tangent equation that matches the graph.

FIGURE 6.37

For the tangent line the difficult part is to obtain the value of A, therefore we will find the other three values first.

The period is 3π from asymptote to asymptote therefore the value of $B = \frac{\pi}{3\pi} = \frac{1}{3}$

The horizontal shift value $\frac{C}{B} = \frac{\pi}{2}\ \ right$

The vertical shift value $D = 4\ \ down$

To find the value of A we need to find half the distance of the middle point to the asymptote and see what factor we need to multiply by to get that point. In this case from the middle point we go 4.5 squares in the grid to the right and the value is 2 units up from the dotted line, therefore the value of $A = 2$

Finally the equation that will match this graph will be

$$amp; f(\theta) = 2\tan\frac{1}{3}(\theta - \frac{\pi}{2}) - 4$$

Vocabulary

Amplitude

Is the height from the mean, or rest value of the function to its maximum or minimum.

Period

Is the distance required for a function to complete one full cycle.

Horizontal Shift

A left or right translation of a function sometimes called phase shift.

Vertical Shift

A translation up and down of a function

In Summary

We have learned to obtain a sine, cosine, and tangent equation given the graph of a function by finding the values of A,B,C,D.

Practice

#########

6.8 Applications of Trigonometric Functions (Harmonic-Sinusoidal Motion)

TEKS

Lesson Objectives

In this lesson you will learn about:
1. Creating an equation for sinusoidal situations.

Introduction

Many situations in real life can be modeled by using a sinusoidal wave, for example average temperature of a certain city through the year from year to year can be approximated by a sinusoidal equation. In this section you will learn how to create a sinusoidal equation and apply it.

Vocabulary

Harmonic motion, sinusoidal waves

Given real life situations we must conclude from the problem if it is harmonic motion problem. If it is then we need to obtain the information from the problem itself to find the values of A,B,C,D to create a sinusoidal equation and finally answer the questions using our created equation.

Example 1.

A weight is at rest hanging from a spring. Then the weight is pulled down 8 cm. and released. The weight oscillates up and down completing one cycle every 3 seconds.

a. Find an equation that models the height of the weight away from the rest point at *t* seconds?
b. What is the height of the spring at 2 seconds?
c. Graph the function.
d. Use the equation to find the value of f(t) = 5 for the first time. What does 5 represent?

Solution

The value of $A = \frac{8--8}{2} = \frac{16}{2} = 8$

Since it takes 3 seconds for one complete cycle then the period is 3 seconds then the value of $B = \frac{2\pi}{3}$.

At **t = 0** seconds we are pulling it down **8 cm**. So if we use a cosine equation there will be no horizontal shift or vertical shift therefore the value of $\frac{C}{B}$ *and* $D = 0$

a) Therefore the function that will give us the height of the weight with respect to time is

$$f(t) = -8\cos\left(\frac{2\pi}{3}t\right)$$

NOTE: The value of A is negative because we are starting at a low point.

b) The height of the spring at two seconds will be:

$$f(2) = -8\cos\left(\frac{2\pi}{3}(2)\right) = 4.01 \quad cm$$

c) The graph that models the height of the weight is

FIGURE 6.38

d) To find when the height of the weight will be 5 cm. we will use:

$$= -8\cos\left(\frac{2\pi}{3}t\right)\frac{5}{-8}$$

The 5 represents a height upward of 5 cm. from the rest point at 1.07 seconds.

Example 2

A pendulum on an old clock swings back and forth to a distance of 3ft. away from the center. It takes one second for the pendulum to get to its farthest point to the right. Assuming that the distance to the right is positive and to the left is negative and the pendulum is starting from the center to the right at t=0 answer the following.

a. Write an equation of that models the distance away from the center of the pendulum with respect to time.
b. Where is the pendulum at 2.5 seconds?
c. Draw a graph of the equation that models the distance away from the center.
d. When will the pendulum be 3ft. to the left for the second time?

Solution

Since the furthest distance away from the center will be 3ft. to the right and to the left therefore the value of $A = 3$.

The period of the pendulum to swing to the right from the middle then to the left and back to the middle will be 4 seconds therefore the value of $B = \frac{2\pi}{4} = \frac{\pi}{2}$ and since the distance at **t = 0** equals **0** then we will choose a sine function.

At **t = 0** the distance to away from the center is **0** and moving right, therefore the value of $\frac{C}{B}$ *and* $D = 0$

a) The function that will model the distance in feet of the pendulum away from the center at t seconds will be

$$f(t) = 3\sin\left(\frac{\pi}{2}t\right)$$

b) To find where the pendulum is at 2.5 second we will use

$$f(2.5) = 3\sin\left(\frac{\pi}{2}(2.5)\right) = -2.12 \ \ ft.$$

This means that it is 2.12 ft. from to the left of the center.

c) The graph that models the distance away from the center after t seconds is

FIGURE 6.39

d) Using the graph the pendulum will be 3ft. to the left for the second time is at 7 seconds.

Example 3

A Ferris wheel that has a diameter of 50ft. makes one full revolution every 2 minutes. When a person gets on the seat, they are 4 feet above the ground.

a. Write a sinusoidal function that gives the height of the person with respect to time (in minutes).
b. What will be the height of the person after 30 seconds and after 45 seconds?
c. Graph the function.
d. When will the person will be 35 ft. above the ground for the first time?

Solution

To find the equation we need to take into consideration the following. The diameter of the Ferris wheel is 50 therefore the lowest position will be 4ft and the highest position is 54 ft therefore $A = \frac{54-4}{2} = \frac{50}{2} = 25$.

The period is 2 min. therefore the value of $B = \frac{2\pi}{2} = \pi$.

Since we are talking about the height the value of $\frac{C}{B} = 0$

The value of $D = 54 - 25 = 29$

a) The function that will give us the height of the person in the Ferris wheel with respect of time in minutes is

$$f(t) = -25\cos(\pi t) + 29$$

We are choosing cosine because its a low point and the value of A will be negative

b) The height of the person after 30 seconds and at 45 seconds will be

$$f(.5) = -25\cos(\pi(.5)) + 29 = 29 \ \ ft.$$
$$f(.75) = -25\cos(\pi(.75)) + 29 = 46.67 \ \ ft.$$

Note that we had to use .5 for 30 sec. and .75 for 45 sec. because the function was done in minutes.

c) The graph that models the height of the person will be:

FIGURE 6.40

d) To find when the height will be 35ft. for the first time we use:

$$= -25\cos(\pi t) + 296$$

The person will reach a height of 35 at about .57714 min. which is about 34.62 sec.

Example 4

The high tide and low tides in meters for the city of Cancun Mexico can be approximated by a sinusoidal wave. The following graph was taken from the page http://www.tablademareas.com/mx/quintana-roo/cancun for April 18,2015.

FIGURE 6.41

Let 12 a.m. be represented by t=0 using the time 9:00 a.m and 9:30 pm be the high tides. Using those two points find the equation of the sinusoidal wave that approximates the high tides and low tides.

The period from high tide to high tide will be 12.5 hrs. Therefore the value of $B = \frac{2\pi}{12.5}$

The max height was .2 meters and the min height was -.1 meters therefore the value of $\frac{.2 - -.1}{2} = \frac{.3}{2} = .15$

To be able to match the 9:00 a.m. height we need a horizontal shift of 9 to the right therefore $\frac{C}{B} = 9$ *right*

Finally the value of $D = .2 - .15 = .05$

The equation that approximates the height of the tide will be

$$f(t) = .15\cos\left(\frac{2\pi}{12.5}(t-9)\right) + .05$$
$$f(t) = .15\cos\left(\frac{2\pi}{12.5}t - 4.543)\right) + .05$$

FIGURE 6.42

compare the graphs, it is a very close approximation specially from 9 a.m. to 9:30 p.m. which is at 21.5 on the graph.

Example 5

The number of hours of daylight in a day can be modeled with a sinusoidal function. The longest day of the year in 2014 which is the summer solstice in the city of El Paso TX. is June 21 (day 172) with a daylight time of 14:04 hrs. while the shortest day of the year the winter solstice is December 22 day(356) with a daylight time of 9:55 hrs. Answer the following questions.

a. Find the sinusoidal function that models the number of daylight hours throughout the year by letting day 1 be January 1st.
b. Graph the function that models the situation.
c. What is the number of daylight hours on August 9? (day 221)
d. When is the number of daylight hours is equal to the number of night time hours?

The following list of numbers are the day number for the end of each month for example 31 is Jan 31 and 90 is March 31.

31,59,90,120,151,181,212,243,273,304,334,365

Solution

First we need to convert the time to decimal hours 14:04 hrs is equivalent to 14.0666 hrs and 9:55 is equivalent to 9.9166 hrs. Therefore the value of $A = \frac{14.0666-9.9166}{2} = \frac{4.15}{2} = 2.075$

From day 172 to day 356 will be half a period, therefore $356 - 172 = 184$ is half a period so the period will be $2(184) = 368$.

The period is 368 therefore the value of $B = \frac{2\pi}{368} = \frac{\pi}{184}$

Since we have a maximum point at day 172 we will choose a cosine function

The Horizontal shift will be $\frac{C}{B} = 172$ *right*

The vertical shift will be $D = 14.0666 - 2.075 = 11.9916$ *up*

a) The function that models the hours of daylight in El Paso, TX. in 2014 will be

$$amp; f(d) = 2.075\cos\left(\frac{\pi}{184}(d-172)\right) + 11.9916$$

b) The graph that models the hours of daylight in El Paso, TX in 2014.

FIGURE 6.43

c) The number of daylight hours will be on August 9 day (221) is?

$$f(221) = 2.075\cos\left(\frac{\pi}{184}(221) - \frac{43\pi}{46}\right) + 11.9916 = 13.38 \ \ hours$$

Converting the decimal hours into actual time it will be 13:22:48 13 hours 22 minutes and 48 seconds which will be about 13 hours and 23 minutes

d) There will be the same amount of daylight hours and night time hours when the number of daylight hours is 12. therefore

$$= 2.075\cos\left(\frac{\pi}{184}d - \frac{43\pi}{46}\right) + 11.991612 - 11.9916$$

Since the function was a cosine function when we take the inverse we obtain 1.566, but since it was cosine our values are ± 1.566

By taking -1.566 first we obtain

$$= \frac{\pi}{184}d - \frac{43\pi}{46} - 1.566 + \frac{43\pi}{46}$$

The first day will be March 21st.

By taking 1.566 we obtain:

$$= \frac{\pi}{184}d - \frac{43\pi}{46}1.566 + \frac{43\pi}{46}$$

The day will be about September 20

Vocabulary

Harmonic Motion

A periodic vibration in which the motions are symmetrical about a region of equilibrium.

Sinusoidal Wave

A mathematical curve that describes a smooth repetitive oscillation, named after the sine function.

In Summary

We have used sinusoidal waves to real life application problems. We have learned to find the values of A,B,C,D to create a sinusoidal function that matches a given situation.

Practice

####################

CHAPTER 7 Analytic Trigonometry

Chapter Outline

Here you will focus on the algebraic manipulations of trigonometric functions. You will learn the trigonometric identities, which are a list of equivalent trigonometric statements. These will serve as a toolbox that, together with algebra, will help you to combine, reduce, and simplify trigonometric expressions.

7.1 Verifying with Trigonometric Identities

TEKS

P.1.B
P.1.G
P.5.M

Lesson Objectives

In this lesson you will learn about:

1. The basic trigonometric identities.
2. How to verify trigonometric identities by changing to sines and cosines.
3. How to use factoring to verify identities.
4. How to use other methods to verify identities.

Introduction

Now that we know the six trigonometric functions and how they are used, we can use these concepts to identify some new properties. Once we know the basic properties, also called identities, we can verify that an equation is true. We do this in a similar fashion to algebraic proofs.

Vocabulary

Verify/Prove: **Reciprocal Identities, Quotient Identities, Pythagorean Identities, Even/Odd Identities.**

1. **Derive Basic Trigonometric Identities**

We know from the unit circle that:

TABLE 7.1:

$sin\theta = y$	$csc\theta = \frac{1}{y}$
$cos\theta = x$	$sec\theta = \frac{1}{x}$
$tan\theta = \frac{y}{x}$	$cot\theta = \frac{x}{y}$

We also know:

$x = cos\theta$

$y = sin\theta$

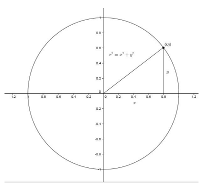
From this we get:

TABLE 7.2:

$csc\theta = \frac{1}{sin\theta}$	$sin\theta = \frac{1}{csc\theta}$
$sec\theta = \frac{1}{cos\theta}$	$cos\theta = \frac{1}{sec\theta}$
$cot\theta = \frac{1}{tan\theta} = \frac{cos\theta}{sin\theta}$	$tan\theta = \frac{1}{cot\theta} = \frac{sin\theta}{cos\theta}$

Also, we have that given the unit circle and a right circle:

TABLE 7.3:

	Pythagorean Theorem: $leg^2 + leg^2 = hypotenuse^2$ $x^2 + y^2 = r^2$ $cos^2\theta + sin^2\theta = 1^2$ $cos^2\theta + sin^2\theta = 1$

In summary the formal names for these basic yet important identities are as follows:

Reciprocal Identities:

TABLE 7.4:

$\sin x = \frac{1}{\csc x}$	$\csc x = \frac{1}{\sin x}$
$\cos x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$
$\tan x = \frac{1}{\cot x}$	$\cot x = \frac{1}{\tan x}$

Quotient Identities:

TABLE 7.5:

$tanx = \frac{sinx}{cosx}$	$cotx = \frac{cosx}{sinx}$

Pythagorean Identities:

TABLE 7.6:

$cosx^2x + sin^2x = 1$	$1 + tan^2x = sec^2x$	$1 + cot^2x = csc^2x$
$1 - sin^2x = cos^2x$	$tan^2x = sec^2x - 1$	$cot^2x = csc^2x - 1$
$1 - cos^2x = sin^2x$	$tan^2x - sec^2x = -1$	$cot^2x - csc^2x = -1$

Even/Odd Identities:

TABLE 7.7:

$sin(-x) = -sinx$	$csc(-x) = -secx$

TABLE 7.7: (continued)

$cos(-x) = cosx$	$sec(-x) = secx$
$tan(-x) = -tanx$	$cot(-x) = -cotx$

2. **Verifying by Changing to Sines and Cosines**

Example: Verify.

A. $sinxsecx = tanx$

$$amp; sinxsecx$$
$$amp; sinx\frac{1}{cosx}$$
$$\frac{sinx}{cosx}$$
$$amp; tanx = tanx$$

B.$secxcotx = cscx$

$$amp; secxcotx$$
$$\frac{1}{cosx}\frac{cosx}{sinx}$$
$$\frac{cosx}{cosxsinx}$$
$$\frac{1}{sinx}$$
$$amp; cscx = cscx$$

C. $sinxtanx = secx - cosx$

$$amp; sinxtanx$$
$$amp; sinx\frac{sinx}{cosx}$$
$$\frac{sin^2x}{cosx}$$
$$\frac{1-cos^2x}{cosx}$$
$$\frac{1}{cosx} - \frac{cos^2x}{cosx}$$
$$amp; secx - cosx = secx - cosx$$

D.$cscx - sinx = cosxcotx$

$$
\begin{aligned}
& \csc x - \sin x \\
&\frac{1}{\sin x} - \sin x \\
&\frac{1}{\sin x} - \frac{\sin x}{\sin x} \sin x \\
&\frac{1}{\sin x} - \frac{\sin^2 x}{\sin x} \\
&\frac{1 - \sin^2 x}{\sin x} \\
&\frac{\cos^2 x}{\sin x} \\
&\frac{\cos x}{\sin x} \cos x \\
& \cot x \cos x \\
& \cos x \cot x = \cos x \cot x
\end{aligned}
$$

3. **Using Factoring to Verify**

We know from algebra how to factor a binomial expression like:

$x^2 + 2x \longleftrightarrow x(x+2)$

Similarly we can do the same for trigonometric expressions:

$\sin^2 x + 2\sin x \longleftrightarrow \sin x(\sin x + 2)$

We can also factor trinomial:

$x^2 + 4x + 4 \longleftrightarrow (x+2)(x+2)$

With trigonometric expressions:

$\sin^2 x + 4\sin x + 4 \longleftrightarrow (\sin x + 2)(\sin x + 2)$

Example: Verify.

A. $\sec x - \sec x \sin^2 x = \cos x$

$$
\begin{aligned}
& \sec x - \sec x \sin^2 x \\
& \sec x(1 - \sin^2 x) \\
& \sec x \cos^2 x \\
&\frac{1}{\cos x} \cos^2 x \\
&\frac{\cos^2 x}{\cos x} \\
& \cos x = \cos x
\end{aligned}
$$

B. $\sin^2\theta + \sin^2\theta \cot^2\theta = 1$

$$sin^2\theta + sin^2\theta cot^2\theta$$
$$sin^2\theta(1+cot^2\theta)$$
$$sin^2\theta csc^2\theta$$
$$sin^2\theta\frac{1}{sin^2\theta}$$
$$\frac{sin^2\theta}{sin^2\theta}$$
$$1 = 1$$

3. **Other Methods for Verifying**

Just like simplifying algebraic expressions we can use algebra facts and tools we already know to verify identities.

Example: Verify.

A. $\frac{1+sin\theta}{cos\theta} = sec\theta + tan\theta$

$$\frac{1+sin\theta}{cos\theta}$$
$$\frac{1}{cos\theta}+\frac{sin\theta}{cos\theta}$$
$$amp; sec\theta + tan\theta = sec\theta + tan\theta$$

B. $\frac{sinx}{1+cosx}+\frac{1+cosx}{sinx} = 2cscx$

$$\frac{sinx}{1+cosx}+\frac{1+cosx}{sinx}$$
$$\frac{sinx}{sinx}\frac{sinx}{1+cosx}+\frac{1+cosx}{1+cosx}\frac{1+cosx}{sinx}$$
$$\frac{sin^2x}{(sinx)(1+cosx)}+\frac{1+2cosx+cos^2x}{(sinx)(1+cosx)}$$
$$\frac{sin^2x+cos^2x+1+2cosx}{(sinx)(1+cosx)}$$
$$\frac{1+1+2cosx}{(sinx)(1+cosx)}$$
$$\frac{2+2cosx}{(sinx)(1+cosx)}$$
$$\frac{2(1+cosx)}{(sinx)(1+cosx)}$$
$$\frac{2}{sinx}$$
$$2\frac{1}{sinx}$$
$$2cscx = 2cscx$$

C. $\frac{sinx}{1+cosx} = \frac{1-cosx}{sinx}$

$$\begin{aligned}
\frac{sinx}{1+cosx} &= \frac{1-cosx}{sinx} \\
\frac{sinx}{sinx}\frac{sinx}{1+cosx} &= \frac{1+cosx}{1+cosx}\frac{1-cosx}{sinx} \\
\frac{sin^2x}{(sinx)(1+cosx)} &= \frac{1+cos^2x}{(sinx)(1+cosx)} \\
\frac{sin^2x}{(sinx)(1+cosx)} &= \frac{sin^2x}{(sinx)(1+cosx)}
\end{aligned}$$

D. $\frac{1}{1+sin\theta} + \frac{1}{1-sin\theta} = 2+2tan^2\theta$

$$\begin{aligned}
&\frac{1}{1+sin\theta} + \frac{1}{1-sin\theta} \\
&\frac{1-sin\theta}{1-sin\theta}\frac{1}{1+sin\theta} + \frac{1+sin\theta}{1+sin\theta}\frac{1}{1-sin\theta} \\
&\frac{1-sin\theta}{1-sin^2\theta} + \frac{1+sin\theta}{1-sin^2\theta} \\
&\frac{1-sin\theta+1+sin\theta}{cos^2\theta} \\
&\frac{2}{cos^2\theta} \\
&2\frac{1}{cos^2\theta} \\
&2sec^2\theta \\
&2(1+tan^2\theta) \\
&2+2tan^2\theta = 2+2tan^2\theta
\end{aligned}$$

Vocabulary

Identity

Is an equality relation in which A=B, such that A and B contain some variables and A and B product the same value as each other regardless of the input.

Proof

The process of making one side of and equation be exactly the same as the other side by a process of using identities.

In Summary

We learned to verify/prove trigonometric identities by using the six basic trigonometric identities, the Pythagorean Identities, Reciprocal Identities, Quotient identities, and Even/Odd identities.

Check for Understanding:

MEDIA

Click image to the left or use the URL below.

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7.2 Sum and Difference Formulas

TEKS

P.5.M
P.1.C
P.1.D

Lesson Objectives

In this lesson you will learn about:

1. The sum and difference formulas for cosine.
2. The sum and difference formulas for sine.
3. The sum and difference formulas for tangent.

Introduction

What happens if we want to find an exact value of a trigonometric function that is not one of the special angles as listed on the unit circle? There has to be a way to apply what we know about the unit circle and right triangles to find exact values for different angles that are not listed as special angles.

Vocabulary

Sum and Difference Formulas for Sine, Cosine and Tangent.

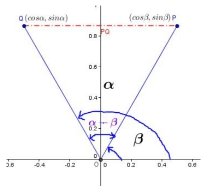
1. **The Sum and Difference formulas for Cosine**

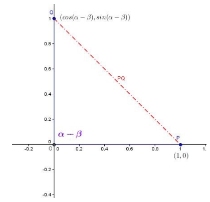
   Let's investigate and derive the formula for sum and difference for cosine using what we know about algebra and trigonometric functions and the unit circle:

TABLE 7.8:

1.	2.

TABLE 7.8: (continued)





$$PQ_1 = \sqrt{(cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2}$$

$$\begin{aligned}
PQ_1 &= \sqrt{(cos\alpha - cos\beta)^2 - (sin\alpha - sin\beta)^2} \\
&= \sqrt{cos^2\alpha - 2cos\alpha cos\beta + cos^2\beta + sin^2\alpha - 2sin\alpha sin\beta + sin^2\beta} \\
&= \sqrt{cos^2\alpha + sin^2\alpha + cos^2\beta + sin^2\beta - 2cos\alpha cos\beta - 2sin\alpha sin\beta} \\
&= \sqrt{2 - 2cos\alpha cos\beta - 2sin\alpha sin\beta} \\
&= \sqrt{2 - 2(cos\alpha cos\beta - sin\alpha sin\beta}
\end{aligned}$$

$$\begin{aligned}
PQ_2 &= \sqrt{(cos(\alpha - \beta) - 1)^2 + (sin(\alpha - \beta) - 0)^2} \\
&= \sqrt{cos^2(\alpha - \beta) - 2cos(\alpha - \beta) + 1 + sin^2(\alpha - \beta)} \\
&= \sqrt{cos^2(\alpha - \beta) + sin^2(\alpha - \beta) + 1 - 2cos(\alpha - \beta)} \\
&= \sqrt{1 + 1 - 2cos(\alpha - \beta)} \\
&= \sqrt{2 - 2cos(\alpha - \beta)}
\end{aligned}$$

$$\begin{aligned}
PQ_2 &= PQ_1 \\
\sqrt{2 - 2cos(\alpha - \beta)} &= \sqrt{2 - 2(cos\alpha cos\beta + sin\alpha sin\beta)} \\
2 - 2cos(\alpha - \beta) &= 2 - 2(cos\alpha cos\beta + sin\alpha sin\beta) \\
-2cos(\alpha - \beta) &= -2(cos\alpha cos\beta + sin\alpha sin\beta) \\
cos(\alpha - \beta) &= cos\alpha cos\beta + sin\alpha sin\beta
\end{aligned}$$

The same method can be used to derive the formula for$cos(\alpha + \beta)$

Sum and Difference Formulas for Cosine

$cos(\alpha \pm \beta) = cos\alpha cos\beta \mp sin\alpha sin\beta$

Example: Use the sum and difference formulas for cosine.

A. What is the exact value of $cos15^o$ using the difference formula for cosine?

$$\begin{aligned} cos15^o &= cos(45^o - 30^o) \\ cos(45^o - 30^o) &= cos45^o cos30^o + sin45^o sin30^o \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{2}(\sqrt{3}+1)}{4} \end{aligned}$$

B. What is the exact value of $cos70^o cos40^o + sin70^o sin40^o$?

$$\begin{aligned} cos70^o cos40^o + sin70^o sin40^o &= cos(70^o - 40^o) \\ &= cos30^o \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

C. What is the exact value of $cos75^o$ using the sum formula for cosine?

$$\begin{aligned} cos75^o &= cos(45^o + 30^o) \\ &= cos45^o cos30^o - sin45^o sin30^o \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{2}(\sqrt{3}-1)}{4} \end{aligned}$$

D. What is the exact value of $cos75^o cos15^o - sin75^o sin15^o$?

$$\begin{aligned} cos75^o cos15^o - sin75^o sin15^o &= cos(75^o + 15^o) \\ &= cos90^o \\ &= 0 \end{aligned}$$

E. What is the exact value for $cos(\alpha+\beta)$ if $sin\alpha = \frac{5}{7}$ and α is in quadrant one, and $sin\beta = -\frac{3}{5}$with β in quadrant three.

We start by examining what we have and what we need to figure out the value for $cos(\alpha+\beta)$. The formula says $cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$. We are give the value for $sin\alpha$ and $sin\beta$, we need $cos\alpha$ and $cos\beta$.

Let's start with finding $cos\alpha$. Since α is in quadrant one we know that the values for both x and y are positive, and r is always positive. Since $x^2 + y^2 = r^2$ and $sin\theta = \frac{y}{r}$we have:

$$\begin{aligned} x^2+y^2&=r^2\\ x^2+5^2&=7^2\\ x^2+25&=49\\ x^2&=24\\ x&=\sqrt{24}\\ x&=2\sqrt{6} \end{aligned}$$

Now that we know the value of x we can use $cos\theta = \frac{x}{r}$to find $cos\alpha$

$cos\alpha = \frac{2\sqrt{6}}{7}$

Now we can find $cos\beta$ the same way. Since β is in quadrant three we have that both x and y are negative, therefore we need to take into account the negative when finding the value for x.

$$\begin{aligned} x^2+y^2&=r^2\\ x^2+(-3)^2&=5^2\\ x^2+9&=25\\ x^2&=16\\ x&=-\sqrt{16}\\ x&=-4 \end{aligned}$$

Now that we know the value of x we know that

$cos\beta = -\frac{4}{5}$

Now we know all the parts for the formula so we just substitute in values.

$$\begin{aligned} cos(\alpha+\beta) &= cos\alpha cos\beta - sin\alpha sin\beta\\ &= \left(\frac{2\sqrt{6}}{7}\right)\left(-\frac{4}{5}\right)-\left(\frac{5}{7}\right)\left(-\frac{3}{5}\right)\\ &= -\frac{8\sqrt{6}}{35}+\frac{15}{35}\\ &= \frac{-8\sqrt{6}+15}{35} \end{aligned}$$

2. **The Sum and Difference formulas for Sine**

The formulas for the sum and difference for Sine can be found in a similar fashion to get:

$sin(\alpha\pm\beta) = sin\alpha cos\beta \pm cos\alpha sin\beta$

Example: Find the exact values:

A. $sin75^o$

$$\begin{aligned} sin75^o &= sin(45^o+30^o) \\ &= sin45^o cos30^o + cos45^o sin30^o \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{2}(\sqrt{3}+1)}{4} \end{aligned}$$

B. $sin15^o cos15^o + cos15^o sin15^o$

$$\begin{aligned} sin15^o cos15^o + cos15^o sin15^o &= sin15^o + 15^o \\ &= sin30^o \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

C. $sin(\alpha-\beta)$ if $sin\alpha = \frac{3}{5}$ and α in quadrant two, and $sin\beta = -\frac{4}{5}$ with β in quadrant four.

We approach this problem the exact same way we approached it for cosine. Start by looking at what we have and what we need. The formula we are going to use is $sin(\alpha-\beta) = sin\alpha cos\beta - cos\alpha sin\beta$and we are given $sin\alpha$ and $sin\beta$. So let's first find $cos\alpha$.

α is in quadrant two, so x is negative and y is positive. We are given y and r already so we use the formula:

$$\begin{aligned} x^2+y^2 &= r^2 \\ x^2+3^2 &= 5^2 \\ x^2+9 &= 25 \\ x^2 &= 16 \\ x &= \sqrt{16} \\ x &= -4 \end{aligned}$$

Which means that:

$$\begin{aligned} cos\alpha &= \frac{x}{r} \\ cos\alpha &= -\frac{4}{5} \end{aligned}$$

β is in quadrant four which means that x is positive and y is negative. We are given y and r already so we use the formula:

$$\begin{aligned}
x^2+y^2 &= r^2 \\
x^2+-4^2 &= 5^2 \\
x^2+16 &= 25 \\
x^2 &= 9 \\
x &= \sqrt{9} \\
x &= 3
\end{aligned}$$

Which means that:

$$\begin{aligned}
cos\beta &= \frac{x}{r} \\
cos\beta &= \frac{3}{5}
\end{aligned}$$

Using the formula and substituting we have our final answer:

$$\begin{aligned}
sin(\alpha-\beta) &= sin\alpha cos\beta - cos\alpha sin\beta \\
&= \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)-\left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right) \\
&= \frac{9+16}{25} \\
&= \frac{25}{25} \\
&= 1
\end{aligned}$$

2. **The Sum and Difference formulas for Tangent**

From our basic identities we know that $tan\theta = \frac{sin\theta}{cos\theta}$, we can use this idea to derive the formulas for the sum and difference of tangent.

$$\begin{aligned}
tan(\alpha-\beta) &= \frac{sin(\alpha-\beta)}{cos(\alpha-\beta)} \\
&= \frac{sin\alpha cos\beta - sin\beta cos\alpha}{sin\alpha sin\beta + cos\alpha cos\beta} \\
&= \frac{sin\alpha cos\beta - sin\beta cos\alpha}{sin\alpha sin\beta + cos\alpha cos\beta} \times \frac{\frac{1}{cos\alpha cos\beta}}{\frac{1}{cos\alpha cos\beta}} \\
&= \frac{sin\alpha cos\beta - sin\alpha \cos\alpha}{cos\alpha cos\beta} \div \frac{sin\alpha sin\beta + cos\alpha cos\beta}{cos\alpha cos\beta} \\
&= \left(\frac{sin\alpha cos\beta}{cos\alpha cos\beta} - \frac{sin\beta cos\alpha}{cos\alpha cos\beta}\right) \div \left(\frac{sin\alpha sin\beta}{cos\alpha cos\beta} + \frac{cos\alpha cos\beta}{cos\alpha cos\beta}\right) \\
&= \left(\frac{sin\alpha}{cos\alpha} - \frac{sin\beta}{cos\beta}\right) \div \left(\frac{sin\alpha}{cos\alpha}\frac{sin\beta}{cos\beta} + 1\right) \\
&= (tan\alpha - tan\beta) \div (1 + tan\alpha tan\beta) \\
&= \frac{tan\alpha - tan\beta}{1 + tan\alpha tan\beta}
\end{aligned}$$

In summary the formulas for sum and difference for tangent are:

$tan(\alpha \pm \beta) = \frac{tan\alpha \pm tan\beta}{1 \mp tan\alpha tan\beta}$

Example: Find the exact values.

A. $tan285^o$

$$\begin{aligned}
tan285^o &= tan(225^o + 60^o) \\
&= \frac{tan225^o + tan60^o}{1 - tan225^o tan60^o} \\
&= \frac{-1+\sqrt{3}}{1-(-1)(\sqrt{3})} \\
&= \frac{-1+\sqrt{3}}{1+\sqrt{3}} \\
&= \left(\frac{-1+\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) \\
&= \frac{-1+\sqrt{3}+\sqrt{3}-\left(\sqrt{3}\right)^2}{1-\sqrt{3}+\sqrt{3}-\left(\sqrt{3}\right)^2} \\
&= \frac{-1+2\sqrt{3}-3}{1-3} \\
&= \frac{-4+2\sqrt{3}}{-2} \\
&= 2-\sqrt{3}
\end{aligned}$$

B. $\frac{tan40^o + tan110^o}{1 - tan40^o tan110^o}$

$$\begin{aligned}\frac{tan40^o + tan110^o}{1 - tan40^o tan110^o} &= tan\left(40^o + 110^o\right)\\ &= tan150^o\\ &= -\frac{\sqrt{3}}{3}\end{aligned}$$

C. Find $tan(\alpha+\beta)$ given $cos\alpha = -\frac{7}{25}$ and α in quadrant two and $sin\beta = -\frac{\sqrt{21}}{5}$ and β is in quadrant three.

Since α is in quadrant two, x is negative and y is positive.

$$\begin{aligned}x^2 + y^2 &= r^2\\ (-7)^2 + y^2 &= 25^2\\ 49 + y^2 &= 625\\ y^2 &= 576\\ y &= \sqrt{576}\\ y &= 24\end{aligned}$$

Then: $tan\alpha =$

$$\begin{aligned}tan\alpha &= \frac{y}{x}\\ &= -\frac{24}{7}\end{aligned}$$

Since β is in quadrant three both x and y are negative.

$$\begin{aligned}x^2 + y^2 &= r^2\\ x^2 + -\sqrt{21}^2 &= 5^2\\ x^2 + 21 &= 25\\ x^2 &= 4\\ x &= -\sqrt{4}\\ x &= -2\end{aligned}$$

Then:

$$\begin{aligned}tan\beta &= \frac{y}{x}\\ &= \frac{-\sqrt{21}}{-2}\\ &= \frac{\sqrt{21}}{2}\end{aligned}$$

Using the formula $tan(\alpha+\beta) = \frac{tan\alpha+tan\beta}{1-tan\alpha tan\beta}$

$$\begin{aligned}
tan(\alpha+\beta) &= \frac{tan\alpha+tan\beta}{1-tan\alpha tan\beta}\\
&= \frac{-\frac{24}{7}+\frac{\sqrt{21}}{2}}{1-\left(-\frac{24}{7}\right)\left(\frac{\sqrt{21}}{2}\right)}\\
&= \left(-\frac{24}{7}+\frac{\sqrt{21}}{2}\right)\div\left(1-\left(-\frac{24}{7}\right)\left(\frac{\sqrt{21}}{2}\right)\right)\\
&= \frac{-48+7\sqrt{21}}{14}\div\left(1+\frac{24\sqrt{21}}{14}\right)\\
&= \left(\frac{-48+7\sqrt{21}}{14}\right)\div\left(\frac{14+24\sqrt{21}}{14}\right)\\
&= \left(\frac{-48+7\sqrt{21}}{14}\right)\times\left(\frac{14}{14+24\sqrt{21}}\right)\\
&= \frac{-48+7\sqrt{21}}{14+24\sqrt{21}}\\
&= \frac{-48+7\sqrt{21}}{2\left(7+12\sqrt{21}\right)}\\
&= \left(\frac{-48+7\sqrt{21}}{2\left(7+12\sqrt{21}\right)}\right)\left(\frac{7-\sqrt{21}}{7-\sqrt{21}}\right)\\
&= \frac{-336+576\sqrt{21}+49\sqrt{21}-1764}{2\,(49-252)}\\
&= \frac{-2100+625\sqrt{21}}{-406}\\
&= \frac{25\left(-84+25\sqrt{21}\right)}{-406}
\end{aligned}$$

Vocabulary

Sum and Difference formulas

$cos(\alpha\pm\beta) = cos\alpha cos\beta \mp sin\alpha sin\beta$

$sin(\alpha\pm\beta) = sin\alpha cos\beta \pm cos\alpha sin\beta$

$tan(\alpha\pm\beta) = \frac{tan\alpha\pm tan\beta}{1\mp tan\alpha tan\beta}$

In Summary

We have learned how to use the sum and difference formulas of sine, cosine and tangent.

Check for Understanding

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/162887

7.3 Double Angle Formulas

TEKS

P.5.M

Lesson Objectives

In this lesson you will learn about:

1. Deriving the double angle formulas.
2. Applying the double angle formulas.

Introduction

Knowing what we know about verifying identities and the sum and difference formulas we will investigate what happens when we double an angle.

Vocabulary

<no new vocabulary>

1. **Derive the Double Angle Formulas**

We are going to use the sum identities for sine and cosine to derive the double angle formulas.

Take $sin(\alpha+\beta) = sin\alpha cos\beta + cos\alpha sin\beta$, now suppose that $\alpha = \beta$ then we have:

$$\begin{aligned}
sin(\alpha+\alpha) &= sin\alpha cos\alpha + cos\alpha sin\alpha \\
&= sin\alpha cos\alpha + sin\alpha cos\alpha \\
sin(2\alpha) &= 2sin\alpha cos\alpha
\end{aligned}$$

Now we will use $cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$and suppose the same that $\alpha = \beta$ then:

$$cos(\alpha+\alpha) = cos\alpha cos\alpha - sin\alpha sin\alpha$$
$$cos(2\alpha) = cos^2\alpha - sin^2\alpha$$

In addition to this formula we can use the Pythagorean Identities to come up with two alternate versions of the double angle identity for cosine:

$$cos^2\alpha - (1-cos^2\alpha)$$
$$cos^2\alpha - 1 + cos^2\alpha$$
$$2cos^2\alpha - 1$$

OR

$$(1-sin^2\alpha) - sin^2\alpha$$
$$1 - sin^2\alpha - sin^2\alpha$$
$$1 - 2sin^2\alpha$$

We know that $tan(\alpha+\beta) = \frac{tan\alpha+tan\beta}{1-tan\alpha tan\beta}$, again if $\alpha = \beta$ then we have:

$$tan(\alpha+\alpha) = \frac{tan\alpha+tan\alpha}{1-tan\alpha tan\alpha}$$
$$tan(2\alpha) = \frac{2tan\alpha}{1-tan^2\alpha}$$

So, in summary we have:

TABLE 7.9:

Double Angle Formula: Sine
$sin(2\alpha) = 2sin\alpha cos\alpha$
Double Angle Formulas: Cosine
$cos(2\alpha) = cos^2\alpha - sin^2\alpha$
$cos(2\alpha) = 2cos^2\alpha - 1$
$cos(2\alpha) = 1 - 2sin^2\alpha$
Double Angle Formula: Tangent
$tan(2\alpha) = \frac{2tan\alpha}{1-tan^2\alpha}$

2. **Apply the Double Angle Formulas**

When finding an exact value of an angle θ that is a double of one of our special angles, we can find what the special angle α is by dividing by two, since it is double.

$$\theta = 2\alpha$$
$$\frac{\theta}{2} = \frac{2\alpha}{2}$$
$$\frac{\theta}{2} = \alpha$$

That constitutes our first step in finding the exact value, divide by two to find the special angle.

Example: Find the exact values.

A. If $cos\theta = -\frac{3}{15}$ and θ is in quadrant two, find $sin(2\theta)$, $cos(2\theta)$, and $tan(2\theta)$.

First we have to find the values for $sin\theta$ and $tan\theta$.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-3)^2 + y^2 &= 15^2 \\ 9 + y^2 &= 225 \\ y^2 &= 216 \\ y &= \sqrt{216} \\ y &= 6\sqrt{6} \end{aligned}$$

$$\begin{aligned} sin\theta &= \frac{y}{r} \\ &= \frac{6\sqrt{6}}{15} \\ &= \frac{2\sqrt{6}}{5} \end{aligned}$$

$$\begin{aligned} tan\theta &= \frac{y}{x} \\ &= \frac{6\sqrt{6}}{-3} \\ &= -2\sqrt{6} \end{aligned}$$

Using the double angle formulas we get:

$$\begin{aligned}
sin(2\theta) &= 2cos\theta sin\theta \\
&= 2\left(-\frac{3}{15}\right)\left(\frac{2\sqrt{6}}{5}\right) \\
&= 2\left(-\frac{1}{5}\right)\left(\frac{2\sqrt{6}}{5}\right) \\
&= 2\left(\frac{2\sqrt{6}}{25}\right) \\
&= \frac{4\sqrt{6}}{25}
\end{aligned}$$

$$\begin{aligned}
cos(2\theta) &= cos^2\theta - sin^2\theta \\
&= \left(-\frac{3}{15}\right)^2 - \left(\frac{2\sqrt{6}}{5}\right)^2 \\
&= \frac{1}{25} - \frac{24}{25} \\
&= -\frac{23}{25}
\end{aligned}$$

$$\begin{aligned}
tan(2\theta) &= \frac{2tan\theta}{1 - tan^2\theta} \\
&= \frac{2\left(-2\sqrt{6}\right)}{1-(-2\sqrt{6})^2} \\
&= \frac{-4\sqrt{6}}{1-24} \\
&= \frac{-4\sqrt{6}}{-23} \\
&= \frac{4\sqrt{6}}{23}
\end{aligned}$$

B. Use the double angle formula to find $sin 90^o$.

We first have to take half of the angle by dividing by 2:

$\frac{90^o}{2} = 45^o$, we use this angle in the formula for the double angle of sine:

$$\begin{aligned} sin(2\theta) &= 2cos\theta sin\theta \\ &= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= 2\left(\frac{2}{4}\right) \\ &= \frac{4}{4} \\ &= 1 \end{aligned}$$

C. $\frac{2tan15^o}{1-tan^2 15^o}$

This is the exact form of the double angle formula for tangent:

$tan(2\theta) = \frac{2tan\theta}{1-tan^2\theta}$, in this case $\theta = 15^o$ so if we double that we get $2\theta = 30^o$. Which means we are actually looking for the exact value of $tan30^o$, which is one of our special angles. Using the unit circle we can see that:

$$\begin{aligned} tan30^o &= \frac{y}{x} \\ &= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \\ &= \frac{1}{2} \div \frac{\sqrt{3}}{2} \\ &= \frac{1}{2} \times \frac{2}{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \\ &= \left(\frac{\sqrt{3}}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

Vocabulary

No new vocabulary

In Summary

We have learned to derive and to apply the double angle formulas

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7.4 Solving Trigonometric Equations

TEKS

P.5.M

Lesson Objectives

In this lesson you will learn about:

1. Using the period to find all solutions.
2. Solving equations with multiple angles.
3. Solving equations in quadratic form.
4. Using factoring to simplify (with multiple trigonometric functions).
5. Using identities to simply and solve trigonometric equations.
6. Solving using a calculator.

Introduction

Now that we know how to simplify and some special properties of trigonometric functions, let's apply what we know and solve some equations! You will find this is very simple, we just apply tools we already know how to do from Algebra in a new way.

Vocabulary

No new vocabulary

1. **Using the Period to find all Solutions**

   We must consider all possible solutions for an angle given a trigonometric function value because trigonometric functions are periodic (they are infinite with a "looping" or "repeating" pattern)

   Recall:

   Sine and cosine are both 2π periodic (it takes 2π to complete one full cycle), and tangent is π periodic.

   If an interval is not specified in order to include ALL possible solutions we add $2k\pi$ or $k\pi$ where $k \geq 0$ and an integer depending on the function.

Steps to Solve a Trigonometric Function:

1. Isolate the trigonometric function to one side.

2. Solve for the variable

Example: Solve.

A.$3cosx+2=5cosx+1$

First we have to isolate the trigonometric function by subtracting to the other side.

$$\begin{aligned}3cosx+2&=5cosx+1\\2&=2cosx+1\\1&=2cosx\\\frac{1}{2}&=cosx\end{aligned}$$

Then we ask ourselves, for what values of x does $cosx=\frac{1}{2}$?

$x=\frac{\pi}{3}$ and $x=\frac{5\pi}{3}$, however since we are looking for ALL possible solutions we note that cosine is a 2π periodic function therefore our final solutions are:

$$\begin{aligned}x&=\frac{\pi}{3}+2k\pi\\&=\frac{\pi}{3}+\frac{6k\pi}{3}\\&=\frac{\pi(1+6k)}{3}\end{aligned}$$

$$\begin{aligned}x&=\frac{5\pi}{3}+2k\pi\\&=\frac{5\pi}{3}+\frac{6k\pi}{3}\\&=\frac{\pi(5+6k)}{3}\end{aligned}$$

B.$4sinx=2sinx+\sqrt{3}$

Again to solve this problem, we need to isolate the trigonometric function.

$$\begin{aligned}4sinx&=2sinx+\sqrt{3}\\2sinx&=\sqrt{3}\\sinx&=\frac{\sqrt{3}}{2}\end{aligned}$$

Then we ask ourselves, for what values of x dies $sinx = \frac{\sqrt{3}}{2}$ and note that sine is a 2π periodic function:

$$\begin{aligned} x &= \frac{\pi}{3} + 2k\pi \\ &= \frac{\pi}{3} + \frac{6k\pi}{3} \\ &= \frac{\pi(1+6k)}{3} \end{aligned}$$

$$\begin{aligned} x &= \frac{2\pi}{3} + 2k\pi \\ &= \frac{2\pi}{3} + \frac{6k\pi}{3} \\ &= \frac{2\pi(1+3k)}{3} \end{aligned}$$

2. **Solving Equations with Multiple Angles**

Sometimes we find ourselves not only solving for x or θ, we can also get problems where we have to solve for $3x$ or 5θ. In these cases, if there is a specified interval, we must substitute k with integers starting at 0 until we reach the end of the interval so that we can have named ALL possible solutions.

Example: Solve each equation on the interval $[0, 2\pi)$ **.**

A. $sin(2x) = \frac{\sqrt{3}}{2}$

First we note the angle values for which sine is $\frac{\sqrt{3}}{2}$:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2k\pi \\ x &= \left(\frac{\pi}{3} + 2k\pi\right) \div 2 \\ &= \frac{\pi}{6} + k\pi \\ &= \frac{\pi(1+6k)}{6} \end{aligned}$$

and

$$\begin{aligned} 2x &= \frac{2\pi}{3} + 2k\pi \\ x &= \left(\frac{2\pi}{3} + 2k\pi\right) \div 2 \\ &= \frac{\pi}{3} + k\pi \\ &= \frac{\pi(1+3k)}{3} \end{aligned}$$

Now that we know the values for x, we must test values for k starting at 0. As soon as we get a value that is greater than the indicated interval $[0,2\pi)$ we stop because our values for x cannot exceed the interval.

Note that: $2\pi = \frac{12\pi}{6}$ and $2\pi = \frac{6\pi}{3}$

$$\begin{aligned}
k &= 0\\
x_1 &= \frac{\pi(1+6(0))}{6}\\
&= \frac{\pi}{6}\\
x_2 &= \frac{\pi(1+3(0))}{3}\\
&= \frac{\pi}{3}
\end{aligned}$$

$$\begin{aligned}
k &= 1\\
x_1 &= \frac{\pi(1+6(1))}{6}\\
&= \frac{7\pi}{6}\\
x_2 &= \frac{\pi(1+3(1))}{3}\\
&= \frac{4\pi}{3}
\end{aligned}$$

$$\begin{aligned}
k &= 2\\
x_1 &= \frac{\pi(1+6(2))}{6}\\
&= \frac{13\pi}{6} > \frac{12\pi}{6}\\
x_2 &= \frac{\pi(1+3(2))}{3}\\
&= \frac{7\pi}{3} > \frac{6\pi}{3}
\end{aligned}$$

So the solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}\right\}$

B.$sin\left(\frac{2\theta}{3}\right) = -1$

There is only one value where sine is -1.

$$\begin{aligned}
\frac{2\theta}{3} &= \frac{3\pi}{2} + 2k\pi \\
2\theta &= \left(\frac{3\pi}{2} + 2k\pi\right) \times 3 \\
2\theta &= \frac{9\pi}{2} + 6k\pi \\
\theta &= \left(\frac{9\pi}{2} + 6k\pi\right) \div 2 \\
&= \frac{9\pi}{4} + 3k\pi \\
&= \frac{9\pi + 12k\pi}{4} \\
&= \frac{3\pi(3+4k)}{4}
\end{aligned}$$

Note that: $2\pi = \frac{8\pi}{4}$

$$\begin{aligned}
&k = 0 \\
&\qquad \theta = \frac{3\pi(3+4(0))}{4} \\
&\qquad\quad = \frac{9\pi}{4} > \frac{8\pi}{4}
\end{aligned}$$

Since right away with $k = 0$ the value exceeds 2π there is no solution for θ that will make this equation true.

3. **Solving Equations in Quadratic Form**

We know that there are many different ways to solve a quadratic; Factoring, Completing the Square, and Quadratic Formula. We can also use these methods to solve a trigonometric equation that looks like a quadratic.

We can use a method most commonly known as "u" substitution, where we substitute the trigonometric function with "u" and solve.

Example: Solve each equation on the interval $[0, 2\pi)$

A.$sin^2x + 2sinx + 1 = 0$

First let $u = sinx$, then

$$\begin{aligned} u^2+2u+1&=0\\ (u+1)(u+1)&=0\\ (u+1)^2&=0\\ u+1&=0\\ u&=-1 \end{aligned}$$

Since $u = sinx$ we can substitute and get

$sinx = -1$ and on the interval $[0,2\pi)$ the only value for x that makes this statement true is $\frac{3\pi}{2}$

B.$2cos^2x+3cosx+1=0$

Again we can let $u = cosx$ and

$2u^2+3u+1=0$
$(2u+1)(u+1)=0$
then $2u+1=0$
$2u=-1$
$u=-\frac{1}{2}$
and $u+1=0$
$u=-1$

Substitute back in $cosx$:

$cosx=-\frac{1}{2}$ and $cosx=-1$

The solution set for x on the interval $[0,2\pi)$ is $\left\{\frac{2\pi}{3},\frac{4\pi}{3},\pi\right\}$.

4. **Using Factoring to Simplify (With Multiple Trigonometric Functions)**

Just like with Algebra, we can factor out a greatest common factor to in a sense "separate" two different trigonometric functions that are present in one equation.

Example: Solve each equation on the interval $[0,2\pi)$.

A.$sinx+2sinxcosx=0$

Since both terms have a common factor $sinx$, we can factor it out:

$sinx(1+2cosx)=0$

Now we can separate the two factors because at least one of the two factors is equal to 0:

$sinx=0$ or $1+2cosx=0$
$2cosx=-1$
$cosx=-\frac{1}{2}$

So the values of x that make these two equations true are $\left\{\frac{2\pi}{3},\frac{4\pi}{3},\pi,0\right\}$

B.$tan^2xcosx=tan^2x$

Both terms have tan^2x as a common factor, but before we can factor it out we must move all trigonometric functions to one side of the equation and make it equal to 0.

$$\begin{aligned} tan^2xcosx &= tan^2x \\ tan^2xcosx - tan^2x &= 0 \\ tan^2x(cosx-1) &= 0 \end{aligned}$$

We then separate the two factors and solve:

$tan^2x = 0$
$tanx = 0$ and $cosx - 1 = 0$
$cosx = 1$

The only values of x that make these two equations true are $\{0, \pi\}$.

5. **Using Identities to Simplify and Solve Trigonometric Equations**

Sometimes we will recognize an identity such as $sin^2x + cos^2x$ in an equation. When we see this we can use the identity to help us simplify the equation and solve for the variable.

Example: Solve each equation on the interval $[0, 2\pi)$.

A.$2cos^2x + sinx - 1 = 0$

We know using the Pythagorean identities that $cos^2x = 1 - sin^2x$, if we substitute this into our equation we have only one trigonometric function to solve for.

$$\begin{aligned} 2cos^2x + sinx - 1 &= 0 \\ 2(1 - sin^2x) + sinx - 1 &= 0 \\ 2 - 2sin^2x + sinx - 1 &= 0 \\ -2sin^2x + sinx + 1 &= 0 \end{aligned}$$

Let $u = sinx$, then:

$-2u^2 + u + 1 = 0$
$(-2u-1)(u-1) = 0$
and $-2u - 1 = 0$
$-2u = 1$
$u = -\frac{1}{2}$ and $u - 1 = 0$
$u = 1$

Substituting back in $sinx$ we have that $sinx = -\frac{1}{2}$ and $sinx = 1$. The solutions that make these two equations true are $\left\{\frac{7\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}\right\}$.

B.$sin2x = cosx$

We know that $sin2x = 2sinxcosx$, so we can substitute this in and solve:

$$
\begin{aligned}
sin2x &= cosx \\
2sinxcosx &= cosx \\
2sinxcosx - cosx &= 0 \\
cosx(2sinx - 1) &= 0
\end{aligned}
$$

Then

$cosx = 0$ and

$$
\begin{aligned}
2sinx - 1 &= 0 \\
2sinx &= 1 \\
sinx &= \frac{1}{2}
\end{aligned}
$$

The only values for x that make these equations true are $\left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

6. **How to Solve Using a Calculator**

We can also use technology when we don't have a special angle to work with and find the approximation value of the variable. We just have to use the inverse trigonometric functions.

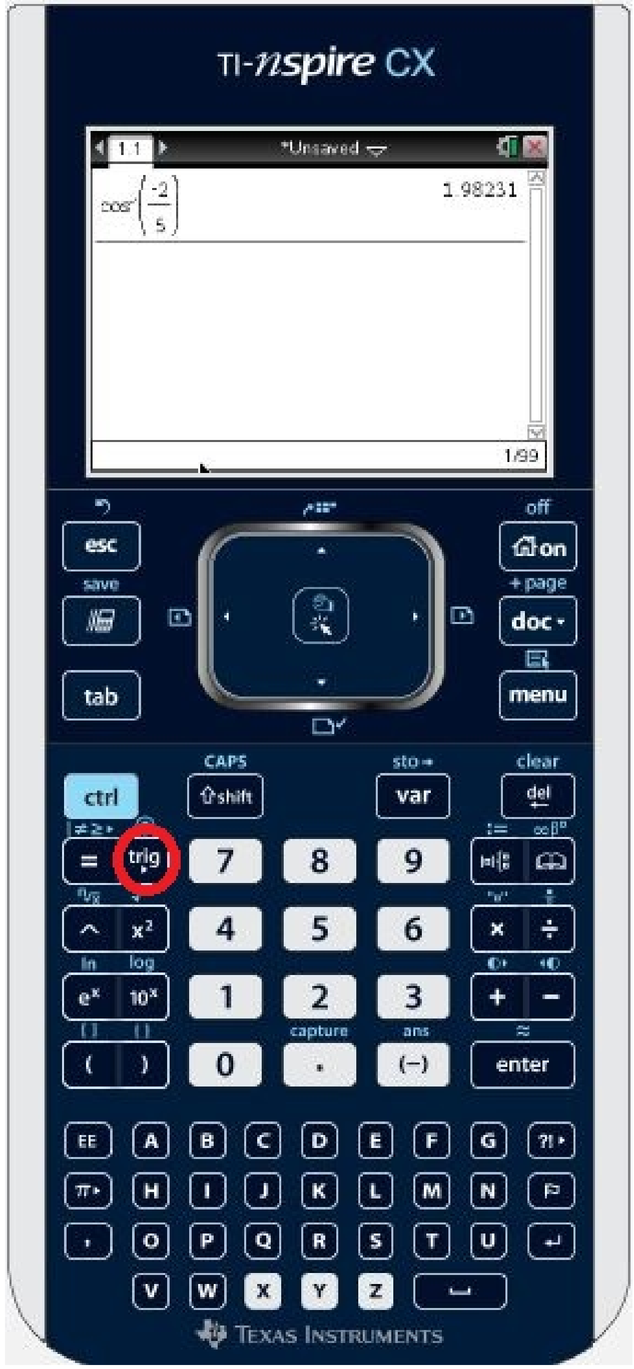
Example: Use the calculator to solve each equation, round to four decimal places.

A.$cosx = -\frac{2}{5}$

We will be using inverse cosine to solve this problem on the calculator.

$$
\begin{aligned}
cosx &= -\frac{2}{5} \\
cos^{-1}(cosx) &= cos^{-1}\left(-\frac{2}{5}\right) \\
x &= cos^{-1}\left(-\frac{2}{5}\right)
\end{aligned}
$$

On the TI-83 or the Nspire:



The solution is $x = 1.9823$ radians.

B.$5sin^2 x - 1 = 0$

First we have to solve for sine.

$$\begin{aligned}
5sin^2x-1&=0\\
5sin^2x&=1\\
sin^2x&=\frac{1}{5}\\
sinx&=\sqrt{\frac{1}{5}}\\
sinx&=\frac{\sqrt{5}}{5}\\
sin^{-1}(sinx)&=sin^{-1}\left(\frac{\sqrt{5}}{5}\right)\\
x&=sin^{-1}\left(\frac{\sqrt{5}}{5}\right)
\end{aligned}$$

Then on the TI-83 or Nspire:

The solution is $x = 0.4636$ radians.

Vocabulary

No new vocabulary

In Summary

We have learned to solve trigonometric equation that involve multiple angles, quadratic form, solving by factoring, and finding all solutions algebraically and using a calculator.

Check for Understanding

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The various trigonometric identities were introduced, proved and added to your toolbox. You learned what proofs look like in trigonometry and used the identities to derive the identities themselves. You solved trigonometric equations and bridged the connection between analytic trigonometry, which is very algebraic, with graphical representations.

CHAPTER 8 Polars and Complex Numbers

Chapter Outline

The Cartesian coordinate system with (x, y) coordinates is great for representing some functions and relations, but is very limiting in some ways. The polar coordinate system allows you to easily graph spirals and other shapes that are not functions. Parametric equations allow you to have both x and y depend on a third variable t. Here you will explore both polar and parametric equations.

8.1 Polar and Rectangular Coordinates

TEKS

1. P.3.D

Lesson Objectives

In this lesson you will learn about:

1. How to plot Polar Coordinates.
2. Finding multiple sets of points for a polar coordinate.
3. How to convert coordinates between polar and rectangular.
4. How to convert equations between polar and rectangular.

Introduction

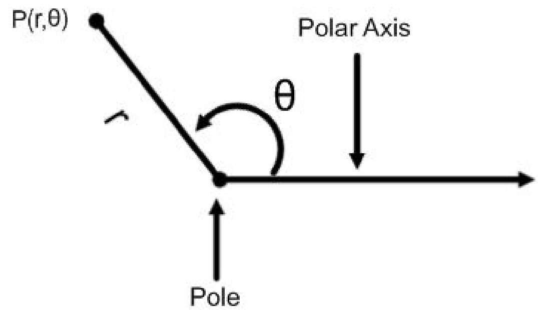
We have been using the rectangular or Cartesian coordinate grid to graph functions. Now we will move on to another type of coordinate system known as the polar coordinate system.

Vocabulary

Pole, Polar Coordinate, Polar Axis,Polar Equation.

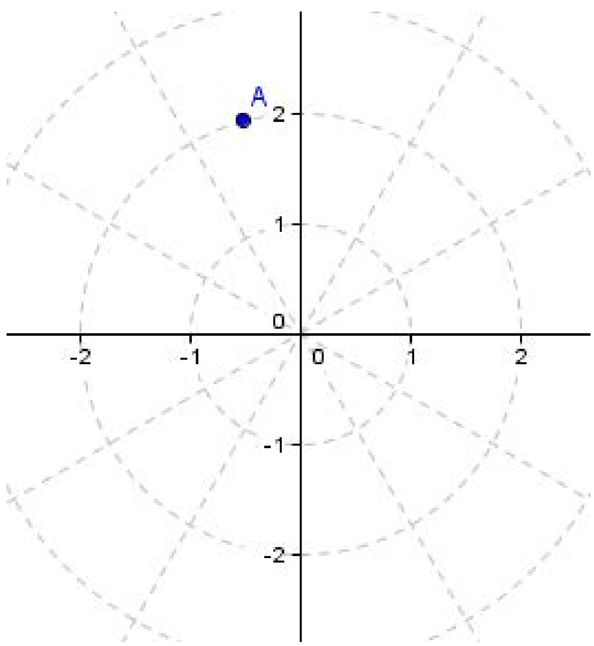
1. **Plotting Polar Coordinates:**

In the Cartesian (rectangular) coordinate system points are ordered pairs (x,y) where x is along the horizontal axis and y is along the vertical axis. In the polar coordinate system our ordered pairs are (r,θ) and θ is the angle of rotation from the polar axis and r is the directed distance from the pole to the point.

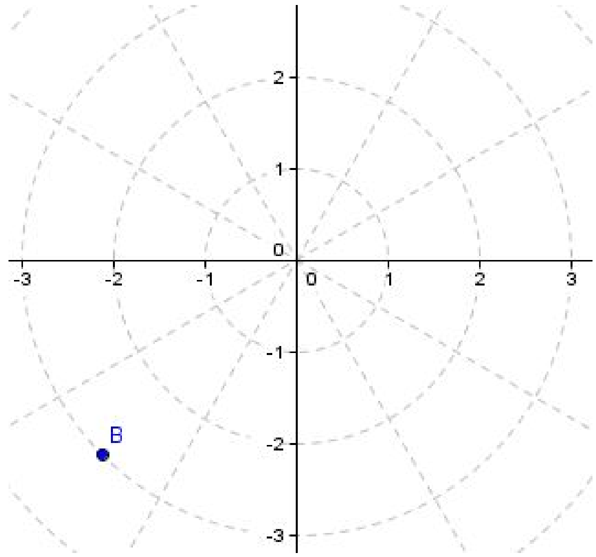


Example:

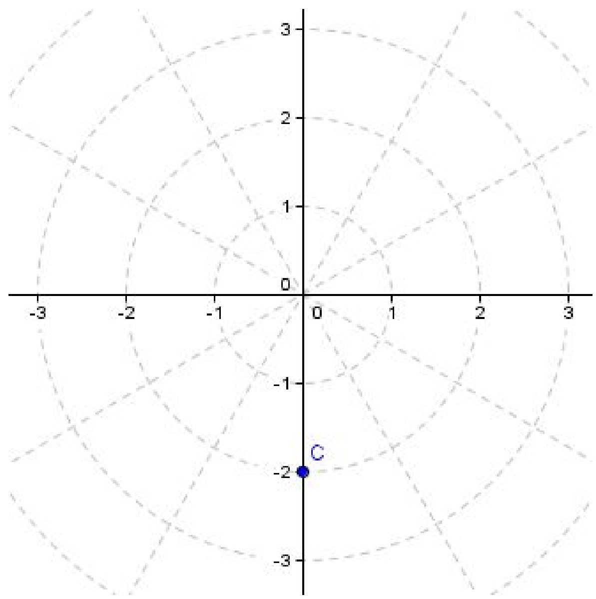
A. $(2, 105^o)$



B. $\left(-3, \frac{\pi}{4}\right)$



C. $\left(-2, \frac{-3\pi}{2}\right)$



2. **Multiple Representations of a Point:**

Any point in a polar axis can be represented infinitely many ways. Because we can add 2π or 360^o for an extra full rotation and end up on the same point:

Likewise if we negate r, we are essentially rotating half a circle which is π or 180^o so in order for the point to be at the same place, we have to add π or 180^o:

We can do this infinitely many times: If k is an integer the point $P(r,\theta)$ can be represented as:

$(r,\theta) = (r,\theta \pm 2k\pi)$
$(r,\theta) = (-r,\theta \pm k\pi)$

Example: Find three other representations of the given point that satisfy the conditions below:

1. $r > 0$ and $2\pi < \theta < 4\pi$

2. $r < 0$ and $0 < \theta < 2\pi$

3. $r > 0$ and $-2\pi < \theta < 0$

A. $\left(1, \frac{\pi}{2}\right)$

1. $\left(1, \frac{\pi}{2} + 2\pi\right) = \left(1, \frac{\pi}{2} + \frac{4\pi}{2}\right) = \left(1, \frac{5\pi}{2}\right)$
2. $\left(-1, \frac{\pi}{2} + \pi\right) = \left(-1, \frac{\pi}{2} + \frac{2\pi}{2}\right) = \left(-1, \frac{3\pi}{2}\right)$
3. $\left(1, \frac{\pi}{2} - 2\pi\right) = \left(1, \frac{\pi}{2} - \frac{4\pi}{2}\right) = \left(1, \frac{-3\pi}{2}\right)$

B. $\left(3, \frac{\pi}{4}\right)$

1. $\left(3, \frac{\pi}{4} + 2\pi\right) = \left(3, \frac{\pi}{4} + \frac{8\pi}{4}\right) = \left(3, \frac{9\pi}{4}\right)$
2. $\left(-3, \frac{\pi}{4} + \pi\right) = \left(-3, \frac{\pi}{4} + \frac{4\pi}{4}\right) = \left(-3, \frac{5\pi}{4}\right)$
3. $\left(3, \frac{\pi}{4} - 2\pi\right) = \left(3, \frac{\pi}{4} - \frac{8\pi}{4}\right) = \left(3, \frac{-7\pi}{4}\right)$

3. **Convert Between Polar and Rectangular Coordinates:**

We can use what we know about x, y, r, and θ to convert:

$cos\theta\frac{x}{r} \rightarrow rcos\theta = x$

$sin\theta = \frac{y}{r} \rightarrow rsin\theta = y$

$tan\theta = \frac{y}{x} \rightarrow \theta = tan^{-1}\left(\frac{y}{x}\right)$

$x^2 + y^2 = r^2$

Example: Convert the polar coordinate to rectangular using the above rules.

A. $(3, \pi)$

$r = 3$
$\theta = \pi \quad x = rcos\theta \rightarrow x = 3cos\pi = 3(-1) = -3$
$y = rsin\theta \rightarrow y = 3sin\pi = 3(0) = 0$

Therefore the rectangular coordinate is $(-3, 0)$

B. $\left(-10, \frac{\pi}{6}\right)$

$r = -10$
$\theta = \frac{\pi}{6}$
$x = rcos\theta \rightarrow x = -10cos\frac{\pi}{6} = -10\left(\frac{\sqrt{3}}{2}\right) = -5\sqrt{3}$
$y = rsin\theta \rightarrow y = -10cos\frac{\pi}{6} = -10\left(\frac{1}{2}\right) = -5$

Therefore the rectangular coordinate is $(-5\sqrt{3}, -5)$

Example: Convert the rectangular coordinate to polar using the above rules.

A. $(1, -\sqrt{3})$

This point is located in quadrant 4, this is important in finding the value for θ.

$x = 1$
$y = -\sqrt{3} \;\; x^2 + y^2 = r^2$
$1^2 + (-\sqrt{3})^2 = r^2$
$1 + 3 = r^2$
$4 = r^2$
$\sqrt{4} = r$
$2 = r \quad tan\theta = \frac{y}{x}$

$= \frac{-\sqrt{3}}{1}$
$= -\sqrt{3}$

There are two possibilities for θ, $\frac{2\pi}{3}$ or $\frac{5\pi}{3}$, since the point is in the fourth quadrant $\theta = \frac{5\pi}{3}$.

Therefore the polar coordinate is $\left(2, \frac{5\pi}{3}\right)$

B. $(-2, 2)$

The point is located in quadrant 2, this is important to finding the value for θ.

$x = -2$
$y = 2$
$x^2 + y^2 = r^2$
$(-2)^2 + 2^2 = r^2$
$4 + 4 = r^2$
$8 = r^2$
$\sqrt{8} = r$
$2\sqrt{2} = r \quad tan\theta = \frac{y}{x}$
$= \frac{2}{-2}$
$= -1$

There are two possibilities for θ, $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$, since the point is in the second quadrant $\theta = \frac{3\pi}{4}$

Therefore the polar coordinate is $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$

4. **Convert Between Polar and Rectangular Coordinates:**

A polar equation is a n equation whose variables are r and θ:

Some examples of polar equations are: $r = \frac{3}{cos\theta}$ and $r = 3tan\theta$

We use the same conversion rules as converting points.

Example: Convert the rectangular equation to polar.

A. $x + y = 4$

$$\begin{aligned} x &= rcos\theta \\ y &= rsin\theta \\ x + y &= 4 \\ rcos\theta + rsin\theta &= 4 \\ r(cos\theta + sin\theta) &= 4 \\ r &= \frac{4}{(cos\theta + sin\theta)} \end{aligned}$$

B. $x^2 + (y+1)^2 = 1$

$$\begin{aligned}
x^2+(y+1)^2 &= r^2\\
rcos\theta^2+(rsin\theta+1)^2 &= r^2\\
r^2cos^2\theta+r^2sin^2\theta+2rsin\theta+1 &= 1\\
r^2\left(cos^2\theta+sin^2\theta\right)+2rsin\theta &= 0\\
r^2+2rsin\theta &= 0\\
r(r+2sin\theta) &= 0
\end{aligned}$$

$r = 0$

$$\begin{aligned}
r+2sin\theta &= 0\\
r &= -2sin\theta
\end{aligned}$$

Example: Convert the polar equation to rectangular.

A. $r = 7$

$$\begin{aligned}
r &= 7\\
r^2 &= x^2+y^2\\
r^2 &= 49\\
49 &= x^2+y^2
\end{aligned}$$

B. $\theta = \frac{\pi}{3}$

$$\begin{aligned}
\theta &= \frac{\pi}{3}\\
tan\theta &= \frac{y}{x}\\
tan\left(\frac{\pi}{3}\right) &= \frac{y}{x}\\
tan\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \div \frac{1}{2}\\
tan\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \times \frac{2}{1}\\
tan\left(\frac{\pi}{3}\right) &= \sqrt{3}\\
\sqrt{3} &= \frac{y}{x}\\
x\sqrt{3} &= y
\end{aligned}$$

C. $r = 2sec\theta$

$$r = 2sec\theta$$
$$r = 2 \times \frac{1}{cos\theta}$$
$$rcos\theta = 2$$
$$x = 2$$

D. $r = -4cos\theta$

$$r = -4cos\theta$$
$$r^2 = -4rcos\theta$$
$$x^2 + y^2 = -4x$$
$$x^2 + 4x + \quad + y^2 = 0 +$$
$$x^2 + 4x + 4 + y^2 = 4$$
$$(x+2)^2 + y^2 = 4$$

Vocabulary

Pole

Fixed point on a polar coordinate system (origin) in from which we rotate.

Polar Coordinate

Is a two dimensional coordinate system in which each point P in the plane is determined by a distance ***r*** from a fixed point **O** called the pole and the angle θ from a fixed direction. We express the coordinate as **(r, θ).**

Polar Axis

The fixed line from which the radius made by the angle vector is measured in a polar coordinate system. (usually a horizontal line)

Polar Equation

An equation of a curve in terms of polar coordinates.

In Summary

We have learned what polar coordinates are and how to plot polar coordinates, to find multiple representations for the same polar coordinate, how to convert between polar and rectangular coordinates and to convert polar equations to rectangular equations.

Check for Understanding

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8.2 Graphing Polar Equations

TEKS

1. P.3.D
2. P.3.E

Lesson Objectives

In this lesson you will learn about:

1. How to graph using point plotting.
2. How to graph using symmetry.
3. Graphs of polar equations.

Introduction

Now that we know how to plot polar coordinates and know what polar equations are we can graph polar equations using point plotting and symmetry. Polar graphs have specific names just like rectangular graphs.

Vocabulary

Limacon, Cardioid, Rose Curve, Petal, Lemniscate.

1. **Graphing Using Point Plotting**

Just like when we graph rectangular equations we can make a table of values but instead of (x,y) we use (r,θ) and evaluate.

Example: Graph the polar equation using point plotting.

$r = 2cos\theta$

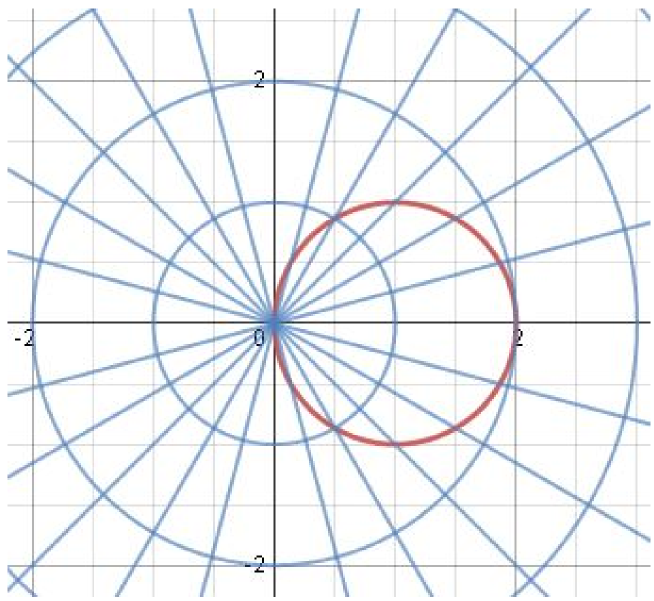
TABLE 8.1:

θ	$r = 2cos\theta$	r	(r,θ)
0	$2cos(0) = 2(1) = 2$	2	$(2,0)$

TABLE 8.1: (continued)

$\frac{\pi}{6}$	$2cos\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$	$\sqrt{3}$	$\left(\sqrt{3}, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$2cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$	1	$\left(1, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$2cos\left(\frac{\pi}{2}\right) = 2(0) = 0$	0	$\left(0, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$2cos\left(\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$	-1	$\left(-1, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	$2cos\left(\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$	$-\sqrt{3}$	$\left(-\sqrt{3}, \frac{5\pi}{6}\right)$
π	$2cos(-\pi) = 2(-1) = -2$	-2	$(-2, \pi)$

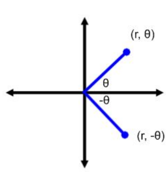
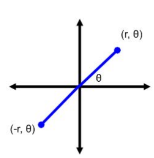
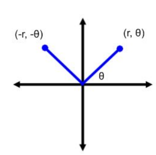
After π values repeat. Take the coordinates and plot.



2. **Graphing Using Symmetry**

There are three types of symmetry for polar equations, just like there are for rectangular equations. Rectangular equations have the three symmetries; origin symmetric, x-axis symmetric, and y-axis symmetric. Polar equations have the three symmetries; pole symmetry, polar axis symmetry, and $\theta = \frac{\pi}{2}$.

TABLE 8.2:

Polar Axis: Replace θ with $-\theta$	The Pole: Replace r with $-r$	$\theta = \frac{\pi}{2}$ Replace (r, θ) with $(-r, -\theta)$
		

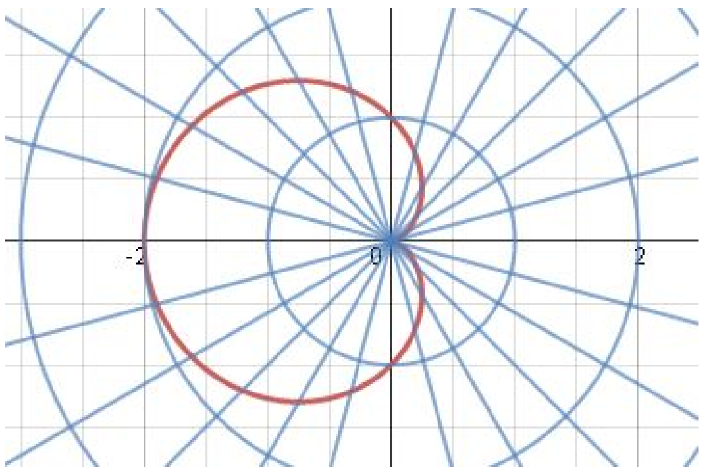
Example: Check for symmetry then graph.

$r = 1 - cos\theta$

TABLE 8.3:

Polar Axis	The Pole	$\theta = \frac{\pi}{2}$
r	=1-cos(-θ)	
r	=1-cos(θ)✓	$-r$
=1-cosθ		
r	=-1+cosθ×	$-r$
=1-cos(-θ)		
-r	=1-cosθ	
r	=-1+cosθ×	

$r = 1 - cos\theta$ is symmetric to the polar axis.



3. **Graphs of Polar Equations**

There are three categories of graphs for polar equations; limacons, rose curves and lemniscates.

Limacons:

Graphs in the form:

$r = a \pm bsin\theta$ for $a > 0$ and $b > 0$

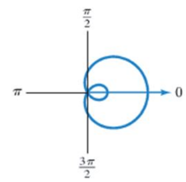
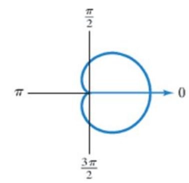
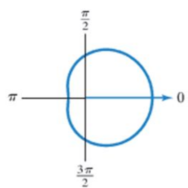
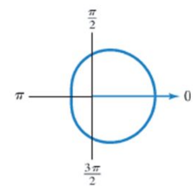
$r = a \pm bcos\theta$

The ratio $\frac{a}{b}$ determines the following shapes:

TABLE 8.4:

Inner Loop	Cardioid
$\frac{a}{b} < 1$	$\frac{a}{b} = 1$

TABLE 8.4: (continued)

	
Dimple, No Inner Loop $1 < \frac{a}{b} < 2$	No Dimple, No Inner Loop $\frac{a}{b} \geq 2$
	

Rose Curves:

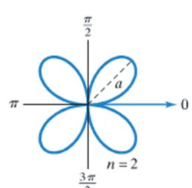
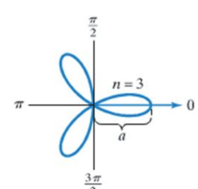
Graphs in the form:

$r = asin(n\theta)$ or $r = acos(n\theta)$ and $a \neq 0$.

If n is even the rose has $2n$ petals, if n is odd the rose has n petals.

a is the "length" of the petal from the pole.

TABLE 8.5:

$r = asin(2\theta)$ $2(n) = 2(2) = 4$ petals	$r = acos(3\theta)$ $n = 3$ petals
	

Lemniscates:

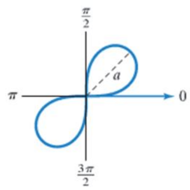
Graphs in the form:

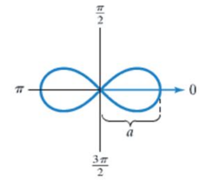
$r^2 = a^2sin(2\theta)$ or $r^2 = a^2cos(2\theta)$ for $a \neq 0$.

TABLE 8.6:

$r^2 = a^2sin(2\theta)$ Symmetric to the pole	$r^2 = a^2cos(2\theta)$ Symmetric to polar axis, the pole, and $\theta = \frac{\pi}{2}$

TABLE 8.6: (continued)





Example: Graph, check the symmetry and name the shape.

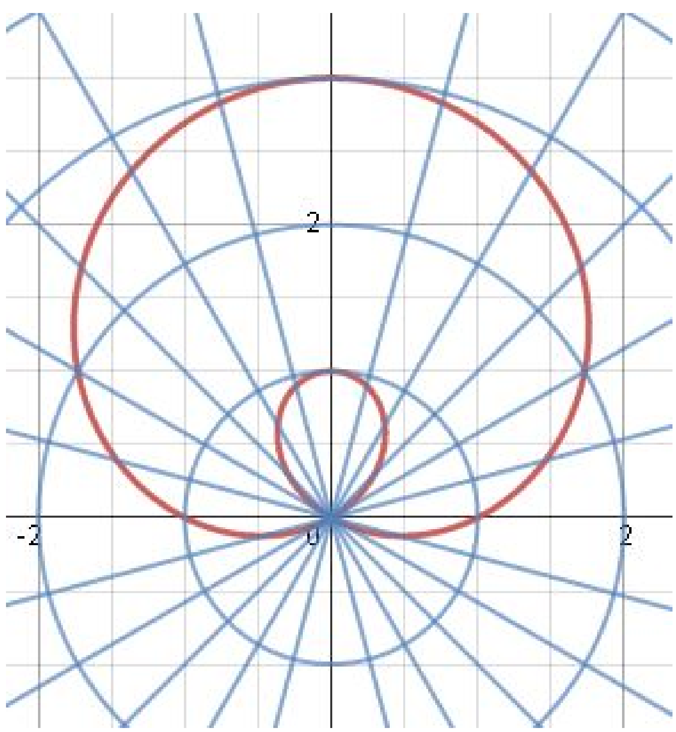
A. $r = 1 + 2sin\theta$

Limacon of the form $r = a + bsin\theta$, where $a = 1$, $b = 2$, $\frac{a}{b} = \frac{1}{2} < 1$, so this is Limacon with an inner loop.

TABLE 8.7:

Polar Axis	Pole	$\theta = \frac{\pi}{2}$
r	=1+2sin(-θ)	
r	=2-2sinθ×	$-r$
=1+2sin(θ)		
r	=-1+2sinθ×	$-r$
=1+2sin(-θ)		
-r	=1-2sinθ	
r	=-1+2sinθ×	

Not Symmetric



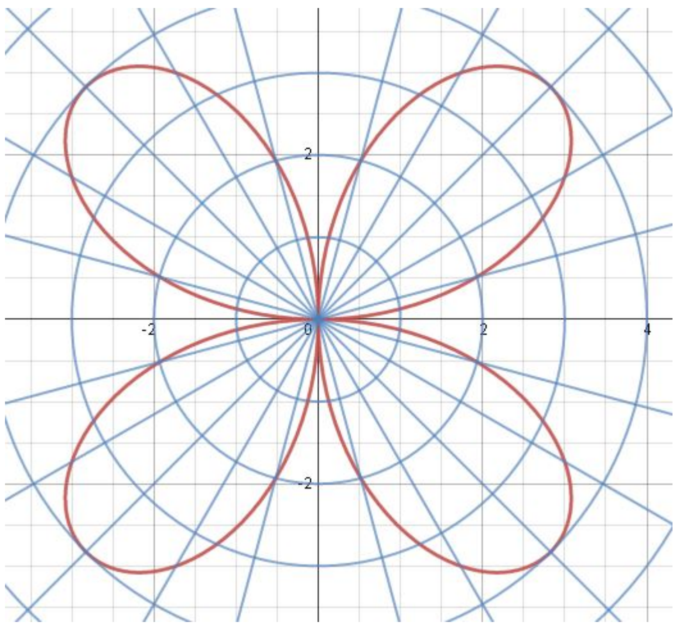
B. $r = 4sin(2\theta)$

Rose Curve of the form $r = asin(n\theta)$ where $a = 4$, $n = 2$, and $2(n) = 2(2) = 4$ petals.

TABLE 8.8:

Polar Axis	Pole	$\theta = \frac{\pi}{2}$
r	=4sin2(-θ)	
r	=-4sin(2θ)×	$-r$
=4sin(2θ)		
r	=-4sin(2θ)×	$-r$
=4sin2(-θ)		
-r	=-4sin(2θ)	
r	=4sin(2θ)✓	

Symmetric to $\theta = \frac{\pi}{2}$



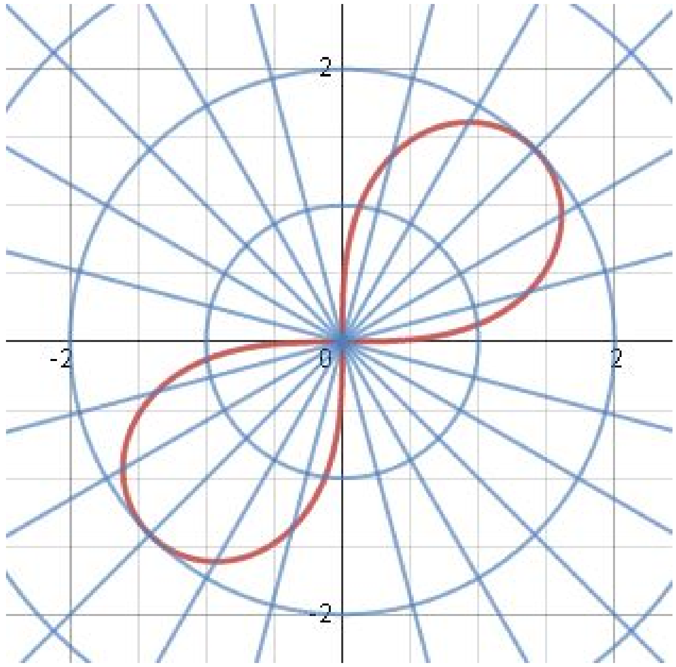
C. $r^2 = 4sin(2\theta)$

Lemniscate of the form $r^2 = a^2 sin(2\theta)$ where $a = 2$.

TABLE 8.9:

Polar Axis	The Pole	$\theta = \frac{\pi}{2}$
r^2	=4sin2(-θ)	
r^2	=-4sin(2θ)×	$(-r)^2$
=4sin(2θ)		
r^2	=4sin(2θ)✓	$(-r)^2$
=4sin2(-θ)		
(-r)^2	=-4sin(2θ)	
r^2	=-4sin(2θ)×	

Symmetric to the pole.



Vocabulary

Limacon

a plane curve whose equation in polar coordinates has one of the forms $\rho = a \cos \theta \pm b$ or $\rho = a \sin \theta \pm b$ and which reduces to a cardioid when $a = b$.

Cardioid

A *cardioid* (from the Greek καρδ´α "heart") is a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius. It is therefore a type of limaçon.

Rose Curve

A smooth curve with leaves arranged symmetrically about a common center.

Lemniscate

a plane curve generated by the locus of the point at which a variable tangent to a rectangular hyperbola intersects a perpendicular from the center to the tangent. Equation: $r^2 = 2a^2 \cos\theta$. Also called Bernoulli's *lemniscate*

In Summary

We have learned how to graph polar coordinates by using point plotting, how to graph using symmetry and the different kinds of graphs of polar equations.

Check for Understanding

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8.3 Complex Numbers in Polar Form

Here you will represent equations and graphs in a new way with parametric equations.

TEKS

1. P.1.D
2. P.1.F
3. P.1.G

Lesson Objective

In this section you will learn about:

1. Plotting complex numbers in the complex plane.
2. Finding the absolute value of a complex number.
3. Convert a complex number to polar form.
4. Multiply and Divide in polar form
5. DeMoiver's Theorem (powers of complex numbers in polar form).

Introduction

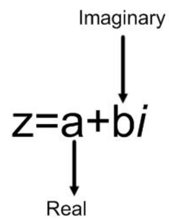
We learned about polar equations and points in the polar coordinate system and how these points and equations can be converted to rectangular and visa versa. Now we can investigate complex numbers and their relationship to polar equations and points.

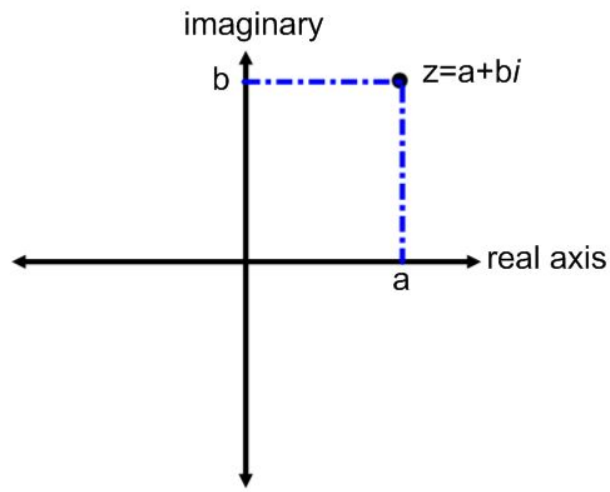
Vocabulary

DeMoivre's Theorem, Complex Plane, Real Axis, Imaginary Axis.

1. **Plotting Complex Numbers in the Complex Plane**

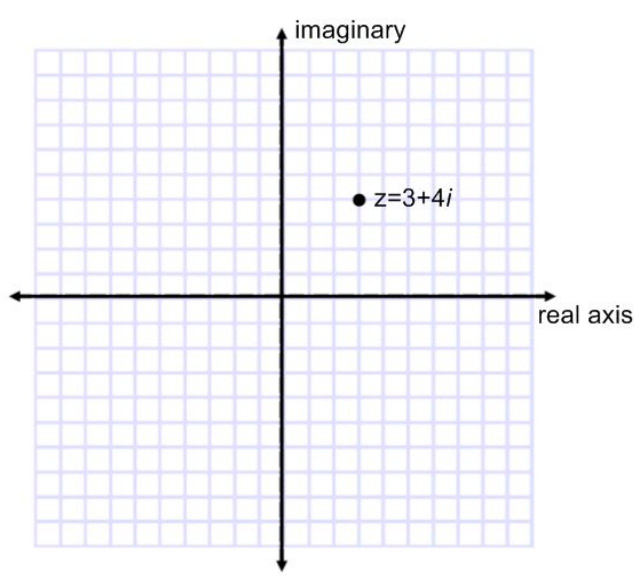
   The complex plane, like the Cartesian plane has two axes, however instead of x and y axes, the axes are real and imaginary. A complex number consists of two parts real and imaginary.



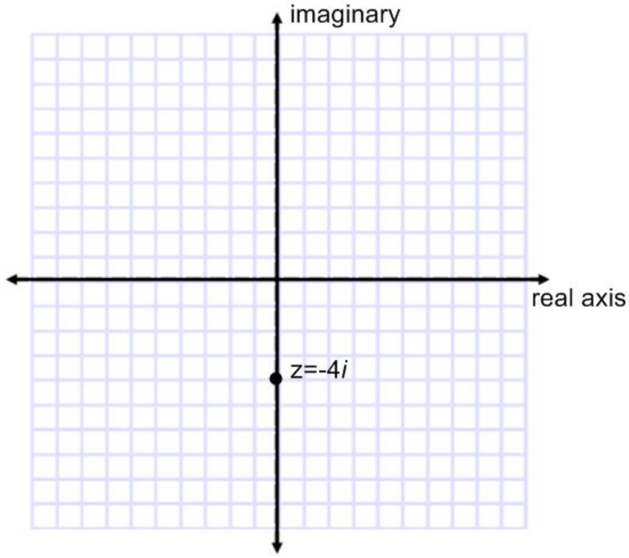


Example: Plot the number.

A. $z = 3+4i$



B. $z = -4i$



2. **Absolute Value of Complex Numbers**

To find the absolute value of a complex number, z, we use the definition of the absolute value. Because the absolute value is the distance from zero, we use the distance formula:

$$\begin{aligned}|z| &= \sqrt{(a-0)^2+(b-0)^2}\\ &= \sqrt{a^2+b^2}\end{aligned}$$

Example: Determine the absolute value of the following.

A. $z = 3+4i$

$$\begin{aligned}|z| &= \sqrt{a^2+b^2}\\ &= \sqrt{3^2+4^2}\\ &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

B. $z = -1-2i$

$$\begin{aligned}|z| &= \sqrt{a^2+b^2}\\ &= \sqrt{(-1)^2+(-2)^2}\\ &= \sqrt{1+4}\\ &= \sqrt{5}\end{aligned}$$

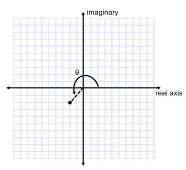
3. **Polar Form of Complex Numbers**

To convert a complex number to polar form we must first plot it in the complex plane and find the r value by using the absolute value. We then know that $tan\theta = \frac{y}{x} = \frac{vertical}{horizontal} = \frac{imaginary}{real} = \frac{b}{a}$ then find the angle θ and substitute into the formula $z = r(cos\theta + isin\theta)$

Example: Write z in polar form.

A. $z = -2-2i$

TABLE 8.10:

	r	$= \sqrt{(-2)^2+(-2)^2}$
r	$= \sqrt{4+4}$	
r	$= \sqrt{8}$	
r	$=2\sqrt{2}$	$tan\theta = \frac{b}{a} = \frac{-2}{-2} = 1$
$\theta = \frac{5\pi}{4}$		

$z = 2\sqrt{2}\left(cos\frac{5\pi}{4} + isin\frac{5\pi}{4}\right)$

B. $z = 2\left(cos60^o + isin60^o\right)$

$$z = r(cos\theta + isin\theta)$$
$$z = 2\left(cos60^o + isin60^o\right)$$
$$z = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$z = 2\left(\frac{1+\sqrt{3}i}{2}\right)$$
$$z = 1+\sqrt{3}i$$

4. **Multiply and Divide in Polar Form**

The rules for multiplication and division are:

$z_1 \times z_2 = r_1 \times r_2[cos(\theta_1+\theta_2) + isin(\theta_1+\theta_2)]$

$z_1 \div z_2 = r_1 \div r_2[cos(\theta_1-\theta_2) + isin(\theta_1-\theta_2)]$

Example: Find the product or quotient.

A. $z_1 \times z_2$ if$z_1 = 4(cos50^o + isin50^o)$ and $z_2 = 7(cos100^o + isin100^o)$

$$\begin{aligned} z_1 \times z_2 &= (4\times 7)[cos(50^o+100^o) + isin(50^o+100^o)] \\ &= 28[cos150^o + isin150^o] \end{aligned}$$

B. $z_1 \div z_2$ if $z_1 = 12\left(cos\frac{3\pi}{4} + isin\frac{3\pi}{4}\right)$ and $z_2 = 4\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$

$$\begin{aligned} z_1 \div z_2 &= (12 \div 4)\left[cos\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) + isin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\right] \\ &= 3\left[cos\frac{\pi}{2} + isin\frac{\pi}{2}\right] \end{aligned}$$

5. Powers of Complex Numbers in Polar Form (DeMoivre's Theorem)

We use the famous DeMoivre's Theorem that states:

Let$z = r(cos\theta + isin\theta)$be a complex number in polar form. If n is a positive integer then:

$z^n = [r(cos\theta + isin\theta)]^n = r^n(cos(n\theta) + isin(n\theta))$

Example: Use DeMoivre's Theorem.

A. $[2cos20^o + isin20^o]^3$

$$\begin{aligned} z^3 &= [2(cos20^o + isin20^o]^3 \\ &= 2^3\left[cos(3)\left(20^o\right) + isin(3)\left(20^o\right)\right] \\ &= 8(cos60^o + isin60^o) \end{aligned}$$

B. $\sqrt[4]{16(cos120^o + isin120^o)}$

$$\begin{aligned} z^{\frac{1}{4}} &= \sqrt[4]{16(cos120^o + isin120^o)} \\ &= [16(cos120^o + isin120^o)]^{\frac{1}{4}} \\ &= 16^{\frac{1}{4}}(cos\left(\frac{1}{4}\right)(120^o) + isin\left(\frac{1}{4}\right)(120^o)) \\ &= 2(cos30^o + isin30^o) \end{aligned}$$

Vocabulary

DeMoivre's Theorem

is used the find the roots of a complex number for any power n, given that n is an integer. **DeMoivre's theorem** can be derived from Euler's equation, and is important because it connects trigonometry to complex numbers.

$z^n = [r(cos\theta + isin\theta)]^n = r^n(cos(n\theta) + isin(n\theta))$

Complex Plane

The field of **complex numbers** is represented as points or vectors in the two-dimensional **plane**. If z = (x,y) = x+iy is a **complex number**, then x is represented on the horizontal, y on the vertical axis.

Real Axis

In the complex plane the x axis is the real axis.

Imaginary Axis

In the complex plane, the y-axis is the imaginary axis.

In Summary

We have learned about plotting complex numbers in the complex plane. We have also learned about finding the absolute value of complex numbers, to change complex numbers to polar form, to multiply and divide numbers in polar form and the powers of complex numbers in polar form.

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8.4 Parametric Equations

TEKS

1. P.3.A
2. P.3.B
3. P.3.C

Lesson Objectives

In this section you will learn about:

1. Graphing plane curves.
2. Eliminating a parameter.
3. Finding parametric equations for functions.
4. Applying parametric equations to projectile motion.

Introduction

Parametric equations can be used in many things, for example we can graph the DNA Helix by using parametric equations.

Vocabulary

Parameter, Parametric Equation, Plane Curve, Orientation, Euclidean Plane, Projectile Motion.

1. **Graphing Plane Curves**

A plane curve is a curve in a Euclidean plan rather than a space curve. Recall from Geometry that the Euclidean plane is different from the Cartesian plane because it accounts for the curve in the Earth's surface. We will be graphing these plane curves using point plotting.

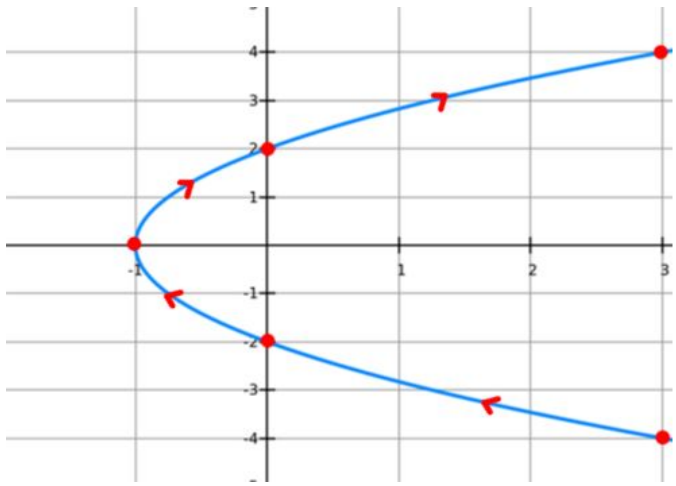
Example: Graph the plane curve defined by the parametric equations.

A. $x = t^2 - 1$ and $y = 2t$ over the interval $-2 \leq t \leq 2$

TABLE 8.11:

t	$x = t^2 - 1$	$y = 2t$	(x, y)
-2	$(-2)^2 - 1 = 4 - 1 = 3$	$2(-2) = -4$	$(3, -4)$
-1	$(-1)^2 - 1 = 1 - 1 = 0$	$2(-1) = -2$	$(0, -2)$
0	$(0)^2 - 1 = 0 - 1 = -1$	$2(0) = 0$	$(-1, 0)$
1	$(1)^2 - 1 = 1 - 1 = 0$	$2(1) = 2$	$(0, 2)$
2	$(2)^2 - 1 = 4 - 1 = 3$	$2(2) = 4$	$(3, 4)$

When we plot the points, we plot them in ascending t order, putting arrows to label the direction.

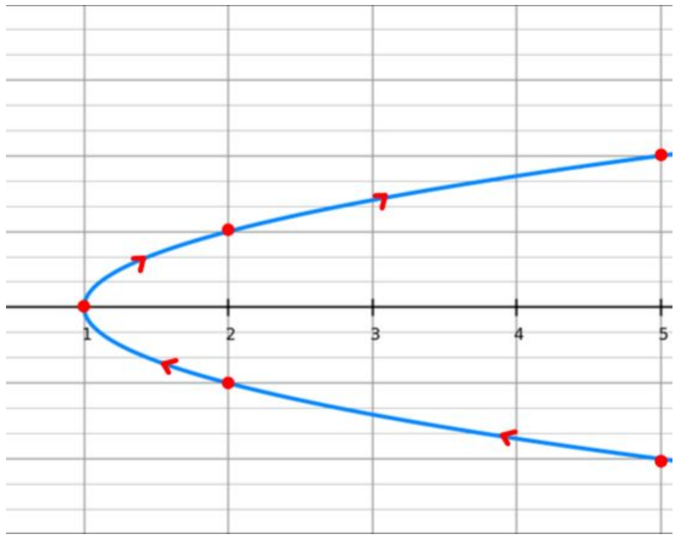


B. $x = t^2 + 1$ and $y = 3t$ over the interval $-2 \leq t \leq 2$

TABLE 8.12:

t	$x = t^2 + 1$	$y = 3t$	(x, y)
-2	$(-2)^2 + 1 = 4 + 1 = 5$	$3(-2) = -6$	$(5, -6)$
-1	$(-1)^2 + 1 = 1 + 1 = 2$	$3(-1) = -3$	$(2, -3)$
0	$(0)^2 + 1 = 0 + 1 = 1$	$3(0) = 0$	$(1, 0)$
1	$(1)^2 + 1 = 1 + 1 = 2$	$3(1) = 3$	$(2, 3)$
2	$(2)^2 + 1 = 4 + 1 = 5$	$3(2) = 6$	$(5, 6)$

We plot the points in ascending t order, putting arrows to label the direction



2. **Eliminating a Parameter**

The parameter that defines x and y can be eliminated to convert a set of parametric equations to rectangular equations. We do this by solving for the parameter and substitution.

Example: Eliminate the parameter to convert to rectangular.

A. $x = \sqrt{t}$ and $y = \frac{1}{2}t + 1$

Start by solving for t in the x equation:

$x = \sqrt{t}$
$x^2 = t$
where $t \geq 0$

Substitute into the y equation:

$$y = \frac{1}{2}t + 1$$
$$y = \frac{1}{2}x^2 + 1$$

B. $x = \sqrt{t}$ and $y = 2t - 1$

Start by solving for t in the x equation:

$x = \sqrt{t}$
$x^2 = t$
where $t \geq 0$

Substitute into the y equation:

$$y = 2t - 1$$
$$y = 2x^2 - 1$$

3. **Find Parametric Equations for Functions**

Finding a set of parametric equations for a rectangular equation is very similar to defining a set of functions that create a composition of functions, there are many different solutions to one problem.

Example: Find a set of parametric equations for the parabola whose equations are given.

A. $y = 9 - x^2$

Answers will vary:

$x = t$
$y = 9 - t^2$ or $x = t^3$
$y = 9 - t^6$ or $x = t + 1$
$y = 8 - t^2 - 2t$
or $x = \frac{t}{2}$
$y = 9 - \frac{t^2}{4}$

B. $y = x^2 - 25$

Answers will vary:

$x = t$
$y = t^2 - 25$ or $x = t^3$
$y = t^6 - 25$ or $x = t + 1$
$y = t^2 + 2t - 25$ or $x = \frac{t}{2}$
$y = \frac{t^2}{4} - 25$

4. **Apply Parametric Equations to Projectile Motion**

Projectile motion is the motion of an object that is launched, thrown, tossed or bounced to create a curve that defines the object's motion path. We can use parametric equations to find horizontal distance between the starting point and the ending point and vertical distance to find height from start to finish using the parameter t for time during the motion.

Example: The path of an object that is launched h feet above the ground with an initial velocity of v_o feet per second and at an angle θ with the horizontal distance and vertical height parametric equations defined by: $x = (v_o cos\theta)t$ and $y = h + (v_o sin\theta)t - 16t^2$ where t is the time in seconds after the object was launched.

A ball was hit with an initial velocity of 180 feet per second at an angle of 40^o to the horizontal. The ball was hit at a height of 3 feet off the ground.

A. Find the parametric equations that describe the position of the ball as a function of time.

$$x = (180cos40^o)t$$
$$y = 3 + (180sin40^o) - 16t^2$$

B. Describe the ball's position after 1, 2, and 3 seconds. Round to the nearest tenth of a foot.

After 1 second:

$$\begin{aligned} x &= (180cos40^o)t \\ &= (180cos40^o)(1) \\ &= 137.888 \\ &= 137.9 \end{aligned}$$

and

$$\begin{aligned} y &= 3+(180sin40^o)t-16t^2 \\ &= 3+(180sin40^o(1)-16(1)^2 \\ &= 102.702 \\ &= 102.7 \end{aligned}$$

After 2 seconds:

$$\begin{aligned} x &= (180cos40^o)t \\ &= (180cos40^o)(2) \\ &= 275.776 \\ &= 275.8 \end{aligned}$$

and

$$\begin{aligned} y &= 3+(180sin40^o)t-16t^2 \\ &= 3+(180sin40^o(2)-16(2)^2 \\ &= 170.404 \\ &= 170.4 \end{aligned}$$

After 3 seconds:

$$\begin{aligned} x &= (180cos40^o)t \\ &= (180cos40^o)(3) \\ &= 413.664 \\ &= 413.7 \end{aligned}$$

and

$$\begin{aligned} y &= 3+(180sin40^o)t-16t^2 \\ &= 3+(180sin40^o(3)-16(3)^2 \\ &= 206.105 \\ &= 206.1 \end{aligned}$$

C. How long to the nearest tenth of a second is the ball in flight? What is the total horizontal distance that the ball travels before it lands?

To find how long the ball is in flight we use the vertical distance equation and set it equal to zero for when the ball hits the ground:

$0 = 3 + (180 sin 40^o)t - 16t^2$

This is not factorable so we use the quadratic formula:

$$\begin{aligned} t &= \frac{-115.702 \pm \sqrt{115.702^2 - 4(-16)(3)}}{2(-16)} \\ &= \frac{-115.702 \pm \sqrt{15306.9}}{-32} \\ &= \frac{-115.702 \pm 123.721}{-32} \end{aligned}$$

Which gives the two solutions:

$$\begin{aligned} t &= -0.025836 \\ t &= 7.2572 \end{aligned}$$

Since time cannot be negative t the so the ball is in flight a total of 7.3 seconds. The horizontal distance from the starting point to when it hits the ground is found by substituting the value 7.3 seconds into the equation:

$$\begin{aligned} x &= (180 cos 40^o)t \\ &= (180 cos 40^o)(7.3) \\ &= 1006.6 \end{aligned}$$

So the horizontal distance is 1006.6 feet.

Vocabulary

Parameter

A *parameter* is a quantity that influences the output or behavior of a *mathematical* object but is viewed as being held constant.

Parametric Equation

Any of a set of **equations** that express the coordinates of the points of a curve as functions of one parameter

Orientation

The *orientation* of the curve represents the direction of travel of the object.

Euclidean Plane

The Euclidean plane is the 2 dimensional x-y coordinate plane that we know.

Projectile Motion

Projectile Motion. Falling Bodies. A formula used to model the vertical *motion* of an object that is dropped, thrown straight up, or thrown straight down

In Summary

We have leaned about graphing parametric equations, eliminating the parameter from a set of parametric equations, finding parametric equations and applying it to projectile motion.

Check for Understanding

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You learned that the polar coordinate system identifies points by their angle (θ) and distance to the origin (r). Polar equations allow you to more easily represent circles and spirals with equations. Parametric equations are a set of two functions, each dependent on the same third variable. You saw that parametric equations are especially useful when working with an object in flight where its vertical and horizontal components don't theoretically interact.

CHAPTER 9 Vectors

Chapter Outline

Graphically, vectors are arrows in the coordinate plane. They have length and direction. Algebraically, they allow a whole new way to think about points, lines, angles. Most importantly, they provide an opportunity for you to apply your knowledge of trigonometry in context.

9.1 Basic Properties of Vectors

Here you will find out what a vector is algebraically and graphically.

TEKS

P.4.I
P.4.J
P.4.K

Lesson Objectives

In this lesson you will learn about:
1. The definition of a vector and what it is.
2. Vector notation.
3. Vector force diagrams.
4. Finding a vector given two points.
5. Vector component form.

Introduction

When you are driving on the highway and you encounter a strong cross wind what happens to your car? When an airplane is flying does wind changes the course of the airplane? These kinds of situations can be modeled by the use of vectors. In this section you will learn about vectors.

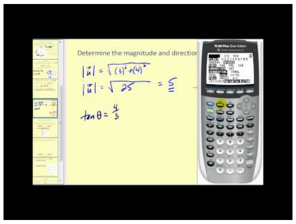
Vocabulary

Tail of Vector, Head of Vector, Directional Bearing, Magnitude, Force Diagrams.

An airplane being pushed off course by wind and a swimmer's movement across a moving river are both examples of vectors in action. Points in the coordinate plane describe location. Vectors, on the other hand, have no location and indicate only direction and magnitude. Vectors can describe the strength of forces like gravity or speed and direction of a ship at sea. Vectors are extremely useful in modeling complex situations in the real world.

What are other differences between vectors and points?

Watch This



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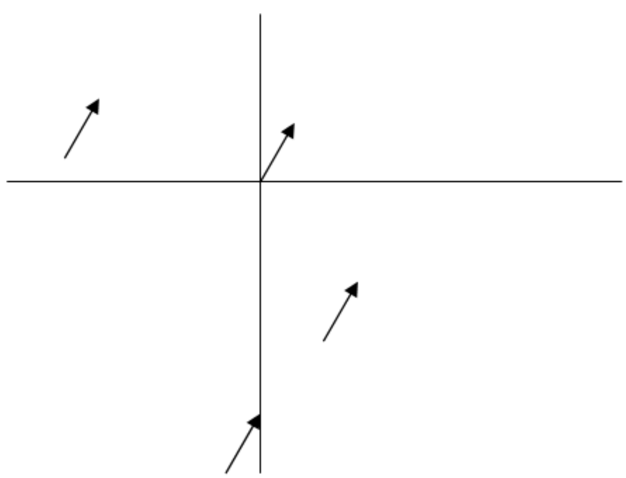
http://www.youtube.com/watch?v=IKzR0Odurm0 James Sousa: Introduction to Vectors

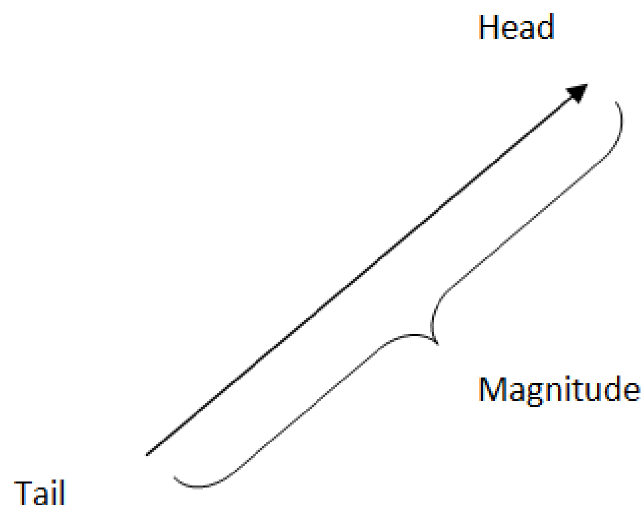
Guidance

A two dimensional vector is represented graphically as an arrow with a tail and a head. The head is the arrow and is also called the terminal point. When finding the vector between two points start with the terminal point and subtract the initial point (the tail).

The two defining characteristics of a vector are its magnitude and its direction. The magnitude is shown graphically by the length of the arrow and the direction is indicated by the angle that the arrow is pointing. Notice how the following vector is shown multiple times on the same coordinate plane. This emphasizes that the location on the coordinate plane does not matter and is not unique. Each representation of the vector has identical direction and magnitude.



One way to define a vector is as a line segment with a direction. Vectors are said to be equal if they have the ***same magnitude*** and the ***same direction.*** The **absolute value of a vector** is the same as the **length of the line segment** or the **magnitude of the vector**. Magnitude can be found by using the Pythagorean Theorem or the distance formula.



There are a few different ways to write a vector v.

$$v, \overrightarrow{v}, \vec{v}, \text{ or } v \text{ with a } \sim \text{ underneath}$$

When you write about vectors algebraically there are a few ways to describe a specific vector. First, you could describe its angle and magnitude as r, θ. Second, you could describe it as an ordered pair: $< x, y >$. Notice that when discussing vectors you should use the brackets $<>$ instead of parentheses because it helps avoid confusion between a vector and a point. Vectors can be multidimensional. Also depending on the author, sometimes the symbol used for magnitude is | | but we will be using || || double lines to avoid confusion with the absolute value symbol

Vector component form from two points.

Given the two points $A = (x_1, y_1) B = (x_2, y_2)$ then the vector components are:

$$\overrightarrow{AB} =$$
$$\overrightarrow{v} = (x_2 - x_1)i + (y_2 - y_1)j$$

Where i is the horizontal component and j is the vertical component.

Another notation for the vector $\overrightarrow{v}$ is

$$\overrightarrow{v} = ai + bj$$

where

$$a = (x_2 - x_1) \ \ and \ \ b = (y_2 - y_1)$$

Magnitude

Given the the two points $A = (x_1, y_1) B = (x_2, y_2)$ Then the magnitude is given by:

$$\| \overrightarrow{AB} \| = \sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$or \quad \| \overrightarrow{AB} \| = \sqrt{a^2 + b^2}$$

Which is actually the distance formula.

Vector component form using the vector magnitude and direction.

Let $\overrightarrow{v}$ be a vector with angle measured from the positive x axis, then the vector can be expressed as

$$\overrightarrow{v} = \| v \| \cos \theta i + \| v \| \sin \theta j$$

Direction Angle

The reference angle will be given by the equation

$$\tan \theta = \frac{|b|}{|a|}$$
$$\theta = \tan^{-1} \left(\frac{|b|}{|a|} \right)$$

Note: This is the reference angle therefore you need to be careful in which quadrant it lies on by the signs of a and b.

Example A

Consider the points: $A(1,3), B(-4,-6), C(5,-13)$. Find the vectors in component form of $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC}, \overrightarrow{CB}$. and their magnitudes.

Solution

Remember that when finding the vector between two points, start with the terminal point and subtract the initial point.

Therefore for the vector $\overrightarrow{AB}$ we have the initial and terminal point $A(1,3), B(-4,6)$ respectively therefore the vector in component form will be

$$\overrightarrow{AB} = < -4 - 1, -6 - 3 > = < -5, -9 >$$
$$Magnitude = \sqrt{(-5)^2, +(-9)^2} = \sqrt{106} \approx 10.29$$

Similarly for the rest of the vectors here is the component form and the magnitudes.

$$\overrightarrow{AB} =< -5,-9 >, \qquad \| \overrightarrow{AB} \| = \sqrt{(-5)^2+(-9)^2} \approx 10.29$$
$$\overrightarrow{BA} =< 5,9 >, \qquad \| \overrightarrow{AB} \| = \sqrt{(5)^2+(9)^2} \approx 10.29$$
$$\overrightarrow{AC} =< 4,-16 >, \qquad \| \overrightarrow{AB} \| = \sqrt{(4)^2+(-16)^2} \approx 16.49$$
$$\overrightarrow{CB} =< -9,7 >, \qquad \| \overrightarrow{AB} \| = \sqrt{(-9)^2+(7)^2} \approx 11.40$$

Example B

For the same points and vectors in example A, are any of the vectors equal?

Solution

In order for any of the vectors to be the same, the vectors must have the same magnitude and same direction. Therefore the only possibility of vectors being equal would be $\overrightarrow{AB}$ *and* $\overrightarrow{BA}$ because they have the same magnitude, now we need to find if they have the same direction.

By the vector component form we can see that vector $\overrightarrow{AB}$ goes left and down and vector $\overrightarrow{BA}$ goes right and up, therefore the vectors are **not equal** because they are going in opposite directions.

Example C

Given the points $A(1,4)$ *and* $B(-7,-3)$

a) Write the vector in vector component form

b) Find the Magnitude and Direction.

c) Write the vector using the magnitude and direction

Solution

a) To find the vector component form we need to find a and b therefore

$$a = x_2 - x_1 = -7 - 1 = -8$$
$$b = y_2 - y_1 = -3 - 4 = -7$$

Therefore the vector component form of $\overrightarrow{AB} =< -8,-7 >$

b) To find the magnitude of the vector we do

$$\begin{aligned}
\| \overrightarrow{AB} \| &= \sqrt{a^2+b^2} \\
&= \sqrt{(-8)^2+(-7)^2} \\
&= \sqrt{113} \\
&\approx 10.63
\end{aligned}$$

To find the direction we have to do the formula

$$\tan\theta = \frac{|b|}{|a|}$$
$$\tan\theta = \left(\frac{|-7|}{|-8|}\right)$$
$$\theta = \tan^{-1}\left(\frac{7}{8}\right)$$
$$\theta = 41.18^\circ$$

Remember that this angle will be the reference angle between the terminal side and the x-axis.

Our vector lies on the Third quadrant therefore the direction of the angle will be $41.18^\circ + 180^\circ = 221.18^\circ$ from the x-axis look at the following picture

c) Writing the vector using magnitude and direction.

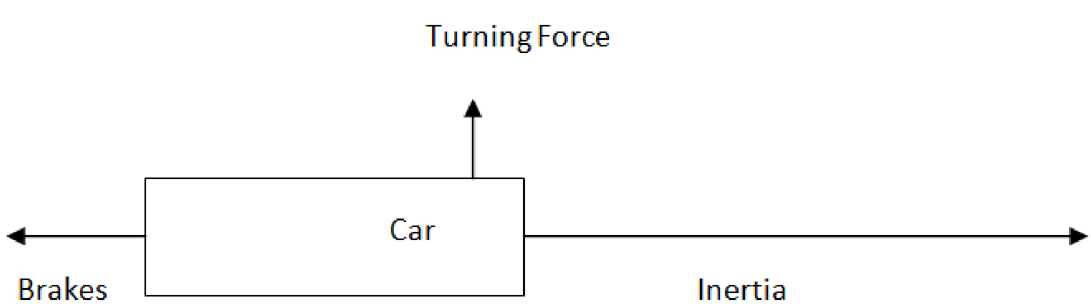
Since the $\| \overrightarrow{AB} \| = \sqrt{113}$ and the angle is 221.18° therefore the vector using magnitude and direction will be

$$\overrightarrow{AB} = \sqrt{113}\cos 221.18^\circ i + \sqrt{113}\sin 221.18^\circ j$$

Example D

A car driving 40 mph brakes and turns around a corner. Draw the approximate force vectors acting on the car as if you were looking down on a map.

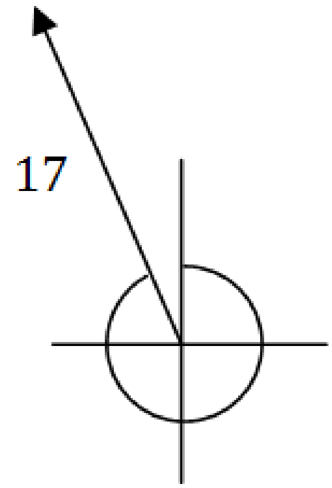
Solution: The primary force is the car's inertia. This is the force acting on the car to keep it moving forward in a straight line. There are three other forces that affect the car. First is gravity, but since this vector is perpendicular to the other vectors it would require a third dimension which will not be considered at this time. The second force is the brakes that act to slow the car down. This force is not as strong as the inertia force. The third force is the act of turning which nudges the front portion of the car to one side. The picture below shows a sketch of the forces acting on the car.



Example E

A ship is traveling NNW at 17 knots (nautical mph). Describe this ship's movement in a vector.

Solution: NNW is halfway between NW and N. When describing ships at sea, it is best to use bearing which has 0° as due North and 270° as due West. This makes NW equal to 315° and NNW equal to 337.5°.



When you see this picture, it turns into a basic trig question to find the x and y components of the vector. Note that the reference angle that the vector makes with the negative portion of the x axis is 67.5°.

$$\cos 67.5^\circ = \frac{x}{17}, \sin 67.5^\circ = \frac{y}{17}$$
$$\vec{v} = 17\cos 67.5^\circ i + 17\sin 67.5^\circ j$$
$$< x, y > \approx < -6.5, 15.7 >$$

therefore for this example the magnitude of the vector will be **:**

$\| < x, y > \| = \sqrt{(-6.5)^2 + (15.7)^2} \approx 16.99$

Concept Problem Revisited

There are many differences between points and vectors. Points are locations and vectors are made up of distance and angles. Parentheses are used for points and <> are used for vectors. One relationship between vectors and points is that a point plus a vector will yield a new point. It is as if there is a starting place and then a vector tells you where to go from that point. Without the starting point, the vector could start from anywhere.

Guided Practice

1. A father is pulling his daughter up a hill. The hill has a 20° incline. The daughter is on a sled which sits on the ground and has a rope that the father pulls as he walks. The rope makes a 39° angle with the slope. Draw a force diagram showing how these forces act on the daughter's center of gravity:

a. The force of gravity.

b. The force holding the daughter in the sled to the ground.

c. The force pulling the daughter backwards down the slope.

d. The force of the father pulling the daughter up the slope.

2. Center the force diagram from the previous question into the origin and identify the angle between each consecutive force vector.

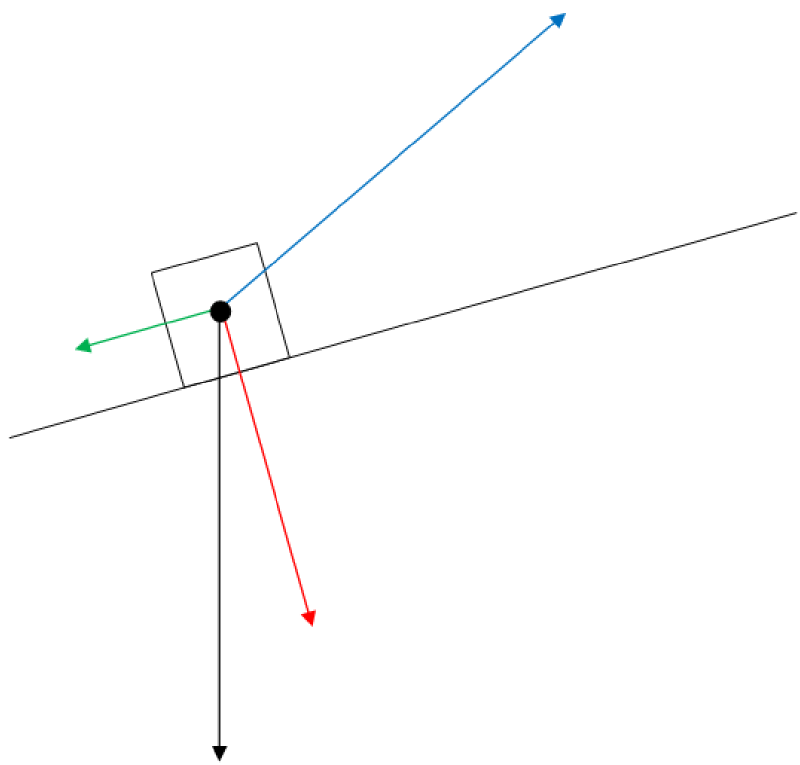
3. Given the following vectors and point, compute the sum.

$A = (1,3), \overrightarrow{v} =< 4,8 >, \overrightarrow{u} =< -1,-5 >$

$A + \overrightarrow{v} + \overrightarrow{u} = ?$

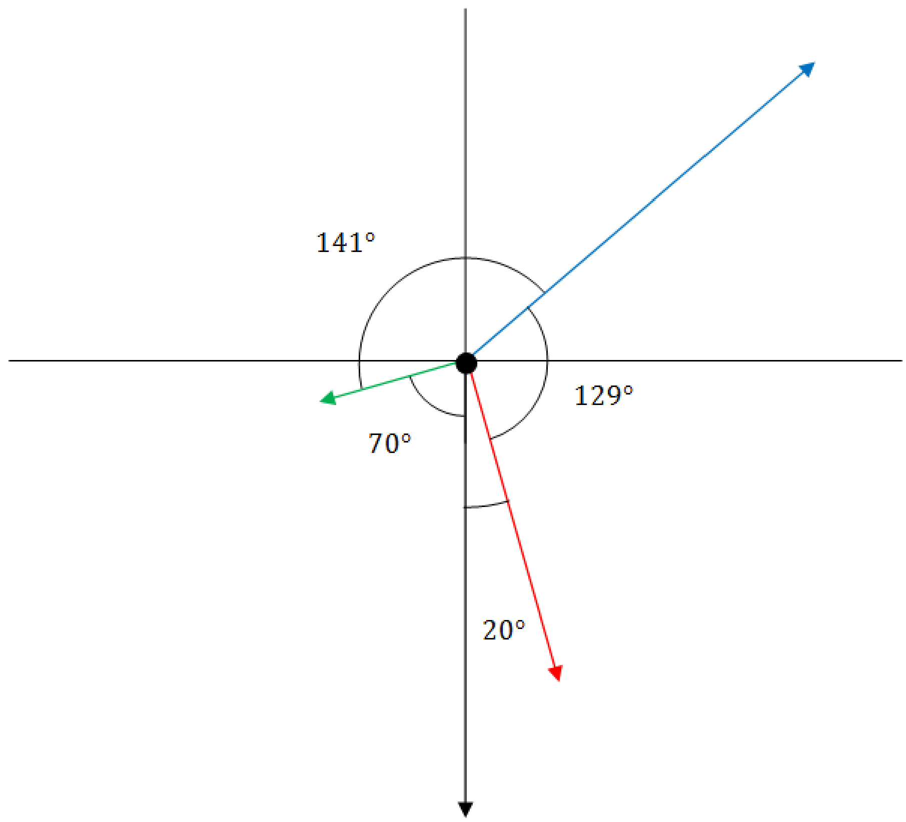
Answers:

1. The girl's center of gravity is represented by the black dot. The force of gravity is the black arrow straight down. The green arrow is gravity's effect pulling the girl down the slope. The red arrow is gravity's effect pulling the girl straight into the slope. The blue arrow represents the force that the father is exerting as he pulls the girl up the hill.



Notice that the father's force vector (blue) is longer than the force pulling the girl down the hill. This means that over time they will make progress and ascend the hill. Also note that the father is wasting some of his energy lifting rather than just pulling. If he could pull at an angle directly opposing the force pulling the girl down the hill, then he would be using all of his energy efficiently.

2. The x and y axis are included as reference and note that the gravity vector overlaps with the negative y axis. In order to find each angle, you must use your knowledge of supplementary, complementary and vertical angles and all the clues from the question. To check, see if all the angles sum to be 360°.



3. $A = (1,3), \overrightarrow{v} = <4,8>, \overrightarrow{u} = <-1,-5>$. $A + \overrightarrow{v} + \overrightarrow{u} = (4,6)$.

Vocabulary

A ***vector*** is a set of instructions indicating direction and magnitude.

The ***tail of a vector*** is the ***initial point*** where the vector starts.

The ***head of a vector*** is the ***terminal point*** where it ends.

A ***force diagram*** is a collection of vectors that each represent a force like gravity or wind acting on an object.

Magnitude refers to the length of the vector and is associated with the strength of the force or the speed of the object.

Bearing is measured with 0° as due North, 90° as East, 180° as South and 270° as West. It is the clockwise angle measured from North.

In Summary

We have learned about what a vector is. We also learned how to find the vector given two points and also to draw diagrams. Finally we learned how to find the magnitude of a vector given the vector components.

Practice

1. Describe what a vector is and give a real-life example of something that a vector could model.

Consider the points: $A(3,5), B(-2,-4), C(1,-12), D(-5,7)$. Find the vectors in component form of:

2. $\overrightarrow{AB}$

3. $\overrightarrow{BA}$

4. $\overrightarrow{AC}$

5. $\overrightarrow{CB}$

6. $\overrightarrow{AD}$

7. $\overrightarrow{DA}$

8. What is $C+\overrightarrow{CB}$? Compute this algebraically and describe why the answer makes sense.

9. Use your answer to the previous problem to help you determine $D+\overrightarrow{DA}$ without doing any algebra.

10. A ship is traveling SSW at 13 knots. Describe this ship's movement in a vector.

11. A vector that describes a ships movement is $<5\sqrt{2}, 5\sqrt{2}>$. What direction is the ship traveling in and what is its speed in knots?

For each of the following vectors, draw the vector on a coordinate plane starting at the origin and find its magnitude.

12. $<3,7>$

13. $<-3,4>$

14. $<-5,10>$

15. $<6,-8>$

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9.2 Operations with Vectors

Here you will add and subtract vectors with vectors and vectors with points.

TEKS

P.4.I
P.4.J
P.4.K

Lesson Objectives

In this lesson you will learn about:
1. How to add and subtract vectors.
2. How to perform scalar multiplication of vectors.
3. How to find the dot product of vectors.
4. How to find the resultant vector when two vectors are combined.
5. How to determine if vectors are perpendicular or parallel.

Introduction

When two or more forces are acting on the same object, they combine to create a new force. A bird flying due south at 10 miles an hour in a headwind of 2 miles an hour only makes headway at a rate of 8 miles per hour. These forces directly oppose each other. In real life, most forces are not parallel. What will happen when the headwind has a slight crosswind as well, blowing NE at 2 miles per hour. How far does the bird get in one hour?

Vocabulary

Resultant Vector, Scaling Vector.

When two or more forces are acting on the same object, they combine to create a new force. A bird flying due south at 10 miles an hour in a headwind of 2 miles an hour only makes headway at a rate of 8 miles per hour. These forces directly oppose each other. In real life, most forces are not parallel. What will happen when the headwind has a slight crosswind as well, blowing NE at 2 miles per hour. How far does the bird get in one hour?

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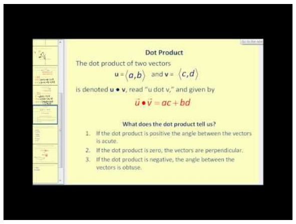

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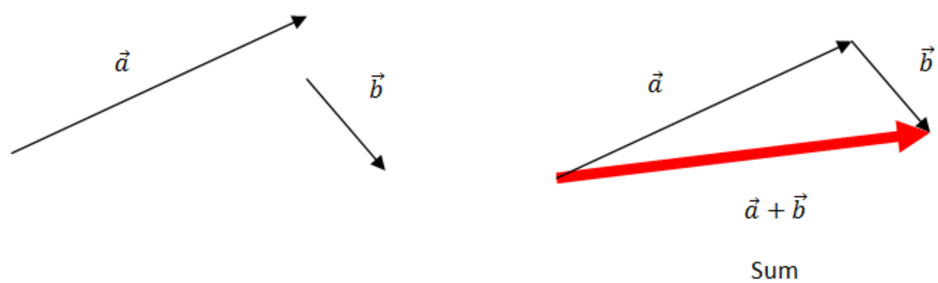
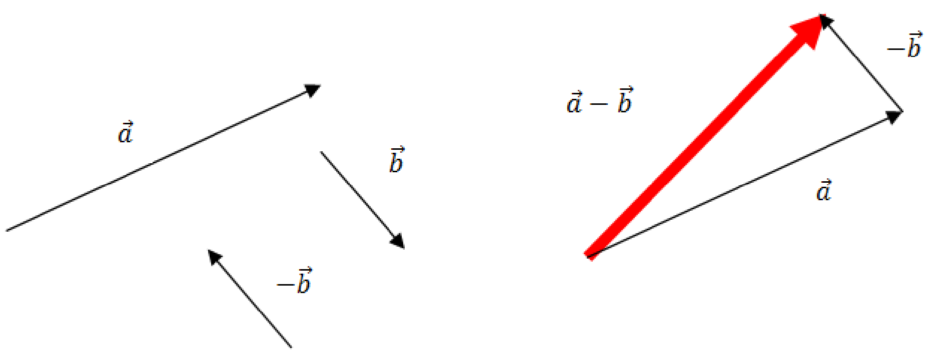
http://www.youtube.com/watch?v=EYIxFJXoUvA James Sousa: Vector Operations

Guidance

When adding vectors, place the tail of one vector at the head of the other. This is called the **tail-to-head rule.** The vector that is formed by joining the tail of the first vector with the head of the second is called the **resultant vector**.



Vector subtraction reverses the direction of the second vector. $\overrightarrow{a}-\overrightarrow{b}=\overrightarrow{a}+(-\overrightarrow{b})$:



Scalar multiplication means to multiply a vector by a number. This changes the magnitude of the vector, but not its direction. If $\overrightarrow{v} =< 3,4 >$, then $2\overrightarrow{v} =< 6,8 >$.

Adding/Subtracting vectors in terms of i and j

If you have two vectors $\overrightarrow{v} = \quad and \quad \overrightarrow{w} =$ then the sum or difference will be

$$\overrightarrow{v}+\overrightarrow{w}=(a_1+a_2)i+(b_1+b_2)j$$
$$\overrightarrow{v}-\overrightarrow{w}=(a_1-a_2)i+(b_1-b_2)j$$

This means add/subtract the i parts together and the j parts together

Scalar Multiplication

Let k be any scalar multiplying the vector $\overrightarrow{v} =$ therefore the scalar multiplication to a vector will be

$$k \cdot \overrightarrow{v} =$$

Direction of a vector

The direction of a vector in component form $v = <a, b>$ is given by the formula

$$\tan\theta = \frac{|b|}{|a|}$$
$$\theta = \tan^{-1}\left(\frac{|b|}{|a|}\right)$$

Note: This will give you the reference angle you need to obtain the correct angle depending on which quadrant it lies on.

The Law of Cosines

In a triangle with sides a,b,c and corresponding angles A,B,C then the relationship of a side with its opposite angle is

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos C$$
$$b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot \cos B$$
$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos A$$

Example A

Given vectors $\vec{v} =< -2,4 > \; and \; \vec{w} =< 3,5 >$ find the resultant vectors.

$a)\ 2\vec{v} - 3\vec{w} \qquad b)\ 3\vec{w} - 5\vec{v} \qquad c)\ 4\vec{v} + 6\vec{w}$

Solution

a) We have to perform the operation

$$\begin{aligned} 2\vec{v} - 3\vec{w} &= 2 < -2,4 > - \; 3 < 3,5 > \\ &= \; < -4,8 > - < 9,15 > \\ &= (-4-9)i + (8-15)j \\ &= -13i - 7j \\ 2\vec{v} - 3\vec{w} &=< -13,-7 > \end{aligned}$$

b) We have to perform the operation

$$\begin{aligned}3\vec{w}-5\vec{v} &= 3<3,5>-\ 5<-2,4>\\ &= \ <9,15>-<-10,20>\\ &= (9--10)i+(15-20)j\\ &= 19i-5j\\ 3\vec{w}-5\vec{v} &=<19,-5>\end{aligned}$$

c) We have to perform the operation

$$\begin{aligned}4\vec{v}+6\vec{w} &= 4<-2,4>+\ 6<3,5>\\ &= \ <-8,16>+<18,30>\\ &= (-8+18)i+(16+30)j\\ &= 10i+46j\\ 4\vec{v}+6\vec{w} &=<10,46>\end{aligned}$$

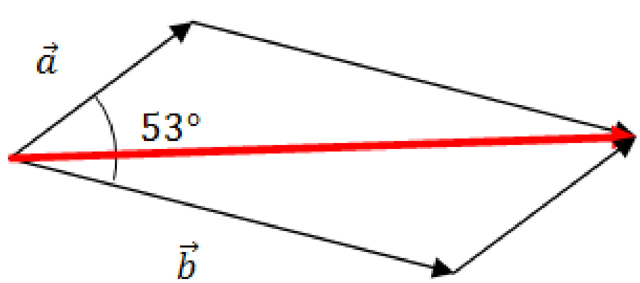
From a geometry course we learned that the sum of the interior angles in any quadrilateral is 360°, more importantly if we have a parallelogram opposite angles are congruent (equal) and if we have one angle θ then the measure of an adjacent angle α will be $\alpha = 180° - \theta$ look at the following image

You will need the previous information to solve some of the following examples.

Example B

Two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, have magnitudes of 5 and 9 respectively. The angle between the vectors is 53°. Find $|\overrightarrow{a}+\overrightarrow{b}|$.

Solution: Adding vectors can be done in either order (just like with regular numbers). Subtracting vectors must be done in a specific order or else the vector will be negative (just like with regular numbers). In either case, use geometric reasoning and the law of cosines with the parallelogram that is formed to find the magnitude of the resultant vector.



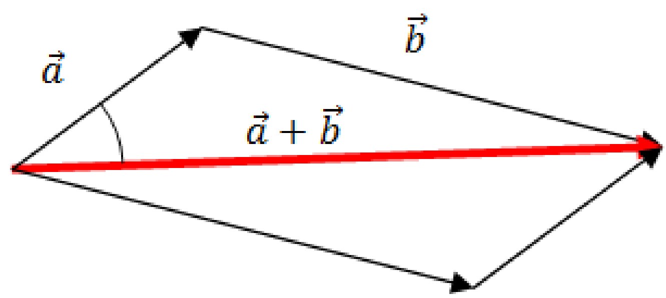
In order to fine the magnitude of the resulting vector (x), note the triangle on the bottom that has sides 9 and 5 with included angle 127°. This is the angle on the top of the image.

$$x^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos\theta$$
$$x^2 = 9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 127^\circ$$
$$x = \sqrt{9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 127^\circ}$$
$$x \approx 12.66$$

Example C

Using the picture from Example B, what is the angle that the sum $\overrightarrow{a} + \overrightarrow{b}$ makes with $\overrightarrow{a}$?

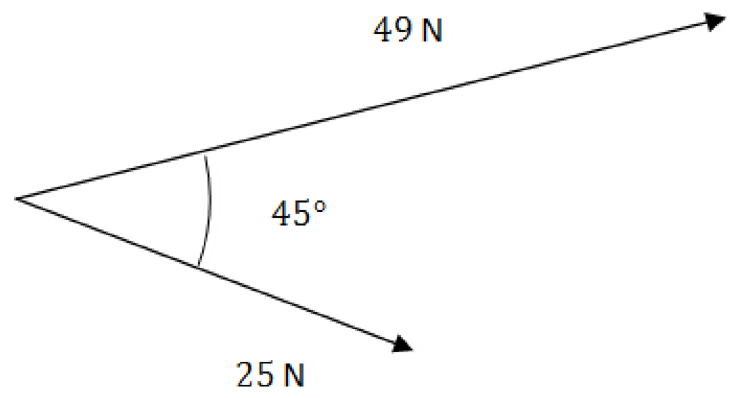
Solution: Start by drawing a good picture and labeling what you know. $|\overrightarrow{a}| = 5, |\overrightarrow{b}| = 9, |\overrightarrow{a} + \overrightarrow{b}| \approx 12.66$. Since you know three sides of the triangle and you need to find one angle, this is the SSS application of the Law of Cosines.



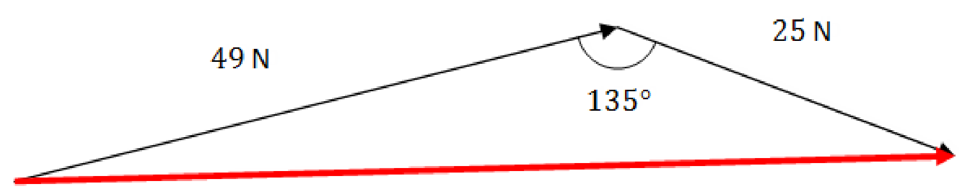
$$9^2 = 12.66^2 + 5^2 - 2 \cdot 12.66 \cdot 5 \cdot \cos\theta$$
$$\theta = 34.6^\circ$$

Example D

Elaine started a dog walking business. She walks two dogs at a time named Elvis and Ruby. They each pull her in different directions at a 45° angle with different forces. Elvis pulls at a force of 25 N and Ruby pulls at a force of 49 N. How hard does Elaine need to pull so that she can stay balanced? *Note: N stands for Newtons which is the standard unit of force.*



Solution: Even though the two vectors are centered at Elaine, the forces are added which means that you need to use the tail-to-head rule to add the vectors together. Finding the angle between each component vector requires logical use of supplement angles.

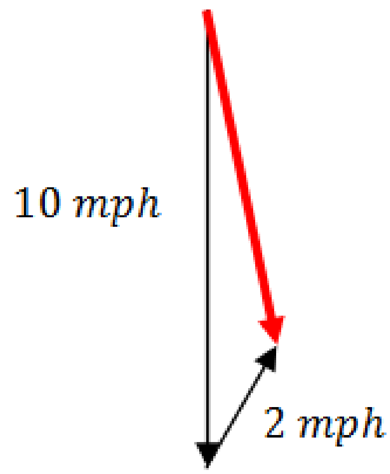


$$x^2 = 49^2 + 25^2 - 2 \cdot 49 \cdot 25 \cdot \cos 135^\circ$$
$$x = \sqrt{49^2 + 25^2 - 2 \cdot 49 \cdot 25 \cdot \cos 135^\circ}$$
$$x \approx 68.98\ N$$

In order for Elaine to stay balanced, she will need to counteract this force with an equivalent force of her own in the exact opposite direction.

Concept Problem Revisited

A bird flying due south at 10 miles an hour with a cross headwind of 2 mph heading NE would have a force diagram that looks like this:



The angle between the bird's vector and the wind vector is 45° which means this is a perfect situation for the Law of Cosines. Let $x =$ the red vector.

$$x^2 = 10^2 + 2^2 - 2 \cdot 10 \cdot 2 \cdot \cos 45^\circ$$
$$x \approx 8.7$$

The bird is blown slightly off track and travels only about 8.7 miles in one hour.

Guided Practice

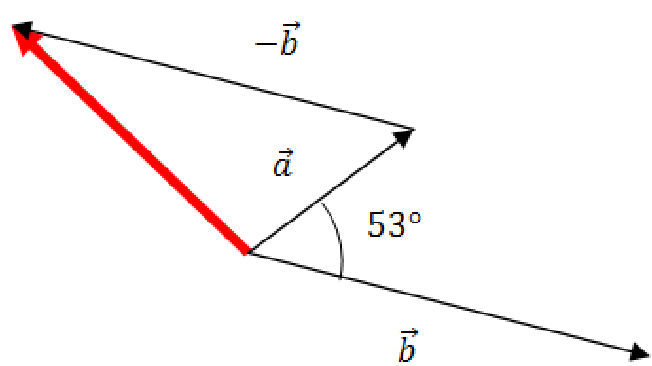
1. Find the magnitude of $|\vec{a} - \vec{b}|$ from Example B

2. Consider vector $\overrightarrow{v} =< 2,5 >$ and vector $\overrightarrow{u} =< -1,9 >$. Determine the component form of the following: $3\overrightarrow{v} - 2\overrightarrow{u}$.

3. An airplane is flying at a bearing of 270° at 400 mph. A wind is blowing due south at 30 mph. Does this cross wind affect the plane's speed?

Answers:

1.



The angle between $-\vec{b}$ and $\vec{a}$ is $53°$ because in the diagram $\vec{b}$ is parallel to $-\vec{b}$ so you can use the fact that alternate interior angles are congruent. Since the magnitudes of vectors $\vec{a}$ and $-\vec{b}$ are known to be 5 and 9, this becomes an application of the Law of Cosines.

$$\begin{aligned} y^2 &= 9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 53° \\ y &\approx 7.2 \end{aligned}$$

2. Do multiplication first for each term, followed by vector subtraction.

$$\begin{aligned} 3 \cdot \vec{v} - 2 \cdot \vec{u} &= 3 \cdot < 2,5 > -2 \cdot < -1,9 > \\ &= < 6,15 > - < -2,18 > \\ &= < 8,-3 > \end{aligned}$$

3. Since the cross wind is perpendicular to the plane, it pushes the plane south as the plane tries to go directly east. As a result the plane still has an airspeed of 400 mph but the ground speed (true speed) needs to be calculated.

$$\begin{aligned} 400^2 + 30^2 &= x^2 \\ \sqrt{400^2 + 30^2} &= x \\ x &\approx 401 \end{aligned}$$

Vocabulary

A ***resultant vector*** is the vector that is produced when two or more vectors are summed or subtracted. It is also what is produced when a single vector is scaled by a constant.

Scaling a vector means that the components are each multiplied by a common scale factor. For example: $4 \cdot < 3,2 > = < 12,8 >$

In Summary

We have learned about finding the resultant force of when two vectors are being added knowing the angle between them and using the law of cosines. We have also learned to multiply a vector by a scalar.

Practice

Consider vector $\vec{v} = < 1,3 >$ and vector $\vec{u} = < -2,4 >$.

1. Determine the component form of $5\overrightarrow{v} - 2\overrightarrow{u}$.
2. Determine the component form of $-2\overrightarrow{v} + 4\overrightarrow{u}$.
3. Determine the component form of $6\overrightarrow{v} + \overrightarrow{u}$.
4. Determine the component form of $3\overrightarrow{v} - 6\overrightarrow{u}$.
5. Find the magnitude of the resultant vector from #1.
6. Find the magnitude of the resultant vector from #2.
7. Find the magnitude of the resultant vector from #3.
8. Find the magnitude of the resultant vector from #4.
9. The vector $< 3, 4 >$ starts at the origin. What is the direction of the vector?
10. The vector $< -1, 2 >$ starts at the origin. What is the direction of the vector?
11. The vector $< 3, -4 >$ starts at the origin. What is the direction of the vector?
12. A bird flies due south at 8 miles an hour with a cross headwind blowing due east at 15 miles per hour. How far does the bird get in one hour?
13. What direction is the bird in the previous problem actually moving?
14. A football is thrown at 50 miles per hour due north. There is a wind blowing due east at 8 miles per hour. What is the actual speed of the football?
15. What direction is the football in the previous problem actually moving?

Multimedia

MEDIA

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9.3 Resolution of Vectors into Components

Here you will find unit vectors and you will convert vectors into linear combinations of standard unit vectors and component vectors.

TEKS

P.4.F
P.4.I
P.4.J
P.4.K

Lesson Objectives

In this lesson you bill learn about:
1. Finding a unit vector.
2. Changing vectors into vector component form.
3. Finding resultant magnitudes by adding different vectors and their direction/bearing.

Introduction

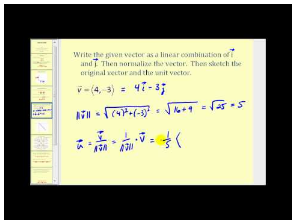
Sometimes working with horizontal and vertical components of a vector can be significantly easier than working with just an angle and a magnitude. This is especially true when combining several forces together.

Vocabulary

Unit Vector, Component Form, Standard Unit Vector, Linear Combination, Bearing.

Consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 lb of force at an angle of 41°. Connie pulls with 10 lb of force at an angle of 100°. Cynthia pulls with 12 lb of force at an angle of 200°. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?

Watch This



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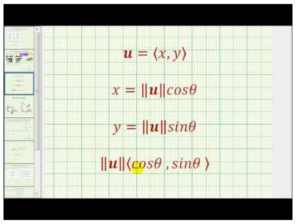
http://www.youtube.com/watch?v=Ect0fBnBlLc James Sousa: The Unit Vector



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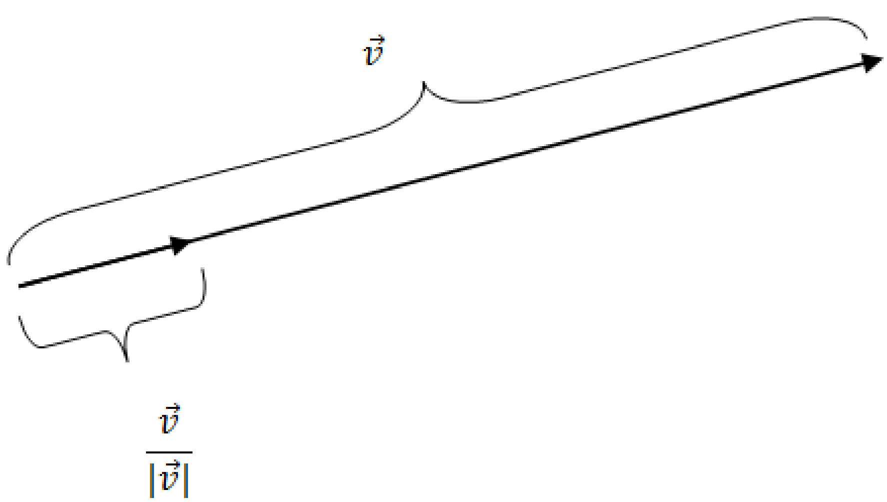
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/61372

http://www.youtube.com/watch?v=WZ3xzVHT0mc James Sousa: Find the Component Form of a Vector Given Magnitude and Direction

Guidance

A unit vector is a vector of length one. Sometimes you might wish to scale a vector you already have so that it has a length of one. If the length was five, you would scale the vector by a factor of $\frac{1}{5}$ so that the resulting vector has magnitude of 1. Another way of saying this is that a unit vector in the direction of vector $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$.



There are two standard unit vectors that make up all other vectors in the coordinate plane. They are $\vec{i}$ which is the vector $< 1,0 >$ and $\vec{j}$ which is the vector $< 0,1 >$. These two unit vectors are perpendicular to each other. A linear combination of $\vec{i}$ and $\vec{j}$ will allow you to uniquely describe any other vector in the coordinate plane. For instance the vector $< 5,3 >$ is the same as $5\vec{i} + 3\vec{j}$.

Often vectors are initially described as an angle and a magnitude rather than in component form. Working with vectors written as an angle and magnitude requires extremely precise geometric reasoning and excellent pictures. One advantage of rewriting the vectors in component form is that much of this work is simplified.

Bearing is the angle measured in a clockwise direction measured from due North.

Example A

A plane has a bearing of 60° and is going 350 mph. Find the component form of the velocity of the airplane. Look at the figure below.

Solution: A bearing of 60° is the same as a 30° on the unit circle which corresponds to the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. When written as a vector $< \frac{\sqrt{3}}{2}, \frac{1}{2} >$ is a unit vector because it has magnitude 1. Now you just need to scale by a factor of 350 and you get your answer of $< 175\sqrt{3}, 175 >$.

Another way to obtain the component form will be

$$\vec{v} = 350\cos 30^\circ i + 350\sin 30^\circ j$$
$$\vec{v} = 350\left(\frac{\sqrt{3}}{2}\right)i + 350\left(\frac{1}{2}\right)j$$
$$\vec{v} = 175\sqrt{3}i + 175j$$
$$\vec{v} =< 175\sqrt{3}, 175 >$$

Example B

Consider the plane flying in Example A. If there is wind blowing with the bearing of 300° at 45 mph, what is the component form of the total velocity of the airplane? In the following figure, the green vector is the solution.

Solution:

A bearing of 300° is the same as 150° on the unit circle which corresponds to the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. You can now write and then scale the wind vector.

$45 \cdot < -\frac{\sqrt{3}}{2}, \frac{1}{2} > = < -\frac{45\sqrt{3}}{2}, \frac{45}{2} >$

Since both the wind vector and the velocity vector of the airplane are written in component form, you can simply sum them to find the component vector of the total velocity of the airplane.

$$< 175\sqrt{3}, 175 > + < -\frac{45\sqrt{3}}{2}, \frac{45}{2} > = < \frac{305\sqrt{3}}{2}, \frac{395}{2} > = < 152.5\sqrt{3}, 197.5 >$$

Another form of obtaining the resultant vector is performing the following. Let $\vec{v}$ be the plane vector and $\vec{w}$ be the wind vector. Therefore the resultant vector will be

$$\vec{v} = 350\cos 30^\circ i + 350\sin 30^\circ j$$
$$\vec{v} = 350\left(\frac{\sqrt{3}}{2}\right) i + 350\left(\frac{1}{2}\right) j$$
$$\vec{v} = 175\sqrt{3}i + 175j$$
$$\vec{v} = < 175\sqrt{3}, 175 >$$

$$\vec{w} = 45\cos 150^\circ i + 45\sin 150^\circ j$$
$$\vec{w} = 45\left(-\frac{\sqrt{3}}{2}\right) i + 45\left(\frac{1}{2}\right) j$$
$$\vec{w} = -22.5\sqrt{3}i + 22.5j$$
$$\vec{w} = < -22.5\sqrt{3}, 22.5 >$$

Therefore $\vec{v}+\vec{w} = < 175\sqrt{3} + -22.5\sqrt{3}, 175 + 22.5 > = < 152.5\sqrt{3}, 197.5 >$ which is the same solution.

Example C

Consider the plane and wind in Example A and Example B. Find the actual ground speed and direction of the plane (as a bearing).

Solution:

Since you already know the component vector of the total velocity of the airplane, you should remember that these components represent an x distance and a y distance and the question asks for the hypotenuse. Therefore we need to find the magnitude of the resultant vector (green vector) in the figure for example B. Therefore our resultant vector was

$$\vec{v}+\vec{w} = < 152.5\sqrt{3}, 197.5 >$$

and our magnitude will be

$$\begin{aligned} \| \vec{v}+\vec{w} \| &= \sqrt{(152.5\sqrt{3})^2 + (197.5)^2} \\ &= \sqrt{108775} \\ &\approx 329.8 \ \ mph \end{aligned}$$

The airplane is traveling at about 329.8 mph.

Since you know the x and y components, you can use tangent to find the angle. Then convert this angle into bearing.

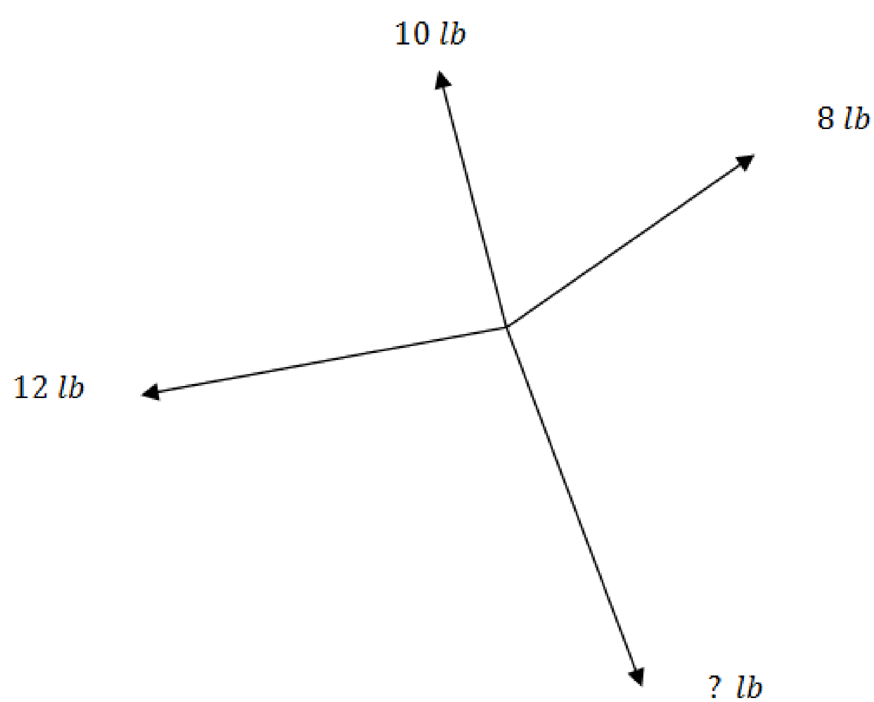
$$\tan\theta = \frac{\left(\left|\frac{395}{2}\right|\right)}{\left(\left|\frac{305\sqrt{3}}{2}\right|\right)}$$
$$\theta = \tan^{-1}\left(\frac{\left(\left|\frac{395}{2}\right|\right)}{\left(\left|\frac{305\sqrt{3}}{2}\right|\right)}\right)$$
$$\theta \approx 36.8^\circ$$

Remember that this is a reference angle an it lies in quadrant one therefore an angle of 36.8° on the unit circle is equivalent to a bearing of 53.2°. Look at figure below.

Note that you can do the entire problem in bearing by just switching sine and cosine, but it is best to truly understand what you are doing every step of the way and this will probably involve always going back to the unit circle.

Concept Problem Revisited

Consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 lb of force at an angle of 41°. Connie pulls with 10 lb of force at an angle of 100°. Cynthia pulls with 12 lb of force at an angle of 200°. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?



To add the three vectors together would require several iterations of the Law of Cosines. Instead, write each vector in component form and set equal to a zero vector indicating that the candy does not move.

$$\overrightarrow{L} + \overrightarrow{CON} + \overrightarrow{CYN} + \overrightarrow{T} = <0,0>$$

$$\begin{aligned}&< 8\cdot\cos 41^\circ, 8\cdot\sin 41^\circ >\\ &+ < 10\cdot\cos 100^\circ, 10\cdot\sin 100^\circ >\\ &+ < 12\cdot\cos 200^\circ, 12\cdot\sin 200^\circ >\\ &+ \overrightarrow{T}\\ &=< 0, 0 >\end{aligned}$$

Use a calculator to add all the x components and bring them to the far side and the y components and then subtract from the <0,0>to obtain

$$\overrightarrow{T} \approx < 6.98, -10.99 >$$

Turning this component vector into an angle and magnitude yields how hard and in what direction he would have to pull. Terry will have to pull with about 13 lb of force at an angle of 302.4°.

Guided Practice

$\overrightarrow{v} =< 2, -5 >, \overrightarrow{u} =< -3, 2 >, \overrightarrow{t} =< -4, -3 >, \overrightarrow{r} =< 5, y >$

$B = (4, -5), P = (-3, 8)$

1. Solve for y in vector $\overrightarrow{r}$ to make $\overrightarrow{r}$ perpendicular to $\overrightarrow{t}$.
2. Find the unit vectors in the same direction as $\overrightarrow{u}$ and $\overrightarrow{t}$.
3. Find the point 10 units away from B in the direction of P.

Answers:

1. $\overrightarrow{t}$ has slope $\frac{3}{4}$ which means that $\overrightarrow{r}$ must have slope $-\frac{4}{3}$. A vector's slope is found by putting the y component over the x component just like with $\frac{rise}{run}$.

$$\begin{aligned}\frac{y}{5} &= -\frac{4}{3}\\ y &= -\frac{20}{3}\end{aligned}$$

2. To find a unit vector, divide each vector by its magnitude.

$$\frac{\overrightarrow{u}}{|\overrightarrow{u}|} =< \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} >, \frac{\overrightarrow{t}}{|\overrightarrow{t}|} =< \frac{-4}{5}, \frac{-3}{5} >$$

3. The vector $\overrightarrow{BP}$ is $< -7, 13 >$. First take the unit vector and then scale it so it has a magnitude of 10.

$$\begin{aligned}\frac{BP}{|BP|} &=< \frac{-7}{\sqrt{218}}, \frac{13}{\sqrt{218}} >\\ 10\cdot\frac{BP}{|BP|} &=< \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} >\end{aligned}$$

You end up with a vector that is ten units long in the right direction. The question asked for a point from B which means you need to add this vector to point B.

$(4, -5) + < \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} > \approx (-0.74, 3.8)$

Vocabulary

A ***unit vector*** is a vector of magnitude one.

Component form means in the form $< x, y >$. To translate from magnitude r and direction θ, use the relationship $< r \cdot \cos\theta, r \cdot \sin\theta > = < x, y >$.

The ***standard unit vectors*** are $\overrightarrow{i}$ which is the vector $< 1, 0 >$ and $\overrightarrow{j}$ which is the vector $< 0, 1 >$.

A ***linear combination*** of vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ means a multiple of one plus a multiple of the other.

A **Bearing** is an angle measurement from from due North in a clockwise direction.

In Summary

We have learned how to find a unit vector, and how to convert a vector into a vector component form. Also we have learned that when more than one different form is acting on an object, it is easier to convert the vector into component form and add the *i* components and the *j* components to find the resultant vector. Finally we learned to find the magnitude of the force and the direction of the force.

Practice

$\overrightarrow{v} = < 1, -3 >, \overrightarrow{u} = < 2, 5 >, \overrightarrow{t} = < 9, -1 >, \overrightarrow{r} = < 2, y >$

$A = (-3, 2), B = (5, -2)$

1. Solve for y in vector $\overrightarrow{r}$ to make $\overrightarrow{r}$ perpendicular to $\overrightarrow{t}$.

2. Find the unit vector in the same direction as $\overrightarrow{u}$.

3. Find the unit vector in the same direction as $\overrightarrow{t}$.

4. Find the unit vector in the same direction as $\overrightarrow{v}$.

5. Find the unit vector in the same direction as $\overrightarrow{r}$.

6. Find the point exactly 3 units away from A in the direction of B.

7. Find the point exactly 6 units away from B in the direction of A.

8. Find the point exactly 5 units away from A in the direction of B.

9. Jack and Jill went up a hill to fetch a pail of water. When they got to the top of the hill, they were very thirsty so they each pulled on the bucket. Jill pulled at 30° with 20 lbs of force. Jack pulled at 45° with 28 lbs of force. What is the resulting vector for the bucket?

10. A plane is flying on a bearing of 60° at 400 mph. Find the component form of the velocity of the plane. What does the component form tell you?

11. A baseball is thrown at a 70° angle with the horizontal with an initial speed of 30 mph. Find the component form of the initial velocity.

12. A plane is flying on a bearing of 200° at 450 mph. Find the component form of the velocity of the plane.

13. A plane is flying on a bearing of 260° at 430 mph. At the same time, there is a wind blowing at a bearing of 30° at 60 mph. What is the component form of the velocity of the plane?

14. Use the information from the previous problem to find the actual ground speed and direction of the plane.

15. Wind is blowing at a magnitude of 40 mph with an angle of 25° with respect to the east. What is the velocity of the wind blowing to the north? What is the velocity of the wind blowing to the east?

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9.4 Dot Product and Angle Between Two Vectors

Here you will compute the dot product between two vectors and interpret its meaning.

TEKS

P.4.I
P.4.J
P.4.K

Lesson Objectives

In this lesson you will learn about:
1. Performing dot product given two vectors.
2. Find the angle between two vectors.

Introduction

While two vectors cannot be strictly multiplied like numbers can, there are two different ways to find the product between two vectors. The cross product between two vectors results in a new vector perpendicular to the other two vectors. You can study more about the cross product between two vectors when you take Linear Algebra. The second type of product is the dot product between two vectors which results in a regular number. This number represents ***how much of one vector goes in the direction of the other***. In one sense, it indicates how much the two vectors agree with each other. This concept will focus on the dot product between two vectors.
What is the dot product between $< -1, 1 >$ and $< 4, 4 >$? What does the result mean?

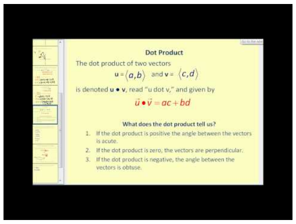
Vocabulary

Dot Product, Orthogonal.

What is the dot product between $< -1, 1 >$ and $< 4, 4 >$? What does the result mean?

Watch This

Watch the portion of this video focusing on the dot product:



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http://www.youtube.com/watch?v=EYIxFJXoUvA James Sousa: Vector Operations

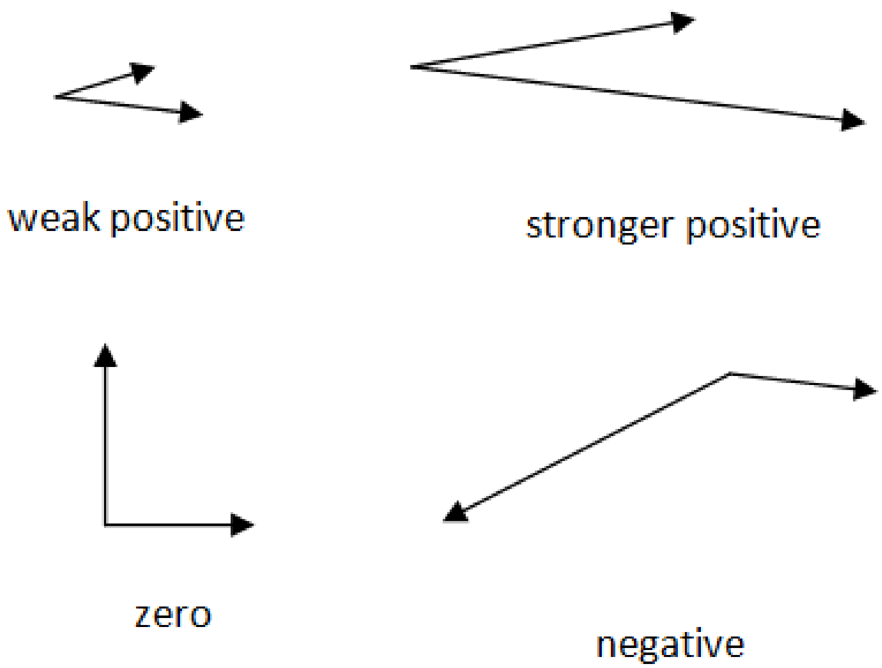
Guidance

The dot product is defined as:

$u \cdot v = < u_1, u_2 > \cdot < v_1, v_2 >= u_1 v_1 + u_2 v_2$

This procedure states that you multiply the corresponding values and then sum the resulting products. It can work with vectors that are more than two dimensions in the same way.

Before trying this procedure with specific numbers, look at the following pairs of vectors and relative estimates of their dot product.



Notice how vectors going in generally the same direction have a positive dot product. Think of two forces acting on a single object. A positive dot product implies that these forces are working together at least a little bit. Another way of saying this is the angle between the vectors is less than 90°.

There are a many important properties related to the dot product that you will prove in the examples, guided practice and practice problems. The two most important are 1) what happens when a vector has a dot product with itself and 2) what is the dot product of two vectors that are perpendicular to each other.

- $v \cdot v = \parallel v \parallel^2$
- v and u are perpendicular if and only if $v \cdot u = 0$

The dot product can help you determine the angle between two vectors using the following formula. Notice that in the numerator the dot product is required because each term is a vector. In the denominator only regular multiplication is required because the magnitude of a vector is just a regular number indicating length.

Angle Between two vectors formula

$$\cos\theta = \frac{u \cdot v}{\| u \| \| v \|}$$

Finally also two vectors are parallel if they have the same slope.

Example A

Show the commutative property holds for the dot product between two vectors. In other words, show that $u \cdot v = v \cdot u$.

Solution: This proof is for two dimensional vectors although it holds for any dimensional vectors.

Start with the vectors in component form.

$$\begin{aligned} u &= < u_1, u_2 > \\ v &= < v_1, v_2 > \end{aligned}$$

Then apply the definition of dot product and rearrange the terms. The commutative property is already known for regular numbers so we can use that.

$$\begin{aligned} u \cdot v &= < u_1, u_2 > \cdot < v_1, v_2 > \\ &= u_1 v_1 + u_2 v_2 \\ &= v_1 u_1 + v_2 u_2 \\ &= < v_1, v_2 > \cdot < u_1, u_2 > \\ &= v \cdot u \end{aligned}$$

Example B

Find the dot product between the following vectors: $< 3, 1 > \cdot < 5, -4 >$

Solution:

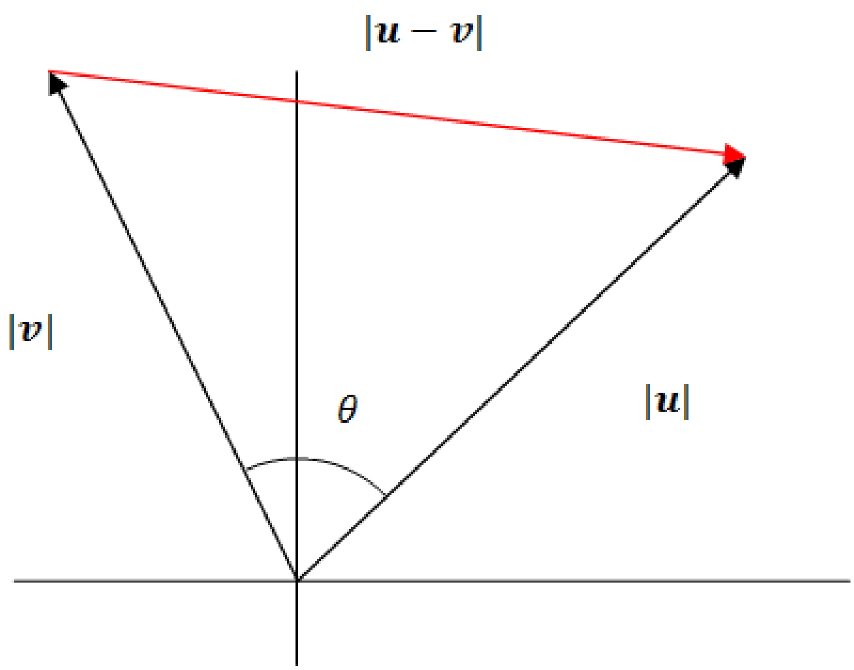
$< 3, 1 > \cdot < 5, -4 >= 3 \cdot 5 + 1 \cdot (-4) = 15 - 4 = 11$

Example C

Prove the angle between two vectors formula:

$\cos\theta = \frac{u \cdot v}{|u||v|}$

Solution: Start with the law of cosines.



$$\begin{aligned}
|u-v|^2 &= |v|^2+|u|^2-2|v||u|\cos\theta \\
(u-v)\cdot(u-v) &= \\
u\cdot u-2u\cdot v+v\cdot v &= \\
|u|^2-2u\cdot v+|v|^2 &= \\
-2u\cdot v &= -2|v||u|\cos\theta \\
\frac{u\cdot v}{|u||v|} &= \cos\theta
\end{aligned}$$

Example D

Solve for the value of b *given the* vectors $\vec{v} =< 5,-7 >$ *and* $\vec{w} =< -3,b >$ to find make vector $\vec{w}$ perpendicular and a parallel to vector

Solution

To find a perpendicular vector we need use the fact that if $\vec{v}\cdot\vec{w} = 0$ then the vectors are perpendicular. Therefore we have.

$$\begin{aligned}
\vec{v}\cdot\vec{w} = a_1a_2+b_1b_2 &= 0 \\
(5)(-3)+(-7)(b) &= 0 \\
-15-7b &= 0 \\
-7b &= 15 \\
b &= -\frac{15}{7}
\end{aligned}$$

therefore the perpendicular vector (orthogonal vector) is $\vec{w} =< -3, -\frac{15}{7} >$

To obtain a parallel vector therefore both vectors must have the same slope so we set up the vector components as follows

$$\begin{aligned}
\frac{b_1}{a_1} &= \frac{b_2}{a_2} \\
\frac{-7}{5} &= \frac{b}{-3} \\
(-7)(-3) &= (5)(b) \\
21 &= 5b \\
\frac{21}{5} &= b
\end{aligned}$$

therefore the parallel vector is $\vec{w} =< -3, \frac{21}{5} >$

Example E

Find the angle between the vectors $\vec{v} =< -2, 5 > \quad and \quad \vec{w} =< 3, -6 >$

Solution

The formula to find the angle between two vectors is $\quad \cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$

therefore we need to find the following:

$$\begin{aligned}
u \cdot v = (-2)(3) + (5)(-6) = -6 - 30 &= -36 \\
\| v \| = \sqrt{(-2)^2 + (5)^2} &= \sqrt{29} \\
\| w \| = \sqrt{(3)^2 + (6)^2} &= \sqrt{45}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\cos\theta &= \frac{-36}{\sqrt{29}\sqrt{45}} \\
\cos\theta &= -.9965457582 \\
\theta &= \cos^{-1}(-.9965457582) \\
\theta &= 175.2^\circ
\end{aligned}$$

Look at the figure below

Concept Problem Revisited

The dot product between the two vectors $< -1, 1 >$ and $< 4, 4 >$ can be computed as:

$(-1)(4) + 1(4) = -4 + 4 = 0$

The result of zero makes sense because these two vectors are perpendicular to each other.

Guided Practice

1. Show the distributive property holds under the dot product **.**

$$u \cdot (v+w) = uv+uw$$

2. Find the dot product between the following vectors.

$$(4i-2j) \cdot (3i-8j)$$

3. What is the angle between $v = <3,5>$ and $u = <2,8>$?

Answers:

1. This proof will work with two dimensional vectors although the property does hold in general.

$u = <u_1,u_2>, v = <v_1,v_2>, w = <w_1,w_2>$

$$\begin{aligned}
u \cdot (v+w) &= u \cdot (<v_1,v_2> + <w_1,w_2>) \\
&= u \cdot <v_1+w_1, v_2+w_2> \\
&= <u_1,u_2> \cdot <v_1+w_1, v_2+w_2> \\
&= u_1(v_1+w_1)+u_2(v_2+w_2) \\
&= u_1v_1+u_1w_1+u_2v_2+u_2w_2 \\
&= u_1v_1+u_2v_2+u_1w_1+u_2w_2 \\
&= u \cdot v + u \cdot w
\end{aligned}$$

2. The standard unit vectors can be written as component vectors.

$$<4,-2> \cdot <3,-8> = 12+(-2)(-8) = 12+16 = 28$$

3. Use the angle between two vectors formula.

$v = <3,5>$ and $u = <2,8>$

$$\begin{aligned}
\frac{u \cdot v}{|u||v|} &= \cos\theta \\
\frac{<3,5> \cdot <2,8>}{\sqrt{34} \cdot \sqrt{68}} &= \cos\theta \\
\frac{6+35}{\sqrt{34} \cdot \sqrt{68}} &= \cos\theta \\
\cos^{-1}\left(\frac{41}{\sqrt{34} \cdot \sqrt{68}}\right) &= \theta \\
31.49 &\approx \theta
\end{aligned}$$

Vocabulary

The ***dot product***

is also known as **inner product** and **scalar product**. It is one of two kinds of products taken between vectors. It produces a number that can be interpreted to tell how much one vector goes in the direction of the other.

Orthogonal Vector

It is a perpendicular vector. Orthogonal means perpendicular.

In Summary

We have learned how to perform dot product of two vectors. We have also learned the meaning of the dot product. Finally we have learned how to find the angle between two vectors given the vectors.

Practice

Find the dot product for each of the following pairs of vectors.

1. $< 2,6 > \cdot < -3,5 >$
2. $< 5,-1 > \cdot < 4,4 >$
3. $< -3,-4 > \cdot < 2,2 >$
4. $< 3,1 > \cdot < 6,3 >$
5. $< -1,4 > \cdot < 2,9 >$

Find the angle between each pair of vectors below.

6. $< 2,6 > \cdot < -3,5 >$
7. $< 5,-1 > \cdot < 4,4 >$
8. $< -3,-4 > \cdot < 2,2 >$
9. $< 3,1 > \cdot < 6,3 >$
10. $< -1,4 > \cdot < 2,9 >$
11. What is $v \cdot v$?
12. How can you use the dot product to find the magnitude of a vector?
13. What is $0 \cdot v$?
14. Show that $(cu) \cdot v = u \cdot (cv)$ where c is a constant.
15. Show that $< 2,3 >$ is perpendicular to $< 1.5,-1 >$.

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9.5 Vector Projection

Here you will project one vector onto another and apply this technique as it relates to force.

TEKS

P.4.I
P.4.J
P.4.K

Lesson Objectives

In this section you will learn about:
1. Vector projection.
2. Finding work.

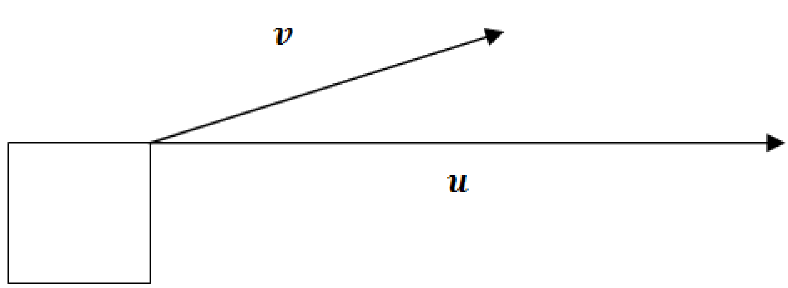
Introduction

Projecting one vector onto another explicitly answers the question: "how much of one vector goes in the direction of the other vector?" The dot product is useful because it produces a scalar quantity that helps to answer this question. In this concept, you will produce an actual vector not just a scalar.

Vocabulary

Vector Projection, Scalar Projection, Work.

Why is vector projection useful when considering pulling a box in the direction of v instead of horizontally in the direction of u?



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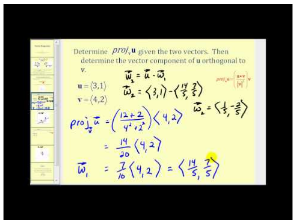

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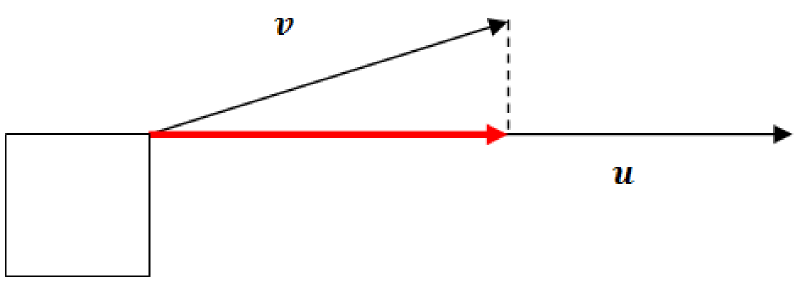
http://www.youtube.com/watch?v=3VlxQPeNJFs James Sousa: Vector Projection

Guidance

Consider the question from the concept.



The definition of vector projection for the indicated red vector is the called $proj_u v$. When you read $proj_u v$ you should say "the vector projection of v onto u ." This implies that the new vector is going in the direction of u. Conceptually, this means that if someone was pulling the box at an angle and strength of vector v then some of their energy would be wasted pulling the box up and some of the energy would actually contribute to pulling the box horizontally.

The definition of scalar projection is simply the length of the vector projection. When the scalar projection is positive it means that the angle between the two vectors is less than 90°. When the scalar projection is negative it means that the two vectors are heading in opposite directions.

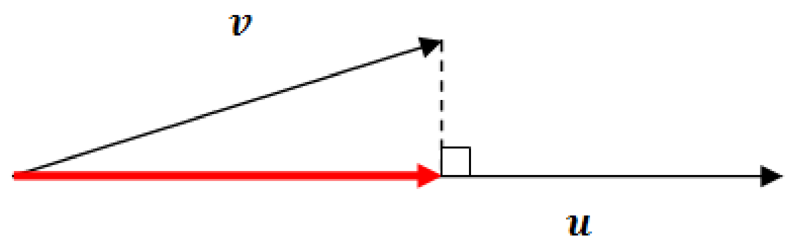
The vector projection formula can be written two ways. The version on the left is most simplified, but the version on the right makes the most sense conceptually. The proof is demonstrated in Example A.

$$proj_u v = \left(\frac{v \cdot u}{\| u \|^2}\right) u = \left(\frac{v \cdot u}{\| u \|}\right) \frac{u}{\| u \|}$$

Example A

Prove the vector projection formula.

Solution: Given two vectors u, v, what is $proj_u v$?



First note that the projected vector in red will go in the direction of u. This means that it will be a product of the unit vector $\frac{u}{|u|}$ and the length of the red vector (the scalar projection). In order to find the scalar projection, note the right triangle, the unknown angle θ between the two vectors and the cosine ratio.

$\cos\theta = \frac{scalar\ projection}{|v|}$

Recall that $\cos\theta = \frac{u \cdot v}{|u||v|}$. Now just substitute and simplify to find the length of the scalar projection.

$$\begin{aligned}
\cos\theta &= \frac{scalar\ projection}{|v|} \\
\frac{u \cdot v}{|u||v|} &= \frac{scalar\ projection}{|v|} \\
\frac{u \cdot v}{|u|} &= scalar\ projection
\end{aligned}$$

Now you have the length of the vector projection and the direction you want it to go:

$proj_u v = \left(\frac{u \cdot v}{|u|}\right)\frac{u}{|u|}$

Example B

Find the scalar projection of vector $v = <3,4>$ onto vector $u = <5,-12>$.

Solution:

As noted in Example A, the scalar projection is the magnitude of the vector projection. This was shown to be $\left(\frac{u \cdot v}{|u|}\right)$ where u is the vector being projected onto.

$\frac{u \cdot v}{|u|} = \frac{<5,-12> \cdot <3,4>}{13} = \frac{15-48}{13} = -\frac{33}{13}$

Example C

Find the vector projection of vector $v = <3,4>$ onto vector $u = <5,-12>$

Solution:

Since the scalar projection has already been found in Example B, you should multiply the scalar by the "onto" unit vector.

$-\frac{33}{13} < \frac{5}{13}, -\frac{12}{13} > = < -\frac{165}{169}, \frac{396}{169} >$

Solution 2:

We can use the formula straight from the beginning to find the vector projection $proj_u v = \frac{v \cdot u}{||u||^2} u$ Therefore we have the following information.

$$\begin{aligned} v \cdot u &= (3)(5) + (4)(-12) = 15 - 48 = -33 \\ ||u|| &= \sqrt{(5)^2 + (-12)^2} = \sqrt{169} \\ ||u||^2 &= 169 \end{aligned}$$

$$proj_u v = \frac{v \cdot u}{||u||^2} u = \frac{-33}{169} < 5, -12 > = < -\frac{165}{169}, \frac{396}{169} >$$

which is the same solution.

Definition of Work

The work W, done by a force F moving and object from A to B is

$$W = F \cdot \overrightarrow{AB}$$

or work is equal to

$$W = ||F|| ||\overrightarrow{AB}|| \cos \theta$$

where ||F|| is the magnitude of the force, ||AB|| is the distance over which the force is applied and θ is the angle between the force and the direction of motion.

Example D

A child pulls a box along a level ground exerting a force of 25 pounds on a rope that makes an angle of 37 degrees with the horizontal. How much work is done pulling the sled 150 ft?

Solution:

The work done will be given by the formula $W = ||F|| ||\overrightarrow{AB}|| \cos \theta$

therefore

$$\begin{aligned} W &= ||F|| ||\overrightarrow{AB}|| \cos \theta \\ W &= (25)(150) \cos 37^\circ \\ W &= 2994.88 foot - pounds \end{aligned}$$

Therefore the amount of work will be about 2995 foot-pounds **.**

Example E

A wagon is pulled up a ramp with a force of 50 pounds on a rope that makes a 32 degree angle with the horizontal. The ramp has an angle of 15 degrees. How much work is done pulling the box 25 ft. up the ramp.

Solution

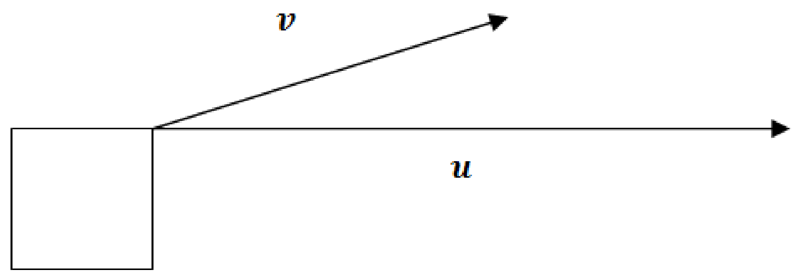
We know that the being applied is 50 pounds and the distance is 25 ft. The angle will be 32-15 = 17 degrees. This is the angle between the rope and the ramp. Therefore the work done will be

$$W = ||F||||\overrightarrow{AB}||\cos\theta$$
$$W = (50)(25)\cos 17^\circ$$
$$W = 1195.38 foot - pounds$$

Therefore the work done is about 1195 foot-pounds.

Concept Problem Revisited

Vector projection is useful in physics applications involving force and work.



When the box is pulled by vector v some of the force is wasted pulling up against gravity. In real life this may be useful because of friction, but for now this energy is inefficiently wasted in the horizontal movement of the box.

Guided Practice

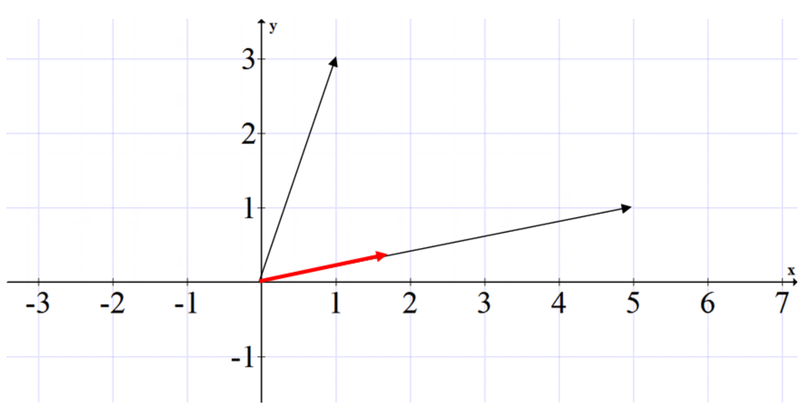
1. Sketch vectors $< 1,3 >$ and $< 5,2 >$. What is the vector projection of $< 1,3 >$ onto $< 5,2 >$? Sketch the projection.

2. Sketch the vector $< -2,-2 >$ and $< 4,-2 >$. Explain using a sketch why a negative scalar projection of $< -2,-2 >$ onto $< 4,-2 >$ makes sense.

3. A father is pulling his daughter up a hill. The hill has a 20° incline. The daughter is on a sled that sits on the ground and has a rope that the father pulls with a force of 100 lb as he walks. The rope makes a 39° angle with the slope. What is the effective force that the father exerts moving his daughter and the sled up the hill?

Answers:

1.

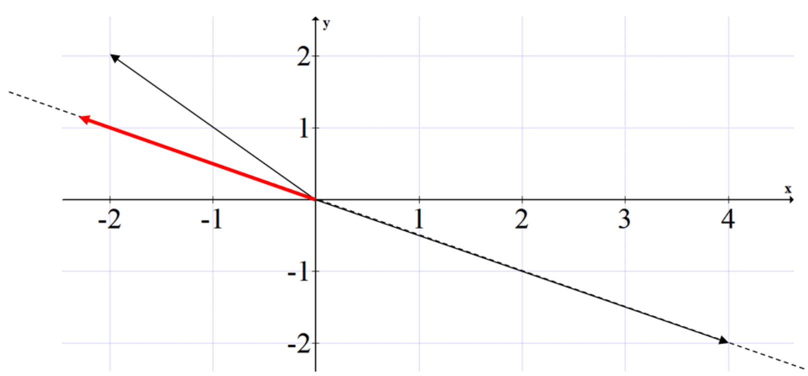


The formula for vector projection where u is the onto vector is:

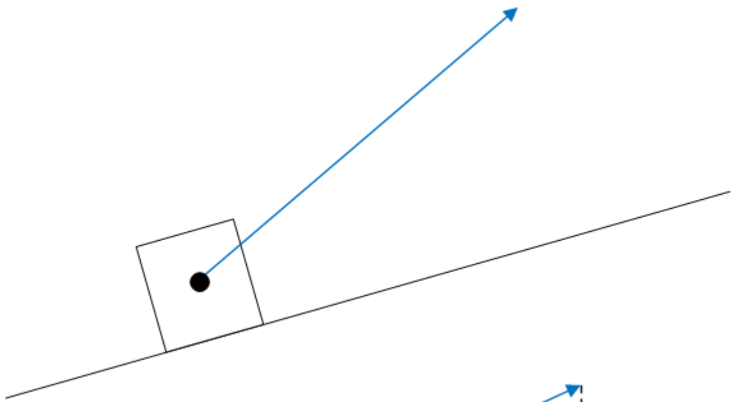
$$
\begin{aligned}
proj_u v = \left(\frac{u \cdot v}{\parallel u \parallel}\right)\frac{u}{\parallel u \parallel} &= \left(\frac{<1,3> \cdot <5,1>}{\sqrt{26}}\right)\frac{<5,1>}{\sqrt{26}} \\
&= \left(\frac{5+3}{26}\right) <5,1> \\
&= \frac{4}{13} <5,1> \\
&=< \frac{20}{13}, \frac{4}{13} >
\end{aligned}
$$

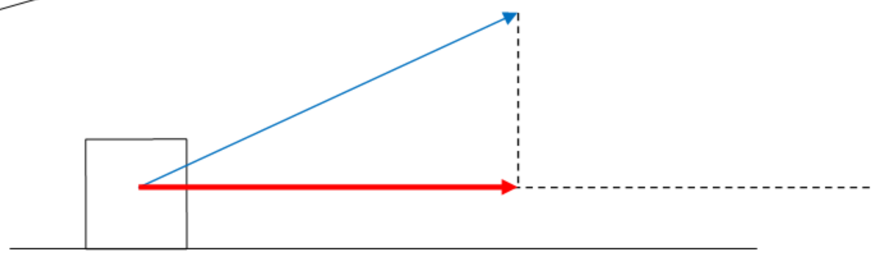
The graph confirms the result of the vector projection.

2. First plot the two vectors and extend the "onto" vector. When the vector projection occurs, the vector $<-2,2>$ goes in the opposite direction of the vector $<4,-2>$. This will create a vector projection going in the opposite direction of $<4,-2>$.



3. The box represents the girl and the sled. The blue arrow indicates the father's 100 lb force. Notice that the question asks for simply the amount of force which means scalar projection. Since this is not dependent on the slope of this hill, we can rotate our perspective and still get the same scalar projection.





The components of the father's force vector is $100 < \cos 39°, \sin 39° >$ and the "onto" vector is any vector horizontally to the right. Since we are only looking for the length of the horizontal component and you already have the angle between the two vectors, the scalar projection is:

$100 \cdot \cos 39° \approx 77.1\ lb$

Vocabulary

The ***vector projection*** is the vector produced when one vector is resolved into two component vectors, one that is parallel to the second vector and one that is perpendicular to the second vector. The parallel vector is the vector projection.

The ***scalar projection*** is the length of the vector projection. When the scalar projection is negative it means that the two vectors are heading in opposite directions and the angle between the vectors is greater than 90°.

In Summary

We have learned how to perform vector projection and how to apply it.

Practice

1. Sketch vectors $< 2,4 >$ and $< 2,1 >$.
2. What is the vector projection of $< 2,4 >$ onto $< 2,1 >$? Sketch the projection.
3. Sketch vectors $< -2,1 >$ and $< -1,3 >$.
4. What is the vector projection of $< -1,3 >$ onto $< -2,1 >$? Sketch the projection.
5. Sketch vectors $< 6,2 >$ and $< 8,1 >$.
6. What is the vector projection of $< 6,2 >$ onto $< 8,1 >$? Sketch the projection.
7. Sketch vectors $< 1,7 >$ and $< 6,3 >$.
8. What is the vector projection of $< 1,7 >$ onto $< 6,3 >$? Sketch the projection.
9. A box is on the side of a hill inclined at 30°. The weight of the box is 40 pounds. What is the magnitude of the

force required to keep the box from sliding down the hill?

10. Sarah is on a sled on the side of a hill inclined at 60°. The weight of Sarah and the sled is 125 pounds. What is the magnitude of the force required for Sam to keep Sarah from sliding down the hill.

11. A 1780 pound car is parked on a street that makes an angle of 15° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

12. A 1900 pound car is parked on a street that makes an angle of 10° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

13. A 30 pound force that makes an angle of 32° with an inclined plane is pulling a box up the plane. The inclined plane makes a 20° angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

14. A 22 pound force that makes an angle of 12° with an inclined plane is pulling a box up the plane. The inclined plane makes a 25° angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

15. Anne pulls a wagon on a horizontal surface with a force of 50 pounds. The handle of the wagon makes an angle of 30° with the ground. What is the magnitude of the effective force pulling the wagon?

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You learned that you can add and subtract vectors just like you can numbers. You saw that vectors also have other algebraic properties and essentially create an entirely new algebraic structure. There are two ways the product of vectors can be taken and you studied one way called the dot product. Lastly, you learned to calculate how much of one vector goes into the direction of another vector.

9.6 References

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CHAPTER 10 Conic Sections

Chapter Outline

Conics are an application of analytic geometry. Here you will get a chance to work with shapes like circles that you have worked with before. You will also get to see the equations and definitions that turn circles into curved ellipses and the rest of the conic sections.

10.1 General Form of a Conic

Here you will see how each conic section is the intersection of a plane and a cone, review completing the square and start working with the general equation of a conic.

TEKS

P.3.F
P.3.G
P.3.H
P.3.I

Lesson Objectives

In this lesson you will learn about:
1. The definition of a conic.
2. How to identify a conic from the general equation.
3. How to complete the square to write the equations in standard form.

Introduction

Conics are a family of graphs that include parabolas, circles, ellipses and hyperbolas. All of these graphs come from the same general equation and by looking and manipulating a specific equation you can learn to tell which conic it is and how it can be graphed.
What is the one essential skill that enables you to manipulate the equation of a conic in order to sketch its graph?

Vocabulary

Circle, Ellipse, Parabola, Hyperbola, Completing the square.

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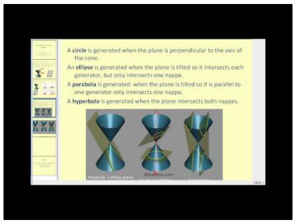

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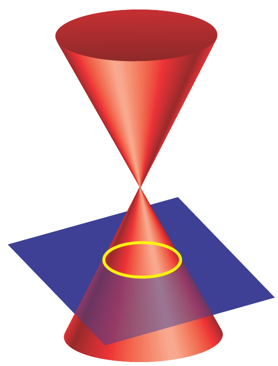
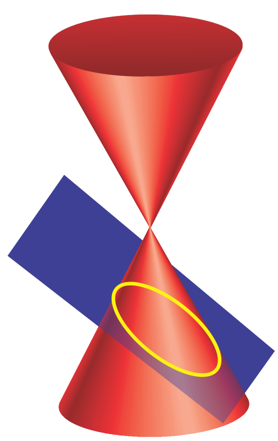
http://www.youtube.com/watch?v=iJOcn9C9y4w James Sousa: Introduction to Conic Sections

Guidance

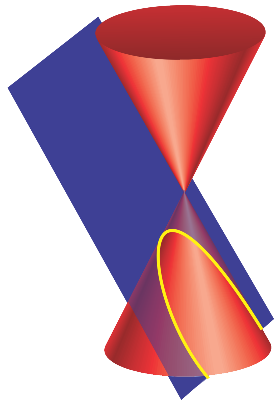
The word conic comes from the word cone which is where the shapes of parabolas, circles, ellipses and hyperbolas originate. Consider two cones that open up in opposite directions and a plane that intersects it horizontally. A flat intersection would produce a perfect circle.



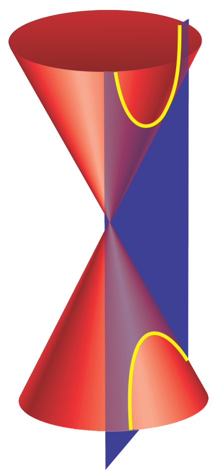
To produce an ellipse, tilt the plane so that the circle becomes elongated and oval shaped. Notice that the angle that the plane is tilted is still less steep than the slope of the side of the cone.



As you tilt the plane even further and the slope of the plane equals the slope of the cone edge you produce a parabola. Since the slopes are equal, a parabola only intersects one of the cones.



Lastly, if you make the plane steeper still, the plane ends up intersecting both the lower cone and the upper cone creating the two parts of a hyperbola.



The intersection of three dimensional objects in three dimensional space to produce two dimensional graphs is quite challenging. In practice, the knowledge of where conics come from is not widely used. It will be more important for you to be able to manipulate an equation into standard form and graph it in a rectangular coordinate plane. The general form of a conic is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Before you start manipulating the general form of a conic equation you should be able to recognize whether it is a circle, ellipse, parabola or hyperbola. In standard form, the two coefficients to examine are A and C.

- For **circles,** the coefficients of x^2 and y^2 are the same sign and the same value: $A = C$
- For **ellipses,** the coefficients of x^2 and y^2 are the same sign and different values: $A, C > 0$, $A \neq C$
- For **hyperbolas,** the coefficients of x^2 and y^2 are opposite signs: $C < 0 < A$ or $A < 0 < C$
- For **parabolas,** either the coefficient of x^2 or y^2 must be zero: $A = 0$ or $C = 0$

Each specific type of conic has its own graphing form, but in all cases the technique of completing the square is essential. The examples review completing the square and recognizing conics.

Example A

Complete the square in the expression $x^2 + 6x$. Demonstrate graphically what completing the square represents.

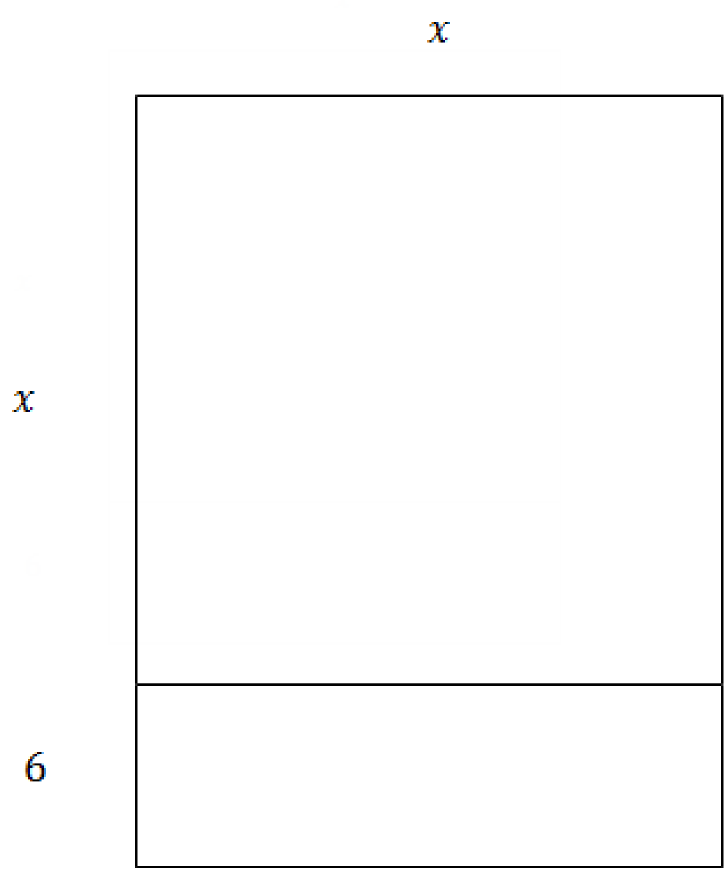
Solution: Algebraically, completing the square just requires you to divide the coefficient of x by 2 and square the result. In this case $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Since you cannot add nine to an expression without changing its value, you must simultaneously add nine and subtract nine so the net change will be zero.

$x^2+6x+9-9$

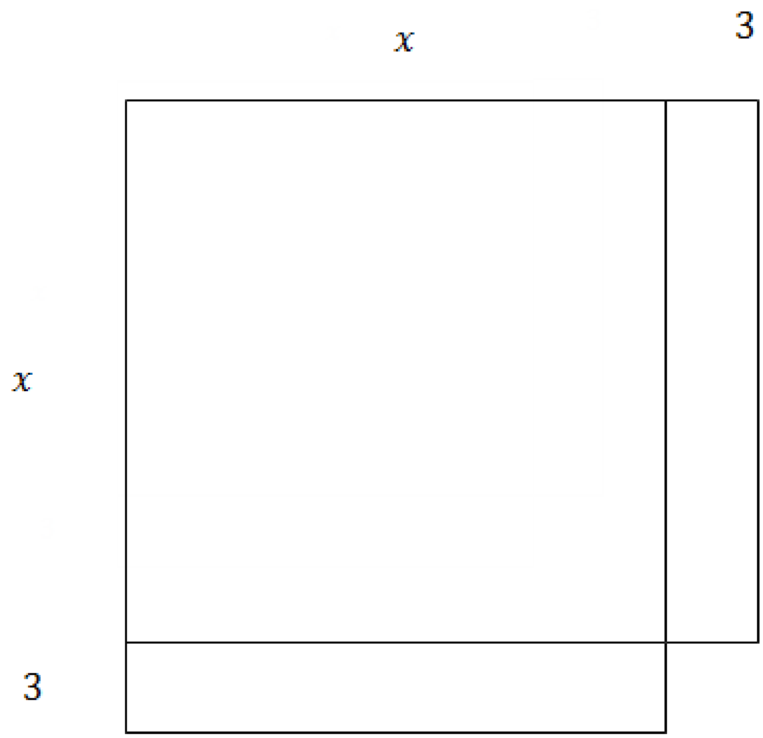
Now you can factor by recognizing a perfect square.

$(x+3)^2-9$

Graphically the original expression x^2+6x can be represented by the area of a rectangle with sides x and $(x+6)$.



The term "complete the square" has visual meaning as well algebraic meaning. The rectangle can be rearranged to be more square-like so that instead of small rectangle of area $6x$ at the bottom, there is a rectangle of area $3x$ on two sides of the x^2 square.



Notice what is missing to make this shape a perfect square? A little corner square of 9 is missing which is why the 9 should be added to make the perfect square of $(x+3)(x+3)$.

Example B

What type of conic is each of the following relations?

a. $5y^2 - 2x^2 = -25$
b. $x = -\frac{1}{2}y^2 - 3$
c. $4x^2 + 6y^2 = 36$
d. $x^2 - \frac{1}{4}y = 1$
e. $-\frac{x^2}{8} + \frac{y^2}{4} = 1$
f. $-x^2 + 99y^2 = 12$

Solution:

a. Hyperbola because the x^2 and y^2 coefficients are different signs.
b. Parabola (sideways) because the x^2 term is missing.
c. Ellipse because the x^2 and y^2 coefficients are different values but the same sign.
d. Parabola (upright) because the y^2 term is missing.
e. Hyperbola because the x^2 and y^2 coefficients are different signs.
f. Hyperbola because the x^2 and y^2 coefficients are different signs.

Example C

Complete the square for both the x and y terms in the following equation.

$x^2 + 6x + 2y^2 + 16y = 0$

Solution: First write out the equation with space so that there is room for the terms to be added to both sides. Since this is an equation, it is appropriate to add the values to both sides instead of adding and subtracting the same value simultaneously. As you rewrite with spaces, factor out any coefficient of the x^2 or y^2 terms since your algorithm for completing the square only works when this coefficient is one.

$x^2+6x+\_\_+2(y^2+8y+\_\_)=0$

Next complete the square by adding a nine and what looks like a 16 on the left (it is actually a 32 since it is inside the parentheses).

$x^2+6x+9+2(y^2+8y+16)=9+32$

Factor.

$(x+3)^2+2(y+4)^2=41$

Concept Problem Revisited

The one essential skill that you need for conics is completing the square. If you can do problems like Example C then you will be able to graph every type of conic.

Guided Practice

1. Identify the type of conic in each of the following relations.

a. $3x^2=3y^2+18$
b. $y=4(x-3)^2+2$
c. $x^2+y^2=4$
d. $y^2+2y+x^2-6x=12$
e. $\frac{x^2}{6}+\frac{y^2}{12}=1$
f. $x^2-y^2+4=0$

2. Complete the square in the following expression.

$6y^2-36y+4$

3. Complete the square for both x and y in the following equation.

$-3x^2-24x+4y^2-32y=8$

Answers:

1. a. The relation is a hyperbola because when you move the $3y^2$ to the left hand side of the equation, it becomes negative and then the coefficients of x^2 and y^2 have opposite signs.

b. Parabola

c. Circle

d. Circle

e. Ellipse

f. Hyperbola

2.

$$\begin{aligned}
&6y^2 - 36y + 4\\
&6(y^2 - 6y + __) + 4\\
&6(y^2 - 6y + 9) + 4 - 54\\
&6(y-3)^2 - 50
\end{aligned}$$

3.

$$\begin{aligned}
-3x^2 - 24x \quad + 4y^2 - 32y &= 8\\
-3(x^2 + 8x + __) + 4(y^2 - 8y + __) &= 8\\
-3(x^2 + 8x + 16) + 4(y^2 - 8y + 16) &= 8 - 48 + 64\\
-3(x+4)^2 \quad + \quad 4(y-4)^2 &= 24
\end{aligned}$$

Vocabulary

Completing the square is a procedure that enables you to combine squared and linear terms of the same variable into a perfect square of a binomial.

Conics are a family of graphs (not functions) that come from the same general equation. This family is the intersection of a two sided cone and a plane in three dimensional space.

In Summary

We have learned what a conic is. We have learned how to identify a conic from the general form and how to complete the square, an essential process needed to be able to graph conics.

Practice

Identify the type of conic in each of the following relations.

1. $3x^2 + 4y^2 = 12$

2. $x^2 + y^2 = 9$

3. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

4. $y^2 + x = 11$

5. $x^2 + 2x - y^2 + 6y = 15$

6. $x^2 = y - 1$

Complete the square for x and/or y in each of the following expressions.

7. $x^2 + 4x$

8. $y^2 - 8y$

9. $3x^2 + 6x + 4$

10. $3y^2 + 9y + 15$

11. $2x^2 - 12x + 1$

Complete the square for x and/or y in each of the following equations.

12. $4x^2 - 16x + y^2 + 2y = -1$

13. $9x^2 - 54x + y^2 - 2y = -81$

14. $3x^2 - 6x - 4y^2 = 9$

15. $y = x^2 + 4x + 1$

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10.2 Parabolas (Optional)

Here you will define a parabola in terms of its directrix and focus, graph parabolas vertically and horizontally, and use a new graphing form of the parabola equation.

TEKS

P.3.F
P.3.G

Lesson Objectives

In this lesson you will learn about:
1. The definition of a parabola.
2. How to identify a parabola.
3. How to graph a parabola.
4. How to obtain the graph of a parabola.

Introduction

When working with parabolas in the past you probably used vertex form and analyzed the graph by finding its roots and intercepts. There is another way of defining a parabola that turns out to be more useful in the real world. One of the many uses of parabolic shapes in the real world is satellite dishes. In these shapes it is vital to know where the receptor point should be placed so that it can absorb all the signals being reflected from the dish.

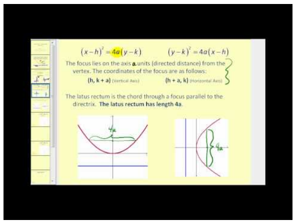
Vocabulary

Parabola, Vertex, Focus, Directrix,

Concept Problem

Where should the receptor be located on a satellite dish that is four feet wide and nine inches deep?

Watch This



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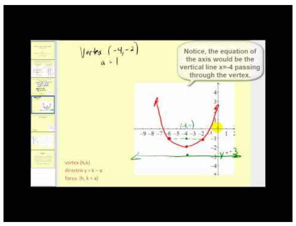
http://www.youtube.com/watch?v=k7wSPisQQYs James Sousa: Conic Sections: The Parabola part 1 of 2



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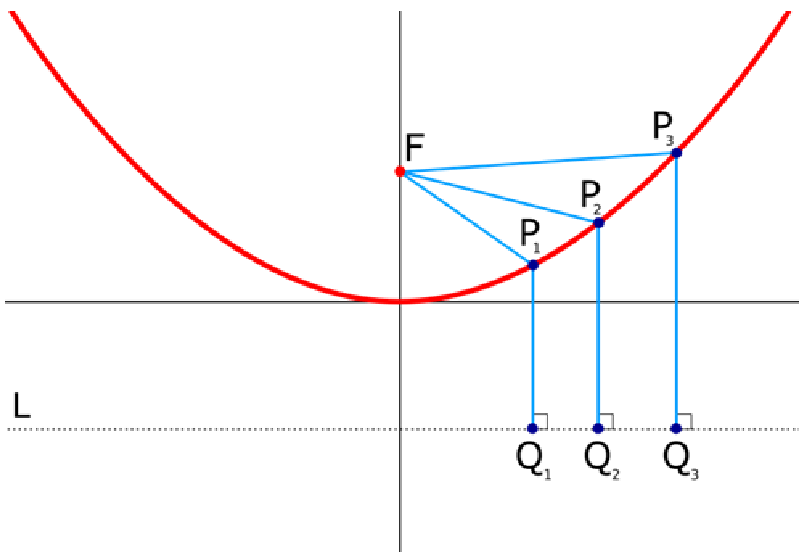
URL: http://www.ck12.org/flx/render/embeddedobject/61847

http://www.youtube.com/watch?v=CKepZr52G6Y James Sousa: Conic Sections: The Parabola part 2 of 2

Guidance

The definition of a parabola is the collection of points equidistant from a point called the focus and a line called the directrix.



Notice how the three points P_1, P_2, P_3 are each connected by a blue line to the focus point F and the directrix line L.

$$\overline{FP_1} = \overline{P_1Q_1}$$
$$\overline{FP_2} = \overline{P_2Q_2}$$
$$\overline{FP_3} = \overline{P_3Q_3}$$

There are two graphing equations for parabolas that will be used in this concept. The only difference is one equation graphs parabolas opening vertically and one equation graphs parabolas opening horizontally.

There are three different forms of having the equation of a parabola

TABLE 10.1:

Polynomial Form	Vertex Form	Standard Form
$y = ax^2 + bx + c$	$y = a(x-h)^2 + k$	$(x-h)^2 = 4p(y-k)$

Notice that the standard form is the vertex form but we subtract k to the left side and divide by a

You can recognize the parabolas opening vertically because they have an x^2 term.

Likewise, parabolas opening horizontally have a y^2 term.

The general equation for a parabola opening vertically is

$$(x-h)^2 = \pm 4p(y-k)$$

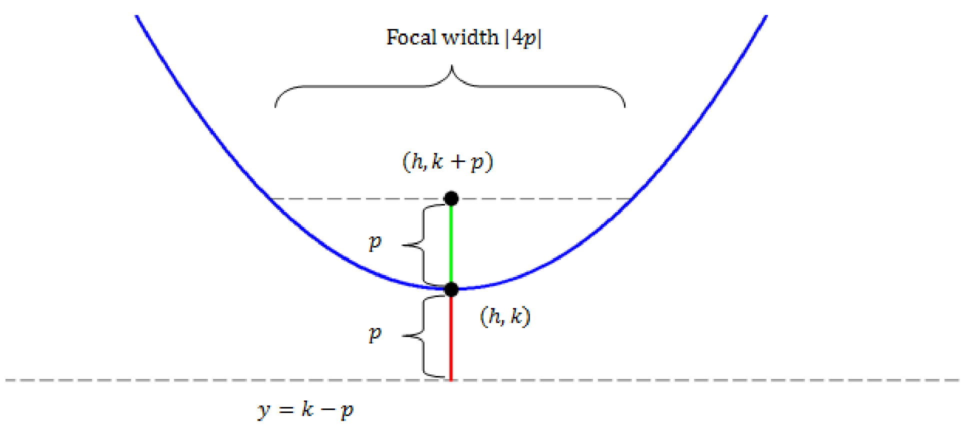
.

The general equation for a parabola opening horizontally is

$$(y-k)^2 = \pm 4p(x-h)$$

.

Where p is the distance from the vertex of the parabola to the focus.



Note that the vertex is still (h,k). The parabola opens upwards or to the right if the $4p$ is positive.

The parabola opens down or to the left if the $4p$ is negative.

The focus is just a point that is distance p away from the vertex.

The directrix is just a line that is distance p away from the vertex in the opposite direction.

You can sketch how wide the parabola is by noting the focal width is $|4p|$.

Once you put the parabola into this graphing form you can sketch the parabola by plotting the vertex, identifying p and plotting the focus and directrix and lastly determining the focal width and sketching the curve.

Example A

Identify the following conic, put it into graphing form and identify its vertex, focal length (p), focus, directrix and focal width.

$2x^2+16x+y=0$

$$\begin{aligned} x^2+8x &= -\frac{1}{2}y \\ x^2+8x+16 &= -\frac{1}{2}y+16 \\ (x+4)^2 &= -\frac{1}{2}(y-32) \\ (x+4)^2 &= -4\cdot\frac{1}{8}(y-32) \end{aligned}$$

The vertex is (-4, 32). The focal length is $p=\frac{1}{8}$.

This parabola opens down which means that the focus is at $\left(-4, 32-\frac{1}{8}\right)$ and the directrix is horizontal at $y=32+\frac{1}{8}$. The focal width is $\frac{1}{2}$.

Here is a graph of the parabola

Example B

Sketch the following parabola and identify the important pieces of information.

$(y+1)^2=4\cdot\frac{1}{2}\cdot(x+3)$

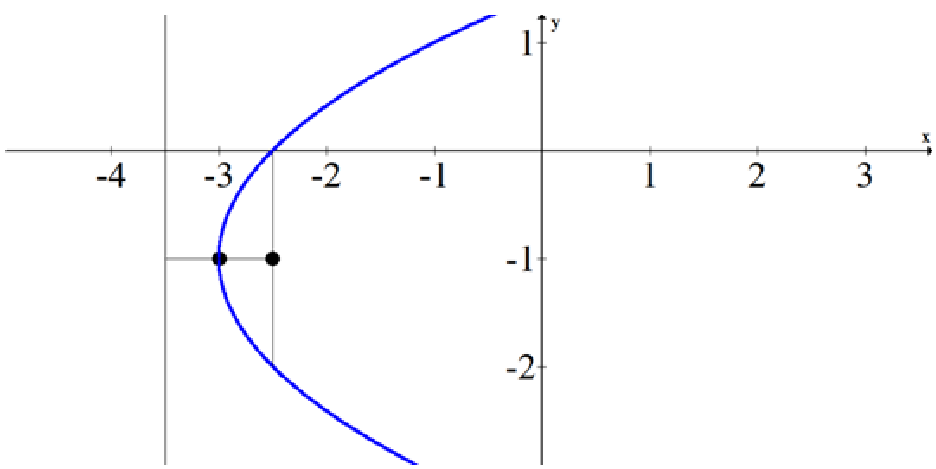
Solution:

The vertex is at (-3, -1). The parabola is sideways because there is a y^2 term.

The parabola opens to the right because the $4p$ is positive.

The focal length is $p=\frac{1}{2}$ which means the focus is $\frac{1}{2}$ to the right of the vertex at (-2.5, -1) and the directrix is $\frac{1}{2}$ to the left of the vertex at $x=-3.5$.

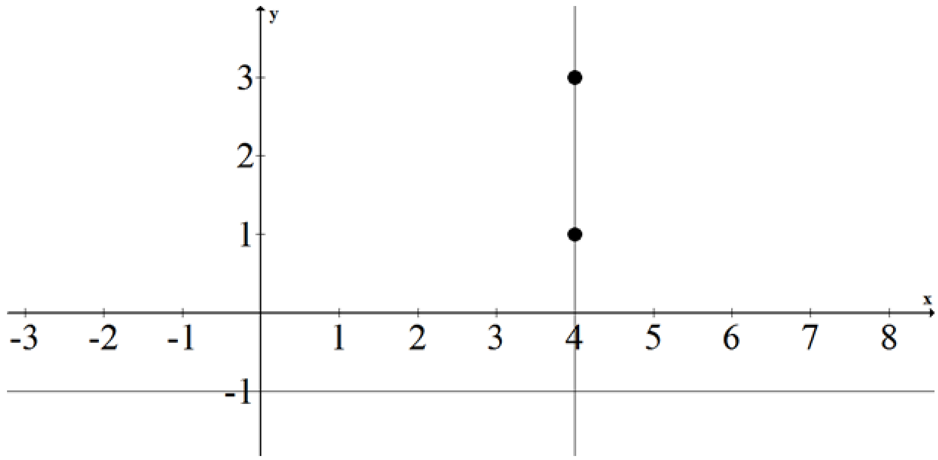
The focal width is 2 which is why the width of the parabola stretches from (-2.5, 0) to (-2.5, -2).



Example C

What is the equation of a parabola that has a focus at (4, 3) and a directrix of $y=-1$?

Solution: It would probably be useful to graph the information that you have in order to reason about where the vertex is.



The vertex must be halfway between the focus and the directrix. This places it at (4, 1).

The focal length is 2. The parabola opens upwards. This is all the information you need to create the equation.

$(x-4)^2 = 4 \cdot 2 \cdot (y-1)$ OR $(x-4)^2 = 8(y-1)$

Concept Problem Revisited

Where should the receptor be located on a satellite dish that is four feet wide and nine inches deep?

Since real world problems do not come with a predetermined coordinate system, you can choose to make the vertex of the parabola at (0, 0). Also we need to change the feet into inches.

Then, if everything is done in inches, another point on the parabola will be (24, 9). *(Many people might mistakenly believe the point* (48, 9) *is on the parabola but remember that half this width stretches to* (-24, 9) *as well.)* Using these two points, the focal width can be found.

$$\begin{aligned}(x-0)^2 &= 4p(y-0)\\(24-0)^2 &= 4p(9-0)\\\frac{24^2}{4\cdot 9} &= p\\16 &= p\end{aligned}$$

The receptor should be sixteen inches away from the vertex of the parabolic dish. The figure below is a representation of the satellite dish. Point B is the focus (0,16) and point C is the point (24,9).

Guided Practice

1. What is the equation of a parabola with focus at (2, 3) and directrix at $y = 5$?

2. What is the equation of a parabola that opens to the right with focal width from (6, -7) to (6, 12)?

3. Sketch the following conic by putting it into graphing form and identifying important information.

$y^2 - 4y + 12x - 32 = 0$

Answers:

1. The vertex must lie directly between the focus and the directrix, so it must be at (2, 4). The focal length is therefore equal to 1. The parabola opens downwards.

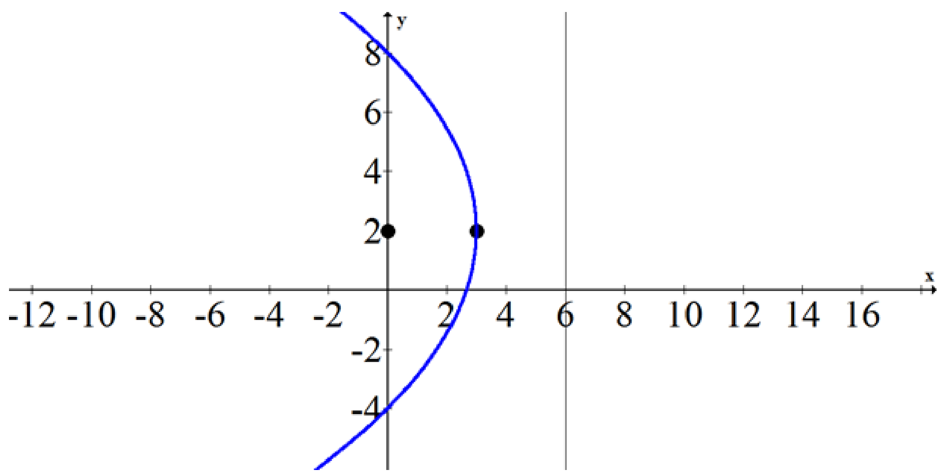
$(x-2)^2 = -4 \cdot 1 \cdot (y-4)$

2. The focus is in the middle of the focal width. The focus is $\left(6, \frac{5}{2}\right)$. The focal width is 19 which is four times the focal length so the focal length must be $\frac{19}{4}$. The vertex must be a focal length to the left of the focus, so the vertex is at $\left(6 - \frac{19}{4}, \frac{5}{2}\right)$. This is enough information to write the equation of the parabola.

$\left(y - \frac{5}{2}\right)^2 = 4 \cdot \frac{19}{4} \cdot \left(x - 6 + \frac{19}{4}\right)$

3. $y^2 - 4y + 12x - 32 = 0$

$$y^2 - 4y = -12x + 32$$
$$y^2 - 4y + 4 = -12x + 32 + 4$$
$$(y-2)^2 = -12(x-3)$$
$$(y-2)^2 = -4 \cdot 3 \cdot (x-3)$$



The vertex is at (3, 2). The focus is at (0, 2). The directrix is at $x = 6$.

Vocabulary

A **parabola** is the set of all points that are equidistant from a point called the **focus** and a line called the **directrix**

The ***focus*** of a parabola is the point that the parabola seems to curve around.

The ***directrix*** of a parabola is the line that the parabola seems to curve away from.

The **vertex** is the highest or lowest point of a parabola opening up or down and the rightmost or leftmost point if the parabola is opening left or right.

In Summary

We have learned what a parabola is. we have learned the standard form of a parabola, how to find the focal distance and the directrix equation. Also we have learned how to sketch a parabola, how to obtain its equation, and how to apply it to real life situations.

Practice

1. What is the equation of a parabola with focus at (1, 4) and directrix at $y = -2$?

2. What is the equation of a parabola that opens to the left with focal width from (-2, 5) to (-2, -7)?

3. What is the equation of a parabola that opens to the right with vertex at (5, 4) and focal width of 12?

4. What is the equation of a parabola with vertex at (1, 8) and directrix at $y = 12$?

5. What is the equation of a parabola with focus at (-2, 4) and directrix at $x = 4$?

6. What is the equation of a parabola that opens downward with a focal width from (-4, 9) to (16, 9)?

7. What is the equation of a parabola that opens upward with vertex at (1, 11) and focal width of 4?

Sketch the following parabolas by putting them into graphing form and identifying important information.

8. $y^2 + 2y - 8x + 33 = 0$

9. $x^2 - 8x + 20y + 36 = 0$

10. $x^2 + 6x - 12y - 15 = 0$

11. $y^2 - 12y + 8x + 4 = 0$

12. $x^2 + 6x - 4y + 21 = 0$

13. $y^2 + 14y - 2x + 59 = 0$

14. $x^2 + 12x - \frac{8}{3}y + \frac{92}{3} = 0$

15. $x^2 + 2x - \frac{4}{5}y + 1 = 0$

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10.3 Circles

Here you will formalize the definition of a circle, translate a conic from standard form into graphing form, and graph circles.

TEKS

P.3.F
P.3.G
P.3.H

Lesson Objectives

In this lesson you will learn about:
1. The definition of a Circle.
2. Graphing circles.
3. Convert to standard form equation of a circle.
4. Identify the center and the radius of a circle.

Introduction

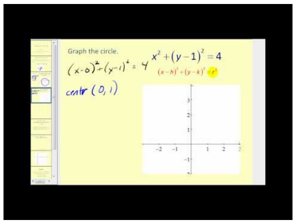
The circles are most common conic used in real life. From the tires in our cars, to the gears in the motors. In this section you will learn the standard form of a circle.

Vocabulary

Circle, Radius, Diameter.

A circle is the collection of points that are the same distance from a single point. What is the connection between the Pythagorean Theorem and a circle?

Watch This



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URL: http://www.ck12.org/flx/render/embeddedobject/61851

http://www.youtube.com/watch?v=g1xa7PvYV3I Conic Sections: The Circle

Guidance

A circle is the collection of points that are equidistant from a single point. This single point is called the center of the circle. A circle does not have a focus or a directrix, instead it simply has a center. Circles can be recognized immediately from the general equation of a conic when the coefficients of x^2 and y^2 are the same sign and the same value. Circles are not functions because they do not pass the vertical line test. The distance from the center of a circle to the edge of the circle is called the radius of the circle. The distance from one end of the circle through the center to the other end of the circle is called the diameter. The diameter is equal to twice the radius.

The graphing form of a circle is:

$(x-h)^2+(y-k)^2=r^2$

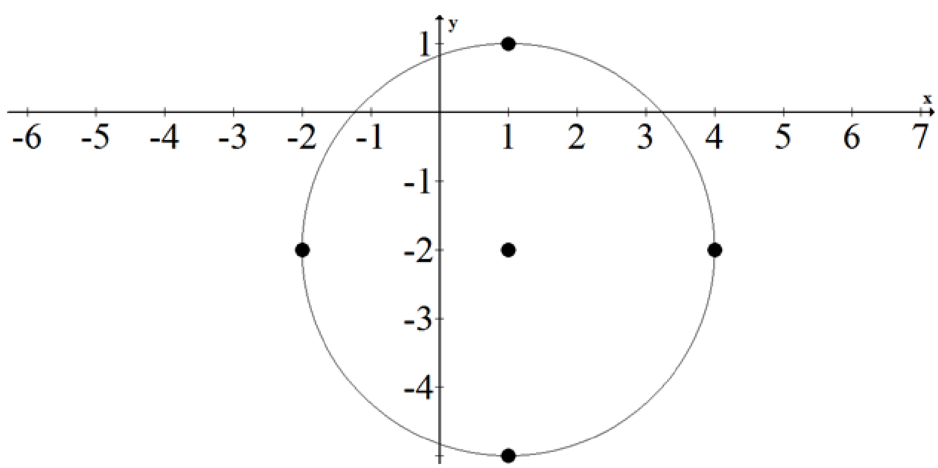
The center of the circle is at (h,k) and the radius of the circle is r. Note that this looks remarkably like the Pythagorean Theorem.

Example A

Graph the following circle.

$(x-1)^2+(y+2)^2=9$

Solution: Plot the center and the four points that are exactly 3 units from the center.



Example B

Turn the following equation into graphing form for a circle. Identify the center and the radius.

$36x^2+36y^2-24x+36y-275=0$

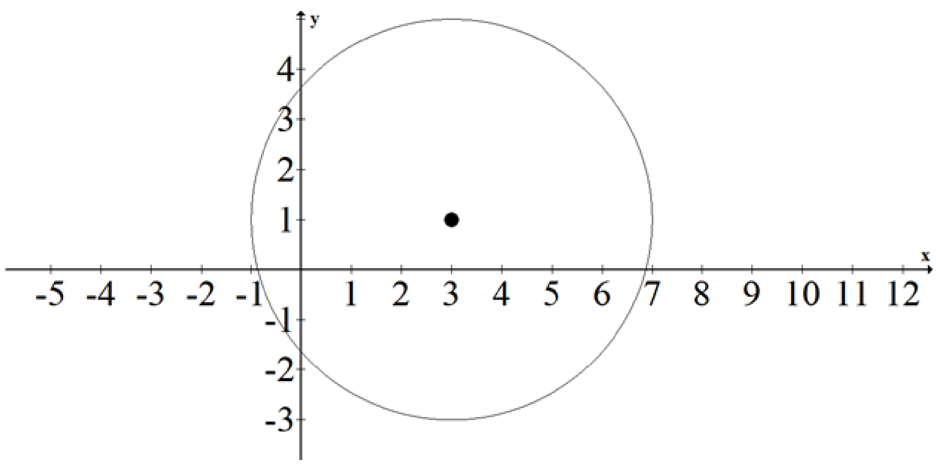
Solution: Complete the square and then divide by the coefficient of x^2 and y^2

$$36x^2 - 24x + 36y^2 + 36y = 275$$
$$36\left(x^2 - \frac{2}{3}x + \_\_\right) + 36(y^2 + y + \_\_) = 275$$
$$36\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 36\left(y^2 + y + \frac{1}{4}\right) = 275 + 4 + 9$$
$$36\left(x - \frac{1}{3}\right)^2 + 36\left(y + \frac{1}{2}\right)^2 = 288$$
$$\left(x - \frac{1}{3}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 8$$

The center is $\left(\frac{1}{3}, -\frac{1}{2}\right)$. The radius is $\sqrt{8} = 2\sqrt{2}$.

Example C

Write the equation of the following circle.



Solution:

The center of the circle is at (3, 1) and the radius of the circle is $r = 4$. The equation is $(x-3)^2 + (y-1)^2 = 16$.

Concept Problem Revisited

The reason why the graphing form of a circle looks like the Pythagorean Theorem is because each x and y coordinate along the outside of the circle forms a perfect right triangle with the radius as the hypotenuse.

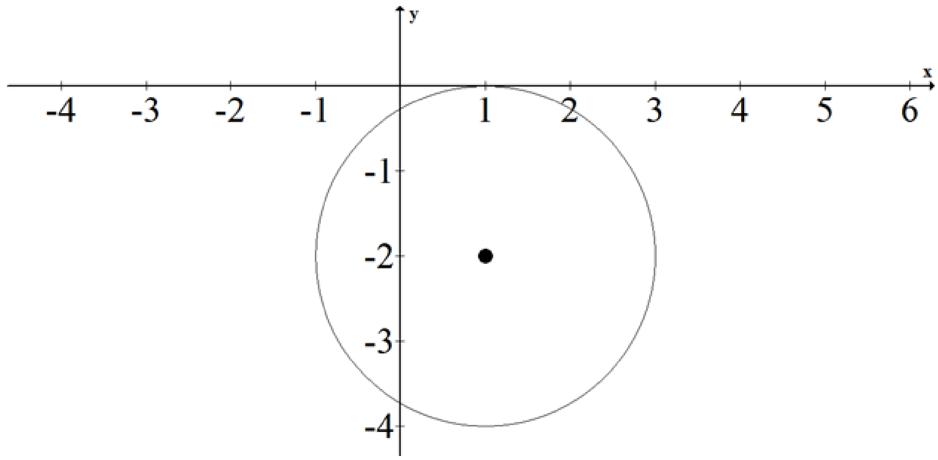
Guided Practice

1. Graph the following conic: $(x+2)^2 + (y-1)^2 = 1$

2. Translate the following conic from standard form to graphing form.

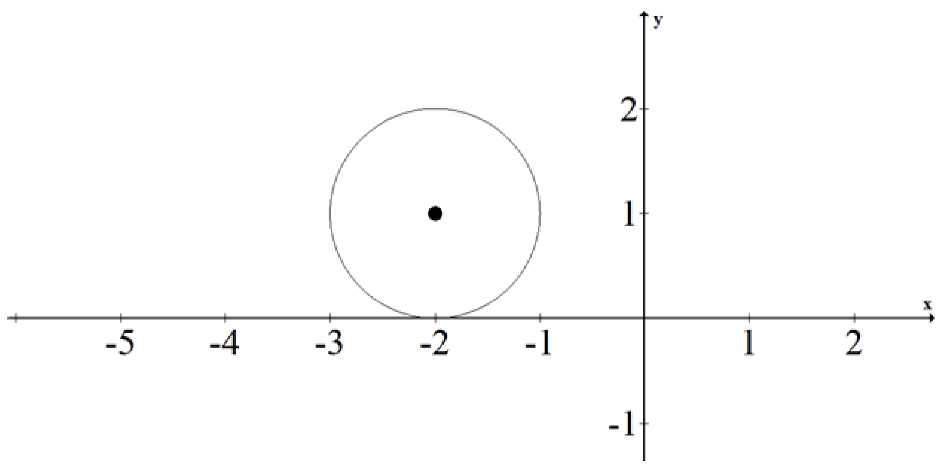
$x^2 - 34x + y^2 + 24y + \frac{749}{2} = 0$

3. Write the equation for the following circle.



Answers:

1.



2.

$$\begin{aligned}
x^2-34x+y^2+24y+\frac{749}{2} &= 0\\
x^2-34x+y^2+24y &= -\frac{749}{2}\\
x^2-34x+289+y^2+24y+144 &= -\frac{749}{2}+289+144\\
(x-17)^2+(y+12)^2 &= \frac{117}{2}
\end{aligned}$$

3.

$(x-1)^2+(y+2)^2=4$

Vocabulary

The ***radius*** of a circle is the distance from the center of the circle to the outside edge.

The ***center*** of a circle is the point that defines the location of the circle.

A ***circle*** is the collection of points that are equidistant from a given point.

The **diameter** of a circle is the distance across the circle passing through the center.

In Summary

We have learned the definition of a circle. We have also learned how to convert from the general form of a conic to the standard form of a circle. Finally we have learned how to graph a circle given the standard form of the circle and how to obtain the equation in standard form given the graph of the circle.

Practice

Graph the following conics:

1. $(x+4)^2+(y-3)^2=1$

2. $(x-7)^2+(y+1)^2=4$

3. $(y+2)^2+(x-1)^2=9$

4. $x^2+(y-5)^2=8$

5. $(x-2)^2+y^2=16$

Translate the following conics from standard form to graphing form.

6. $x^2-4x+y^2+10y+18=0$

7. $x^2+2x+y^2-8y+1=0$

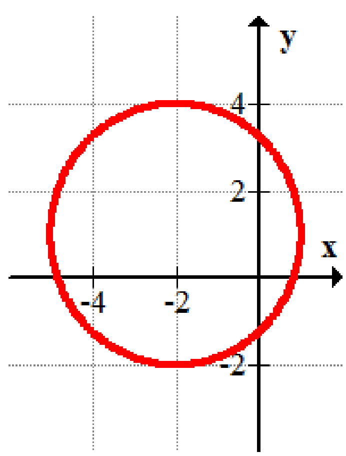
8. $x^2-6x+y^2-4y+12=0$

9. $x^2+2x+y^2+14y+25=0$

10. $x^2-2x+y^2-2y=0$

Write the equations for the following circles.

11.



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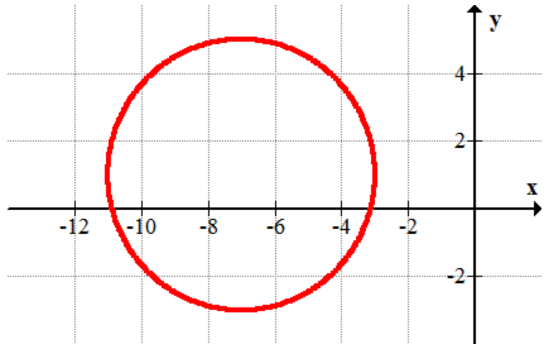

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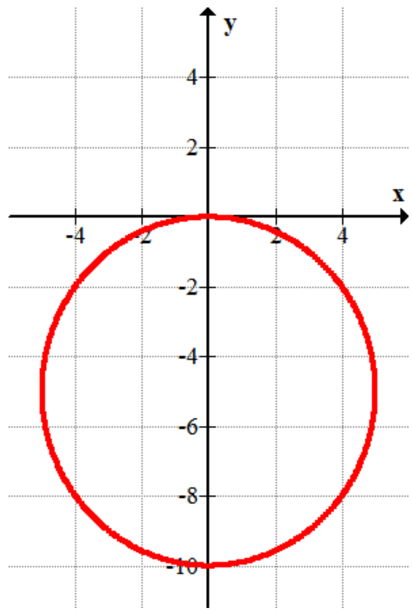

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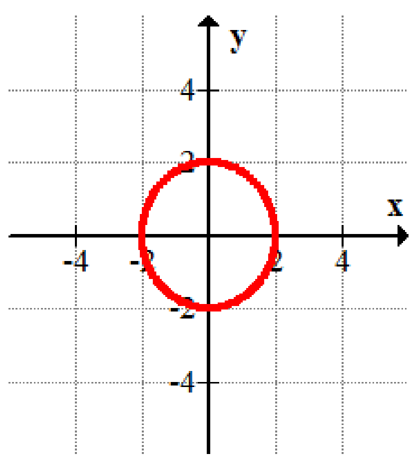

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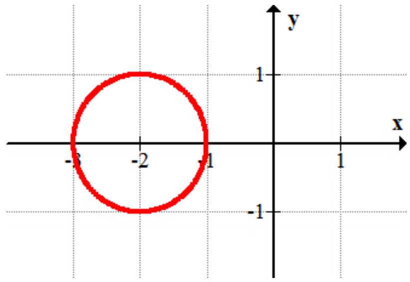

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10.4 Ellipses

Here you will translate ellipse equations from standard conic form to graphing form, graph ellipses and identify the different axes. You will also identify eccentricity and solve word problems involving ellipses.

TEKS

P.3.F
P.3.G
P.3.H

Lesson Objectives

In this lesson you will learn about:
1. The standard form equation of an ellipse.
2. How to find the foci, and the vertices of an ellipse.
3. How to graph an ellipse.
4. How to convert from the general form to the standard ellipse form.

Introduction

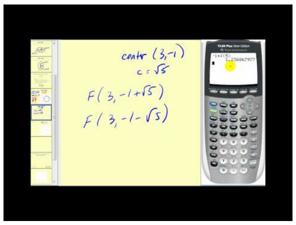
An ellipse is commonly known as an oval. Ellipses are just as common as parabolas in the real world with their own uses. Rooms that have elliptical shaped ceilings are called whisper rooms because if you stand at one focus point and whisper, someone standing at the other focus point will be able to hear you. Most of the car logos are ellipses think about ford, kia, toyota logos.

Vocabulary

Foci, Vertex, Major Axis, Minor Axis, Eccentricity.

Ellipses look similar to circles, but there are a few key differences between these shapes. Ellipses have both an x-radius and a y-radius while circles have only one radius. Another difference between circles and ellipses is that an ellipse is defined as the collection of points that are a set distance from two focus points while circles are defined as the collection of points that are a set distance from one center point. A third difference between ellipses and circles is that not all ellipses are similar to each other while all circles are similar to each other. Some ellipses are narrow and some are almost circular. How do you measure how strangely shaped an ellipse is?

Watch This



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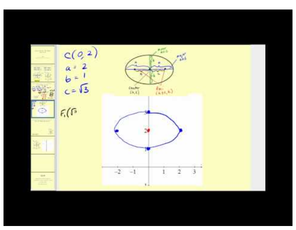
http://www.youtube.com/watch?v=LVumLCx3fQo James Sousa: Conic Sections: The Ellipse part 1



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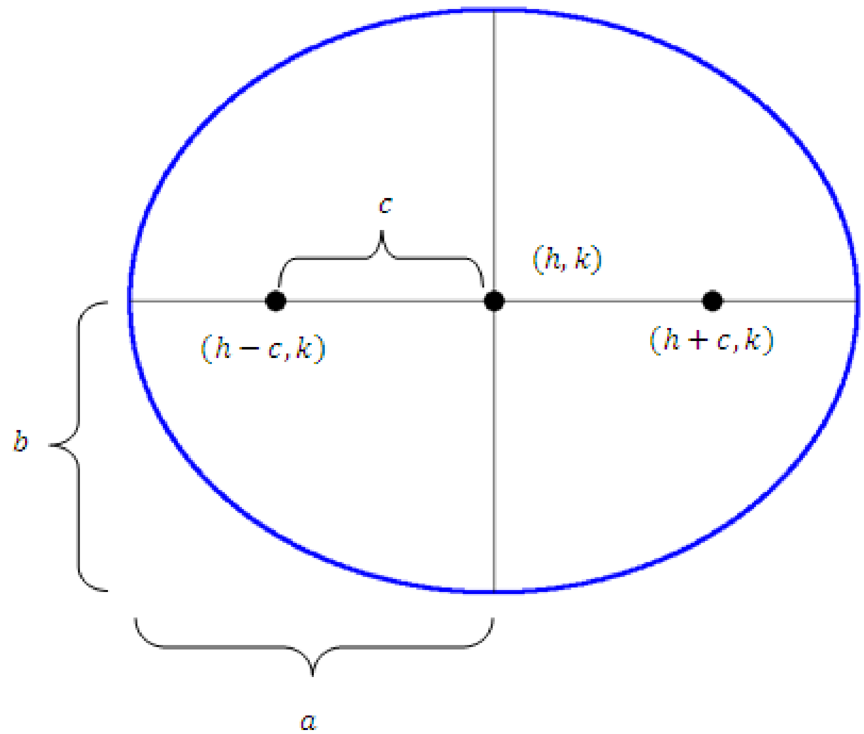
http://www.youtube.com/watch?v=oZB69DY0q9A James Sousa: Conic Sections: The Ellipse part 2

Guidance

An ellipse has two foci. For every point on the ellipse, the sum of the distances to each foci is constant. This is what defines an ellipse. Another way of thinking about the definition of an ellipse is to allocate a set amount of string and fix the two ends of the string so that there is some slack between them. Then use a pencil to pull the string taught and trace the curve all the way around both fixed points. You will trace an ellipse and the fixed end points of the string will be the foci. Foci is the plural form of focus. In the picture below, (h,k) is the center of the ellipse and the other two marked points are the foci.



The Standard form equation for an ellipse is:

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

In this case the major axis is horizontal because a, the x-radius, is larger. If the y-radius were larger, then a and b would reverse. In other words, the coefficient a always comes from the length of the semi major axis (the longer axis) and the coefficient b always comes from the length of the semi minor axis (the shorter axis).

In order to find the locations of the two foci, you will need to find the focal radius represented as c using the following relationship:

$a^2-b^2=c^2$

Since a is the larger number, then a is the distance from the center of the ellipse to the vertex and c is the distance from the center of the ellipse to the foci.

Once you have the focal radius, measure from the center along the major axis to locate the foci. The general shape of an ellipse is measured using eccentricity. Eccentricity is a measure of how oval or how circular the shape is. Ellipses can have an eccentricity between 0 and 1 where a number close to 0 is extremely circular and a number close to 1 is less circular. Eccentricity is calculated by:

$e=\frac{c}{a}$

Example A

Find the vertices (endpoints of the major axis), foci and eccentricity of the following ellipse.

$\frac{x^2}{25}+\frac{y^2}{16}=1$

Solution: The center of this ellipse is at (0, 0). The semi major axis is $a=5$ and travels horizontally. This means that the vertices are at (5, 0) and (-5, 0). The semi-minor axis is $b=4$ and travels vertically.

$$25 - 16 = c^2$$
$$3 = c$$

The focal radius is 3. This means that the foci are at (3, 0) and (-3, 0).

The eccentricity is $e = \frac{3}{5}$.

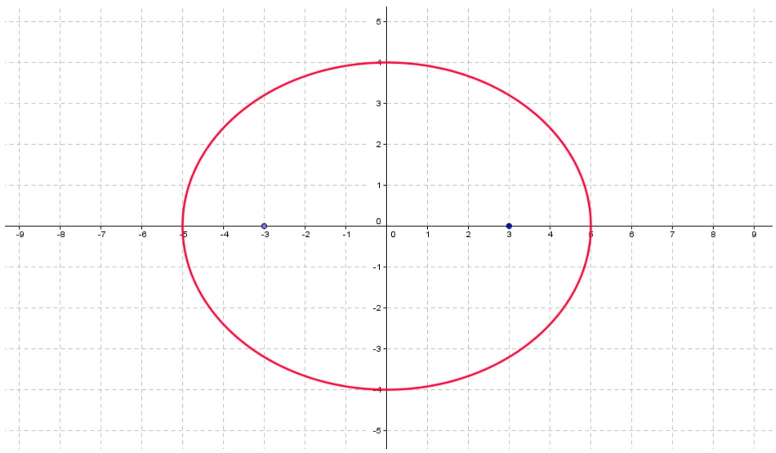


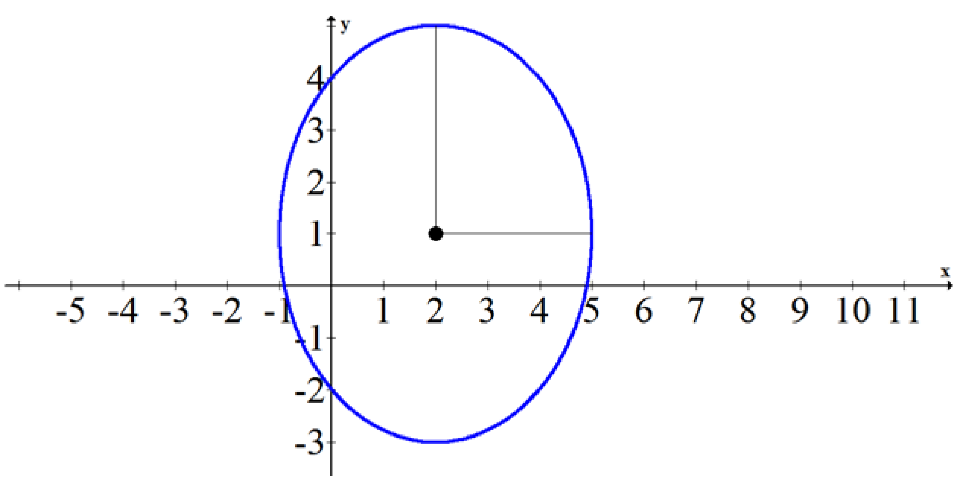
FIGURE 10.1

Example B

Sketch the following ellipse.

$\frac{(y-1)^2}{16} + \frac{(x-2)^2}{9} = 1$

Solution: Plotting the foci are usually important, but in this case the question simply asks you to sketch the ellipse. All you need is the center, x-radius and y-radius.



Example C

Put the following conic into graphing form.

$25x^2 - 150x + 36y^2 + 72y - 639 = 0$

Solution:

$$\begin{aligned}25x^2-150x+36y^2+72y-639&=0\\25(x^2-6x+\quad)+36(y^2+2y+\quad)&=639\\25(x^2-6x+9)+36(y^2+2y+1)&=639+225+36\\25(x-3)^2+36(y+1)^2&=900\\\frac{25(x-3)^2}{900}+\frac{36(y+1)^2}{900}&=\frac{900}{900}\\\frac{(x-3)^2}{36}+\frac{(y+1)^2}{25}&=1\end{aligned}$$

Example D

Given the graph of the following ellipse, write the equation in standard form and find the foci, the vertices.

Solution:

First we need to notice that it is a vertical ellipse. Then it looks like the center of this ellipse will be (-3,2).

Then the distance from the center to the vertex is 10 and the distance from the center to the semi-vertex is 5.

Therefore $a=10, b=5, c=\pm\sqrt{10^2-5^2}=\pm\sqrt{75}\approx\pm 8.66$

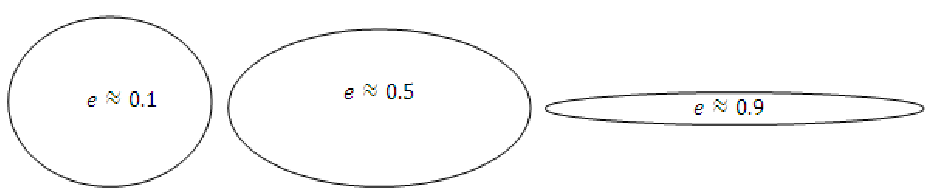
Finally the equation of the ellipse in standard form is $\frac{(x+3)^2}{25}+\frac{(y-2)^2}{100}=1$

The coordinates of the foci are $(-3,2+8.66),(-3,2-8.66)$ *or* $(-3,10.66),(-3,-6.66)$

The coordinates of the vertices are $(-3,2+10),(-3,2-10)$ *or* $(-3,12),(-3,-8)$

Concept Problem Revisited

Ellipses are measured using their eccentricity. Here are three ellipses with estimated eccentricity for you to compare.



Eccentricity is the ratio of the focal radius to the semi major axis: $e=\frac{c}{a}$.

Guided Practice

1. Find the vertices (endpoints of the major axis), foci and eccentricity of the following ellipse.

$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{16}=1$

2. Sketch the following ellipse.

$(x-3)^2+\frac{(y-1)^2}{9}=1$

3. Put the following conic into standard form form.

$9x^2-9x+4y^2+12y+\frac{9}{4}=-8$

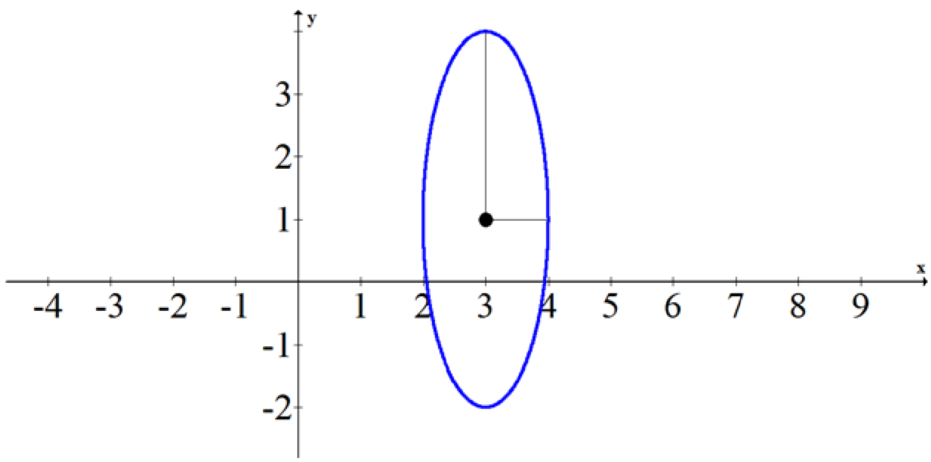
Answers:

1. The center of the ellipse is at (2, -1). The major axis is vertical which means the semi major axis is $a = 4$. The vertices are (2, 3) and (2, -5).

$$\begin{aligned} 16^2 - 4^2 &= c^2 \\ \sqrt{240} &= c \\ 4\sqrt{15} &= c \end{aligned}$$

Thus the foci are $(2, -1 + 4\sqrt{15})$ and $(2, -1 - 4\sqrt{15})$

2.



3.

$$\begin{aligned} 9x^2 - 9x + 4y^2 + 12y + \frac{9}{4} &= -8 \\ 9x^2 - 9x + \frac{9}{4} + 4y^2 + 12y &= -8 \\ 9\left(x^2 - x - \frac{1}{4}\right) + 4(y^2 + 3y) &= -8 \\ 9\left(x - \frac{1}{2}\right)^2 + 4\left(y^2 + 3y + \frac{9}{4}\right) &= -8 + 4 \cdot \frac{9}{4} \\ 9\left(x - \frac{1}{2}\right)^2 + 4\left(y + \frac{3}{2}\right)^2 &= 1 \\ \frac{(x - \frac{1}{2})^2}{\frac{1}{9}} + \frac{(y + \frac{3}{2})^2}{\frac{1}{4}} &= 1 \end{aligned}$$

Vocabulary

The ***semi-major axis*** is the distance from the center of the ellipse to the furthest point on the ellipse. The letter a represents the length of the semi-major axis.

The ***major axis*** is the longest distance from end to end of an ellipse. This distance is twice that of the semi-major axis.

The ***semi-minor axis*** is the distance from the center to the edge of the ellipse on the axis that is perpendicular to the semi-major axis. The letter b represents the length of the semi-minor axis.

An ***ellipse*** is the collection of points whose sum of distances from two foci is constant.

The ***foci*** in an ellipse are the two points that the ellipse curves around.

Eccentricity is a measure of how oval or how circular the shape is. It is the ratio of the focal radius to the semi major axis: $e = \frac{c}{a}$.

In Summary

We have learned what an ellipse is. We have learned how to graph an ellipse given the standard equation, how to convert the general form of a polynomial into the standard form of a polynomial and how to write the equation of an ellipse in standard form given the graph and find the foci, vertices, and eccentricity.

Practice

Find the vertices, foci, and eccentricity for each of the following ellipses.

1. $\frac{(x-1)^2}{4} + \frac{(y+5)^2}{16} = 1$
2. $\frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$
3. $(x-2)^2 + \frac{(y-1)^2}{4} = 1$

Now sketch each of the following ellipses *(note that they are the same as the ellipses in #1 - #3)*.

4. $\frac{(x-1)^2}{4} + \frac{(y+5)^2}{16} = 1$
5. $\frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$
6. $(x-2)^2 + \frac{(y-1)^2}{4} = 1$

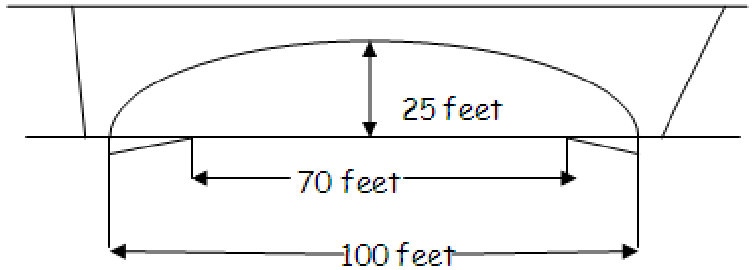
Put each of the following equations into graphing form.

7. $x^2 + 2x + 4y^2 + 56y + 197 = 16$
8. $x^2 - 8x + 9y^2 + 18y + 25 = 9$
9. $9x^2 - 36x + 4y^2 + 16y + 52 = 36$

Find the equation for each ellipse based on the description.

10. An ellipse with vertices (4, -2) and (4, 8) and minor axis of length 6.
11. An ellipse with minor axis from (4, -1) to (4, 3) and major axis of length 12.
12. An ellipse with minor axis from (-2, 1) to (-2, 7) and one focus at (2, 4).
13. An ellipse with one vertex at (6, -15), and foci at (6, 10) and (6, -14).

A bridge over a roadway is to be built with its bottom the shape of a semi-ellipse 100 feet wide and 25 feet high at the center. The roadway is to be 70 feet wide.



14. Find one possible equation of the ellipse that models the bottom of the bridge.
15. What is the clearance between the roadway and the overpass at the edge of the roadway?

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10.5 Hyperbolas

Here you will translate conic equations into graphing form and graph hyperbolas. You will also learn how to measure the eccentricity of a hyperbola and solve word problems.

TEKS

P.3.F
P.3.G
P.3.I

Lesson Objectives

In this lesson you will learn about:
1. The definition of a hyperbola.
2. How to convert from general form of a conic to the standard form of a hyperbola.
3. How to graph a hyperbola in standard form.
4. How to obtain the foci and the asymptotes of a hyperbola.

Introduction

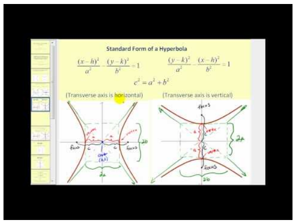
Hyperbolas are relations that have asymptotes. When graphing rational functions you often produce a hyperbola. In this concept, hyperbolas will not be oriented in the same way as with rational functions, but the basic shape of a hyperbola will still be there.

Vocabulary

Hyperbola, Eccentricity.

Hyperbolas can be oriented so that they open side to side or up and down. One of the most common mistakes that you can make is to forget which way a given hyperbola should open. What are some strategies to help?

Watch This



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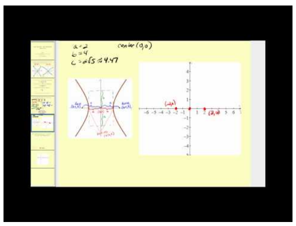
http://www.youtube.com/watch?v=i6vM82SNAUk James Sousa: Conic Sections: The Hyperbola part 1



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http://www.youtube.com/watch?v=6Xahrwp6LkI James Sousa: Conic Sections: The Hyperbola part 2

Guidance

A hyperbola has two foci. For every point on the hyperbola, the difference of the distances to each foci is constant. This is what defines a hyperbola.

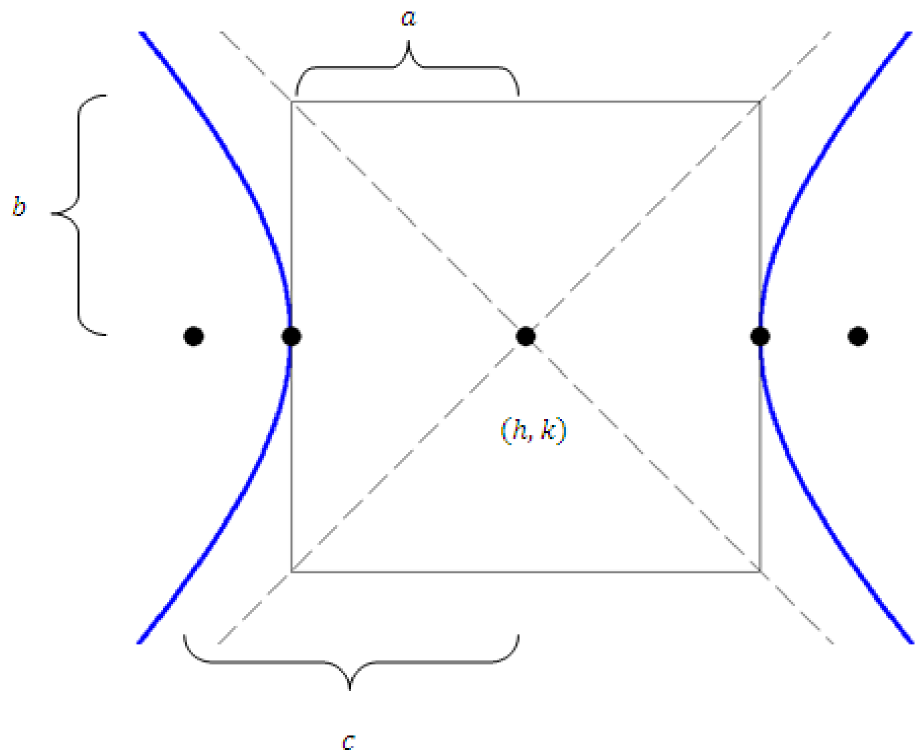
The graphing form of a hyperbola that opens side to side is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

A hyperbola that opens up and down is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Notice that for hyperbolas, a goes with the positive term and b goes with the negative term. It does not matter which constant is larger.



When graphing, the constants a and b enable you to draw a rectangle around the center. The transverse axis travels from vertex to vertex and has length $2a$. The conjugate axis travels perpendicular to the transverse axis through the center and has length $2b$. The foci lie beyond the vertices so the eccentricity, which is measured as $e = \frac{c}{a}$, is larger than 1 for all hyperbolas. Hyperbolas also have two directrix lines that are $\frac{a^2}{c}$ away from the center (not shown on the image).

The focal radius is $a^2 + b^2 = c^2$.

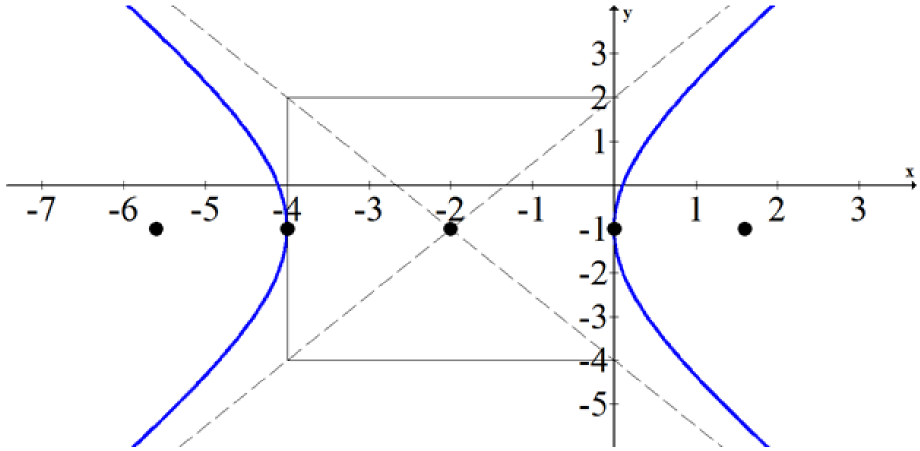
Example A

Put the following hyperbola into graphing form and sketch it.

$9x^2 - 4y^2 + 36x - 8y - 4 = 0$

Solution:

$$\begin{aligned}
9(x^2 + 4x) - 4(y^2 + 2y) &= 4 \\
9(x^2 + 4x + 4) - 4(y^2 + 2y + 1) &= 4 + 36 - 4 \\
9(x+2)^2 - 4(y+1)^2 &= 36 \\
\frac{(x+2)^2}{4} - \frac{(y+1)^2}{9} &= 1
\end{aligned}$$



Example B

Find the equation of the hyperbola with foci at (-3, 5) and (9, 5) and asymptotes with slopes of $\pm\frac{4}{3}$.

Solution: The center is between the foci at (3, 5). The focal radius is $c = 6$. The slope of the asymptotes is always the rise over run inside the box. In this case since the hyperbola is horizontal and a is in the x direction the slope is $\frac{b}{a}$. This makes a system of equations.

$$\frac{b}{a} = \pm\frac{4}{3}$$
$$a^2 + b^2 = 6^2$$

When you solve, you get $a = \sqrt{13}$, $b = \frac{4}{3}\sqrt{13}$.

$\frac{(x-3)^2}{13} - \frac{(y-5)^2}{\frac{16}{9}\cdot 13} = 1$

Example C

Find the equation of a hyperbola that opens left and right with a vertex at the point (3,0), passing through the point (5,5) with center (0,0).

Solution

We know that the hyperbola opens left and right, therefore the form of the hyperbola has to be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We also know that the vertex of the hyperbola is a (3,0) therefore it implies that the value of $a = 3$ therefore the equation now should be

$$\frac{x^2}{9} - \frac{y^2}{b^2} = 1$$

Since we want the hyperbola to pass through the point (5,5), then we substitute those values in the place of x and y yielding us the equation

$$\frac{25}{9}-\frac{25}{b^2}=1$$

Finally our job is to solve for the value of b^2 as follows

$$\begin{aligned}\frac{25}{9}-\frac{25}{b^2}&=1\\-\frac{25}{b^2}&=1-\frac{25}{9}\\-\frac{25}{b^2}&=-\frac{16}{9}\\-225&=-16b^2\\\frac{225}{16}&=b^2\end{aligned}$$

Finally our equation for the hyperbola will be:

$$\frac{x^2}{9}-\frac{y^2}{\frac{225}{16}}=1$$

Also we can obtain that the value of $b=\frac{15}{4}$

The following graph shows graph of this hyperbola confirming that it passes trough the point (5,5)

Example D

Find the equation of the conic that has a focus point at (1, 2), a directrix at $x=5$, and an eccentricity equal to $\frac{3}{2}$. Use the property that the distance from a point on the hyperbola to the focus is equal to the eccentricity times the distance from that same point to the directrix:

$\overline{PF}=e\overline{PD}$

Solution: This relationship bridges the gap between ellipses which have eccentricity less than one and hyperbolas which have eccentricity greater than one. When eccentricity is equal to one, the shape is a parabola.

$\sqrt{(x-1)^2+(y-2)^2}=\frac{3}{2}\sqrt{(x-5)^2}$

Square both sides and rearrange terms so that it is becomes a hyperbola in graphing form.

$$\begin{aligned}x^2-2x+1+(y-2)^2&=\frac{9}{4}(x^2-10x+25)\\x^2-2x+1-\frac{9}{4}x^2+\frac{90}{4}x-\frac{225}{4}+(y-2)^2&=0\\-\frac{5}{4}x^2+\frac{92}{4}x+(y-2)^2&=\frac{221}{4}\\-5x^2+92x+4(y-2)^2&=221\\-5\left(x^2-\frac{92}{5}x\right)+4(y-2)^2&=221\end{aligned}$$

$$
\begin{aligned}
-5\left(x^2-\frac{92}{5}x+\frac{92^2}{10^2}\right)+4(y-2)^2 &= 221-\frac{2116}{5}\\
-5\left(x-\frac{92}{10}\right)^2+4(y-2)^2 &= -\frac{1011}{5}\\
\left(x-\frac{92}{10}\right)^2-(y-2)^2 &= \frac{1011}{100}\\
\frac{\left(x-\frac{92}{10}\right)^2}{\left(\frac{1011}{100}\right)}-\frac{(y-2)^2}{\left(\frac{1011}{100}\right)} &= 1
\end{aligned}
$$

Concept Problem Revisited

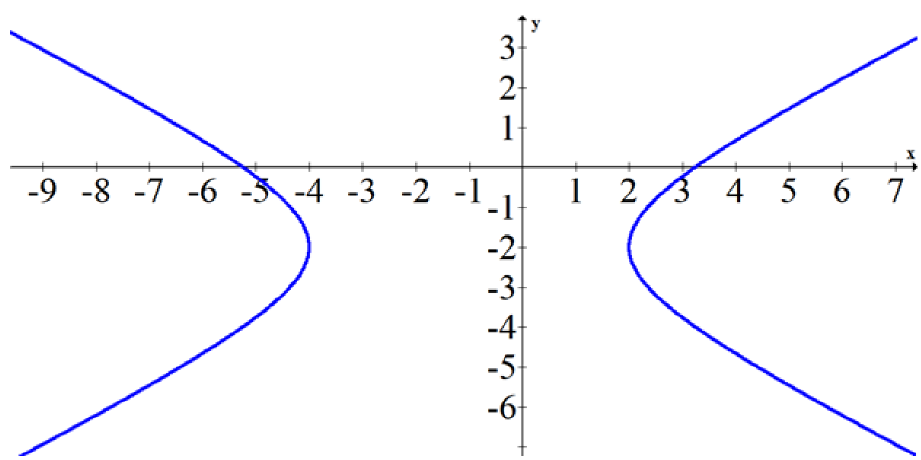
The best strategy to remember which direction the hyperbola opens is often the simplest. Consider the hyperbola $x^2-y^2=1$. This hyperbola opens side to side because x can clearly never be equal to zero. This is a basic case that shows that when the negative is with the y value then the hyperbola opens up side to side.

Guided Practice

1. Completely identify the components of the following conic.

$9x^2-16y^2-18x+96y+9=0$

2. Given the following graph, estimate the equation of the conic.



3. Find the equation of the hyperbola that has foci at (13, 5) and (-17, 5) with asymptote slopes of $\pm\frac{3}{4}$.

Answers:

1. $9x^2-16y^2-18x+96y+9=0$

$$
\begin{aligned}
9(x^2-2x)-16(y^2-6y)&=-9\\
9(x^2-2x+1)-16(y^2-6y+9)&=-9+9-144\\
9(x-1)^2-16(y-3)^2&=144\\
-\frac{(x-1)^2}{16}+\frac{(y-3)^2}{16}&=1\\
\frac{(y-3)^2}{9}-\frac{(x-1)^2}{16}&=1
\end{aligned}
$$

Shape: Hyperbola that opens vertically.

Center: (1, 3)

$a=3$

$b=4$

$c=5$

$e=\frac{c}{a}=\frac{5}{3}$

$d=\frac{a^2}{c}=\frac{9}{5}$

Foci: (1, 8), (1, -2)

Vertices: (1, 6), (1, 0)

Equations of asymptotes: $(x-1)=\pm\frac{3}{4}(y-3)$

Note that it is easiest to write the equations of the asymptotes in point-slope form using the center and the slope.

Equations of directrices: $y=3\pm\frac{9}{5}$

2. Since exact points are not marked, you will need to estimate the slope of asymptotes to get an approximation for a and b. The slope seems to be about $\pm\frac{2}{3}$. The center seems to be at (-1, -2). The transverse axis is 6 which means $a=3$.

$\frac{(x+1)^2}{9}-\frac{(y+2)^2}{4}=1$

3. The center of the conic must be at (-2, 5). The focal radius is $c=15$. The slopes of the asymptotes are $\pm\frac{3}{4}=\frac{b}{a}$.

$a^2+b^2=c^2$

Since 3, 4, 5 is a well known Pythagorean number triple it should be clear to you that $a=12, b=9$.

$\frac{(x+2)^2}{12^2}-\frac{(y-5)^2}{9^2}=1$

Vocabulary

Eccentricity is the ratio between the length of the focal radius and the length of the semi transverse axis. For hyperbolas, the eccentricity is greater than one.

A ***hyperbola*** is the collection of points that share a constant difference between the distances between two focus points.

In Summary

We have learned about the definition of a hyperbola. We have learned how to find the vertices of the hyperbolas, the foci of the hyperbolas and the asymptotes of the hyperbola. Also we have learned how to graph a hyperbola given the equation in standard form, how to convert from general conic form to hyperbola standard form and finally how to obtain the equation of a hyperbola passing through a given point.

Practice

Use the following equation for #1 - #5: $x^2+2x-4y^2-24y-51=0$

1. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.

2. Identify whether the hyperbola opens side to side or up and down.

3. Find the location of the vertices.

4. Find the equations of the asymptotes.

5. Sketch the hyperbola.

Use the following equation for #6 - #10: $-9x^2-36x+16y^2-32y-164=0$

6. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.

7. Identify whether the hyperbola opens side to side or up and down.

8. Find the location of the vertices.

9. Find the equations of the asymptotes.

10. Sketch the hyperbola.

Use the following equation for #11 - #15: $x^2-6x-9y^2-54y-81=0$

11. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.

12. Identify whether the hyperbola opens side to side or up and down.

13. Find the location of the vertices.

14. Find the equations of the asymptotes.

15. Sketch the hyperbola.

MEDIA

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10.6 Degenerate Conics

Here you will discover what happens when a conic equation can't be put into graphing form.

TEKS

P.3.G

Lesson Objectives

In this lesson you will learn about:
1. What degenerate conics are.
2. How to identify conics from the general form of a conic.

Introduction

The general equation of a conic is $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. This form is so general that it encompasses all regular lines, singular points and degenerate hyperbolas that look like an X. This is because there are a few special cases of how a plane can intersect a two sided cone. How are these degenerate shapes formed?

Vocabulary

Degenerate Conics.

Guidance

Degenerate conic equations simply cannot be written in graphing form. There are three types of degenerate conics:

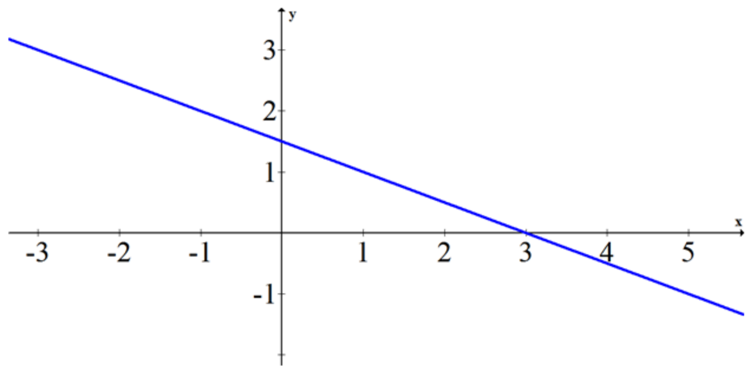
1. ***A singular point***, which is of the form: $\frac{(x-h)^2}{a}+\frac{(y-k)^2}{b}=0$. You can think of a singular point as a circle or an ellipse with an infinitely small radius.
2. ***A line***, which has coefficients $A=B=C=0$ in the general equation of a conic. The remaining portion of the equation is $Dx+Ey+F=0$, which is a line.
3. ***A degenerate hyperbola***, which is of the form: $\frac{(x-h)^2}{a}-\frac{(y-k)^2}{b}=0$. The result is two intersecting lines that make an "X" shape. The slopes of the intersecting lines forming the X are $\pm\frac{b}{a}$. This is because b goes with the y portion of the equation and is the rise, while a goes with the x portion of the equation and is the run.

Example A

Transform the conic equation into standard form and sketch.

$0x^2+0xy+0y^2+2x+4y-6=0$

Solution: This is the line $y=-\frac{1}{2}x+\frac{3}{2}$.



Example B

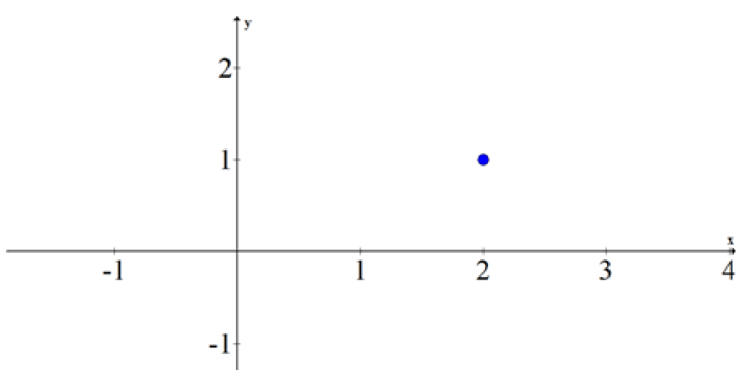
Transform the conic equation into standard form and sketch.

$3x^2-12x+4y^2-8y+16=0$

Solution: $3x^2-12x+4y^2-8y+16=0$

$$\begin{aligned}3(x^2-4x)+4(y^2-2y)&=-16\\3(x^2-4x+4)+4(y^2-2y+1)&=-16+12+4\\3(x-2)^2+4(y-1)^2&=0\\\frac{(x-2)^2}{4}+\frac{(y-1)^2}{3}&=0\end{aligned}$$

The point (2, 1) is the result of this degenerate conic.



Example C

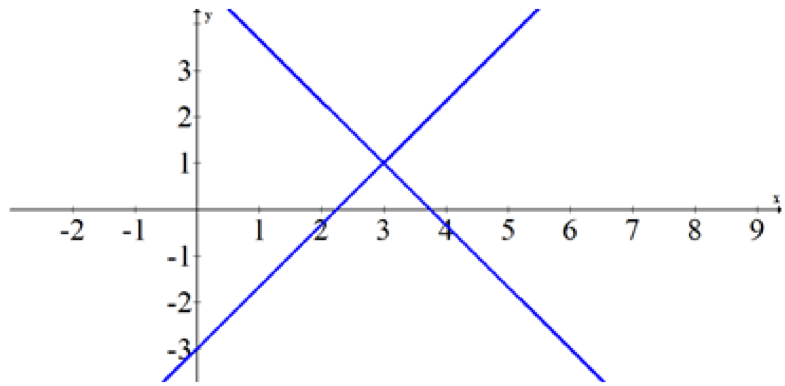
Transform the conic equation into standard form and sketch.

$16x^2-96x-9y^2+18y+135=0$

Solution: $16x^2-96x-9y^2+18y+135=0$

$$\begin{aligned}16(x^2-6x)-9(y^2-2y)&=-135\\16(x^2-6x+9)-9(y^2-2y+1)&=-135+144-9\\16(x-3)^2-9(y-1)^2&=0\\\frac{(x-3)^2}{9}-\frac{(y-1)^2}{16}&=0\end{aligned}$$

This is a degenerate hyperbola.



Concept Problem Revisited

When you intersect a plane with a two sided cone where the two cones touch, the intersection is a **single point**. When you intersect a plane with a two sided cone so that the plane touches the edge of one cone, passes through the central point and continues touching the edge of the other conic, this produces a **line**. When you intersect a plane with a two sided cone so that the plane passes vertically through the central point of the two cones, it produces a **degenerate hyperbola**.

Vocabulary

A ***degenerate conic*** is a conic that does not have the usual properties of a conic. Since some of the coefficients of the general equation are zero, the basic shape of the conic is merely a point, a line or a pair of lines. The connotation of the word degenerate means that the new graph is less complex than the rest of conics.

Guided Practice

1. Create a conic that describes just the point (4, 7).

2. Transform the conic equation into standard form and sketch.

$-4x^2+8x+y^2+4y=0$

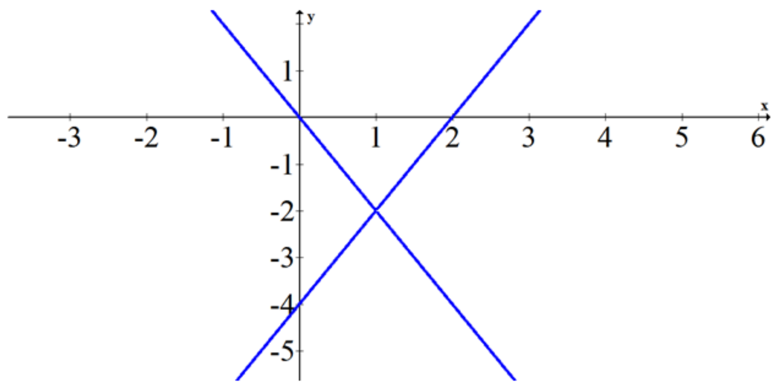
3. Can you tell just by looking at a conic in general form if it is a degenerate conic?

Answers:

1. $(x-4)^2+(y-7)^2=0$

2.

$$
\begin{aligned}
-4x^2+8x+y^2+4y &= 0 \\
-4(x^2-2x)+(y^2+4y) &= 0 \\
-4(x^2-2x+1)+(y^2+4y+4) &= -4+4 \\
-4(x-1)^2+(y+2)^2 &= 0 \\
\frac{(x-1)^2}{1}-\frac{(y+2)^2}{4} &= 0
\end{aligned}
$$



3. In general you cannot tell if a conic is degenerate from the general form of the equation. You can tell that the degenerate conic is a line if there are no x^2 or y^2 terms, but other than that you must always try to put the conic equation into graphing form and see whether it equals zero because that is the best way to identify degenerate conics.

Practice

1. What are the three degenerate conics?

Change each equation into graphing form and state what type of conic or degenerate conic it is.

2. $x^2-6x-9y^2-54y-72=0$

3. $4x^2+16x-9y^2+18y-29=0$

4. $9x^2+36x+4y^2-24y+72=0$

5. $9x^2+36x+4y^2-24y+36=0$

6. $0x^2+5x+0y^2-2y+1=0$

7. $x^2+4x-y+8=0$

8. $x^2-2x+y^2-6y+6=0$

9. $x^2-2x-4y^2+24y-35=0$

10. $x^2-2x+4y^2-24y+33=0$

Sketch each conic or degenerate conic.

11. $\frac{(x+2)^2}{4}+\frac{(y-3)^2}{9}=0$

12. $\frac{(x-3)^2}{9}+\frac{(y+3)^2}{16}=1$

13. $\frac{(x+2)^2}{9}-\frac{(y-1)^2}{4}=1$

14. $\frac{(x-3)^2}{9}-\frac{(y+3)^2}{4}=0$

15. $3x+4y=12$

You learned that a conic section is the family of shapes that are formed by the different ways a flat plane intersects a two sided cone in three dimensional space. Parabolas, circles, ellipses and hyperbolas each have their own graphing form of equations that helped you identify information about them like the focus and the directrix.

10.7 References

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CHAPTER 11 Sequences and Series

Chapter Outline

Discrete math is all about patterns, sequences, summing numbers, counting and probability. Many of these topics you will revisit in classes after Calculus. The goal here is to familiarize you with the important notation and the habits of thinking that accompany a mature way of looking at problems.

11.1 Recursion

Here you will define patterns recursively and use recursion to solve problems.

TEKS

P.5.A
P.5.B

Lesson Objectives

In this lesson you will learn about:
1. What a recursive sequence is.
2. How to find the values of a terms of a sequence using a recursive formula.
3. How to find a recursive formula for a sequence.

Introduction

When you look at a pattern, there are many ways to describe it. You can describe patterns explicitly by stating how each term $a_k$is obtained from the term number k. You can also describe patterns recursively by stating how each new term $a_k$is obtained from the previous term a_{k-1}. Recursion defines an entire sequence based on the first term and the pattern between consecutive terms. The Fibonacci sequence is a famous recursive sequence, but how is it represented using recursion?
$0, 1, 1, 3, 5, 8, 13, 21, 34, \ldots$

Vocabulary

Sequence, Recursive Pattern/Sequence, Explicit Pattern/Sequence.

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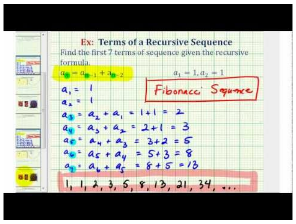

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http://www.youtube.com/watch?v=RjsyEWDEQe0 James Sousa: Finding Terms in a Sequence Given the Recursive Formula

Guidance

When most people see a pattern they see how consecutive terms are related to one another. You might describe patterns with phrases like the ones below:

TABLE 11.1:

Pattern	**Recursive Description**
$3, 6, 12, 24, \ldots$	"Each term is twice as big as the previous term"
$3, 6, 9, 12, \ldots$	"Each term is three more than the previous term"

Each phrase is a sign of recursive thinking that defines each term as a function of the previous term.

$$a_k = f(a_{k-1})$$

In some cases, a recursive formula can be a function of the previous two or three terms. Keep in mind that the downside of a recursively defined sequence is that it is impossible to immediately know the 100^{th} term without knowing the 99^{th} term.

Example A

For the Fibonacci sequence, determine the first eleven terms and the sum of these terms.

Solution: $0+1+1+2+3+5+8+13+21+34+55 = 143$

Example B

Write a recursive definition that fits the following sequence.

$3, 7, 11, 15, 18, \ldots$

Solution: In order to write a recursive definition for a sequence you must define the pattern and state the first term. With this information, others would be able to replicate your sequence without having seen it for themselves.

$a_1 = 3$

$a_k = a_{k-1} + 4$

Example C

What are the first nine terms of the sequence defined by:

$a_1 = 1$

$a_k = \frac{1}{k} + 1?$

Solution: $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{3}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}$

Concept Problem Revisited

The Fibonacci sequence is represented by the recursive definition:

$a_1 = 0$

$a_2 = 1$

$a_k = a_{k-2} + a_{k-1}$

Guided Practice

1. The Lucas sequence is like the Fibonacci sequence except that the starting numbers are 2 and 1 instead of 1 and 0. What are the first ten terms of the Lucas sequence?

2. Zeckendorf's Theorem states that every positive integer can be represented uniquely as a sum of nonconsecutive Fibonacci numbers. What is the Zeckendorf representation of the number 50 and the number 100?

3. Consider the following pattern generating rule:

If the last number is odd, multiply it by 3 and add 1.

If the last number is even, divide the number by 2.

Repeat.

Try a few different starting numbers and see if you can state what you think always happens.

Answers:

1. $2, 1, 3, 4, 7, 11, 18, 29, 47, 76$

2. $50 = 34 + 13 + 3;\ 100 = 89 + 8 + 3$

3. You can choose any starting positive integer you like. Here are the sequences that start with 7 and 15.

$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1 \ldots$

$15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1 \ldots$

You could make the conjecture that any starting number will eventually lead to the repeating sequence 4, 2, 1.

This problem is called the Collatz Conjecture and is an unproven statement in mathematics. People have used computers to try all the numbers up to 5×2^{60} and many mathematicians believe it to be true, but since all natural numbers are infinite in number, this test does not constitute a proof.

Vocabulary

A ***recursively defined pattern or sequence*** is a sequence with terms that are defined based on the prior term(s) in the sequence.

An ***explicit pattern or sequence*** is a sequence with terms that are defined based on the term number.

In Summary

We have learned what a recursive sequence is and how to find the values of each term of the sequence by using the recursive sequence.

Practice

Write a recursive definition for each of the following sequences.

1. $3, 7, 11, 15, 19, \ldots$

2. $3, 9, 27, 81, \ldots$

3. $3, 6, 9, 12, 15, \ldots$

4. $3, 6, 12, 24, 48, \ldots$

5. $1, 4, 16, 64, \ldots$

6. Find the first 6 terms of the following sequence:

$b_1 = 2$

$b_2 = 8$

$b_k = 6b_{k-1} - 4b_{k-2}$

7. Find the first 6 terms of the following sequence:

$c_1 = 4$

$c_2 = 18$

$c_k = 2c_{k-1} + 5c_{k-2}$

Suppose the Fibonacci sequence started with 2 and 5.

8. List the first 10 terms of the new sequence.

9. Find the sum of the first 10 terms of the new sequence.

Write a recursive definition for each of the following sequences. *These are trickier!*

10. $1, 4, 13, 40, \ldots$

11. $1, 5, 17, 53, \ldots$

12. $2, 11, 56, 281, \ldots$

13. $2, 3, 6, 18, 108, \ldots$

14. $4, 6, 11, 18, 30, \ldots$

15. $7, 13, 40, 106, 292, \ldots$

11.2 Arithmetic and Geometric Sequences

Here you will identify different types of sequences and use sequences to make predictions.

TEKS

P.5.A
P.5.B

Lesson Objectives

In this lesson you will learn about:
1. Infinite and finite sequences.
2. What an Arithmetic sequence is.
3. How to find and explicit formula for an arithmetic sequence.
4. What a Geometric sequence is and how to find the formula.

Introduction

A sequence is a list of numbers with a common pattern. The common pattern in an arithmetic sequence is that the same number is added or subtracted to each number to produce the next number. The common pattern in a geometric sequence is that the same number is multiplied or divided to each number to produce the next number. Are all sequences arithmetic or geometric?

Vocabulary

Arithmetic Sequence, Geometric Sequence.

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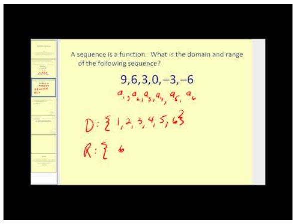

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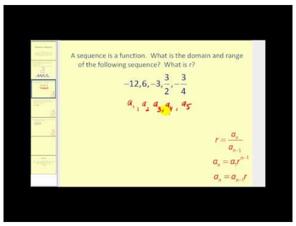
http://www.youtube.com/watch?v=jExpsJTu9o8 James Sousa: Arithmetic Sequences



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URL: http://www.ck12.org/flx/render/embeddedobject/62263

http://www.youtube.com/watch?v=XHyeLKZYb2w James Sousa: Geometric Sequences

Guidance

A sequence is just a list of numbers separated by commas. A sequence can be finite or infinite. If the sequence is infinite, the first few terms are followed by an ellipsis $(\ldots)$ indicating that the pattern continues forever.

An infinite sequence: $1, 2, 3, 4, 5, \ldots$

A finite sequence: $2, 4, 6, 8$

In general, you describe a sequence with subscripts that are used to index the terms. The k^{th} term in the sequence is a_k.

$a_1, a_2, a_3, a_4, \ldots, a_k, \ldots$

Arithmetic sequences are defined by an initial value a_1 and a common difference d.

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= a_1 + d \\
a_3 &= a_1 + 2d \\
a_4 &= a_1 + 3d \\
&\vdots \\
a_n &= a_1 + (n-1)d
\end{aligned}$$

Geometric sequences are defined by an initial value a_1 and a common ratio r.

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= a_1 \cdot r \\
a_3 &= a_1 \cdot r^2 \\
a_4 &= a_1 \cdot r^3 \\
&\vdots \\
a_n &= a_1 \cdot r^{n-1}
\end{aligned}$$

If a sequence does not have a common ratio or a common difference, it is neither an arithmetic or a geometric sequence. You should still try to figure out the pattern and come up with a formula that describes it.

Example A

For each of the following three sequences, determine if it is arithmetic, geometric or neither.

a. $0.135, 0.189, 0.243, 0.297, \ldots$
b. $\frac{2}{9}, \frac{1}{6}, \frac{1}{8}, \ldots$
c. $0.54, 1.08, 3.24, \ldots$

Solution:

a. The sequence is arithmetic because the common difference is 0.054.
b. The sequence is geometric because the common ratio is $\frac{3}{4}$.
c. The sequence is not arithmetic because the differences between consecutive terms are 0.54 and 2.16 which are not common. The sequence is not geometric because the ratios between consecutive terms are 2 and 3 which are not common.

Example B

For the following sequence, determine the common ratio or difference, the next three terms, and the 2013^{th} term.

$\frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \ldots$

Solution: The sequence is arithmetic because the difference is exactly 1 between consecutive terms. The next three terms are $\frac{14}{3}, \frac{17}{3}, \frac{20}{3}$. An equation for this sequence would is:

$a_n = \frac{2}{3} + (n-1) \cdot 1$

Therefore, the 2013^{th} term requires 2012 times the common difference added to the first term.

$a_{2013} = \frac{2}{3} + 2012 \cdot 1 = \frac{2}{3} + \frac{6036}{3} = \frac{6038}{3}$

Example C

For the following sequence, determine the common ratio or difference and the next three terms.

$\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \frac{10}{243}, \ldots$

Solution: This sequence is neither arithmetic nor geometric. The differences between the first few terms are $-\frac{2}{9}, -\frac{2}{9}, -\frac{10}{81}, -\frac{14}{243}$. While there was a common difference at first, this difference did not hold through the sequence. **Always check the sequence in multiple places to make sure that the common difference holds up throughout.**

The sequence is also not geometric because the ratios between the first few terms are $\frac{2}{3}, \frac{1}{2}, \frac{4}{9}$. These ratios are not common.

Even though you cannot get a common ratio or a common difference, it is still possible to produce the next three terms in the sequence by noticing the numerator is an arithmetic sequence with starting term of 2 and a common difference of 2. The denominators are a geometric sequence with an initial term of 3 and a common ratio of 3. The next three terms are:

$\frac{12}{3^6}, \frac{14}{3^7}, \frac{16}{3^8}$

Concept Problem Revisited

Example C shows that some patterns that use elements from both arithmetic and geometric series are neither arithmetic nor geometric. Two famous sequences that are neither arithmetic nor geometric are the Fibonacci sequence and the sequence of prime numbers.

Fibonacci Sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Prime Numbers: $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

Vocabulary

A ***sequence*** is a list of numbers separated by commas.

The common pattern in an ***arithmetic sequence*** is that the same number is added or subtracted to each number to produce the next number. This is called the ***common difference.***

The common pattern in a ***geometric sequence*** is that the same number is multiplied or divided to each number to produce the next number. This is called the ***common ratio.***

Guided Practice

1. What is the tenth term in the following sequence?

$-12, 6, -3, \frac{3}{2}, \ldots$

2. What is the tenth term in the following sequence?

$-1, \frac{2}{3}, \frac{7}{3}, 4, \frac{17}{3}, \ldots$

3. Find an equation that defines the a_k term for the following sequence.

$0, 3, 8, 15, 24, 35, \ldots$

Answers:

1. The sequence is geometric and the common ratio is $-\frac{1}{2}$. The equation is $a_n = -12 \cdot \left(-\frac{1}{2}\right)^{n-1}$. The tenth term is:

$-12 \cdot \left(-\frac{1}{2}\right)^9 = \frac{3}{128}$

2. The pattern might not be immediately recognizable, but try ignoring the $\frac{1}{3}$ in each number to see the pattern a different way.

$-3, 2, 7, 12, 17, \ldots$

You should see the common difference of 5. This means the common difference from the original sequence is $\frac{5}{3}$. The equation is $a_n = -1 + (n-1)\left(\frac{5}{3}\right)$. The 10^{th} term is:

$-1 + 9 \cdot \left(\frac{5}{3}\right) = -1 + 3 \cdot 5 = -1 + 15 = 14$

3. The sequence is not arithmetic nor geometric. It will help to find the pattern by examining the common differences and then the common differences of the common differences. This numerical process is connected to ideas in calculus.

0, 3, 8, 15, 24, 35

3, 5, 7, 9, 11

2, 2, 2, 2

Notice when you examine the common difference of the common differences the pattern becomes increasingly clear. Since it took *two* layers to find a constant function, this pattern is *quadratic* and very similar to the perfect squares.

$1, 4, 9, 16, 25, 36, \dots$

The a_k term can be described as $a_k = k^2 - 1$

Vocabulary

A ***sequence*** is a list of numbers separated by commas.

The common pattern in an ***arithmetic sequence*** is that the same number is added or subtracted to each number to produce the next number. This is called the ***common difference.***

The common pattern in a ***geometric sequence*** is that the same number is multiplied or divided to each number to produce the next number. This is called the ***common ratio.***

In Summary

We have learned about what an infinite and finite sequence are. We also learned what an arithmetic and geometric sequences are and finally how to find the values and formulas.

Practice

Use the sequence $1, 5, 9, 13, \dots$ for questions 1-3.

1. Find the next three terms in the sequence.
2. Find an equation that defines the a_k term of the sequence.
3. Find the 150^{th} term of the sequence.

Use the sequence $12, 4, \frac{4}{3}, \frac{4}{9}, \dots$ for questions 4-6.

4. Find the next three terms in the sequence.
5. Find an equation that defines the a_k term of the sequence.
6. Find the 17^{th} term of the sequence.

Use the sequence $10, -2, \frac{2}{5}, -\frac{2}{25}, \dots$ for questions 7-9.

7. Find the next three terms in the sequence.
8. Find an equation that defines the a_k term of the sequence.
9. Find the 12^{th} term of the sequence.

Use the sequence $\frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \dots$ for questions 10-12.

10. Find the next three terms in the sequence.
11. Find an equation that defines the a_k term of the sequence.
12. Find the 314^{th} term of the sequence.
13. Find an equation that defines the a_k term for the sequence $4, 11, 30, 67, \dots$
14. Explain the connections between arithmetic sequences and linear functions.

15. Explain the connections between geometric sequences and exponential functions.

11.3 Sigma Notation

Here you will learn how to represent the sum of sequences of numbers using sigma notation.

TEKS

P.5.A
P.5.B
P.5.C
P.5.D
P.5.E

Lesson Objectives

In this lesson you will learn about:
1. What sigma notation is.
2. How to write out the terms when you are given sigma notation.
3. To find the sum of the terms given sigma notation.

Introduction

Writing the sum of long lists of numbers that have a specific pattern is not very efficient. Summation notation allows you to use the pattern and the number of terms to represent the same sum in a much more concise way. How can you use sigma notation to represent the following sum?
$1+4+9+16+25+\cdots+144$

Vocabulary

Sigma Notation, Summation Notation.

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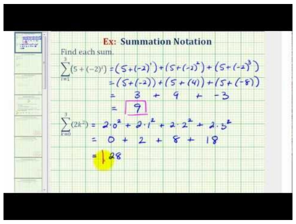

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http://www.youtube.com/watch?v=0L0rU17hHuM James Sousa: Find a Sum Written in Summation/Sigma Notation

Guidance

A **series** is a sum of a sequence. The Greek capital letter sigma is used for summation notation because it stands for the letter S as in sum.

Consider the following general sequence and note that the subscript for each term is an index telling you the term number.

a_1, a_2, a_3, a_4, a_5

When you write the sum of this sequence in a series, it can be represented as a sum of each individual term or abbreviated using a capital sigma.

$a_1 + a_2 + a_3 + a_4 + a_5 = \sum_{i=1}^{5} a_i$

The three parts of sigma notation that you need to be able to read are the argument, the lower index and the upper index. The argument, a_i, tells you what terms are added together. The lower index, $i = 1$, tells you where to start and the upper index, 5, tells you where to end. You should practice reading and understanding sigma notation because it is used heavily in Calculus.

Example A

Write out all the terms of the series.

$\sum_{k=4}^{8} 2k$

Solution:

$\sum_{k=4}^{8} 2k = 2 \cdot 4 + 2 \cdot 5 + 2 \cdot 6 + 2 \cdot 7 + 2 \cdot 8$

Example B

Write the sum in sigma notation: $2+3+4+5+6+7+8+9+10$

Solution:

$2+3+4+5+6+7+8+9+10 = \sum_{i=2}^{10} i$

Example C

Write the sum in sigma notation.

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}$

Solution:

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} = \sum_{i=1}^{7} \frac{1}{i^2}$

Concept Problem Revisited

The hardest part when first using sigma representation is determining how each pattern generalizes to the k^{th} term. Once you know the k^{th} term, you know the argument of the sigma. For the sequence creating the series below, $a_k = k^2$. Therefore, the argument of the sigma is i^2.

$$1+4+9+16+25+\cdots+144 = 1^2+2^2+3^2+4^2+\cdots 12^2 = \sum_{i=1}^{12} i^2$$

Guided Practice

1. Write out all the terms of the sigma notation and then calculate the sum.

$$\sum_{k=0}^{4} 3k-1$$

2. Represent the following infinite series in summation notation.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

3. Is there a way to represent an infinite product? How would you represent the following product?

$$1 \cdot \sin\left(\frac{360}{3}\right) \cdot \sin\left(\frac{360}{4}\right) \cdot \sin\left(\frac{360}{5}\right) \cdot \sin\left(\frac{360}{6}\right) \cdot \sin\left(\frac{360}{7}\right) \cdot \ldots$$

Answers:

1.

$$\begin{aligned}\sum_{k=0}^{4} 3k-1 &= (3\cdot 0-1)+(3\cdot 1-1)+(3\cdot 2-1)+(3\cdot 3-1)+(3\cdot 4-1)\\ &= -1+2+5+8+11\end{aligned}$$

2. There are an infinite number of terms in the series so using an infinity symbol in the upper limit of the sigma is appropriate.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots = \sum_{i=1}^{\infty}\frac{1}{2^i}$$

3. Just like summation uses a capital Greek letter for S, product uses a capital Greek letter for P which is the capital form of π.

$$1 \cdot \sin\left(\frac{360}{2\cdot 3}\right) \cdot \sin\left(\frac{360}{2\cdot 4}\right) \cdot \sin\left(\frac{360}{2\cdot 5}\right) \cdot \sin\left(\frac{360}{2\cdot 6}\right) \cdot \sin\left(\frac{360}{2\cdot 7}\right) \cdot \ldots = \prod_{i=3}^{\infty} \sin\left(\frac{360}{2\cdot i}\right)$$

This infinite product is the result of starting with a circle of radius 1 and inscribing a regular triangle inside the circle. Then you inscribe a circle inside the triangle and a square inside the new circle. The shapes alternate being inscribed within each other as they are nested inwards: circle, triangle, circle, square, circle, pentagon, ... The question that this calculation starts to answer is whether this process reduces to a number or to zero.

Vocabulary

Sigma notation is also known as ***summation notation*** and is a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.

In Summary

We have learned about what sigma notation is. We have learned about how to write the terms given the sigma notation and how to calculate the sums when given sigma notation.

Practice

For 1-5, write out all the terms of the sigma notation and then calculate the sum.

1. $\sum_{k=1}^{5} 2k-3$

2. $\sum_{k=0}^{8} 2^k$

3. $\sum_{i=1}^{4} 2\cdot 3^i$

4. $\sum_{i=1}^{10} 4i-1$

5. $\sum_{i=0}^{4} 2\cdot\left(\frac{1}{3}\right)^i$

Represent the following series in summation notation with a lower index of 0.

6. $1+4+7+10+13+16+19+22$

7. $3+5+7+9+11$

8. $8+7+6+5+4+3+2+1$

9. $5+6+7+8$

10. $3+6+12+24+48+\cdots$

11. $10+5+\frac{5}{2}+\frac{5}{4}$

12. $4-8+16-32+64\cdots$

13. $2+4+6+8+\cdots$

14. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$

15. $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\cdots$

11.4 Arithmetic Series

Here you will learn to compute finite arithmetic series more efficiently than just adding the terms together one at a time.

TEKS

P.5.C
P.5.D

Lesson Objectives

In this lesson you will learn about:
1. Finding sums of arithmetic sequences.

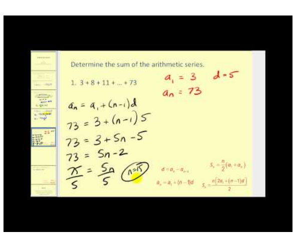
Introduction

While it is possible to add arithmetic series one term at a time, it is not feasible or efficient when there are a large number of terms. What is a clever way to add up all the whole numbers between 1 and 100?

Vocabulary

Arithmetic Series.

Watch This



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http://www.youtube.com/watch?v=Dj1JZIdIwwo James Sousa: Arithmetic Series

Guidance

The key to adding up a finite arithmetic series is to pair up the first term with the last term, the second term with the second to last term and so on. The sum of each pair will be equal. Consider a generic series:

$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots a_n$

When you pair the first and the last terms and note that $a_n = a_1 + (n-1)k$ the sum is:

$a_1 + a_n = a_1 + a_1 + (n-1)k = 2a_1 + (n-1)k$

When you pair up the second and the second to last terms you get the same sum:

$a_2 + a_{n-1} = (a_1 + k) + (a_1 + (n-2)k) = 2a_1 + (n-1)k$

The next logical question to ask is: how many pairs are there? If there are n terms total then there are exactly $\frac{n}{2}$ pairs. If n happens to be even then every term will have a partner and $\frac{n}{2}$ will be a whole number. If n happens to be odd then every term but the middle one will have a partner and $\frac{n}{2}$ will include a $\frac{1}{2}$ pair that represents the middle term with no partner. Here is the general formula for arithmetic series:

$\sum_{i=1}^{n} a_i = \frac{n}{2}(2a_1 + (n-1)k)$ where k is the common difference for the terms in the series.

Example A

Add up the numbers between one and ten (inclusive) in two ways.

Solution: One way to add up lists of numbers is to pair them up for easier mental arithmetic.

$$\begin{aligned}1+2+3+4+5+6+7+8+9+10 &= 3+7+11+15+19\\ &= 10+26+19\\ &= 36+19\\ &= 55\end{aligned}$$

Another way is to note that $1+10 = 2+9 = 3+8 = 4+7 = 5+6 = 11$. There are 5 pairs of 11 which total 55.

Example B

Evaluate the following sum.

$\sum_{k=0}^{5} 5k - 2$

Solution: The first term is -2, the last term is 23 and there are 6 terms making 3 pairs. A common mistake is to forget to count the 0 index.

$\sum_{k=0}^{5} 5k - 2 = \frac{6}{2} \cdot (-2 + 23) = 3 \cdot 21 = 63$

Example C

Try to evaluate the sum of the following geometric series using the same technique as you would for an arithmetic series.

$\frac{1}{8} + \frac{1}{2} + 2 + 8 + 32$

Solution:

The real sum is: $\frac{341}{8}$

When you try to use the technique used for arithmetic sequences you get: $3\left(\frac{1}{8}+32\right)=\frac{771}{8}$

It is important to know that geometric series have their own method for summing. The method learned in this concept only works for arithmetic series.

Concept Problem Revisited

Gauss was a mathematician who lived hundreds of years ago and there is an anecdote told about him when he was a young boy in school. When misbehaving, his teacher asked him to add up all the numbers between 1 and 100 and he stated 5050 within a few seconds.

You should notice that $1+100=2+99=\cdots=101$ and that there are exactly 50 pairs that sum to be 101. $50\cdot 101=5050$.

Guided Practice

1. Sum the first 15 terms of the following arithmetic sequence.

$-1,\frac{2}{3},\frac{7}{3},4,\frac{17}{3}\ldots$

2. Sum the first 100 terms of the following arithmetic sequence.

$-7,-4,-1,2,5,8,\ldots$

3. Evaluate the following sum.

$\sum_{i=0}^{500} 2i-312$

Answers:

1. The initial term is -1 and the common difference is $\frac{5}{3}$.

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \frac{n}{2}(2a_1+(n-1)k)\\ &= \frac{15}{2}\left(2(-1)+(15-1)\frac{5}{3}\right)\\ &= \frac{15}{2}\left(-2+14\cdot\frac{5}{3}\right)\\ &= 160\end{aligned}$$

2. The initial term is -7 and the common difference is 3.

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \frac{n}{2}(2a_1+(n-1)k)\\ &= \frac{100}{2}(2(-7)+(100-1)3)\\ &= 14150\end{aligned}$$

3. The initial term is -312 and the common difference is 2.

$$\sum_{i=0}^{500} 2i - 312 = \frac{501}{2}(2(-312) + (501 - 1)2)$$
$$= 94188$$

Vocabulary

An ***arithmetic series*** is a sum of numbers whose consecutive terms form an arithmetic sequence.

In Summary

We have learned how to find the sum of an arithmetic sequence without performing the sum term by term.

Practice

1. Sum the first 24 terms of the sequence $1, 5, 9, 13, \dots$
2. Sum the first 102 terms of the sequence $7, 9, 11, 13, \dots$
3. Sum the first 85 terms of the sequence $-3, -1, 1, 3, \dots$
4. Sum the first 97 terms of the sequence $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \dots$
5. Sum the first 56 terms of the sequence $-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}, \dots$
6. Sum the first 91 terms of the sequence $-8, -4, 0, 4, \dots$

Evaluate the following sums.

7. $\sum_{i=0}^{300} 3i + 18$
8. $\sum_{i=0}^{215} 5i + 1$
9. $\sum_{i=0}^{100} i - 15$
10. $\sum_{i=0}^{85} -13i + 1$
11. $\sum_{i=0}^{212} -2i + 6$
12. $\sum_{i=0}^{54} 6i - 9$
13. $\sum_{i=0}^{167} -5i + 3$
14. $\sum_{i=0}^{341} 6i + 102$
15. $\sum_{i=0}^{452} -7i - \frac{5}{2}$

11.5 Geometric Series

Here you will sum infinite and finite geometric series and categorize geometric series as convergent or divergent.

TEKS

P.5.D
P.5.E

Lesson Objectives

In this lesson you will learn about:
1. Find the finite sum of a geometric sequence.

Introduction

An advanced factoring technique allows you to rewrite the sum of a finite geometric series in a compact formula. An infinite geometric series is more difficult because sometimes it sums to be a number and sometimes the sum keeps on growing to infinity. When does an infinite geometric series sum to be just a number and when does it sum to be infinity?

Vocabulary

Converge, Diverge, Partial Sum.

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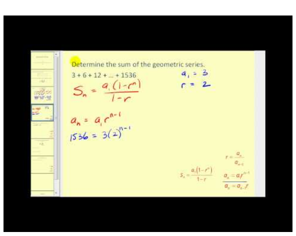

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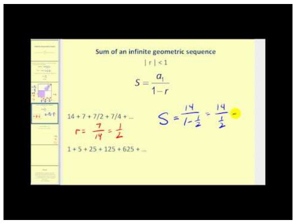
http://www.youtube.com/watch?v=mYg5gKlJjHc James Sousa: Geometric Series



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URL: http://www.ck12.org/flx/render/embeddedobject/62269

http://www.youtube.com/watch?v=RLZXFhvdlV8 James Sousa: Infinite Geometric Series

Guidance

Recall the advanced factoring technique for the difference of two squares and, more generally, two terms of any power (5 in this case).

$$\begin{aligned}
a^2-b^2 &= (a-b)(a+b)\\
a^5-b^5 &= (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)\\
a^n-b^n &= (a-b)(a^{n-1}+\cdots+b^{n-1})
\end{aligned}$$

If the first term is one then $a=1$. If you replace b with the letter r, you end up with:

$1-r^n=(1-r)(1+r+r^2+\cdots r^{n-1})$

You can divide both sides by $(1-r)$ because $r\neq 1$.

$1+r+r^2+\cdots r^{n-1}=\frac{1-r^n}{1-r}$

The left side of this equation is a geometric series with starting term 1 and common ratio of r. Note that even though the ending exponent of r is $n-1$, there are a total of n terms on the left. To make the starting term not one, just scale both sides of the equation by the first term you want, a_1.

$a_1+a_1r+a_1r^2+\cdots a_1r^{n-1}=a_1\left(\frac{1-r^n}{1-r}\right)$

This is the sum of a finite geometric series.

To sum an infinite geometric series, you should start by looking carefully at the previous formula for a finite geometric series. As the number of terms get infinitely large $(n\to\infty)$ one of two things will happen.

$a_1\left(\frac{1-r^n}{1-r}\right)$

Option 1: The term r^n will go to infinity or negative infinity. This will happen when $|r|\geq 1$. When this happens, the sum of the infinite geometric series does not go to a specific number and the series is said to be **divergent.**

Option 2: The term r^n will go to zero. This will happen when $|r|<1$. When this happens, the sum of the infinite geometric series goes to a certain number and the series is said to be **convergent**.

One way to think about these options is think about what happens when you take 0.9^{100} and 1.1^{100}.

$$\begin{aligned}
0.9^{100} &\approx 0.00002656\\
1.1^{100} &\approx 13780
\end{aligned}$$

As you can see, even numbers close to one either get very small quickly or very large quickly.

The formula for calculating the sum of an infinite geometric series that converges is:

$\sum_{i=1}^{\infty} a_1 \cdot r^{i-1} = a_1 \left(\frac{1}{1-r}\right)$

Notice how this formula is the same as the finite version but with $r^n = 0$, just as you reasoned.

Example A

Compute the sum of the following infinite geometric series.

$0.2 + 0.02 + 0.002 + 0.0002 + \cdots$

Solution: You can tell just by looking at the sum that the infinite sum will be the repeating decimal $0.\overline{2}$. You may recognize this as the fraction $\frac{2}{9}$, but if you don't, this is how you turn a repeating decimal into a fraction.

Let $x = 0.\overline{2}$

Then $10x = 2.\overline{2}$

Subtract the two equations and solve for x.

$$\begin{aligned} 10x - x &= 2.\overline{2} - 0.\overline{2} \\ 9x &= 2 \\ x &= \frac{2}{9} \end{aligned}$$

Example B

Why does an infinite series with $r = 1$ diverge?

Solution: If $r = 1$ this means that the common ratio between the terms in the sequence is 1. This means that each number in the sequence is the same. When you add up an infinite number of any finite numbers (even fractions close to zero) you will always get infinity or negative infinity. The only exception is 0. This case is trivial because a geometric series with an initial value of 0 is simply the following series, which clearly sums to 0:

$0 + 0 + 0 + 0 + \cdots$

Example C

What is the sum of the first 8 terms in the following geometric series?

$4 + 2 + 1 + \frac{1}{2} + \cdots$

Solution: The first term is 4 and the common ratio is $\frac{1}{2}$.

$SUM = a_1 \left(\frac{1-r^n}{1-r}\right) = 4\left(\frac{1-\left(\frac{1}{2}\right)^8}{1-\frac{1}{2}}\right) = 4\left(\frac{\frac{255}{256}}{\frac{1}{2}}\right) = \frac{255}{32}$

Concept Problem Revisited

An infinite geometric series converges if and only if $|r| < 1$. Infinite arithmetic series never converge.

Guided Practice

1. Compute the sum from Example A using the infinite summation formula and confirm that the sum truly does converge.

2. Does the following geometric series converge or diverge? Does the sum go to positive or negative infinity?

$-2+2-2+2-2+\cdots$

3. You put \$100 in a bank account at the end of every year for 10 years. The account earns 6% interest. How much do you have total at the end of 10 years?

Answers:

1. The first term of the sequence is $a_1 = 0.2$. The common ratio is 0.1. Since $|0.1| < 1$, the series does converge.

$0.2\left(\frac{1}{1-0.1}\right) = \frac{0.2}{0.9} = \frac{2}{9}$

2. The initial term is -2 and the common ratio is -1. Since the $|-1| \geq 1$, the series is said to diverge. Even though the series diverges, it does not approach negative or positive infinity. When you look at the partial sums (the sums up to certain points) they alternate between two values:

$-2, 0, -2, 0, \ldots$

This pattern does not go to a specific number. Just like a sine or cosine wave, it does not have a limit as it approaches infinity.

3. The first deposit gains 9 years of interest: $100 \cdot 1.06^9$

The second deposit gains 8 years of interest: $100 \cdot 1.06^8$. This pattern continues, creating a geometric series. The last term receives no interest at all.

$100 \cdot 1.06^9 + 100 \cdot 1.06^8 + \cdots 100 \cdot 1.06 + 100$

Note that normally geometric series are written in the opposite order so you can identify the starting term and the common ratio more easily.

$a_1 = 100, r = 1.06$

The sum of the 10 years of deposits is:

$a_1\left(\frac{1-r^n}{1-r}\right) = 100\left(\frac{1-1.06^{10}}{1-1.06}\right) \approx \$1318.08$

Vocabulary

To ***converge*** means the sum approaches a specific number.

To ***diverge*** means the sum does not converge, and so usually goes to positive or negative infinity. It could also mean that the series oscillates infinitely.

A ***partial sum*** of an infinite sum is the sum of all the terms up to a certain point. Considering partial sums can be useful when analyzing infinite sums.

In Summary

We learned what a convergent and a divergent series are. We learned how to find the sum of a geometric sequence that converges.

Practice

Find the sum of the first 15 terms for each geometric sequence below.

1. $5, 10, 20, \ldots$

2. $2, 8, 32, \ldots$

3. $5, \frac{5}{2}, \frac{5}{4}, \ldots$

4. $12, 4, \frac{4}{3}, \ldots$

5. $\frac{1}{3}, 1, 3, \ldots$

For each infinite geometric series, identify whether the series is convergent or divergent. If convergent, find the number where the sum converges.

6. $5+10+20+\cdots$

7. $2+8+32+\cdots$

8. $5+\frac{5}{2}+\frac{5}{4}+\cdots$

9. $12+4+\frac{4}{3}+\cdots$

10. $\frac{1}{3}+1+3+\cdots$

11. $6+2+\frac{2}{3}+\cdots$

12. You put \$5000 in a bank account at the end of every year for 30 years. The account earns 2% interest. How much do you have total at the end of 30 years?

13. You put \$300 in a bank account at the end of every year for 15 years. The account earns 4% interest. How much do you have total at the end of 10 years?

14. You put \$10,000 in a bank account at the end of every year for 12 years. The account earns 3.5% interest. How much do you have total at the end of 12 years?

15. Why don't infinite arithmetic series converge?

11.6 Counting with Permutations and Combinations

Here you will review counting using decision charts, permutations and combinations.

TEKS

P.1.B
P.5.F

Lesson Objectives

In this lesson you will learn about:
1. What permutations are and the formula.
2. What a combination is and a formula.

Introduction

Sometimes it makes sense to count the number of ways for an event to occur by looking at each possible outcome. However, when there are a large number of outcomes this method quickly becomes inefficient. If someone asked you how many possible regular license plates there are for the state of California, it would not be feasible to count each and every one. Instead, you would need to use the fact that on the typical California license plate there are four numbers and three letters. Using this information, about how many license plates could there be?

Vocabulary

Permutations, Combinations.

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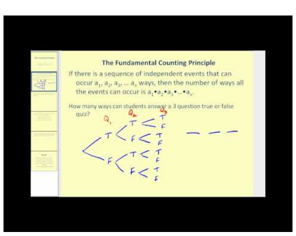

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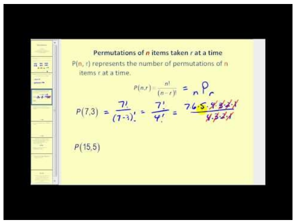

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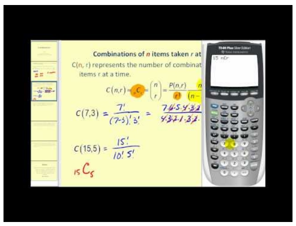

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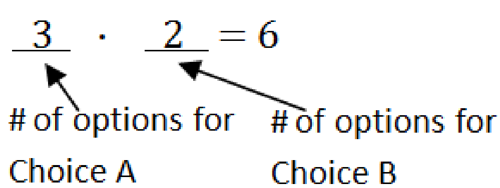
http://www.youtube.com/watch?v=SGn1913lOYM James Sousa: Combinations

Guidance

Consider choice A with three options (A_1, A_2, A_3) and choice B with two options (B_1, B_2). If you had to choose an option from A and then an option from B, the overall total number of options would be $3 \cdot 2 = 6$. The options are $A_1B_1, A_1B_2, A_2B_1, A_2B_2, A_3B_1, A_3B_2$.

You can see where the six comes from by making a decision chart and using the Fundamental Counting Principle. First, determine how many decisions you are making. Here, there are only two decisions to make (1: choose an option from A; 2: choose an option from B), so you will have two "slots" in your decision chart. Next, think about how many possibilities there are for the first choice (in this case there are 3) and how many possibilities there are for the second choice (in this case there are 2). The Fundamental Counting Principle says that you can multiply those numbers together to get the total number of outcomes.



Another type of counting question is when you have a given number of objects, you want to choose some (or all) of them, and you want to know how many ways there are to do this. For example, a teacher has a classroom of 30 students, she wants 5 of them to do a presentation, and she wants to know how many ways this could happen. These types of questions have to do with **combinations** and **permutations.** The difference between combinations and permutations has to do with whether or not the order that you are choosing the objects matters.

- A teacher choosing a group to make a presentation would be a **combination** problem, because **order does not matter.**

- A teacher choosing 1^{st}, 2^{nd}, and 3^{rd} place winners in a science fair would be a **permutation** problem, because the **order matters** (a student getting 1^{st} place vs. 2^{nd} place are different outcomes).

Recall that the factorial symbol, !, means to multiply every whole number up to and including that whole number together. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. The factorial symbol is used in the formulas for permutations and combinations.

Combination Formula: The number of ways to choose k objects from a group of n objects is –

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Permutation Formula: The number of ways to choose **and arrange** k objects from a group of n objects is –

$${}_nP_k = k!\binom{n}{k} = k! \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!}$$

Notice that in both permutation and combination problems you are not allowed to repeat your choices. Any time you are allowed to repeat and order does not matter, you can use a decision chart. (Problems with repetition where order does not matter are more complex and are not discussed in this text.)

Whenever you are doing a counting problem, the first thing you should decide is if the problem is a decision chart problem, a permutation problem, or a combination problem. You will find that permutation problems can also be solved with decision charts. The opposite is not true. There are many decision chart problems (ones where you are allowed to repeat choices) that could not be solved with the permutation formula.

Note: Here you have only begun to explore counting problems. For more information about combinations, permutations, and other types of counting problems, consult a Probability text.

Example A

You are going on a road trip with 4 friends in a car that fits 5 people. How many different ways can everyone sit if you have to drive the whole way?

Solution: A decision chart is a great way of thinking about this problem. You have to sit in the driver's seat. There are four options for the first passenger seat. Once that person is seated there are three options for the next passenger seat. This goes on until there is one person left with one seat.

$$1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example B

How many different ways can the gold, silver and bronze medals be awarded in an Olympic event with 12 athletes competing?

Solution: Since the order does matter with the three medals, this is a permutation problem. You will start with 12 athletes and then choose and arrange 3 different winners.

$${}_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot \ldots}{9 \cdot \ldots} = 12 \cdot 11 \cdot 10 = 1320$$

Note that you could also use a decision chart to decide how many possibilities are there for gold (12) how many possibilities are there for silver (11 since one already has gold) and how many possibilities are there for bronze (10). You can use a decision chart for any permutation problem.

$$12 \cdot 11 \cdot 10 = 1320$$

Example C

You are deciding which awards you are going to display in your room. You have 8 awards, but you only have room to display 4 awards. Right now you are not worrying about how to arrange the awards so the order does not matter. In how many ways could you choose the 4 awards to display?

Solution: Since order does not matter, this is a combination problem. You start with 8 awards and then choose 4.

${}_8C_4 = \binom{8}{4} = \frac{8!}{4!(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70$

Note that if you try to use a decision chart with this question, you will need to do an extra step of reasoning. There are 8 options I could choose first, then 7 left, then 6 and lastly 5.

$8 \cdot 7 \cdot 6 \cdot 5 = 1680$

This number is so big because it takes into account order, which you don't care about. It is the same result you would get if you used the permutation formula instead of the combination formula. To get the right answer, you need to divide this number by the number of ways 4 objects can be arranged, which is $4! = 24$. This has to do with the connection between the combination formula and the permutation formula.

Concept Problem Revisited

A license plate that has 3 letters and 4 numbers can be represented by a decision chart with seven spaces. You can use a decision chart because order definitely does matter with license plates. The first spot is a number, the next three spots are letters and the last three spots are numbers. Note that when choosing a license plate, repetition is allowed.

$$10 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^4 = 175,760,000$$

This number is only approximate because in reality there are certain letter and number combinations that are not allowed, some license plates have extra symbols, and some commercial and government license plates have more numbers, fewer letters or blank spaces.

Guided Practice

1. There are 20 hockey players on a pro NHL team, two of which are goalies. In how many different ways can 5 skaters and 1 goalie be on the ice at the same time?

2. In how many different ways could you score a 70% on a 10 question test where each question is weighted equally and is either right or wrong?

3. How many different 4 digit ATM passwords are there? Assume you can repeat digits.

Answers:

1. The question asks for how many on the ice, implying that order does not matter. This is combination problem with two combinations. You need to choose 1 goalie out of a possible of 2 and choose 5 skaters out of a possible 18.

$\binom{2}{1}\binom{18}{5} = 2 \cdot \frac{18!}{5! \cdot 13!} = 17136$

2. The order of the questions you got right does not matter, so this is a combination problem.

$\binom{10}{7} = \frac{10!}{7!3!} = 120$

3. Order does matter. There are 10 digits and repetition is allowed. You can use a decision chart for each of the four options.

$10 \cdot 10 \cdot 10 \cdot 10 = 10,000$

Vocabulary

A ***combination*** is the number of ways of choosing k objects from a total of n objects (order does not matter).

A ***permutation*** is the number of ways of choosing and arranging k objects from a total of n objects (order does matter)

A ***decision chart*** is a sequence of numbers that multiply together where each number represents the number of possible options for that slot.

In Summary

We learned about what a permutation is and how to find the total number of possible outcomes. Also we learned about what a combination is and how to find the total number of possible outcomes.

Practice

Simplify each of the following expressions so that they do not have a factorial symbol.

1. $\frac{7!}{3!}$

2. $\frac{110!}{105!5!}$

3. $\frac{52!}{49!}$

4. In how many ways can you choose 3 objects from a set of 9 objects?

5. In how many ways can you choose and arrange 4 objects from a set of 15 objects?

First, state whether each problem is a **permutation/decision chart** problem or a **combination** problem. Then, solve.

6. Suppose you need to choose a new combination for your combination lock. You have to choose 3 numbers, each different and between 0 and 40. How many combinations are there?

7. You just won a contest where you can choose 2 friends to go with you to a concert. You have five friends who are available and want to go. In how many ways can you choose the friends?

8. You want to construct a 3 digit number from the digits 4, 6, 8, 9. How many possible numbers are there?

9. There are 12 workshops at a conference and Sam has to choose 3 to attend. In how many ways can he choose the 3 to attend?

10. 9 girls and 5 boys are finalists in a contest. In how many ways can 1^{st}, 2^{nd}, and 3^{rd} place winners be chosen?

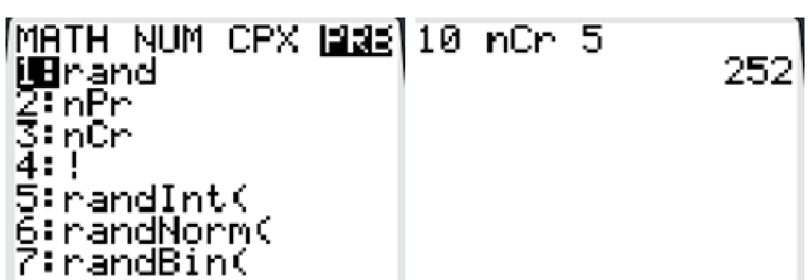
11. For the special at a restaurant you can choose 3 different items from the 10 item menu. How many different combinations of meals could you get?

12. You visit 12 colleges and want to apply to 4 of them. In how many ways could you choose the four to apply to?

13. For the 12 colleges you visited, you want to rank your top five. In how many ways could you rank your top 5?

14. ***Explain why the following problem is not strictly a permutation or combination problem***: The local ice cream shop has 12 flavors. You decide to buy 2 scoops in a dish. In how many ways could you do this if you are allowed to get two of the same scoop?

15. Your graphing calculator has the combination and permutation formulas built in. Push the MATH button and scroll to the right to the PRB list. You should see ${}_nP_r$ and ${}_nC_r$ as options. In order to use these: 1) On your home screen type the value for n; 2) Select ${}_nP_r$ or ${}_nC_r$; 3) Type the value for k (r on the calculator). Use your calculator to verify that ${}_{10}C_5 = 252$.



11.7 Basic Probability

Here you will calculate the probability of simple and compound events.

TEKS

P.1.F

Lesson Objectives

In this lesson you will learn about:
1. Finding probability of an event.

Introduction

Most are familiar with how flipping a coin or rolling dice works and yet probability remains one of the most counterintuitive branches of mathematics for many people. The idea that flipping a coin and getting 10 heads in a row is just as unlikely as getting the following sequence of heads and tails is hard to comprehend.

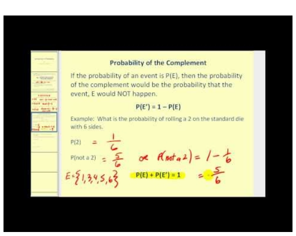
$HHTHTTTHTH$

Assume a plane crashes on average once every 100 days (extremely inaccurate). Given a plane crashed today, what day in the next 100 days is the plane most likely to crash next?

Vocabulary

Probability, Complement of an Event, Independent Event.

Watch this



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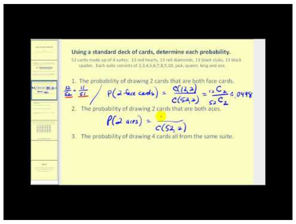
http://www.youtube.com/watch?v=YWt_u5l_jHs James Sousa: Introduction to Probability



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URL: http://www.ck12.org/flx/render/embeddedobject/62259

http://www.youtube.com/watch?v=IZAMLgS5x6w James Sousa: Determining Probability

Guidance

Probability is the chance of an event occurring. Simple probability is defined as the number of outcomes you are looking for (also called successes) divided by the total number of outcomes. The notation $P(E)$ is read "the probability of event E".

$P(E) = \frac{\#\ successes}{\#\ possible\ outcomes}$

Probabilities can be represented with fractions, decimals, or percents. Since the number of possible outcomes is in the denominator, the probability is always between zero and one. A probability of 0 means the event will definitely not happen, while a probability of 1 means the event will definitely happen.

$0 \le P(E) \le 1$

The probability of something not happening is called the **complement** and is found by subtracting the probability from one.

$P(E^C) = 1 - P(E)$

You will often be looking at probabilities of two or more independent experiments. Experiments are independent when the outcome of one experiment has no effect on the outcome of the other experiment. If there are two experiments, one with outcome A and the other with outcome B, then the probability of A and B is:

$P(A\ and\ B) = P(A) \cdot P(B)$

The probability of A or B is:

$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$

Example A

If you are dealt one card from a 52 card deck, what is the probability that you are dealt a heart? What is the probability that you are dealt a 3? What is the probability that you are dealt the three of hearts?

Solution: There are 13 hearts in a deck of 52 cards. $P(heart) = \frac{13}{52} = \frac{1}{4}$

There are 4 threes in the deck of 52. $P(three) = \frac{4}{52} = \frac{1}{13}$

There is only one three of hearts. $P(three\ and\ heart) = \frac{1}{52}$

Example B

Dean and his friend Randy like to play a special poker game with their friends. Dean goes home a winner 60% of the time and Randy goes home a winner 75% of the time.

a. What is the probability that they both win in the same night?
b. What is the probability that Randy wins and Dean loses?
c. What is the probability that they both lose?

Solution: First represent the information with probability symbols.

Let D be the event that Dean wins. Let R be the event that Randy wins. The complement of each probability is when Dean or Randy loses instead.

$P(D) = 0.60, \quad P(D^C) = 0.40$

$P(R) = 0.75, \quad P(R^C) = 0.25$

a. $P(D \text{ and } R) = P(D) \cdot P(R) = 0.60 \cdot 0.75 = 0.45$
b. $P(R \text{ and } D^C) = P(R) \cdot P(D^C) = 0.75 \cdot 0.40 = 0.30$
c. $P(D^C \text{ and } R^C) = P(D^C) \cdot P(R^C) = 0.40 \cdot 0.25 = 0.10$

Example C

If a plane crashes on average once every hundred days, what is the probability that the plane will crash in the next 100 days?

Solution: The naïve and incorrect approach would be to interpret the question as "what is the sum of the probabilities for each of the days?" Since there are 100 days and each day has a probability of 0.01 for a plane crash, then by this logic, there is a 100% chance that a plane crashes. This isn't true because if on average the plane crashes once every hundred days, some stretches of 100 days there will be more crashes and some stretches there will be no crashes. The 100% solution does not hold.

In order to solve this question, you need to rephrase the question and ask a slightly different one that will help as an intermediate step. What is the probability that a plane does not crash in the next 100 days?

In order for this to happen it must not crash on day 1 and not crash on day 2 and not crash on day 3 etc.

The probability of the plane not crashing on any day is $P(no\ crash) = 1 - P(crash) = 1 - 0.01 = 0.99$.

The product of each of these probabilities for the 100 days is:

$0.99^{100} \approx 0.366$

Therefore, the probability that a plane does not crash in the next 100 days is about 36.6%. To answer the original question, the probability that a plane does crash in the next 100 days is $1 - 0.366 = 0.634$ or about 63.4%.

Concept Problem Revisited

Whether or not a plane crashes today does not matter. The probability that a plane crashes tomorrow is $p = 0.01$. The probability that it crashes any day in the next 100 days is equally $p = 0.01$. The key part of the question is the word "next".

The probability that a plane does not crash on the first day and does crash on the second day is a compound probability, which means you multiply the probability of each event.

$P(Day\ 1\ no\ crash\ AND\ Day\ 2\ crash) = 0.99 \cdot 0.01 = 0.0099$

Notice that this probability is slightly smaller than 0.01. Each successive day has a slightly smaller probability of being the <u>next</u> day that a plane crashes. Therefore, the day with the highest probability of a plane crashing <u>next</u> is tomorrow.

Guided Practice

1. Jack is a basketball player with a free throw average of 0.77. What is the probability that in a game where he has 8 shots that he makes all 8? What is the probability that he only makes 1?

2. If it has a 20% chance of raining on Tuesday, your phone has 30% chance of running out of batteries, and there is a 10% chance that you forget your wallet, what is the probability that you are in the rain without money or a phone?

3. Consider the previous question with the rain, wallet and phone. What is the probability that at least one of the three events does occur?

Answers:

1. Let J represent the event that Jack makes the free throw shot and J^C represent the event that Jack misses the shot.

$P(J) = 0.77, \ P(J^C) = 0.23$

The probability that Jack makes all 8 shots is the same as Jack making one shot and making the second shot and making the third shot etc.

$P(J)^8 = 0.77^8 \approx 12.36\%$

There are 8 ways that Jack could make 1 shot and miss the rest. The probability of each of these cases occurring is:

$P(J^C)^7 \cdot P(J) = 0.23^7 \cdot 0.77$

Therefore, the overall probability of Jack making 1 shot and missing the rest is:

$0.23^7 \cdot 0.77 \cdot 8 = 0.0002097 = 0.02097\%$

2. While a pessimist may believe that all the improbable negative events will occur at the same time, the actual probability of this happening is less than one percent:

$0.20 \cdot 0.30 \cdot 0.1 = 0.006 = 0.6\%$

3. The naïve approach would be to simply add the three probabilities together. This is incorrect. The better way to approach the problem is to ask the question: what is the probability that none of the events occur?

$0.8 \cdot 0.7 \cdot 0.9 = 0.504$

The probability that at least one occurs is the complement of none occurring.

$1 - 0.504 = 0.496 = 49.6\%$

Vocabulary

The ***probability*** of an event is the number of outcomes you are looking for (called successes) divided by the total number of outcomes.

The **complement of an event** is the event not happening.

Independent events are events where the occurrence of the first event does not impact the probability of the second event.

In Summary

We have learned about probability and how to find it. We have learned that about the complement of an event and what independent events are.

Practice

A card is chosen from a standard deck.

1. What's the probability that the card is a queen?

2. What's the probability that the card is a queen or a spade?

You toss a nickel, a penny, and a dime.

3. List all the possible outcomes (the elements in the sample space).

4. What is the probability that the nickel comes up heads?

5. What is the probability that none of the coins comes up heads?

6. What is the probability that at least one of the coins comes up heads?

A bag contains 7 red marbles, 9 blue marbles, and 10 green marbles. You reach in the bag and choose 4 marbles, one after the other, without replacement.

7. What is the probability that all 4 marbles are red?

8. What is the probability that you get a red marble, then a blue marble, then 2 green marbles?

You take a 40 question multiple choice test and believe that for each question you have a 55% chance of getting it right.

9. What is the probability that you get all the questions right?

10. What is the probability that you get all of the questions wrong?

A player rolls a pair of standard dice. Find each probability.

11. $P(\textit{sum is even})$

12. $P(\textit{sum is } 7)$

13. $P(\textit{sum is at least } 3)$

14. You want to construct a 3 digit number at random from the digits 4, 6, 8, 9 without repeating digits. What is the probability that you construct the number 684?

15. In poker, a straight is 5 cards in a row (ex. 3, 4, 5, 6, 7), NOT all the same suit (if they are all the same suit it is considered a straight flush or a royal flush). A straight can start or end with an Ace. What's the probability of a straight? *For an even bigger challenge, see if you can calculate the probabilities for all of the poker hands.*

11.8 Binomial Theorem

Here you will apply the Binomial Theorem to expand binomials that are raised to a power. In order to do this you will use your knowledge of sigma notation and combinations.

TEKS

P.5.F

Lesson Objectives

In this lesson you will learn about:
1. The binomial expansion.
2. How to expand a binomial using combination formula.

Introduction

The Binomial Theorem tells you how to expand a binomial such as $(2x-3)^5$ without having to compute the repeated distribution. What is the expanded version of $(2x-3)^5$?

Vocabulary

Binomial Expansion.

The Binomial Theorem tells you how to expand a binomial such as $(2x-3)^5$ without having to compute the repeated distribution. What is the expanded version of $(2x-3)^5$?

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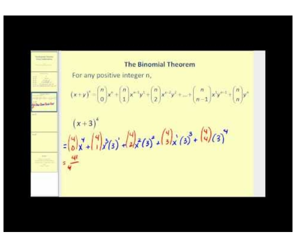

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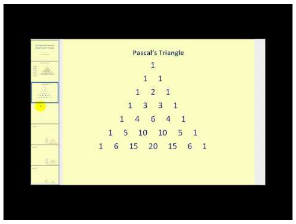
http://www.youtube.com/watch?v=YxysKtqpbVI James Sousa: The Binomial Theorem Using Combinations



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http://www.youtube.com/watch?v=NLQmQGA4a3M James Sousa: The Binominal Theorem Using Pascal's Triangle

Guidance

The Binomial Theorem states:

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Writing out a few terms of the summation symbol helps you to understand how this theorem works:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

Going from one term to the next in the expansion, you should notice that the exponents of a decrease while the exponents of b increase. You should also notice that the coefficients of each term are combinations. Recall that $\binom{n}{0}$ is the number of ways to choose 0 objects from a set of n objects.

Another way to think about the coefficients in the Binomial Theorem is that they are the numbers from Pascal's Triangle. Look at the expansions of $(a+b)^n$ below and notice how the coefficients of the terms are the numbers in Pascal's Triangle.

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$\vdots$$

Be extremely careful when working with binomials of the form $(a-b)^n$. You need to remember to capture the negative with the second term as you write out the expansion: $(a-b)^n = (a+(-b))^n$.

Example A

Expand the following binomial using the Binomial Theorem.

$(m-n)^6$

Solution:

$$\begin{aligned}(m-n)^6 &= \binom{6}{0}m^6+\binom{6}{1}m^5(-n)^1+\binom{6}{2}m^4(-n)^2+\binom{6}{3}m^3(-n)^3\\ &+\binom{6}{4}m^2(-n)^4+\binom{6}{5}m^1(-n)^5+\binom{6}{6}(-n)^6\\ &= 1m^6-6m^5n+15m^4n^2-20m^3n^3+15m^2n^4-6m^1n^5+1n^6\end{aligned}$$

Example B

What is the coefficient of the term x^7y^9 in the expansion of the binomial $(x+y)^{16}$?

Solution: The Binomial Theorem allows you to calculate just the coefficient you need.

$\binom{16}{9}=\frac{16!}{9!7!}=\frac{16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}=11,440$

Example C

What is the coefficient of x^6 in the expansion of $(4-3x)^7$?

Solution: For this problem you should calculate the whole term, since the 3 and the 4 in $(3-4x)$ will impact the coefficient of x^6 as well. $\binom{7}{6}4^1(-3x)^6=7\cdot4\cdot729x^6=20,412x^6$. The coefficient is 20,412.

Concept Problem Revisited

The expanded version of $(2x-3)^5$ is:

$$\begin{aligned}(2x-3)^5 &= \binom{5}{0}(2x)^5+\binom{5}{1}(2x)^4(-3)^1+\binom{5}{2}(2x)^3(-3)^2\\ &+\binom{5}{3}(2x)^2(-3)^3+\binom{5}{4}(2x)^1(-3)^4+\binom{5}{5}(-3)^6\\ &=(2x)^5+5(2x)^4(-3)^1+10(2x)^3(-3)^2\\ &+10(2x)^2(-3)^3+5(2x)^1(-3)^4+(-3)^5\\ &=32x^5-240x^4+720x^3-1080x^2+810x-243\end{aligned}$$

Vocabulary

The ***Binomial Theorem*** is a theorem that states how to expand binomials that are raised to a power using combinations. The ***Binomial Theorem*** is:

$(a+b)^n=\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i}$

Guided Practice

1. What is the coefficient of x^3 in the expansion of $(x-4)^5$?

2. Compute the following summation.

$$\sum_{i=0}^{4}\binom{4}{i}$$

3. Collapse the following polynomial using the Binomial Theorem.

$32x^5-80x^4+80x^3-40x^2+10x-1$

Answers:

1. $\binom{5}{2}\cdot 1^3(-4)^2=160$

2. This is asking for $\binom{4}{0}+\binom{4}{1}+\cdots+\binom{4}{4}$, which are the sum of all the coefficients of $(a+b)^4$.

$1+4+6+4+1=16$

3. Since the last term is -1 and the power on the first term is a 5 you can conclude that the second half of the binomial is $(?-1)^5$. The first term is positive and $(2x)^5=32x^5$, so the first term in the binomial must be $2x$. The binomial is $(2x-1)^5$.

In Summary

We have learned about the binomial expansion theorem. We have learned how to use the combinations to expand a binomial.

Practice

Expand each of the following binomials using the Binomial Theorem.

1. $(x-y)^4$
2. $(x-3y)^5$
3. $(2x+4y)^7$
4. What is the coefficient of x^4 in $(x-2)^7$?
5. What is the coefficient of x^3y^5 in $(x+y)^8$?
6. What is the coefficient of x^5 in $(2x-5)^6$?
7. What is the coefficient of y^2 in $(4y-5)^4$?
8. What is the coefficient of x^2y^6 in $(2x+y)^8$?
9. What is the coefficient of x^3y^4 in $(5x+2y)^7$?

Compute the following summations.

10. $\sum\limits_{i=0}^{9}\binom{9}{i}$
11. $\sum\limits_{i=0}^{12}\binom{12}{i}$
12. $\sum\limits_{i=0}^{8}\binom{8}{i}$

Collapse the following polynomials using the Binomial Theorem.

13. $243x^5-405x^4+270x^3-90x^2=15x-1$
14. $x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$
15. $128x^7-448x^6y+672x^5y^2-560x^4y^3+280x^3y^4-84x^2y^5+14xy^6-y^7$ [U+EFFE]

11.9 Induction Proofs

Here you will learn how to prove statements about numbers using induction.

TEKS

P.1.F

Lesson Objectives

In this lesson you will learn about:
1. What induction is.
2. Doing proofs by Induction.

Introduction

Induction is one of many methods for proving mathematical statements about numbers. The basic idea is that you prove a statement is true for a small number like 1. This is called the base case. Then, you show that if the statement is true for some random number k, then it must also be true for $k+1$.

Vocabulary

Induction, Base Case.

An induction proof is like dominoes set up in a line, where the base case starts the falling cascade of truth. Once you have shown that in general if the statement is true for k then it must also be true for $k+1$, it means that once you show the statement is true for 1, then it must also be true for 2, and then it must also be true for 3, and then it must also be true for 4 and so on.

What happens when you forget the base case?

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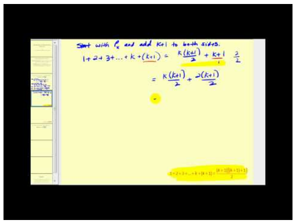

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http://www.youtube.com/watch?v=QHkG0d5kZvE James Sousa: Mathematical Induction

Guidance

Induction is a method of proof usually used to prove statements about positive whole numbers (the natural numbers). Induction has three steps:

1. The base case is where the statement is shown to be true for a specific number. Usually this is a small number like 1.
2. The inductive hypothesis is where the statement is assumed to be true for k.
3. The inductive step/proof is where you show that then the statement must be true for $k+1$.

These three logical pieces will show that the statement is true for every number greater than the base case.

Suppose you wanted to use induction to prove: $n \geq 1, 2+2^2+2^3+\cdots+2^n = 2^{n-1}-2$.

Start with the **Base Case**. Show that the statement works when $n = 1$:

$2^1 = 2$ and $2^{1+1}-2 = 4-2 = 2$. Therefore, $2^1 = 2^{1+1}-2$. (Both sides are equal to 2)

Next, state your **Inductive Hypothesis**. Assume that the statement works for some random number k:

$2+2^2+\cdots+2^k = 2^{k+1}-2$ (You are assuming that this is a true statement)

Next, you will want to use algebra to manipulate the previous statement to **prove** that the statement is also true for $k+1$. So, you will be trying to show that $2+2^2+\cdots+2^{k+1} = 2^{k+1+1}-2$. Start with the inductive hypothesis and multiply both sides of the equation by 2. Then, do some algebra to get the equation looking like you want.

Inductive Hypothesis (starting equation): $2+2^2+\cdots 2^k = 2^{k+1}-2$

Multiply by 2: $2(2+2^2+\cdots+2^k) = 2(2^{k+1}-2)$

Rewrite: $2^2+2^3+\cdots+2^{k+1} = 2^{k+1+1}-4$

Add 2 to both sides: $2+2^2+2^3+\cdots+2^{k+1} = 2^{k+1+1}-4+2$

Simplify: $2+2^2+\cdots+2^{k+1} = 2^{k+1+1}-2$

This is exactly what you were trying to prove! So, first you showed that the statement worked for $n = 1$. Then, you showed that the if the statement works for one number than it must work for the next number. This means, the statement must be true for all numbers greater than or equal to 1.

The idea of induction can be hard to understand at first and it definitely takes practice. One thing that makes induction tricky is that there is not a clear procedure for the "proof" part. With practice, you will start to see some common algebra techniques for manipulating equations to prove what you are trying to prove.

Example A

There is something wrong with this proof. Can you explain what the mistake is?

For $n \geq 1: 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Base Case: $1 = 1^2 = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = \frac{6}{6} = 1$

Inductive Hypothesis: Assume the following statement is true:

$1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$

Proof: You want to show the statement is true for $k+1$.

"Since the statement is assumed true for k, which is any number, then it must be true for $k+1$. *You can just substitute* $k+1$ *in."*

$1^2+2^2+3^2+\cdots+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

Solution: This is the most common fallacy when doing induction proofs. The fact that the statement is assumed to be true for k does not immediately imply that it is true for $k+1$ and you cannot just substitute in $k+1$ to produce what you are trying to show. This is equivalent to assuming true for all numbers and then concluding true for all numbers which is circular and illogical.

Example B

Write the base case, inductive hypothesis and what you are trying to show for the following statement. Do not actually prove it.

$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

Solution:

Base Case: $1^3=\frac{1^2(1+1)^2}{4}$ (Both sides are equal to 1)

Inductive Hypothesis: Assume the following statement is true:

$1^3+2^3+3^3+\cdots+k^3=\frac{k^2(k+1)^2}{4}$

Next, you would want to prove that the following is true:

$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{(k+1)^2((k+1)+1)^2}{4}$

Example C

Prove the following statement: *For* $n\geq 1, 1^3+2^3+3^3+\cdots n^3=(1+2+3+\cdots n)^2$.

Solution:

Base Case(s): Two base cases are shown however only one is actually necessary.

$$1^3=1^2$$
$$1^3+2^3=1+8=9=3^2=(1+2)^2$$

Inductive Hypothesis: Assume the statement is true for some number k . In other words, assume the following is true:

$1^3+2^3+3^3+\cdots k^3=(1+2+3+\cdots k)^2$

Proof: You want to show the statement is true for $k+1$. It is a good idea to restate what your goal is at this point. Your goal is to show that:

$1^3+2^3+3^3+\cdots k^3+(k+1)^3=(1+2+3+\cdots k+(k+1))^2$

You need to start with the assumed case and do algebraic manipulations until you have created what you are trying to show (the equation above):

$1^3+2^3+3^3+\cdots k^3=(1+2+3+\cdots k)^2$

From the work you have done with arithmetic series you should notice:

$1+2+3+4+\cdots+k=\frac{k}{2}(2+(k-1))=\frac{k(k+1)}{2}$

Substitute into the right side of the equation and add $(k+1)^3$ to both sides:

$1^3+2^3+3^3+\cdots k^3+(k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3$

When you combine the right hand side algebraically you get the result of another arithmetic series.

$1^3+2^3+3^3+\cdots k^3+(k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 = (1+2+3+\cdots k+(k+1))^2$

$\therefore$

The symbol $\therefore$ is one of many indicators like QED that follow a proof to tell the reader that the proof is complete.

Concept Problem Revisited

If you forget the base case in an induction proof, then you haven't really proved anything. You can get silly results like this "proof" of the statement: "$1 = 3$"

Base Case: Missing

Inductive Hypothesis: $k = k+1$ where k is a counting number.

Proof: Start with the assumption step and add one to both sides.

$$
\begin{aligned}
k &= k+1 \\
k+1 &= k+2
\end{aligned}
$$

Thus by transitivity of equality:

$$
\begin{aligned}
k &= k+1 = k+2 \\
k &= k+2
\end{aligned}
$$

Since k is a counting number, k could equal 1. Therefore:

$1 = 3$

Guided Practice

1. Write the base case, inductive hypothesis, and what you are trying to show for the following statement. Do not actually prove it.

For $n \geq 1,\ n^3+2n$ is divisible by 3 for any positive integer n

2. Complete the proof for the previous problem.

3. Prove the following statement using induction:

For $n \geq 1,\ 1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$

Answers:

1. **Base Case:** $1^3+2\cdot 1 = 3$ which is divisible by 3.

Inductive Step: Assume the following is true for k:

k^3+2k divisible by 3.

Next, you will want to show the following is true for $k+1$:

$(k+1)^3+2(k+1)$ is divisible by 3.

2. The goal is to show that $(k+1)^3+2(k+1)$ is divisible by 3 if you already know k^3+2k is divisible by 3. Expand $(k+1)^3+2(k+1)$ to see what you get:

$$\begin{aligned}(k+1)^3+2(k+1) &= k^3+3k^2+3k+1+2k+2\\ &= (k^3+2k)+3(k^2+k+1)\end{aligned}$$

k^3+2k is divisible by 3 by assumption (the inductive step) and $3(k^2+k+1)$ is clearly a multiple of 3 so is divisible by 3. The sum of two numbers that are divisible by is also divisible by 3.

$\therefore$

3. **Base Case:** $1 = \frac{1(1+1)}{2} = 1 \cdot \frac{2}{2} = 1$

Inductive Hypothesis: $1+2+3+4+\cdots+k = \frac{k(k+1)}{2}$

Proof: Start with what you know and work to showing it true for $k+1$.

Inductive Hypothesis: $1+2+3+4+\cdots+k = \frac{k(k+1)}{2}$

Add $k+1$ to both sides: $1+2+3+4+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1)$

Find a common denominator for the right side: $1+2+3+4+\cdots+k+(k+1) = \frac{k^2+k}{2}+\frac{2k+2}{2}$

Simplify the right side: $1+2+3+4+\cdots+k+(k+1) = \frac{k^2+3k+2}{2}$

Factor the numerator of the right side: $1+2+3+4+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$

Rewrite the right side: $1+2+3+4+\cdots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$

$\therefore$

Vocabulary

The **base case** is the anchor step. It is the first domino to fall, creating a cascade and thus proving the statement true for every number greater than the base case.

The **inductive hypothesis** is the step where you assume the statement is true for k.

The **inductive step** is the **proof**. It is when you show the statement is true for $k+1$ using only the inductive hypothesis and algebra.

In Summary

We have learned about what induction is and how to perform proofs by induction.

Practice

For each of the following statements: a) show the base case is true; b) state the inductive hypothesis; c) state what you are trying to prove in the inductive step/proof. *Do not prove yet.*

1. For $n \geq 5,\ 4n < 2^n$.

2. For $n \geq 1,\ 8^n - 3^n$ is divisible by 5.

3. For $n \geq 1,\ 7^n - 1$ is divisible by 6.

4. For $n \geq 2,\ n^2 \geq 2n$.

5. For $n \geq 1,\ 4^n + 5$ is divisible by 3.

6. For $n \geq 1,\ 0^2+1^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Now, prove each of the following statements. Use your answers to problems 1-6 to help you get started.

7. For $n \geq 5,\ 4n < 2^n$.

8. For $n \geq 1,\ 8^n - 3^n$ is divisible by 5.

9. For $n \geq 1,\ 7^n - 1$ is divisible by 6.

10. For $n \geq 2,\ n^2 \geq 2n$.

11. For $n \geq 1,\ 4^n + 5$ is divisible by 3.

12. For $n \geq 1,\ 0^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

13. You should believe that the following statement is clearly false. What happens when you try to prove it true by induction?

For $n \geq 2,\ n^2 < n$

14. Explain why the base case is necessary for proving by induction.

15. The principles of inductive proof can be used for other proofs besides proofs about numbers. Can you prove the following statement from geometry using induction?

The sum of the interior angles of any $n-gon$ *is* $180(n-2)$ *for* $n \geq 3$.

You learned that recursion, how most people intuitively see patterns, is where each term in a sequence is defined by the term that came before. You saw that terms in a pattern can also be represented as a function of their term number. You learned about two special types of patterns called arithmetic sequences and geometric sequences that have a wide variety of applications in the real world. You saw that series are when terms in a sequence are added together. A strong understanding of patterns helped you to count efficiently, which in turn allowed you to compute both basic and compound probabilities. Finally, you learned that induction is a method of proof that allows you to prove your own mathematical statements.

11.10 References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA

CHAPTER 12 Concepts of Calculus

Chapter Outline

Newton and Leibniz invented Calculus a few hundred years ago to account for subtle paradoxes in their representations of the physical world. What happens when you add an infinite number of infinitely small numbers together? What happens when you multiply an infinitely small number by another infinitely small number? Calculus is a system of tools and a way of thinking about infinity that helps to make sense of these perceived paradoxes.

12.1 Limit Notation

Here you will write and read limit notation and use limit notation to describe the behavior of a function at a point and at infinity.

When learning about the end behavior of a rational function you described the function as either having a horizontal asymptote at zero or another number, or going to infinity. Limit notation is a way of describing this end behavior mathematically.

You already know that as x gets extremely large then the function $f(x) = \frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ goes to $\frac{8}{3}$ because the greatest powers are equal and $\frac{8}{3}$ is the ratio of the leading coefficients. How is this statement represented using limit notation?

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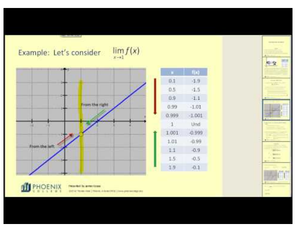

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URL: http://www.ck12.org/flx/render/embeddedobject/62302

http://www.youtube.com/watch?v=ahZ8LLtgu_w James Sousa: Introduction to Limits

Guidance

Limit notation is a way of stating an idea that is a little more subtle than simply saying $x = 5$ or $y = 3$.

$\lim_{x\to a} f(x) = b$

"The limit of f of x as x approaches a is b"

The letter a can be any number or infinity. The function $f(x)$ is any function of x. The letter b can be any number. If the function goes to infinity, then instead of writing "$= \infty$" you should write that the limit does not exist or " *DNE*". This is because infinity is not a number. If a function goes to infinity then it has no limit.

While a function may never actually reach a height of b it will get arbitrarily close to b. One way to think about the concept of a limit is to use a physical example. Stand some distance from a wall and then take a big step to get halfway to the wall. Take another step to go halfway to the wall again. If you keep taking steps that take you halfway to the wall then two things will happen. First, you will get extremely close to the wall but never actually reach the wall regardless of how many steps you take. Second, an observer who wishes to describe your situation would notice that the wall acts as a limit to how far you can go.

Example A

Translate the following statement into limit notation.

The limit of $y = 4x^2$ as x approaches 2 is 16.

Solution: $\lim_{x\to 2} 4x^2 = 16$

Example B

Translate the following mathematical statement into words.

$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{1}{2}\right)^{i}=1$

Solution: The limit of the sum of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ as the number of terms approaches infinity is 1.

Example C

Use limit notation to represent the following mathematical statement.

$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$

Solution:

$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{1}{3}\right)^{i}=\frac{1}{2}$

Concept Problem Revisited

The limit of $\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ as x approaches infinity is $\frac{8}{3}$. This can be written using limit notation as:

$\lim\limits_{x\to\infty}\left(\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}\right)=\frac{8}{3}$

Vocabulary

Limit notation is a way of expressing the fact that the function gets arbitrarily close to a value. In calculus or analysis you may define a limit in terms of the Greek letter epsilon ε and delta δ.

Guided Practice

1. Describe the end behavior of the following rational function at infinity and negative infinity using limits.

$f(x)=\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}$

2. Translate the following limit expression into words.

$\lim\limits_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)=x$

3. What do you notice about the limit expression in # 2?

Answers:

1. Since the function has equal powers of x in the numerator and in the denominator, the end behavior is $-\frac{1}{2}$ as x goes to both positive and negative infinity.

$\lim\limits_{x\to\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=\lim\limits_{x\to-\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=-\frac{1}{2}$

2. The limit of the ratio of the difference between f of quantity x plus h and f of x and h as h approaches 0 is x.

3. You should notice that $h\to 0$ does not mean $h=0$ because if it did then you could not have a 0 in the denominator. You should also note that in the numerator, $f(x+h)$ and $f(x)$ are going to be super close together as h approaches zero. Calculus will enable you to deal with problems that seem to look like $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Practice

Describe the end behavior of the following rational functions at infinity and negative infinity using limits.

1. $f(x)=\frac{2x^4+4x^2-1}{5x^4+3x+9}$

2. $g(x)=\frac{8x^3+4x^2-1}{2x^3+4x+7}$

3. $f(x) = \frac{x^2+2x^3-3}{5x^3+x+4}$

4. $f(x) = \frac{4x+4x^2-5}{2x^2+3x+3}$

5. $f(x) = \frac{3x^2+4x^3+4}{6x^3+3x^2+6}$

Translate the following statements into limit notation.

6. The limit of $y = 2x^2+1$ as x approaches 3 is 19.

7. The limit of $y = e^x$ as x approaches negative infinity is 0.

8. The limit of $y = \frac{1}{x}$ as x approaches infinity is 0.

Use limit notation to represent the following mathematical statements.

9. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3}$

10. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

11. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$

12. $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 1$

Translate the following mathematical statements into words.

13. $\lim_{x\to 0} \frac{5x^2-4}{x+1} = -4$

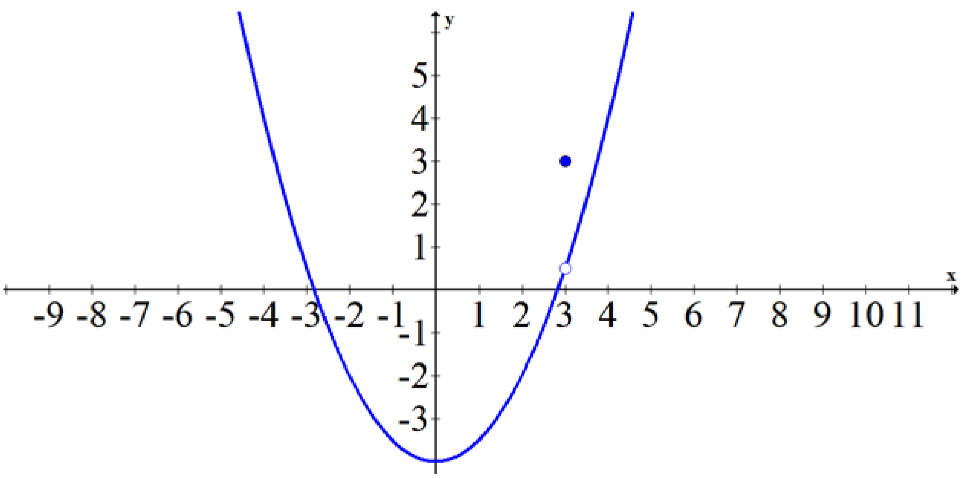
14. $\lim_{x\to 1} \frac{x^3-1}{x-1} = 3$

15. If $\lim_{x\to a} f(x) = b$, is it possible that $f(a) = b$? Explain.

12.2 Graphs to Find Limits

Here you will use graphs to help you evaluate limits and refine your understanding of what a limit represents.

A limit can describe the end behavior of a function. This is called a limit at infinity or negative infinity. A limit can also describe the limit at any normal x value. Sometimes this is simply the height of the function at that point. Other times this is what you would expect the height of the function to be at that point even if the height does not exist or is at some other point. In the following graph, what are $f(3)$, $\lim_{x\to 3} f(x)$, $\lim_{x\to\infty} f(x)$?



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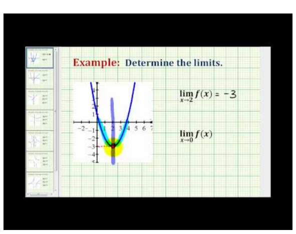

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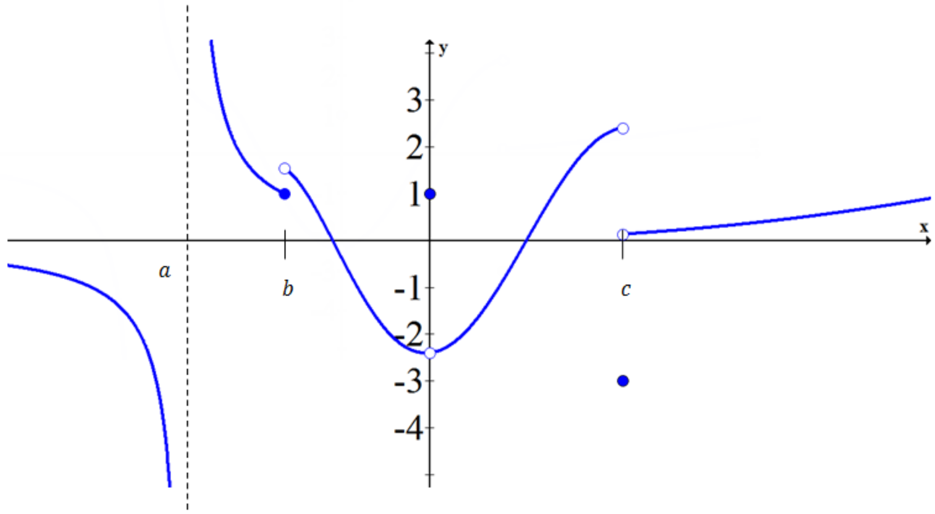
URL: http://www.ck12.org/flx/render/embeddedobject/62304

http://www.youtube.com/watch?v=LdewtuWi7fM James Sousa: Determining Basic Limits Graphically

Guidance

When evaluating the limit of a function from its graph, you need to distinguish between the function evaluated at the point and the limit around the point.



Functions like the one above with discontinuities, asymptotes and holes require you to have a very solid understanding of how to evaluate and interpret limits.

At $x = a$, the function is undefined because there is a vertical asymptote. You would write:

$f(a) = DNE,\ \lim_{x \to a} f(x) = DNE$

At $x = b$, the function is defined because the filled in circle represents that it is the height of the function. This appears to be at about 1. However, since the two sides do not agree, the limit does not exist here either.

$f(b) = 1,\ \lim_{x \to b} f(x) = DNE$

At $x = 0$, the function has a discontinuity in the form of a hole. It is as if the point $(0, -2.4)$ has been lifted up and placed at $(0, 1)$. You can evaluate both the function and the limit at this point, however these quantities will not match. When you evaluate the function you have to give the actual height of the function, which is 1 in this case. When you evaluate the limit, you have to give what the height of the function is supposed to be based solely on the neighborhood around 0. Since the function appears to reach a height of -2.4 from both the left and the right, the limit does exist.

$f(0) = 1,\ \lim_{x \to 0} f(x) = -2.4$

At $x = c$, the limit does not exist because the left and right hand neighborhoods do not agree on a height. On the other hand, the filled in circle represents that the function is defined at $x = c$ to be -3.

$f(c) = -3,\ \lim\limits_{x \to c} f(x) = DNE$

At $x \to \infty$ you may only discuss the limit of the function since it is not appropriate to evaluate a function at infinity (you cannot find $f(\infty)$). Since the function appears to increase without bound, the limit does not exist.

$\lim\limits_{x \to \infty} f(x) = DNE$

At $x \to -\infty$ the graph appears to flatten as it moves to the left. There is a horizontal asymptote at $y = 0$ that this function approaches as $x \to -\infty$.

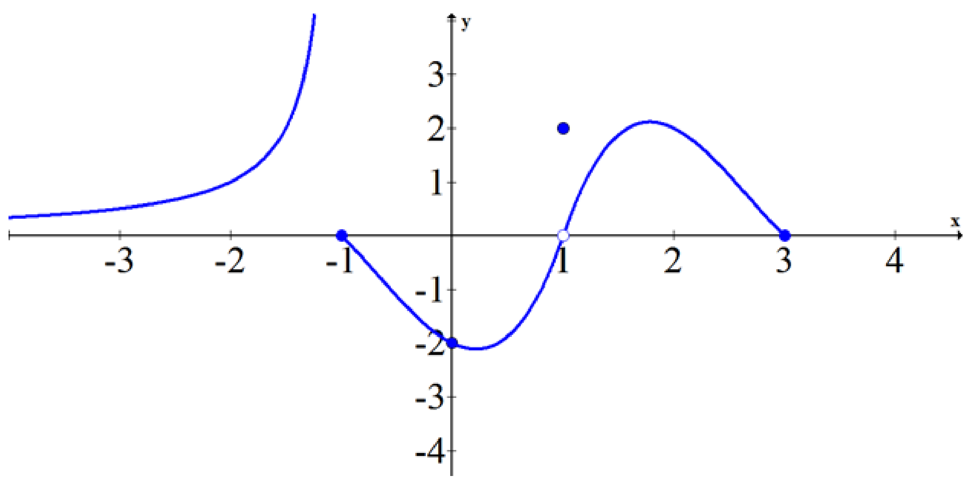
$\lim\limits_{x \to -\infty} f(x) = 0$

When evaluating limits graphically, your main goal is to determine whether the limit exists. The limit only exists when the left and right sides of the functions meet at a specific height. Whatever the function is doing at that point does not matter for the sake of limits. The function could be defined at that point, could be undefined at that point, or the point could be defined at some other height. Regardless of what is happening at that point, when you evaluate limits graphically, you only look at the neighborhood to the left and right of the function at the point.

Example A

Evaluate the following expressions using the graph of the function $f(x)$.

a. $\lim_{x \to -\infty} f(x)$
b. $\lim_{x \to -1} f(x)$
c. $\lim_{x \to 0} f(x)$
d. $\lim_{x \to 1} f(x)$
e. $\lim_{x \to 3} f(x)$
f. $f(-1)$
g. $f(2)$
h. $f(1)$
i. $f(3)$



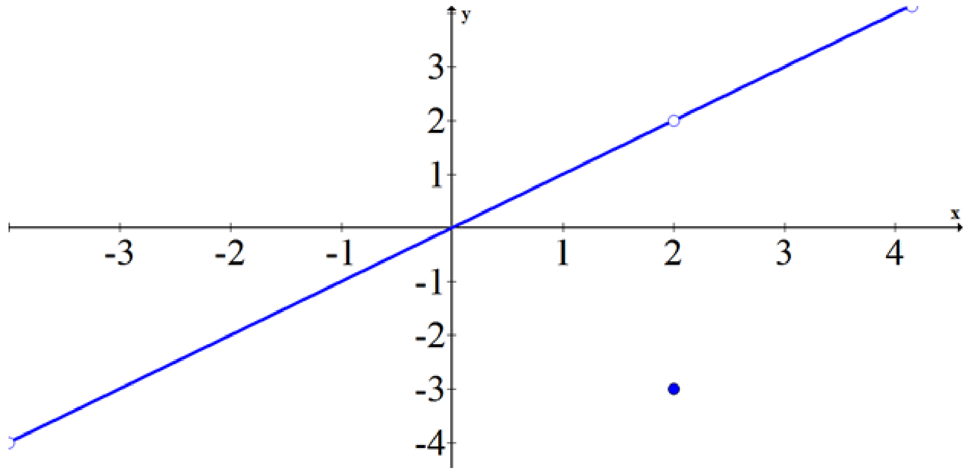
Solution:

a. $\lim_{x \to -\infty} f(x) = 0$
b. $\lim_{x \to -1} f(x) = DNE$
c. $\lim_{x \to 0} f(x) = -2$
d. $\lim_{x \to 1} f(x) = 0$
e. $\lim_{x \to 3} f(x) = DNE$ (This is because only one side exists and a regular limit requires both left and right sides to agree)
f. $f(-1) = 0$
g. $f(0) = -2$
h. $f(1) = 2$
i. $f(3) = 0$

Example B

Sketch a graph that has a limit at $x = 2$, but that limit does not match the height of the function.

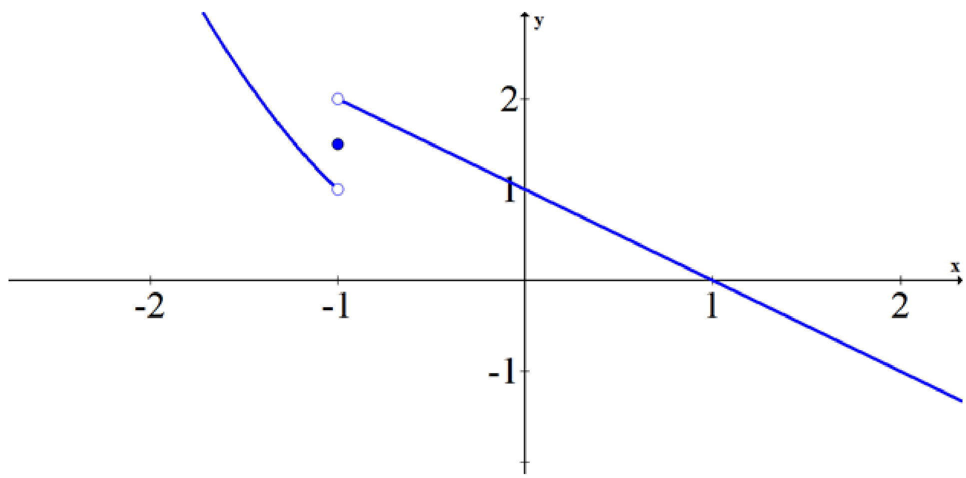
Solution: While there are an infinite number of graphs that fit this criteria, you should make sure your graph has a removable discontinuity at $x = 2$.



Example C

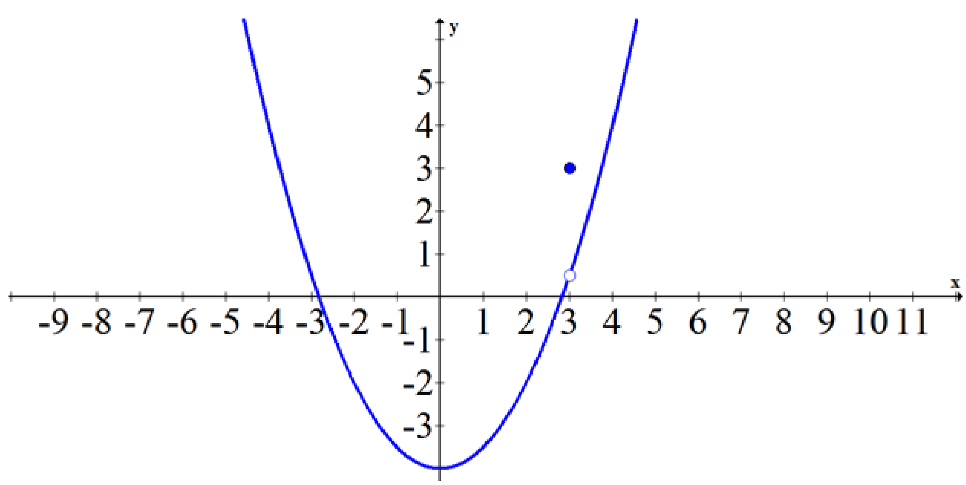
Sketch a graph that is defined at $x = -1$ but $\lim_{x \to -1} f(x)$ does not exist.

Solution: The graph must have either a jump or an infinite discontinuity at $x = -1$ and also have a solid hole filled in somewhere on that vertical line.



Concept Problem Revisited

Given the graph of the function $f(x)$ to be:



$\lim_{x \to 3} f(x) = \frac{1}{2}$

$f(3) = 3$

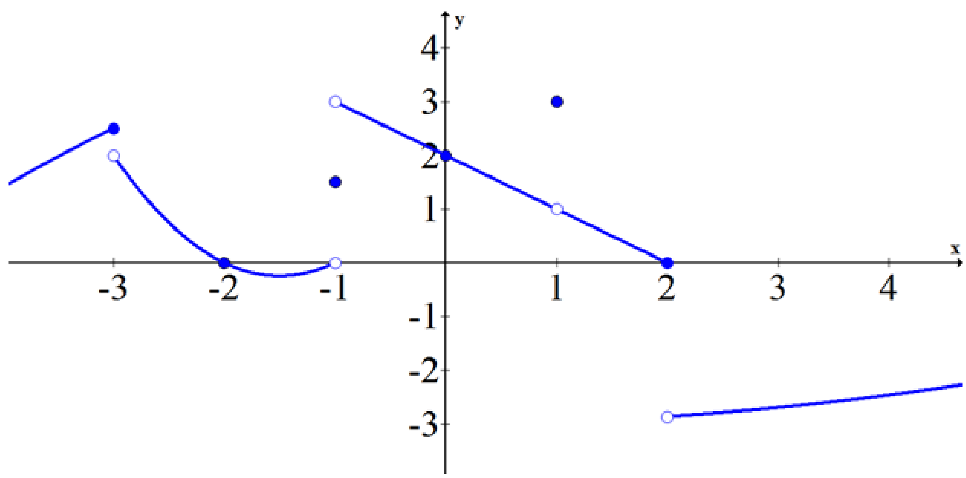
$\lim_{x \to \infty} f(x) = DNE$

Vocabulary

The phrase ***"does not exist"*** or ***"DNE"*** is used with limits to imply that the limit does not approach a particular numerical value. Sometimes this means that the limit continues to grow bigger or smaller to infinity. Sometimes it means that the limit can't decide between two disagreeing values. There are times when instead of indicating the limit does not exist, you might choose to write the limit goes to infinity. This notation is technically not correct, but it does provide more useful information than writing DNE and therefore can be acceptable.

Guided Practice

1. Identify everywhere where the limit does not exist and where the limit does exist in the following function.



2. Evaluate and explain how to find the limits as x approaches 0 and 1 in the previous question.

3. Evaluate the limits of the following piecewise function at -2, 0 and 1.

$$f(x) = \begin{cases} 2 & x < -2 \\ -1 & x = -2 \\ -x-2 & -2 < x \leq 0 \\ x^2 & 0 < x < 1 \\ -2 & x = 1 \\ x^2 & 1 < x \end{cases}$$

Answers:

1. The limit does not exist at $x = -3, -1, 2, +\infty, -\infty$. At every other point (including $x = 1$), the limit does exist.

2. $\lim_{x \to 0} f(x) = 2, \ \lim_{x \to 1} f(x) = 1$

Both of these limits exist because the left hand and right hand neighborhoods of these points seem to approach the same height. In the case of the point $(0,2)$ the function happened to be defined there. In the case of the point $(1,1)$ the function happened to be defined elsewhere, but that does not matter. You only need to consider what the function does right around the point.

3. Since you already know how to graph piecewise functions (graph each function in the x interval indicated) you can then observe graphically the limits at -2, 0 and 1.

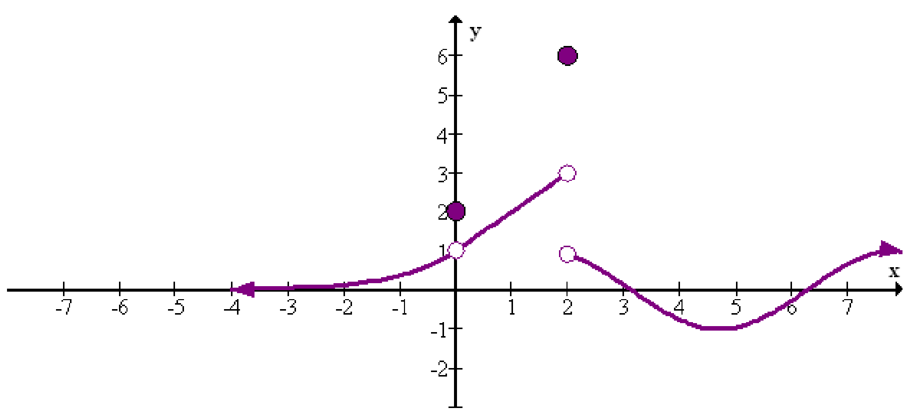
$\lim_{x \to -2} f(x) = 2$

$\lim_{x \to 0} f(x) = DNE$

$\lim_{x \to 1} f(x) = 1$

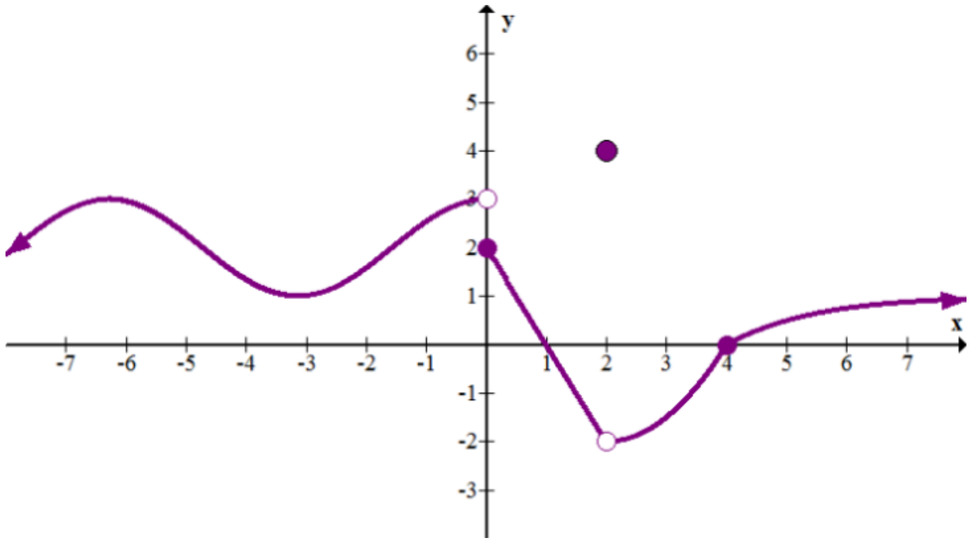
Practice

Use the graph of $f(x)$ below to evaluate the expressions in 1-6.



1. $\lim_{x \to -\infty} f(x)$
2. $\lim_{x \to \infty} f(x)$
3. $\lim_{x \to 2} f(x)$
4. $\lim_{x \to 0} f(x)$
5. $f(0)$
6. $f(2)$

Use the graph of $g(x)$ below to evaluate the expressions in 7-13.



7. $\lim_{x \to -\infty} g(x)$
8. $\lim_{x \to \infty} g(x)$
9. $\lim_{x \to 2} g(x)$
10. $\lim_{x \to 0} g(x)$
11. $\lim_{x \to 4} g(x)$
12. $g(0)$
13. $g(2)$
14. Sketch a function $h(x)$ such that $h(2) = 4$, but $\lim_{x \to 2} h(x) = DNE$.
15. Sketch a function $j(x)$ such that $j(2) = 4$, but $\lim_{x \to 2} j(x) = 3$.

12.3 Tables to Find Limits

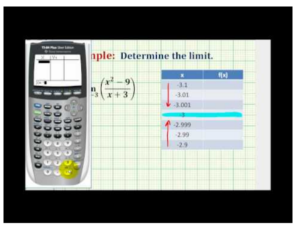
Here you will estimate limits using tables.

Calculators such as the TI-84 have a table view that allows you to make extremely educated guesses as to what the limit of a function will be at a specific point, even if the function is not actually defined at that point.

How could you use a table to calculate the following limit?

$\lim_{x \to -1} \frac{x^2+3x+2}{x+1}$

Watch This



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URL: http://www.ck12.org/flx/render/embeddedobject/62306

http://www.youtube.com/watch?v=l7Tcay720vw James Sousa: Determine a Limit Numerically

Guidance

If you were given the following information organized in a table, how would you fill in the center column?

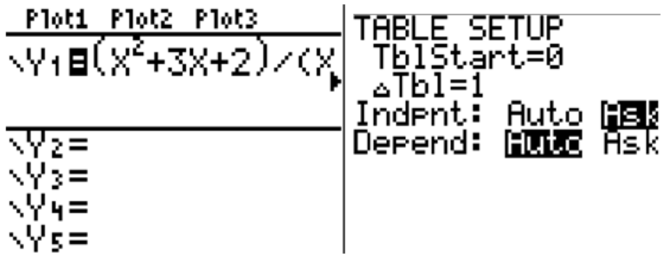
TABLE 12.1:

3.9	3.99	3.999		4.001	4.01	4.1
12.25	12.01	12.00001		11.99999	11.99	11.75

It would be logical to see the symmetry and notice how the top row approaches the number 4 from the left and the right. It would also be logical to notice how the bottom row approaches the number 12 from the left and the right. This would lead you to the conclusion that the limit of the function represented by this table is 12 as the top row approaches 4. It would not matter if the value at 4 was undefined or defined to be another number like 17, the pattern tells you that the limit at 4 is 12.

Using tables to help evaluate limits requires this type of logic. To use a table on your calculator to evaluate a limit:

1. Enter the function on the $y =$ screen
2. Go to table set up and highlight "ask" for the independent variable
3. Go to the table and enter values close to the number that x approaches



Example A

Complete the table and use the result to estimate the limit.

$\lim_{x \to 2} \frac{x-2}{x^2-x-2}$

TABLE 12.2:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

Solution: While it is not necessary to use the table feature in the calculator, it is very efficient. Another option is to substitute the given x values into the expression $\frac{x-2}{x^2-x-2}$ and record your results.

TABLE 12.3:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.34483	0.33445	0.33344	0.33322	0.33223	0.32258

The evidence suggests that the limit is $\frac{1}{3}$.

Example B

Complete the table and use the result to estimate the limit.

$\lim_{x \to 2} \frac{x-2}{x^2-4}$

TABLE 12.4:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

Solution: You can trick the calculator into giving a very exact answer by typing in 1.999999999999 because then the calculator rounds instead of producing an error.

TABLE 12.5:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.25641	0.25063	0.25006	0.24994	0.24938	0.2439

The evidence suggests that the limit is $\frac{1}{4}$.

Example C

Complete the table and use the result to estimate the limit.

$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$

TABLE 12.6:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

Solution:

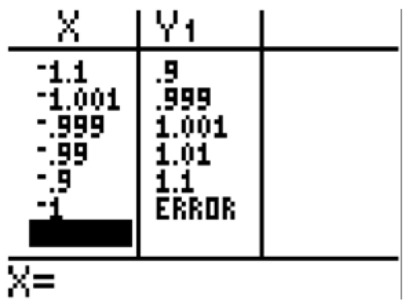
TABLE 12.7:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.29112	0.28892	0.2887	0.28865	0.28843	0.28631

The evidence suggests that the limit is a number between 0.2887 and 0.28865. When you learn to find the limit analytically, you will know that the exact limit is $\frac{1}{2} \cdot 3\frac{1}{2} \approx 0.2886751346$.

Concept Problem Revisited

When you enter values close to -1 in the table you get y values that are increasingly close to the number 1. This implies that the limit as x approaches -1 is 1. Notice that when you evaluate the function at -1, the calculator produces an error. This should lead you to the conclusion that while the function is not defined at $x = -1$, the limit does exist.



Vocabulary

Numerically is a term used to describe one of several different representations in mathematics. It refers to tables where the actual numbers are visible.

Guided Practice

1. Graph the following function and the use a table to verify the limit as x approaches 1.

$f(x) = \frac{x^3-1}{x-1}, x \neq 1$

2. Estimate the limit numerically.

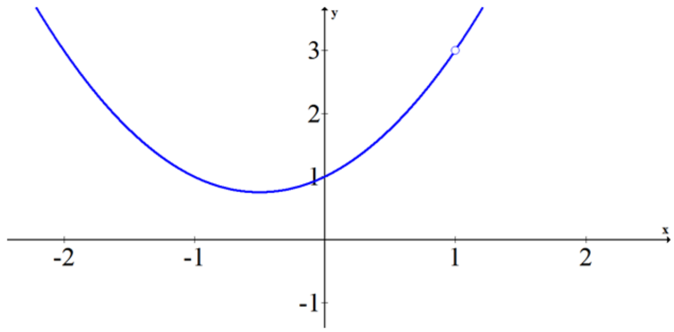
$\lim_{x \to 2} \frac{x^2-3x+2}{x-2}$

3. Estimate the limit numerically.

$\lim_{x \to 0} \frac{\left[\frac{4}{x+2}\right]-2}{x}$

Answers:

1. $\lim_{x \to 1} f(x) = 3$. This is because when you factor the numerator and cancel common factors, the function becomes a quadratic with a hole at the point $(1, 3)$.



You can verify the limit in the table.

TABLE 12.8:

x	$f(x)$
.75	2.3125
.9	2.71
.99	2.9701
.999	2.997
1	Error
1.001	3.003
1.01	3.0301
1.1	3.31
1.25	3.8125

2. $\lim_{x\to 2} f(x) = 1$

TABLE 12.9:

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	0.75	0.9	0.99	0.999	Error	1.001	1.01	1.1	1.25

3. $\lim_{x\to 0} f(x) = 0$

TABLE 12.10:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.20526	0.02005	0.002	-0.002	-0.02	-0.1952

Practice

Estimate the following limits numerically.

1. $\lim_{x\to 5} \frac{x^2-25}{x-5}$

2. $\lim_{x\to -1} \frac{x^2-3x-4}{x+1}$

3. $\lim_{x\to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+2}$

4. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$

5. $\lim_{x\to 3} \left(\frac{1}{x-3} - \frac{9}{x^2-9}\right)$

6. $\lim_{x \to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x \to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x \to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$
9. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$
10. $\lim_{x \to 0} \frac{x^2+5x}{x}$
11. $\lim_{x \to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$
13. $\lim_{x \to 2} \frac{\sqrt{x+3}-2}{x-1}$
14. $\lim_{x \to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x \to -1} \frac{x-2}{x+1}$

12.4 Substitution to Find Limits

Here you will start to find limits analytically using substitution.

Finding limits for the vast majority of points for a given function is as simple as substituting the number that x approaches into the function. Since this turns evaluating limits into an algebra-level substitution, most questions involving limits focus on the cases where substituting does not work. How can you decide if substitution is an appropriate analytical tool for finding a limit?

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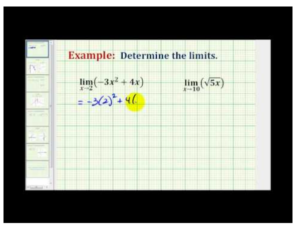

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URL: http://www.ck12.org/flx/render/embeddedobject/62310

http://www.youtube.com/watch?v=VLiMfJHZIpk James Sousa: Determine a Limit Analytically

Guidance

Finding a limit analytically means finding the limit using algebraic means. In order to evaluate many limits, you can substitute the value that x approaches into the function and evaluate the result. This works perfectly when there are no holes or asymptotes at that particular x value. You can be confident this method works as long as you don't end up dividing by zero when you substitute.

If the function $f(x)$ has no holes or asymptote at $x = a$ then: $\lim_{x\to a} f(x) = f(a)$

Occasionally there will be a hole at $x = a$. The limit in this case is the height of the function if the hole did not exist. In other words, if the function is a rational expression with factors that can be canceled, cancel the term algebraically and then substitute into the resulting expression. If no factors can be canceled, it could be that the limit does not exist at that point due to asymptotes.

Example A

Which of the following limits can you determine using direct substitution? Find that limit.

$\lim_{x\to 2} \frac{x^2-4}{x-2}$, $\lim_{x\to 3} \frac{x^2-4}{x-2}$

Solution: The limit on the right can be evaluated using direct substitution because the hole exists at $x = 2$ not $x = 3$.

$\lim_{x\to 3} \frac{x^2-4}{x-2} = \frac{3^2-4}{3-2} = \frac{9-4}{1} = 5$

Example B

Evaluate the following limit by canceling first and then using substitution.

$\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$

Solution:

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}\\ &= \lim_{x\to 2}(x+2)\\ &= 2+2\\ &= 4\end{aligned}$$

Example C

Evaluate the following limit analytically: $\lim_{x\to 4}\frac{x^2-x-12}{x-4}$

Solution:

$$\begin{aligned}\lim_{x\to 4}\frac{x^2-x-12}{x-4} &= \lim_{x\to 4}\frac{(x-4)(x+3)}{(x-4)}\\ &= \lim_{x\to 4}(x+3)\\ &= 4+3\\ &= 7\end{aligned}$$

Concept Problem Revisited

In order to decide whether substitution is an appropriate first step you can always just try it. You'll know it won't work if you end up trying to evaluate an expression with a denominator equal to zero. If this happens, go back and try to factor and cancel, and then try substituting again.

Vocabulary

Substitution is a method of determining limits where the value that x is approaching is substituted into the function and the result is evaluated. This is one way to evaluate a limit ***analytically.***

Guided Practice

1. Evaluate the following limit analytically.

$\lim\limits_{x\to 3}\frac{x^2-9}{x-3}$

2. Evaluate the following limit analytically.

$\lim\limits_{t\to 4}\sqrt{t+32}$

3. Evaluate the following limit analytically.

$\lim\limits_{y\to 4}\frac{3|y-1|}{y+4}$

Answers:

1. $\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)}{(t-3)} = \lim_{x\to 3}(x+3) = 6$

2. $\lim_{t\to 4}\sqrt{t+32} = \sqrt{4+32} = \sqrt{36} = 6$

3. $\lim_{y\to 4}\frac{3|y-1|}{y+4} = \frac{3|4-1|}{4+4} = \frac{3\cdot 3}{8} = \frac{9}{8}$

Practice

Evaluate the following limits analytically.

1. $\lim_{x \to 5} \frac{x^2-25}{x-5}$
2. $\lim_{x \to -1} \frac{x^2-3x-4}{x+1}$
3. $\lim_{x \to 5} \sqrt{5x} - 12$
4. $\lim_{x \to 0} \frac{x^3+3x^2-x}{5x}$
5. $\lim_{x \to 1} \frac{3x|x-4|}{x+1}$
6. $\lim_{x \to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x \to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x \to 0} \frac{5x-1}{2x^2+3}$
9. $\lim_{x \to 1} 4x^2 - 2x + 5$
10. $\lim_{x \to 0} \frac{x^2+5x}{x}$
11. $\lim_{x \to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x \to 0} \frac{5x+1}{x}$
13. $\lim_{x \to 1} \frac{5x+1}{x}$
14. $\lim_{x \to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x \to -1} \frac{x-2}{x+1}$

12.5 Rationalization to Find Limits

Here you will evaluate limits analytically using rationalization.

Some limits cannot be evaluated directly by substitution and no factors immediately cancel. In these situations there is another algebraic technique to try called rationalization. With rationalization, you make the numerator and the denominator of an expression rational by using the properties of conjugate pairs.

How do you evaluate the following limit using rationalization?

$\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}$

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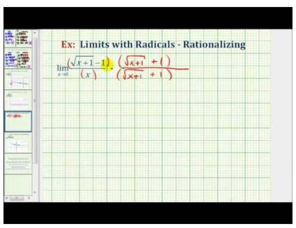

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URL: http://www.ck12.org/flx/render/embeddedobject/62308

http://www.youtube.com/watch?v=ouWAhqeAaik James Sousa: Find a Limit Requiring Rationalizing

Guidance

The properties of conjugates are used in a variety of places in PreCalculus.

Conjugates can be used to simplify expressions with a radical in the denominator:

$$\frac{5}{1+\sqrt{3}} = \frac{5}{\left(1+\sqrt{3}\right)} \cdot \frac{\left(1-\sqrt{3}\right)}{\left(1-\sqrt{3}\right)} = \frac{5-5\sqrt{3}}{1-3} = \frac{5-5\sqrt{3}}{-2}$$

Conjugates can be used to simplify complex numbers with i in the denominator:

$$\frac{4}{2+3i} = \frac{4}{(2+3i)} \cdot \frac{(2-3i)}{(2-3i)} = \frac{8-12i}{4+9} = \frac{8-12i}{13}$$

Here, they can be used to transform an expression in a limit problem that does not immediately factor to one that does immediately factor.

$$\lim_{x \to 16} \frac{\left(\sqrt{x}-4\right)}{(x-16)} \cdot \frac{\left(\sqrt{x}+4\right)}{\left(\sqrt{x}+4\right)} = \lim_{x \to 16} \frac{(x-16)}{(x-16)\left(\sqrt{x}+4\right)}$$

Now you can cancel the common factors in the numerator and denominator and use substitution to finish evaluating the limit.

The rationalizing technique works because when you algebraically manipulate the expression in the limit to an equivalent expression, the resulting limit will be the same. Sometimes you must do a variety of different algebraic manipulations in order avoid a zero in the denominator when using the substitution method.

Example A

Evaluate the following limit: $\lim_{x \to 3} \frac{x^2-9}{\sqrt{x}-\sqrt{3}}$.

Solution:

$$\begin{aligned}
\lim_{x\to 3}\frac{(x-3)(x+3)}{\left(\sqrt{x}-\sqrt{3}\right)}\cdot\frac{\left(\sqrt{x}+\sqrt{3}\right)}{\left(\sqrt{x}+\sqrt{3}\right)} &= \lim_{x\to 3}\frac{(x-3)(x+3)\left(\sqrt{x}+\sqrt{3}\right)}{(x-3)} \\
&= \lim_{x\to 3}(x+3)\left(\sqrt{x}+\sqrt{3}\right) \\
&= 6\cdot 2\sqrt{3} \\
&= 12\sqrt{3}
\end{aligned}$$

Example B

Evaluate the following limit: $\lim_{x\to 25}\frac{x-25}{\sqrt{x}-5}$.

Solution:

$$\begin{aligned}
\lim_{x\to 25}\frac{x-25}{\sqrt{x}-5} &= \lim_{x\to 25}\frac{(x-25)}{\left(\sqrt{x}-5\right)}\cdot\frac{\left(\sqrt{x}+5\right)}{\left(\sqrt{x}+5\right)} \\
&= \lim_{x\to 25}\frac{(x-25)\left(\sqrt{x}+5\right)}{(x-25)} \\
&= \lim_{x\to 25}\left(\sqrt{x}+5\right) \\
&= \sqrt{25}+5 \\
&= 10
\end{aligned}$$

Example C

Evaluate the following limit: $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$.

Solution:

$$\begin{aligned}
\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7} &= \lim_{x\to 7}\frac{\left(\sqrt{x+2}-3\right)}{(x-7)}\cdot\frac{\left(\sqrt{x+2}+3\right)}{\left(\sqrt{x+2}+3\right)} \\
&= \lim_{x\to 7}\frac{(x+2-9)}{(x-7)\cdot\left(\sqrt{x+2}+3\right)} \\
&= \lim_{x\to 7}\frac{(x-7)}{(x-7)\cdot\left(\sqrt{x+2}+3\right)} \\
&= \lim_{x\to 7}\frac{1}{\left(\sqrt{x+2}+3\right)} \\
&= \frac{1}{\sqrt{7+2}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Concept Problem Revisited

In order to evaluate the limit of the following rational expression, you need to multiply by a clever form of 1 so that when you substitute there is no longer a zero factor in the denominator.

$$\begin{aligned}\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16} &= \lim_{x\to 16}\frac{(\sqrt{x}-4)}{(x-16)}\cdot\frac{(\sqrt{x}+4)}{(\sqrt{x}+4)}\\ &= \lim_{x\to 16}\frac{(x-16)}{(x-16)(\sqrt{x}+4)}\\ &= \lim_{x\to 16}(\sqrt{x}+4)\\ &= 4+4\\ &= 8\end{aligned}$$

Vocabulary

Rationalization generally means to multiply a rational function by a clever form of one in order to eliminate radical symbols or imaginary numbers in the denominator. ***Rationalization*** is also a technique used to evaluate limits in order to avoid having a zero in the denominator when you substitute.

Guided Practice

1. Evaluate the following limit: $\lim_{x\to 0}\frac{(2+x)^{-1}-2^{-1}}{x}$.

2. Evaluate the following limit: $\lim_{x\to -3}\frac{\sqrt{x^2-5}-2}{x+3}$.

3. Evaluate the following limit: $\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right)$.

Answers:

1.

$$\begin{aligned}\lim_{x\to 0}\frac{(2+x)^{-1}-2^{-1}}{x} &= \lim_{x\to 0}\frac{\frac{1}{x+2}-\frac{1}{2}}{x}\cdot\frac{(x+2)\cdot 2}{(x+2)\cdot 2}\\ &= \lim_{x\to 0}\frac{2-(x+2)}{2x(x+2)}\\ &= \lim_{x\to 0}\frac{-x}{2x(x+2)}\\ &= \lim_{x\to 0}\frac{-1}{2(x+2)}\\ &= -\frac{1}{2(0+2)}\\ &= -\frac{1}{4}\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x \to -3} \frac{\sqrt{x^2-5}-2}{x+3} &= \lim_{x \to -3} \frac{\left(\sqrt{x^2-5}-2\right)}{(x+3)} \cdot \frac{\left(\sqrt{x^2-5}+2\right)}{\left(\sqrt{x^2-5}+2\right)} \\
&= \lim_{x \to -3} \frac{(x^2-5-4)}{(x+3)} \\
&= \lim_{x \to -3} \frac{(x^2-9)}{(x+3)} \\
&= \lim_{x \to -3} \frac{(x-3)(x+3)}{(x+3)} \\
&= \lim_{x \to -3} (x-3) \\
&= -3-3 \\
&= -6
\end{aligned}$$

3.

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{3}{x\sqrt{9-x}} - \frac{1}{x} \right) &= \lim_{x \to 0} \left(\frac{3}{x\sqrt{9-x}} - \frac{\sqrt{9-x}}{x\sqrt{9-x}} \right) \\
&= \lim_{x \to 0} \left(\frac{3-\sqrt{9-x}}{x\sqrt{9-x}} \right) \\
&= \lim_{x \to 0} \left(\frac{\left(3-\sqrt{9-x}\right)}{x\sqrt{9-x}} \cdot \frac{\left(3+\sqrt{9-x}\right)}{\left(3+\sqrt{9-x}\right)} \right) \\
&= \lim_{x \to 0} \left(\frac{9-(9-x)}{x\sqrt{9-x}} \right) \\
&= \lim_{x \to 0} \frac{x}{x\sqrt{9-x}} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{9-x}} \\
&= \frac{1}{\sqrt{9-0}} \\
&= \frac{1}{3}
\end{aligned}$$

Practice

Evaluate the following limits:

1. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$
2. $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$
3. $\lim_{x \to 1} \frac{\sqrt{x+3}-2}{x-1}$
4. $\lim_{x \to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

5. $\lim_{x\to4}\frac{\sqrt{3x+4}-x}{4-x}$

6. $\lim_{x\to0}\frac{2-\sqrt{x+4}}{x}$

7. $\lim_{x\to0}\frac{\sqrt{x+7}-\sqrt{7}}{x}$

8. $\lim_{x\to16}\frac{16-x}{4-\sqrt{x}}$

9. $\lim_{x\to0}\frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}$

10. $\lim_{x\to2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$

11. $\lim_{x\to1}\frac{1-\sqrt{x}}{1-x}$

12. $\lim_{x\to\frac{1}{9}}\frac{9x-1}{3\sqrt{x}-1}$

13. $\lim_{x\to4}\frac{4x^2-64}{2\sqrt{x}-4}$

14. $\lim_{x\to9}\frac{9x^2-90x+81}{9-3\sqrt{x}}$

15. When given a limit to evaluate, how do you know when to use the rationalization technique? What will the function look like?

12.6 One Sided Limits and Continuity

Here you will determine one sided limits graphically, numerically and algebraically and use the concept of a one sided limit to define continuity.

A one sided limit is exactly what you might expect; the limit of a function as it approaches a specific x value from either the right side or the left side. One sided limits help to deal with the issue of a jump discontinuity and the two sides not matching.

Is the following piecewise function continuous?

$$f(x) = \begin{cases} -x-2 & x < 1 \\ -3 & x = 1 \\ x^2-4 & 1 < x \end{cases}$$

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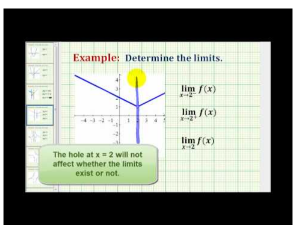

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URL: http://www.ck12.org/flx/render/embeddedobject/62351

http://www.youtube.com/watch?v=3iZUK15aPE0 James Sousa: Determining Limits and One-Sided Limits Graphically

Guidance

A one sided limit can be evaluated either from the left or from the right. Since left and right are not absolute directions, a more precise way of thinking about direction is "from the negative side" or "from the positive side". The notation for these one sided limits is:

$\lim_{x \to a^-} f(x), \lim_{x \to a^+} f(x)$

The negative in the superscript of a is not an exponent. Instead it indicates **from the negative side**. Likewise the positive superscript is not an exponent, it just means **from the positive side**. When evaluating one sided limits, it does not matter what the function is doing at the actual point or what the function is doing on the other side of the number. Your job is to determine what the height of the function should be using only evidence on one side.

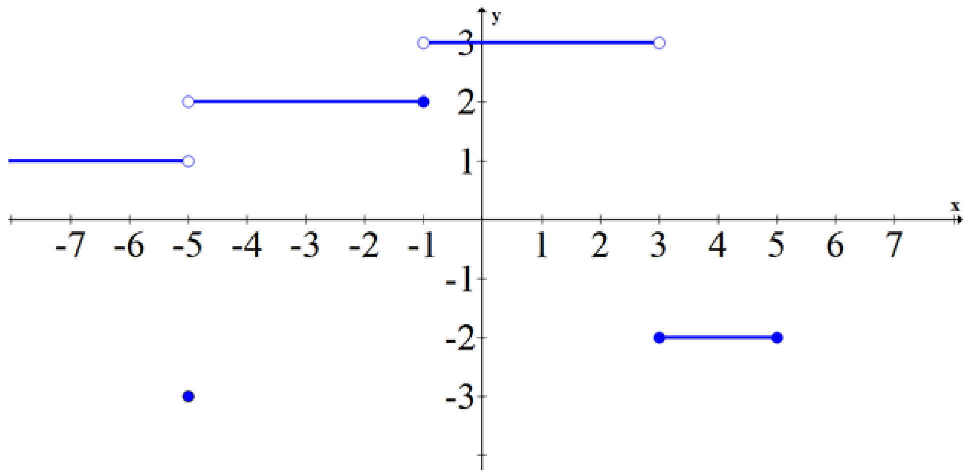
You have defined continuity in the past as the ability to draw a function completely without lifting your pencil off of the paper. You can now define a more rigorous definition of continuity.

If both of the one sided limits equal the value of the function at a given point, then the function is continuous at that point. In other words, a function is continuous at a if:

$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

Example A

What are the one sided limits at -5, -1, 3 and 5?



Solution:

$\lim_{x \to -5^-} f(x) = 1$

$\lim_{x \to -5^+} f(x) = 2$

$\lim_{x \to -1^-} f(x) = 2$

$\lim_{x \to -1^+} f(x) = 3$

$\lim_{x \to 3^-} f(x) = 3$

$\lim_{x \to 3^+} f(x) = -2$

$\lim_{x \to 5^-} f(x) = -2$

$\lim_{x \to -5^+} f(x) = DNE$

Example B

Evaluate the one sided limit at 4 from the negative direction numerically.

$f(x) = \frac{x^2-7x+12}{x-4}$

Solution: When creating the table, only use values that are smaller than 4.

TABLE 12.11:

x	3.9	3.99	3.999
$f(x)$	0.9	0.99	0.999

$\lim_{x \to 4^-} \left(\frac{x^2-7x+12}{x-4} \right) = 1$

Example C

Evaluate the following limits.

a. $\lim_{x \to 3^-} (4x-3)$

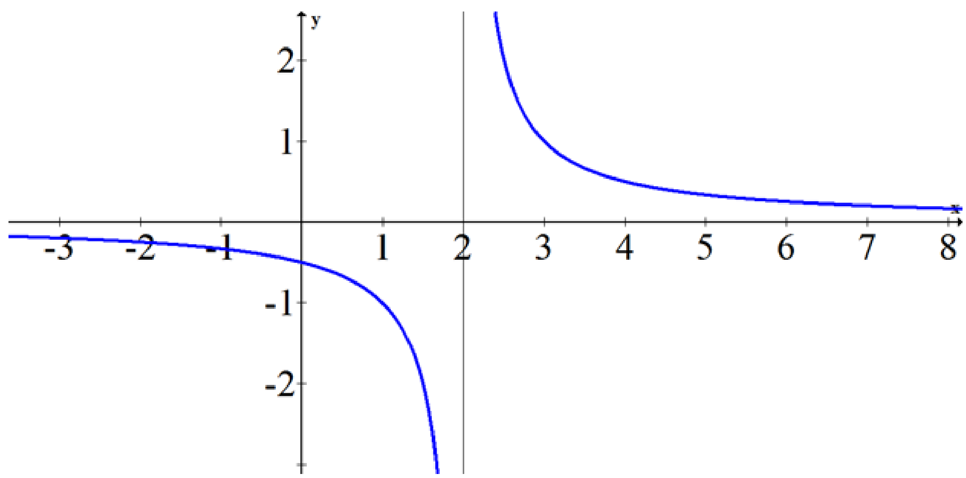
b. $\lim_{x \to 2^+} \left(\frac{1}{x-2}\right)$
c. $\lim_{x \to 1^+} \left(\frac{x^2+2x-3}{x-1}\right)$

Solution: Most of the time one sided limits are the same as the corresponding two sided limit. The exceptions are when there are jump discontinuities, which normally only happen with piecewise functions, and infinite discontinuities, which normally only happen with rational functions.

a. $\lim_{x \to 3^-} (4x-3) = 4 \cdot 3 - 3 = 12 - 3 = 9$

b. $\lim_{x \to 2^+} \left(\frac{1}{x-2}\right) = DNE$ or ∞

The reason why ∞ is preferable in this case is because the two sides of the limit disagree. One side goes to negative infinity and the other side goes to positive infinity (see the graph below). If you just indicate DNE then you are losing some perfectly good information about the nature of the function.



c. $\lim_{x \to 1^+} \left(\frac{x^2+2x-3}{x-1}\right) = \lim_{x \to 1^+} \left(\frac{(x-1)(x+3)}{(x-1)}\right) = \lim_{x \to 1^+} (x+3) = 1+3 = 4$

Concept Problem Revisited

In order to confirm or deny that the function is continuous, graphical tools are not accurate enough. Sometimes jump discontinuities can be off by such a small amount that the pixels on the display of your calculator will not display a difference. Your calculator will certainly not display removable discontinuities.

$$f(x) = \begin{cases} -x-2 & x < 1 \\ -3 & x = 1 \\ x^2-4 & 1 < x \end{cases}$$

You should note that on the graph, everything to the left of 1 is continuous because it is just a line. Next you should note that everything to the right of 1 is also continuous for the same reason. The only point to check is at $x = 1$. To check continuity, explicitly use the definition and evaluate all three parts to see if they are equal.

- $\lim_{x \to a^-} f(x) = -1-2 = -3$
- $f(1) = -3$
- $\lim_{x \to 1^+} f(x) = 1^2 - 4 = -3$

Therefore, $\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$ and the function is continuous at $x = 1$ and everywhere else.

Vocabulary

A ***one sided limit*** is a limit of a function when the evidence from only the positive or only the negative side is used to evaluate the limit.

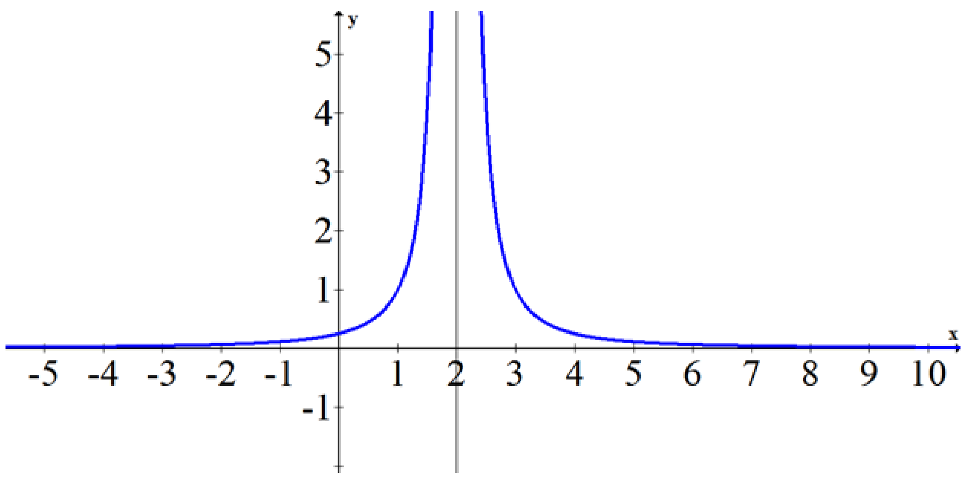
Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For an entire function to be continuous, the function must be continuous at every single point in an unbroken domain.

Guided Practice

1. Megan argues that according to the definition of continuity, the following function is continuous. She says

- $\lim_{x \to 2^-} f(x) = \infty$
- $\lim_{x \to 2^+} f(x) = \infty$
- $f(2) = \infty$

Thus since $\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$, it meets the definition of continuous. How could you use the graph below to clarify Megan's reasoning?



2. Evaluate the following limits.

a. $\lim_{x \to 1^-} (2x - 1)$
b. $\lim_{x \to -3^+} \left(\frac{2}{x+2}\right)$
c. $\lim_{x \to 2^+} \left(\frac{x^3-8}{x-2}\right)$

3. Is the following function continuous?

$$f(x) = \begin{cases} x^2 - 1 & x < -1 \\ 3 & x = -1 \\ -x+3 & -1 < x \end{cases}$$

Answers:

1. Megan is being extremely liberal with the idea of "$=\infty$" because what she really means for the two limits is "DNE". For the function evaluated at 2 the correct response is "undefined". Two things that do not exist cannot be equal to one another.

2.

a. $\lim_{x\to 1^-}(2x-1)=2\cdot 1-1=2-1=1$
b. $\lim_{x\to -3^+}\left(\frac{2}{x+2}\right)=\frac{2}{-3+2}=\frac{2}{-1}=-2$
c. $\lim_{x\to 2^+}\left(\frac{x^3-8}{x-2}\right)=\lim_{x\to 2^+}\left(\frac{(x-2)(x^2+2x+4)}{(x-2)}\right)=\lim_{x\to 2^+}(x^2+2x+4)=2^2+2\cdot 2+4=12$

3. Use the definition of continuity.

- $\lim_{x\to 1^-} f(x)=(-1)^2-1=1-1=0$
- $f(-1)=3$
- $\lim_{x\to -1^+} f(x)=-1+3=2$

$\lim_{x\to a^-} f(x)\neq f(a)\neq \lim_{x\to a^+} f(x)$ so this function is discontinuous at $x=-1$. It is continuous everywhere else.

Practice

Evaluate the following limits.

1. $\lim_{x\to 6^-}(3x^2-4)$
2. $\lim_{x\to 0^-}\frac{3x-1}{x}$
3. $\lim_{x\to 0^+}\frac{3x-1}{x}$
4. $\lim_{x\to 0^+}\frac{x}{|x|}$
5. $\lim_{x\to 0^-}\frac{x}{|x|}$
6. $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{1+\sqrt{x}-1}}$

Consider

$$f(x)=\begin{cases}2x^2-1 & x<1\\ 1 & x=1\\ -x+2 & 1<x\end{cases}$$

7. What is $\lim_{x\to 1^-} f(x)$?

8. What is $\lim_{x\to 1^+} f(x)$?

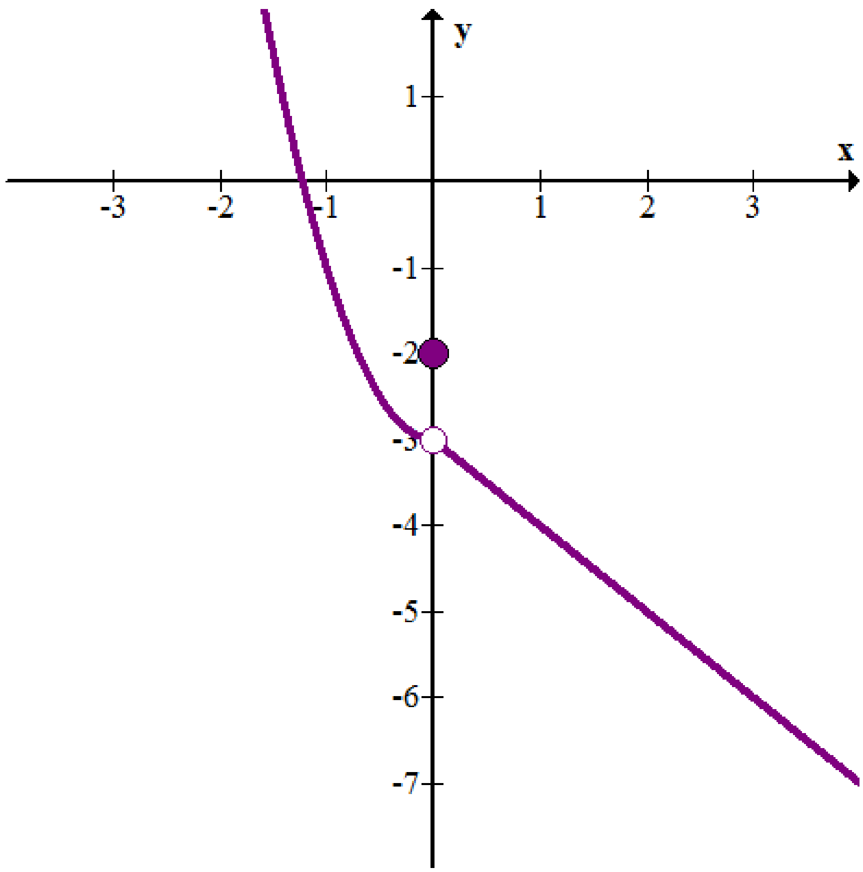
9. Is $f(x)$ continuous at $x=1$?

Consider

$$g(x)=\begin{cases}4x^2+2x-1 & x<-2\\ 8 & x=-2\\ -3x+5 & -2<x\end{cases}$$

10. What is $\lim_{x \to -2^-} g(x)$?
11. What is $\lim_{x \to -2^+} g(x)$?
12. Is $g(x)$ continuous at $x = -2$?

Consider $h(x)$ shown in the graph below.



13. What is $\lim_{x \to 0^-} h(x)$?
14. What is $\lim_{x \to 0^+} h(x)$?
15. Is $h(x)$ continuous at $x = 0$?

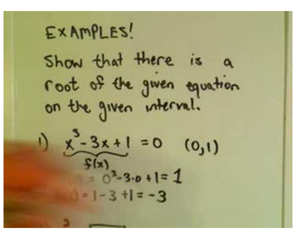
12.7 Intermediate and Extreme Value Theorems

Here you will use continuity to explore the intermediate and extreme value theorems.

While the idea of continuity may seem somewhat basic, when a function is continuous over a closed interval like $x \in [1,4]$, you can actually draw some major conclusions. The conclusions may be obvious when you understand the statements and look at a graph, but they are powerful nonetheless.

What can you conclude using the Intermediate Value Theorem and the Extreme Value Theorem about a function that is continuous over the closed interval $x \in [1,4]$?

Watch This



MEDIA

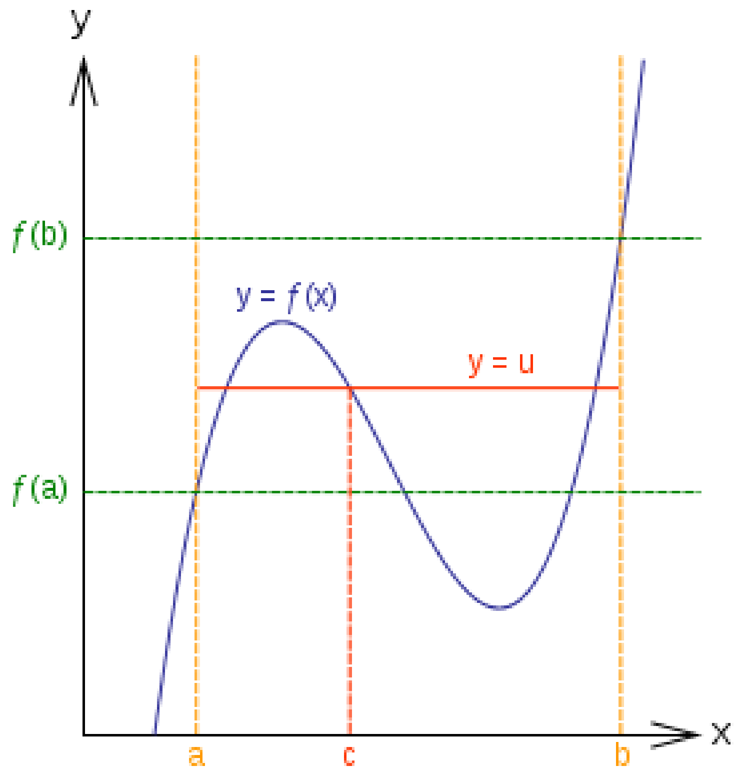
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62353

http://www.youtube.com/watch?v=6AFT1wnId9U Intermediate Value Theorem

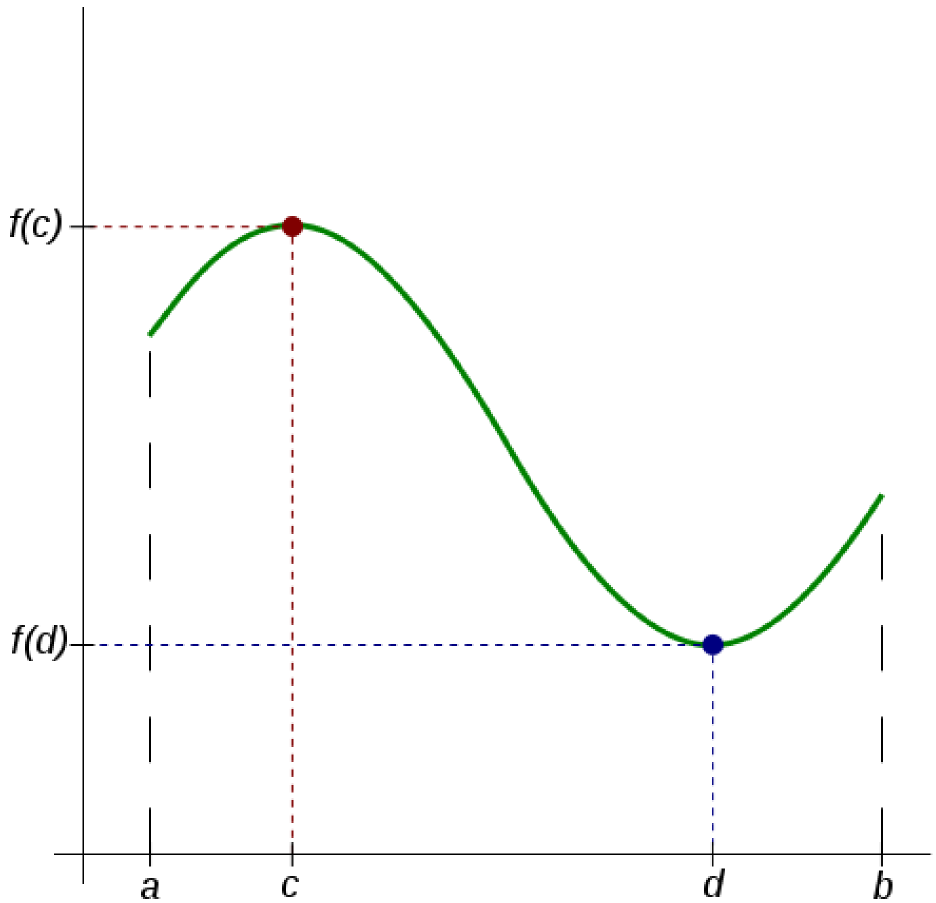
Guidance

The **Intermediate Value Theorem** states that if a function is continuous on a closed interval and u is a value between $f(a)$ and $f(b)$ then there exists a $c \in [a,b]$ such that $f(c) = u$.



Simply stated, if a function is continuous between a low point and a high point, then it must be valued at each intermediate height in between the low and high points.

The Extreme Value Theorem states that in every interval $[a,b]$ where a function is continuous there is at least one maximum and one minimum. In other words, it must have at least two extreme values.

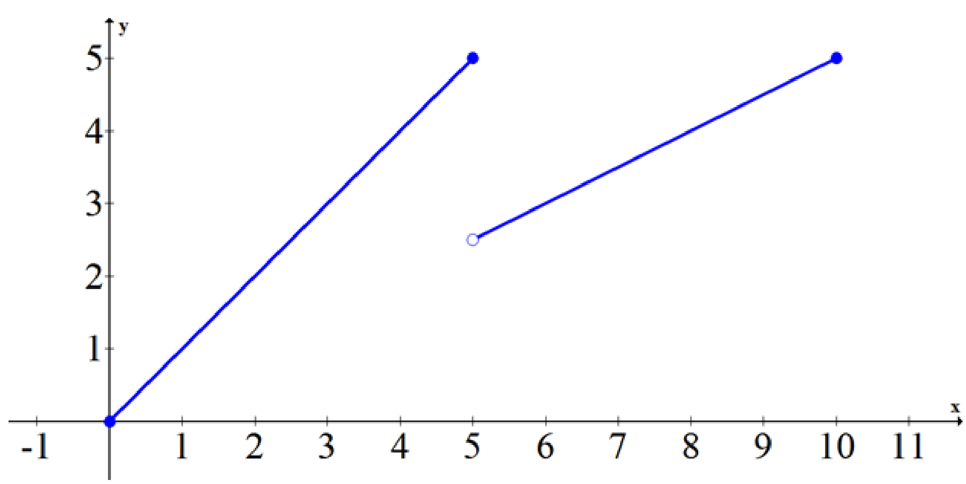


Example A

Show that the converse of the Intermediate Value Theorem is false.

Solution: The converse of the Intermediate Value Theorem is: *If there exists a value* $c \in [a,b]$ such that $f(c) = u$ for every u between $f(a)$ and $f(b)$ then the function is continuous.

In order to show the statement is false, all you need is one counterexample where every intermediate value is hit and the function is discontinuous.



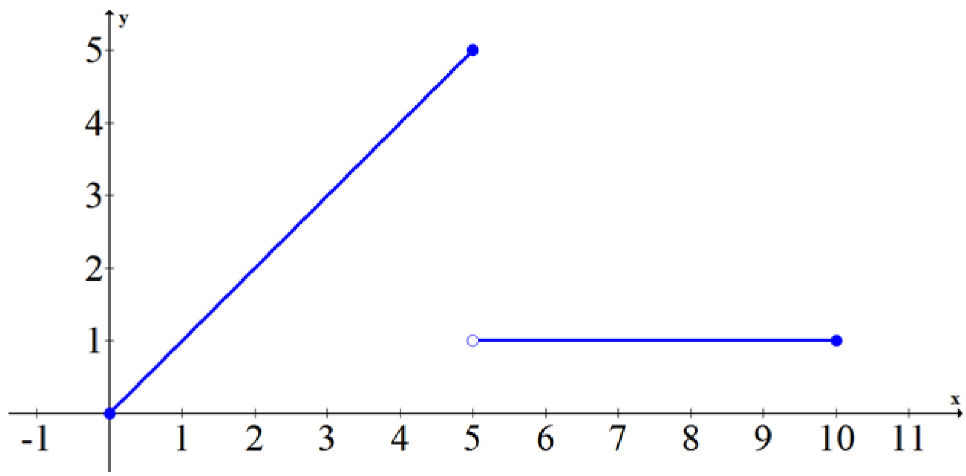
This function is discontinuous on the interval $[0,10]$ but every intermediate value between the first height at $(0,0)$ and the height of the last point $(10,5)$ is hit.

Example B

Show that the converse of the Extreme Value Theorem is false.

Solution: The converse of the Extreme Value Theorem is*:* If there is at least one maximum and one minimum in the closed interval $[a,b]$ *then the function is continuous on* $[a,b]$.

In order to show the statement is false, all you need is one counterexample. The goal is to find a function on a closed interval $[a,b]$ that has at least one maximum and one minimum and is also discontinuous.



On the interval $[0,10]$, the function attains a maximum at $(5,5)$ and a minimum at $(0,0)$ but is still discontinuous.

Example C

Use the Intermediate Value Theorem to show that the function $f(x) = (x+1)^3 - 4$ has a zero on the interval $[0,3]$.

Solution: First note that the function is a cubic and is therefore continuous everywhere.

- $f(0) = (0+1)^3 - 4 = 1^3 - 4 = -3$
- $f(3) = (3+1)^3 - 4 = 4^3 - 4 = 60$

By the Intermediate Value Theorem, there must exist a $c \in [0,3]$ such that $f(c) = 0$ since 0 is between -3 and 60.

Concept Problem Revisited

If a function is continuous on the interval $x \in [1,4]$, then you can conclude by the Intermediate Value Theorem that there exists a $c \in [1,4]$ such that $f(c) = u$ for every u between $f(1)$ and $f(4)$. You can also conclude that on this interval the function has both a maximum and a minimum value.

Vocabulary

The ***converse*** of an if then statement is a new statement with the hypothesis of the original statement switched with the conclusion of the original statement. In other words, the converse is when the if part of the statement and the then part of the statement are swapped. In general, the converse of a statement is not true.

A ***counterexample*** to an if then statement is when the hypothesis (the if part of the sentence) is true, but the conclusion (the then part of the statement) is not true.

Guided Practice

1. Use the Intermediate Value Theorem to show that the following equation has at least one real solution.

$x^8 = 2^x$

2. Show that there is at least one solution to the following equation.

$\sin x = x + 2$

3. When are you not allowed to use the Intermediate Value Theorem?

Answers:

1. First rewrite the equation: $x^8 - 2^x = 0$

Then describe it as a continuous function: $f(x) = x^8 - 2^x$

This function is continuous because it is the difference of two continuous functions.

- $f(0) = 0^8 - 2^0 = 0 - 1 = -1$
- $f(2) = 2^8 - 2^2 = 256 - 4 = 252$

By the Intermediate Value Theorem, there must exist a c such that $f(c) = 0$ because $-1 < 0 < 252$. The number c is one solution to the initial equation.

2. Write the equation as a continuous function: $f(x) = \sin x - x - 2$

The function is continuous because it is the sum and difference of continuous functions.

- $f(0) = \sin 0 - 0 - 2 = -2$
- $f(-\pi) = \sin(-\pi) + \pi - 2 = 0 + \pi - 2 > 0$

By the Intermediate Value Theorem, there must exist a c such that $f(c) = 0$ because $-2 < 0 < \pi - 2$. The number c is one solution to the initial equation.

3. The Intermediate Value Theorem should not be applied when the function is not continuous over the interval.

Practice

Use the Intermediate Value Theorem to show that each equation has at least one real solution.

1. $\cos x = -x$

2. $\ln(x) = e^{-x} + 1$

3. $2x^3 - 5x^2 = 10x - 5$

4. $x^3 + 1 = x$

5. $x^2 = \cos x$

6. $x^5 = 2x^3 + 2$

7. $3x^2 + 4x - 11 = 0$

8. $5x^4 = 6x^2 + 1$

9. $7x^3 - 18x^2 - 4x + 1 = 0$

10. Show that $f(x) = \frac{2x-3}{2x-5}$ has a real root on the interval $[1, 2]$.

11. Show that $f(x) = \frac{3x+1}{2x+4}$ has a real root on the interval $[-1, 0]$.

12. True or false: A function has a maximum and a minimum in the closed interval $[a, b]$; therefore, the function is continuous.

13. True or false: A function is continuous over the interval $[a, b]$; therefore, the function has a maximum and a minimum in the closed interval.

14. True or false: If a function is continuous over the interval $[a,b]$, then it is possible for the function to have more than one relative maximum in the interval $[a,b]$.

15. What do the Intermediate Value and Extreme Value Theorems have to do with continuity?

12.8 Instantaneous Rate of Change

Here you will learn about instantaneous rate of change and the concept of a derivative.

When you first learned about slope you learned the mnemonic device "rise over run" to help you remember that to calculate the slope between two points you use the following formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In Calculus, you learn that for curved functions, it makes more sense to discuss the slope at one precise point rather than between two points. The slope at one point is called the slope of the tangent line and the slope between two separate points is called a secant line.

Consider a car driving down the highway and think about its speed. You are probably thinking about speed in terms of going a given distance in a given amount of time. The units could be miles per hour or feet per second, but the units always have time in the denominator. What happens when you consider the instantaneous speed of the car at one instant of time? Wouldn't the denominator be zero?

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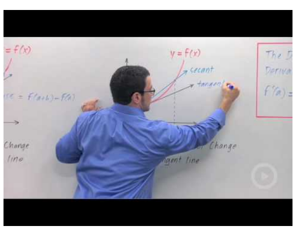

MEDIA

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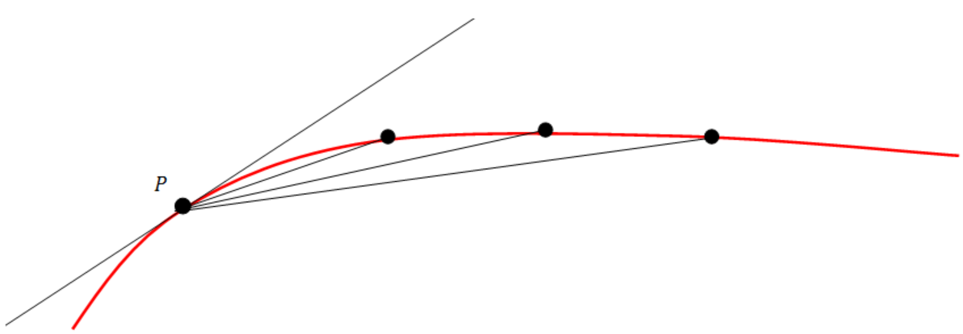
URL: http://www.ck12.org/flx/render/embeddedobject/63172

http://www.youtube.com/watch?v=7CvLzpzGhJI Brightstorm: Definition of a Derivative

Guidance

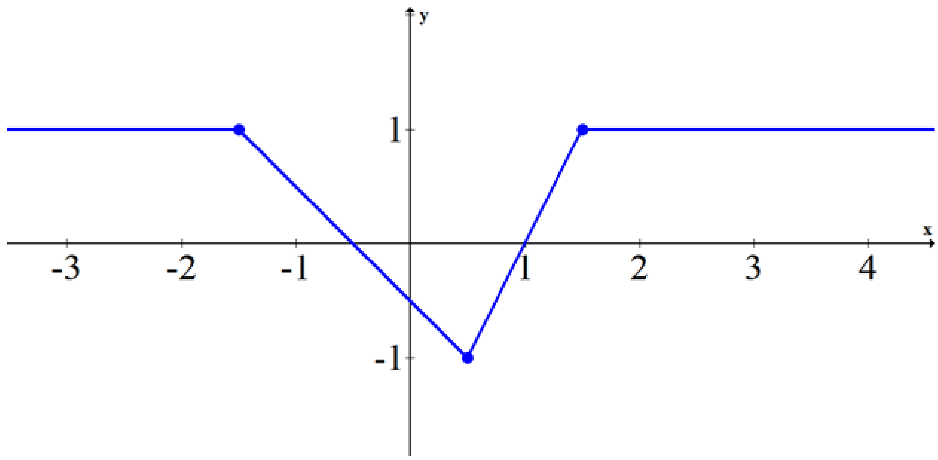
The slope at a point P (also called the slope of the tangent line) can be approximated by the slope of secant lines as the "run" of each secant line approaches zero.



Because you are interested in the slope as the "run" approaches zero, this is a limit question. One of the main reasons that you study limits in calculus is so that you can determine the slope of a curve at a point (the slope of a tangent line).

Example A

Estimate the slope of the following function at -3, -2, -1, 0, 1, 2, 3. Organize the slopes in a table.



Solution: By mentally drawing a tangent line at the following x values you can estimate the following slopes.

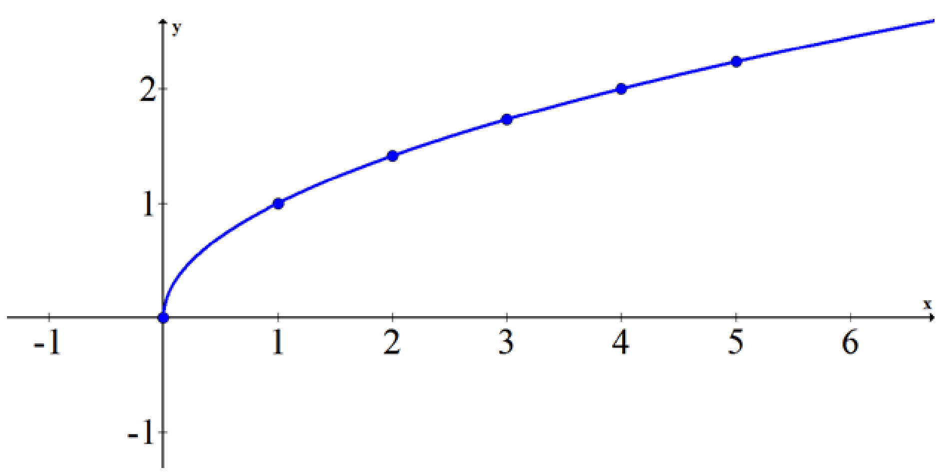
TABLE 12.12:

x	*slope*
-3	0
-2	0
-1	-1
0	-1
1	2
2	0
3	0

If you graph these points you will produce a graph of what's known as the derivative of the original function.

Example B

Estimate the slope of the function $f(x) = \sqrt{x}$ at the point $(1,1)$ by calculating 4 successively close secant lines.



Solution: Calculate the slope between $(1,1)$ and 4 other points on the curve:

- The slope of the line between $\left(5,\ \sqrt{5}\right)$ and $(1,1)$ is: $m_1 = \frac{\sqrt{5}-1}{5-1} \approx 0.309$
- The slope of the line between $(4,\ 2)$ and $(1,1)$ is: $m_2 = \frac{2-1}{4-1} \approx 0.333$
- The slope of the line between $\left(3,\ \sqrt{3}\right)$ and $(1,1)$ is: $m_3 = \frac{\sqrt{3}-1}{3-1} \approx 0.366$
- The slope of the line between $\left(2,\ \sqrt{2}\right)$ and $(1,1)$ is: $m_4 = \frac{\sqrt{2}-1}{2-1} \approx 0.414$

If you had to guess what the slope was at the point $(1,1)$ what would you guess the slope to be?

Example C

Evaluate the following limit and explain its connection with Example B.

$\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)$

Solution: Notice that the pattern in the previous problem is leading up to $\frac{\sqrt{1}-1}{1-1}$. Unfortunately, this cannot be computed directly because there is a zero in the denominator. Luckily, you know how to evaluate using limits.

$$\begin{aligned}
m &= \lim_{x\to 1}\left(\frac{(\sqrt{x}-1)}{(x-1)}\cdot\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}\right)\\
&= \lim_{x\to 1}\left(\frac{(x-1)}{(x-1)(\sqrt{x}+1)}\right)\\
&= \lim_{x\to 1}\left(\frac{1}{(\sqrt{x}+1)}\right)\\
&= \frac{1}{\sqrt{1}+1}\\
&= \frac{1}{2}\\
&= 0.5
\end{aligned}$$

The slope of the function $f(x) = \sqrt{x}$ at the point $(1,1)$ is exactly $m = \frac{1}{2}$.

Concept Problem Revisited

If you write the ratio of distance to time and use limit notation to allow time to go to zero you do seem to get a zero in the denominator.

$\lim_{time\to 0}\left(\frac{distance}{time}\right)$

The great thing about limits is that you have learned techniques for finding a limit even when the denominator goes to zero. Instantaneous speed for a car essentially means the number that the speedometer reads at that precise moment in time. You are no longer restricted to finding slope from two separate points.

Vocabulary

A ***tangent line*** to a function at a given point is the straight line that just touches the curve at that point. The slope of the tangent line is the same as the slope of the function at that point.

A ***secant line*** is a line that passes through two distinct points on a function.

A ***derivative*** is a function of the slopes of the original function.

Guided Practice

1. Sketch a complete cycle of a sine graph. Estimate the slopes at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

2. Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph.

3. Approximate the slope of $y = x^3$ at $(1,1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

Answers:

1.

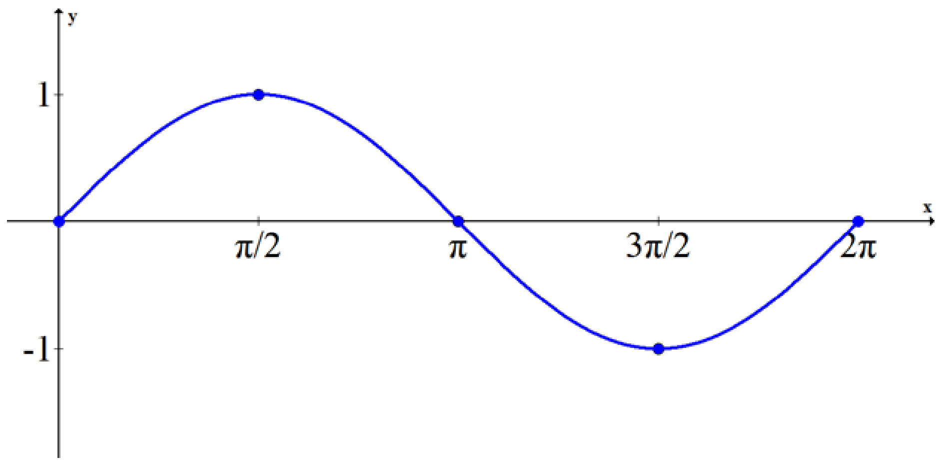
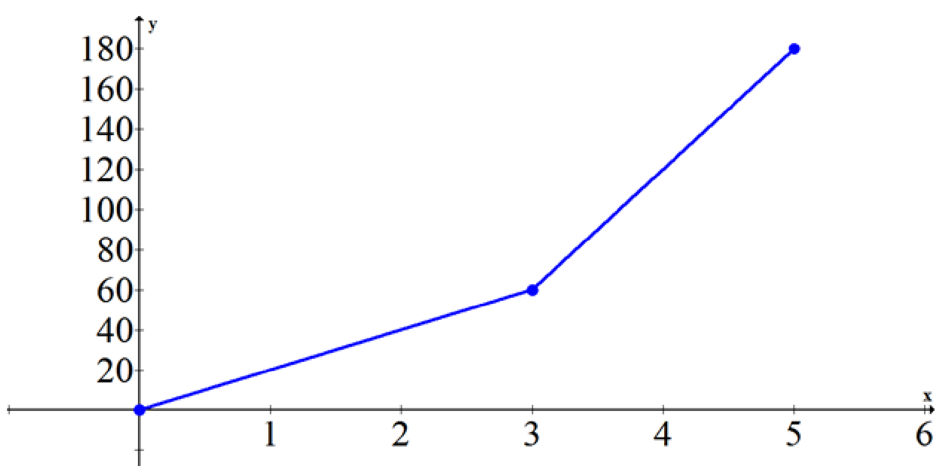


TABLE 12.13:

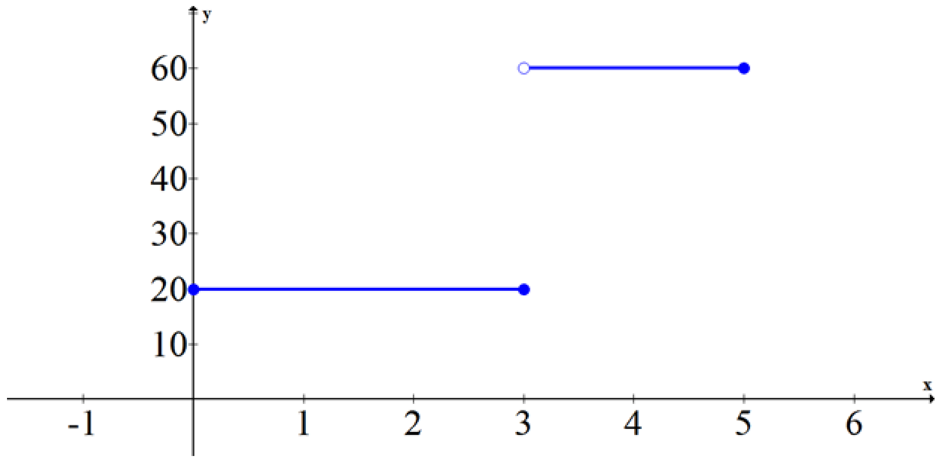
x	***Slope***
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

You should notice that these are the exact values of cosine evaluated at those points.

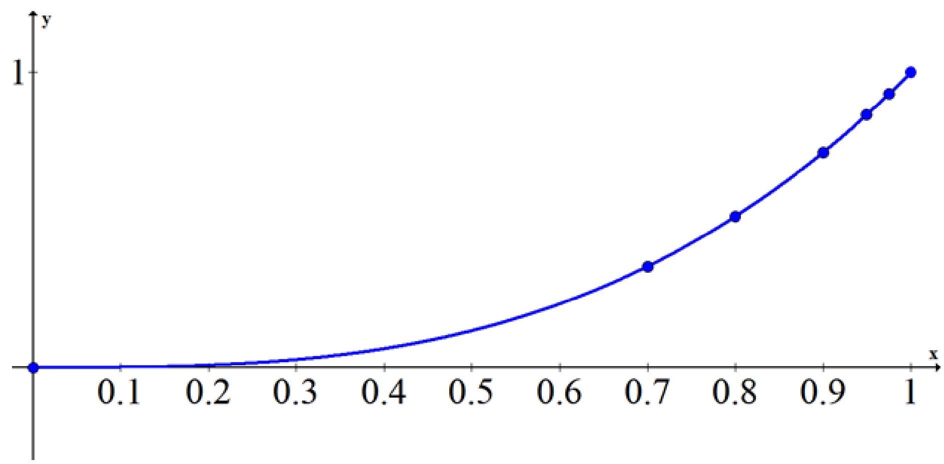
2. Distance vs. Time:



Rate vs. Time: (this is the graph of the derivative of the original function shown above)



3.



- The slope of the line between $(0.7, 0.7^3)$ and $(1,1)$ is: $m_1 = \frac{0.7^3-1}{0.7-1} \approx 2.19$
- The slope of the line between $(0.8, 0.8^3)$ and $(1,1)$ is: $m_2 = \frac{0.8^3-1}{0.8-1} \approx 2.44$
- The slope of the line between $(0.9, 0.9^3)$ and $(1,1)$ is: $m_3 = \frac{0.9^3-1}{0.9-1} \approx 2.71$
- The slope of the line between $(0.95, 0.95^3)$ and $(1,1)$ is: $m_1 = \frac{0.95^3-1}{0.95-1} \approx 2.8525$
- The slope of the line between $(0.975, 0.975^3)$ and $(1,1)$ is: $m_1 = \frac{0.975^3-1}{0.975-1} \approx 2.925625$

The slope at $(1,1)$ will be slightly greater than the estimates because of the way the slope curves. The slope at $(1,1)$ appears to be about 3.

Practice

1. Approximate the slope of $y = x^2$ at $(1,1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

2. Evaluate the following limit and explain how it confirms your answer to #1.

$$\lim_{x \to 1} \left(\frac{x^2-1}{x-1} \right)$$

3. Approximate the slope of $y = 3x^2 + 1$ at $(1,4)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

4. Evaluate the following limit and explain how it confirms your answer to #3.

$\lim_{x \to 1} \left(\frac{3x^2+1-4}{x-1} \right)$

5. Approximate the slope of $y = x^3 - 2$ at $(1,-1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

6. Evaluate the following limit and explain how it confirms your answer to #5.

$\lim_{x \to 1} \left(\frac{x^3-2-(-1)}{x-1} \right)$

7. Approximate the slope of $y = 2x^3 - 1$ at $(1,1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

8. What limit could you evaluate to confirm your answer to #7?

9. Sketch a complete cycle of a cosine graph. Estimate the slopes at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

10. How do the slopes found in the previous question relate to the sine function? What function do you think is the derivative of the cosine function?

11. Sketch the line $y = 2x + 1$. What is the slope at each point on this line? What is the derivative of this function?

12. Logan travels by bike at 30 mph for 2 hours. Then she gets in a car and drives 65 mph for 3 hours. Sketch both the distance vs. time graph and the rate vs. time graph.

13. Explain what a tangent line is and how it relates to derivatives.

14. Why is finding the slope of a tangent line for a point on a function the same as the instantaneous rate of change at that point?

15. What do limits have to do with finding the slopes of tangent lines?

12.9 Area Under a Curve

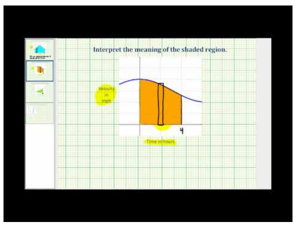
Here you will estimate the area under a curve and interpret its meaning in context.

Calculating the area under a straight line can be done with geometry. Calculating the area under a curved line requires calculus. Often the area under a curve can be interpreted as the accumulated amount of whatever the function is modeling. Suppose a car's speed in meters per second can be modeled by a quadratic for the first 8 seconds of acceleration:

$s(t) = t^2$

How far has the car traveled in 8 seconds?

Watch This



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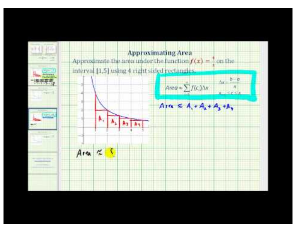
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62355

http://www.youtube.com/watch?v=Z_OHgubPJKA James Sousa: Interpret the Meaning of Area Under a Function



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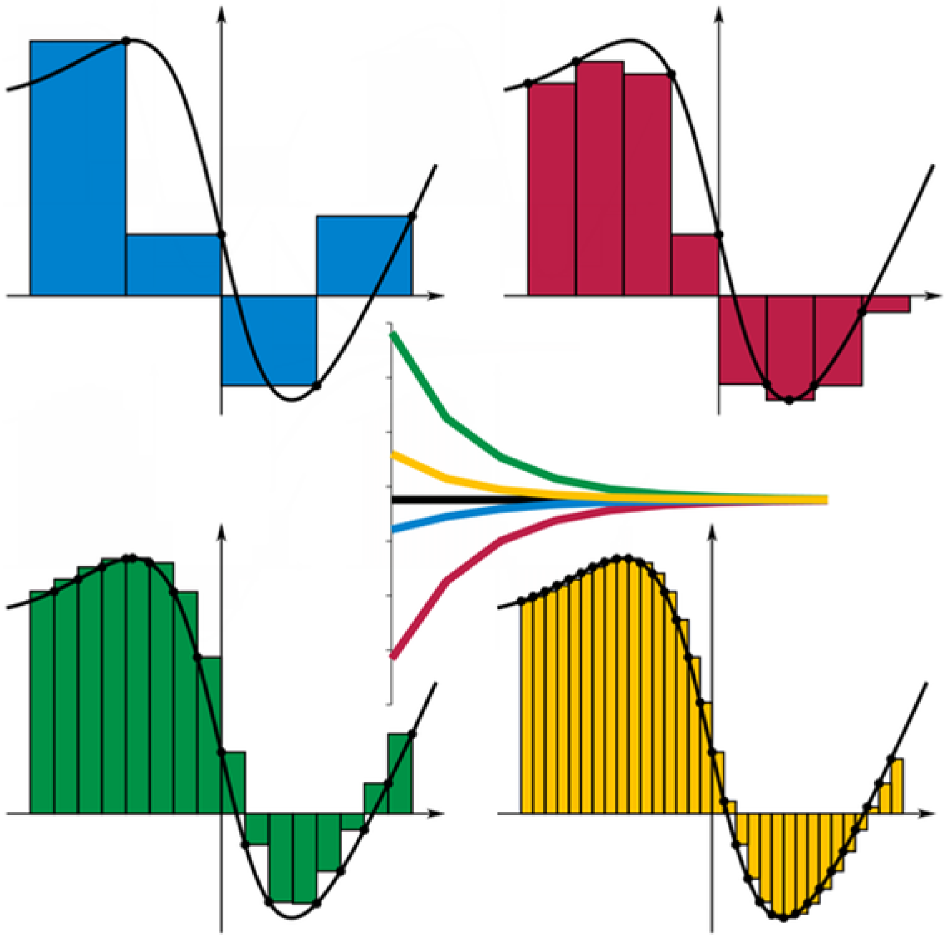
URL: http://www.ck12.org/flx/render/embeddedobject/62357

http://www.youtube.com/watch?v=BvwyTXeuLS0 James Sousa: Approximate the Area Under a Curve with 4 Right Sided Rectangles

Guidance

The area under a curve can be approximated with rectangles equally spaced under a curve as shown below. For consistency, you can choose whether the boxes should hit the curve on the left hand corner, the right hand corner, the maximum value, or the minimum value. The more boxes you use the narrower the boxes will be and thus, the more accurate your approximation of the area will be.



The blue approximation uses right handed boxes. The red approximation assigns the height of the box to be the minimum value of the function in each subinterval. The green approximation assigns the height of the box to be the maximum value of the function in each subinterval. The yellow approximation uses left handed boxes. Rectangles above the x-axis will have positive area and rectangles below the x-axis will have negative area in this context.

All four of these area approximations get better as the number of boxes increase. In fact, the limit of each approximation as the number of boxes increases to infinity is the precise area under the curve.

This is where the calculus idea of an integral comes in. An integral is the limit of a sum as the number of summands increases to infinity.

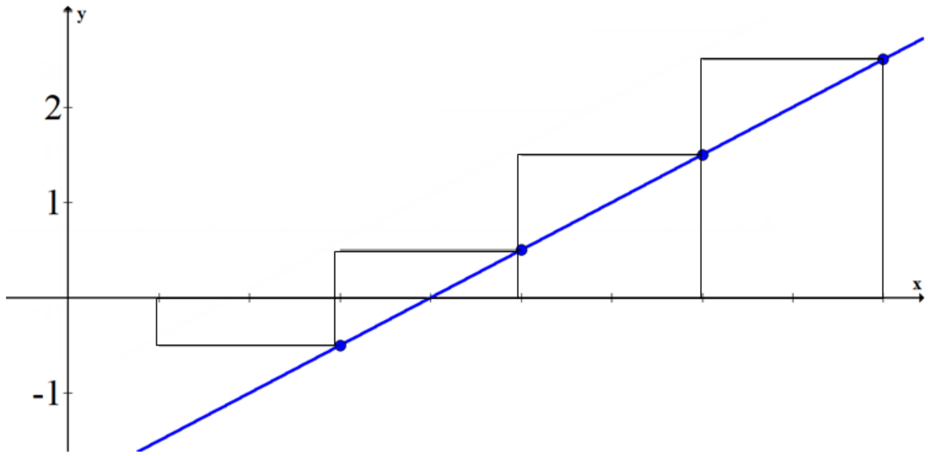
$\int f(x) = \lim_{n \to \infty} \sum_{i=1}^{n} (Area\ of\ box\ i)$

The symbol on the left is the calculus symbol of an integral. Using boxes to estimate the area under a curve is called a Riemann Sum.

Example A

Use four right handed boxes to approximate the area between 1 and 9 of the function $f(x) = \frac{1}{2}x - 2$.

Solution:



The area of the first box is 2 times the height of the function evaluated at 3:

$2 \cdot \left(\frac{1}{2} \cdot 3 - 2\right) = 3 - 4 = -1$

Because this box is under the x-axis, its area is negative.

The area for each of the rest of the boxes is 2 times the height of the function evaluated at 5, 7 and 9.

$$2 \cdot \left(\frac{1}{2} \cdot 5 - 2\right) = 5 - 4 = 1$$

$2 \cdot \left(\frac{1}{2} \cdot 7 - 2\right) = 7 - 4 = 3$

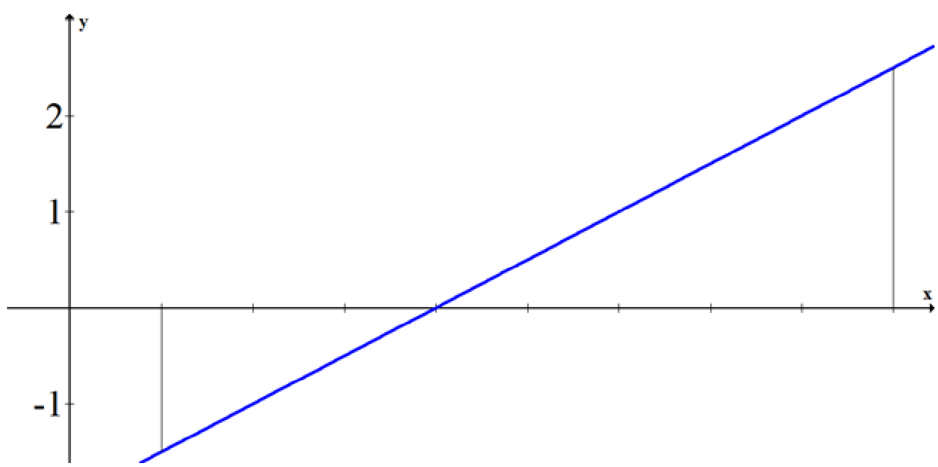
$2 \cdot \left(\frac{1}{2} \cdot 9 - 2\right) = 9 - 4 = 5$

The approximate sum of the total area under the curve is: $-1 + 1 + 3 + 5 = 8$ square units.

Example B

Evaluate the exact area under the curve in Example A using the area formula for a triangle.

Solution: Remember that the area below the x axis is negative while the area above the x axis is positive.



Negative Area: $\frac{1}{2} \cdot 3 \cdot 1.5 = \frac{9}{4}$

Positive Area: $\frac{1}{2} \cdot 5 \cdot 2.5 = \frac{25}{4}$

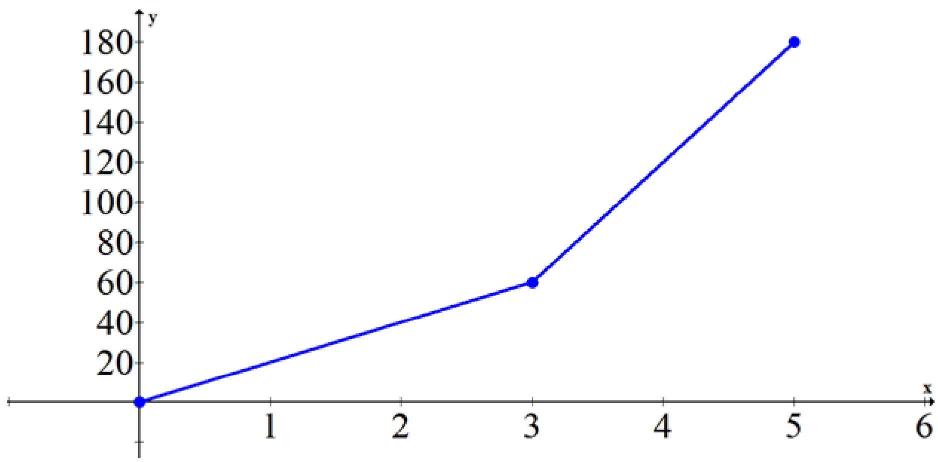
Area under the curve between 1 and 8: $\frac{25}{4} - \frac{9}{4} = \frac{16}{4} = 4$

It appears that approximations that are 2 units wide produce an area with significant error.

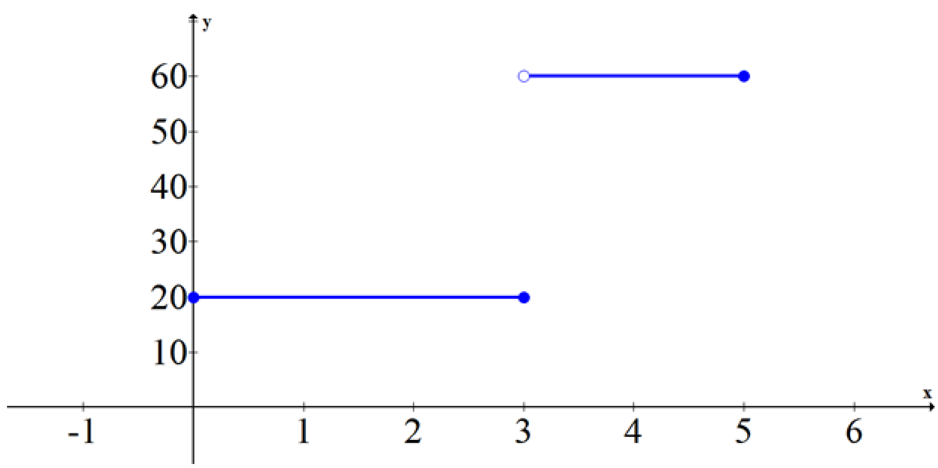
Example C

Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph. Use an area under the curve argument to connect the two graphs.

Solution: Distance vs. Time:



Rate vs. Time:



The slope of the first graph is 20 from 0 to 3 and then 60 from 3 to 5. The second graph is a graph of the slopes from the first graph. If you calculate the area of the second graph at the key points 0, 1, 2, 3, 4 and 5 you will see that they align perfectly with the points on the first graph.

TABLE 12.14:

x	***Area under curve from 0 to*** x
0	0
1	20
2	40
3	60
4	120
5	180

Concept Problem Revisited

You can use the area under the curve to find the total distance traveled in the first 8 seconds. Since the quadratic is a curve you must choose the number of subintervals you want to use and whether you want right or left handed boxes for estimating. Suppose you choose 8 left handed boxes of width one.

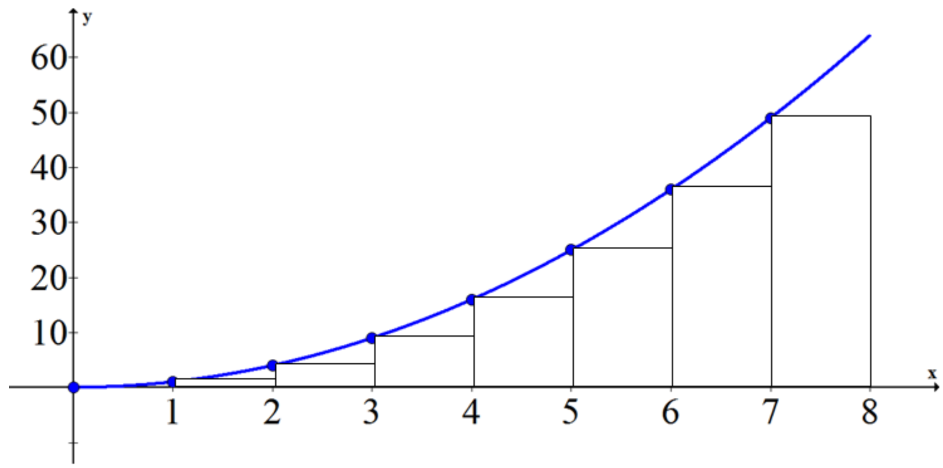


TABLE 12.15:

x	0	1	2	3	4	5	6	7
Area of box to the right	$1 \cdot 0$	$1 \cdot 1$	$1 \cdot 4$	$1 \cdot 9$	$1 \cdot 16$	$1 \cdot 25$	$1 \cdot 36$	$1 \cdot 49$

The approximate sum is $1+4+9+16+25+36+49 = 140$. This means that the car traveled approximately 140 meters in the first 8 seconds.

Vocabulary

Subintervals are created when an interval is broken into smaller equally sized intervals.

An ***integral*** is the limit of a sum as the number of summands increases to infinity.

Using boxes to estimate the area under a curve is called a ***Riemann Sum***.

A ***summand*** is one of many pieces being summed together.

Guided Practice

1. Approximate the area under the curve using eight subintervals and right endpoints.

$f(x) = 3x^2 - 1, \ -1 \le x \le 7$

2. Approximate the area under the curve using eight subintervals and left endpoints.

$f(x) = \frac{4}{x} + 3, \ 2 \le x \le 6$

3. Approximate the area under the curve using twenty subintervals and left endpoints.

$f(x) = x^x, \ 1 \le x \le 3$

Answers:

1. While a graph is helpful to visualize the problem and drawing each box can help give meaning to each summand, it is not always necessary. Since there are going to be 8 subintervals over the total interval of $-1 \leq x \leq 7$, each interval is going to have a width of 1. The height of each interval is going to be at the right hand endpoints of each subinterval (0, 1, 2, 3, 4, 5, 6, 7).

$$\sum height \cdot width = \sum_{i=0}^{7} (3i^2 - 1) \cdot 1 = 412$$

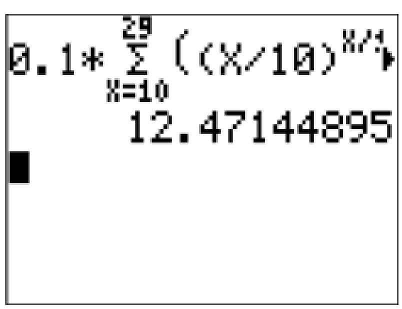
2. Each interval will be only $\frac{1}{2}$ wide which means that the left endpoints have x values of: 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5. Since the index of summation notation does not work with decimals, double each of these to get a good counting sequence: 4, 5, 6, 7, 8, 9, 10, 11 and remember to halve them in the argument of the summation.

$$\begin{aligned} x &= \frac{i}{2} \\ \sum height \cdot width &= \sum_{i=4}^{11} \left(\frac{4}{\left(\frac{i}{2}\right)} \right) \cdot \frac{1}{2} \approx 4.7462 \end{aligned}$$

3. When the number of subintervals gets large and the subintervals get extremely narrow it will be impossible to draw an accurate picture. This is why using summation notation and thinking through what the indices and the argument will be is incredibly important. With 20 subintervals between $[1,3]$, each interval will be 0.1 wide. Left endpoints means that the first box has a height of $f(1)$ and the second box has a height of $f(1.1)$.

$$\begin{aligned} \sum height \cdot width &= f(1) \cdot 0.1 + f(1.1) \cdot 0.1 + f(1.2) \cdot 0.1 + \cdots + f(2.9) \cdot 0.1 \\ &= 0.1(f(1) + f(1.1) + \cdots f(2.9)) \\ &= 0.1 \cdot \sum_{i=10}^{29} f\left(\frac{i}{10}\right) \\ &= 0.1 \cdot \sum_{i=10}^{29} \left(\frac{i}{10}\right)^{\left(\frac{i}{10}\right)} \\ &\approx 12.47144 \end{aligned}$$

Your calculator can compute summations when you go under the math menu.



Practice

1. Approximate the area under the curve using eight subintervals and right endpoints.

$f(x) = x^2 - x + 1, \ 0 \le x \le 8$

2. Approximate the area under the curve using eight subintervals and left endpoints.

$f(x) = x^2 - 2x + 1, \ -4 \le x \le 4$

3. Approximate the area under the curve using twenty subintervals and left endpoints.

$f(x) = \sqrt{x+3}, \ 0 \le x \le 4$

4. Approximate the area under the curve using 100 subintervals and left endpoints. Compare to your answer from #3.

$f(x) = \sqrt{x+3}, \ 0 \le x \le 4$

5. Approximate the area under the curve using eight subintervals and left endpoints.

$f(x) = \cos(x), \ 0 \le x \le 4$

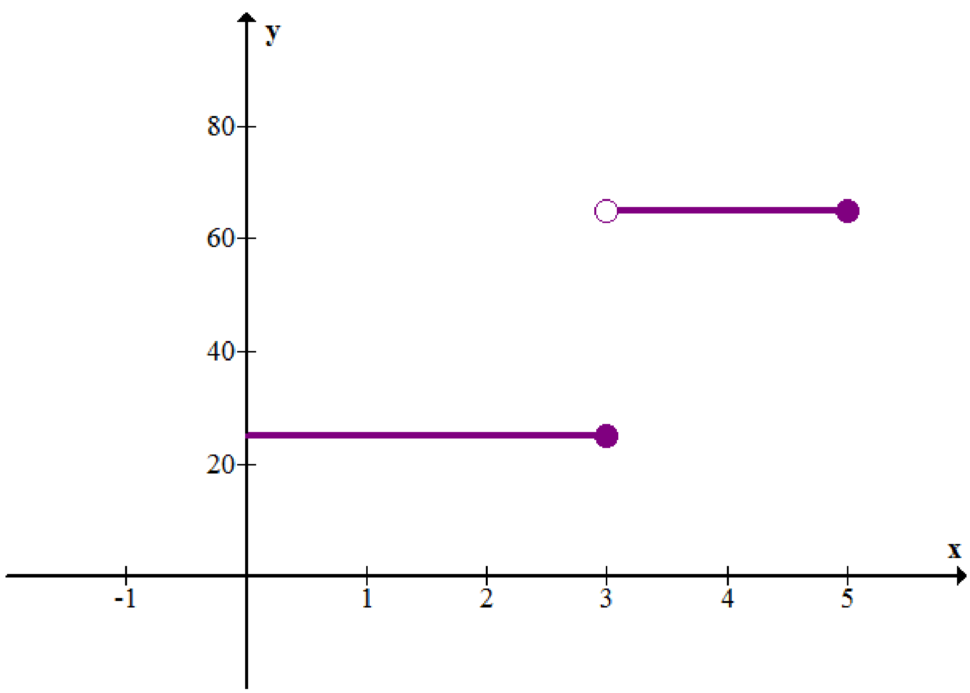
6. Approximate the area under the curve using twenty subintervals and left endpoints.

$f(x) = \cos(x), \ 0 \le x \le 4$

7. Approximate the area under the curve using 100 subintervals and left endpoints.

$f(x) = \cos(x), \ 0 \le x \le 4$

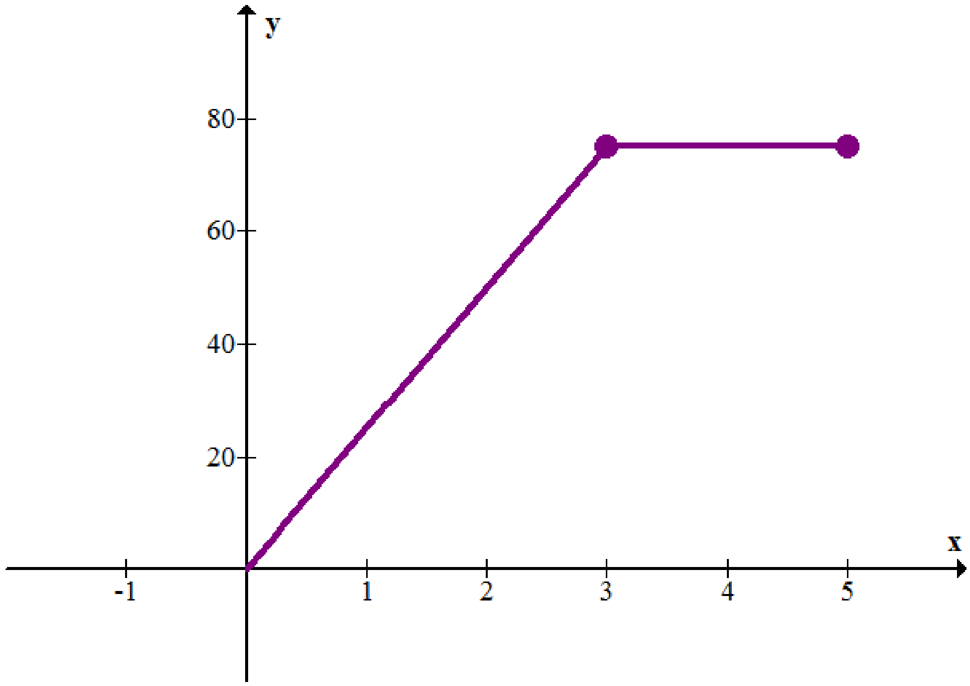
The following graph shows the rate (in miles per hour) vs. time (in hours) for a car.



8. Describe what is happening with the car.

9. How far did the car travel in 5 hours?

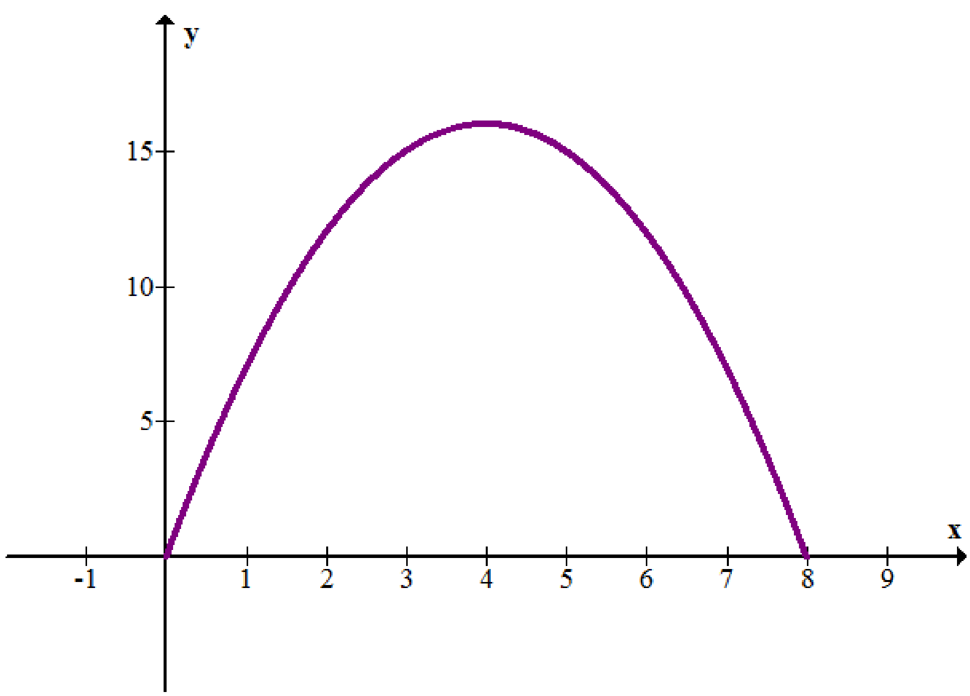
The following graph shows the rate (in feet per second) vs. time (in seconds) for a car.



10. Describe what is happening with the car. In particular, what is happening in the first 3 seconds?

11. How far did the car travel in 5 seconds?

The following graph shows the function $f(x) = -(x-4)^2 + 16$, which represents the rate (in feet per second) vs. time (in seconds) for a runner.



12. Describe what is happening with the runner. In particular, what happens after 4 seconds?

13. Use rectangles to approximate the total distance (in feet) that the runner traveled in the 8 seconds. Try to get as good an approximation as possible.

14. Explain how an integral is like the opposite of a derivative.

15. How do integrals relate to sums?

You learned that limits enable you to work infinitely close to a point without caring what happens at the point itself. Using this subtle approach, you began to answer two of the biggest questions in Calculus. 1) What is the slope of the tangent line at a point on a curve? 2) What is the area under a curve?

12.10 References

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