Solving Equations

SFDR Algebra I

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CHAPTER 1 Solving Equations

CHAPTER OUTLINE

1.1 Solving Two Step Equations

You will be able to solve a two-step equation for the value of an unknown variable. 8.8.C, A.5.A

Example A

$2x + 5 = 17$ — x is being multiplied by 2. Then 5 is being added.
Work backward performing the inverse operations:

$-5 \quad -5$ — Subtract 5 on both sides.

$\frac{2x}{2} = \frac{12}{2}$ — Since x is being multiplied by 2, divide both sides by 2.

$x = 6$

Example B

$-8 = \frac{y}{2} - 3$ — y is being divided by 2. Then 3 is being subtracted.
Work backward performing the inverse operations:

$+3 \quad +3$ — Add 3 to both sides.

$(2) -5 = \frac{y}{2}(2)$ — Since y is being divided by 2, multiply both sides by 2.

$-10 = y$

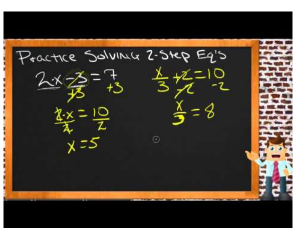
Example C

$\frac{2}{3}n - \frac{1}{2} = \frac{5}{2}$ — n is being multiplied by $\frac{2}{3}$. Then $\frac{1}{2}$ is being subtracted.
Work backward performing the inverse operations:

$+\frac{1}{2} + \frac{1}{2}$ — Add $\frac{1}{2}$ to both sides.

$\left(\frac{3}{2}\right)\frac{2}{3}n = 3\left(\frac{3}{2}\right)$ — Since n is being multiplied by $\frac{2}{3}$, multiply by the reciprocal.

$n = \frac{9}{2}$



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Vocabulary

Equation = a mathematical statement which shows that two expressions are equal

Inverse Operations = opposite operations that undo each other. Addition and subtraction are inverse operations. Multiplication and Division are inverse operations.

Independent Practice.

TABLE 1.1:

1. $5r+2=17$	2. $25=-2w-3$	3. $-7=4y+9$
4. $-3f+19=4$	5. $-22=-x-12$	6. $\frac{y}{3}-8=1$
7. $\frac{y}{3}$ - 8 = 1	8. $\frac{2}{3}h-\frac{1}{4}=\frac{1}{3}$	9. $\frac{-2}{5}=\frac{-1}{4}m+\frac{3}{5}$
10. $12.5=2g-3.5$	11. $6.3=2x-4.5$	12. $-6=\frac{y}{5}=4$
13. $\frac{7}{9}=2n+\frac{1}{9}$	14. $-9y-4.2=13.8$	15. $-1=\frac{b}{4}-7$
16. $0.6x+1.5=4$		

PLIX (Play Learn Interact eXplore)

T-Shirt Equation

Two Step Equations

1.2 Solving Equations by Combining Like Terms

You will be able to solve an equation by combining like terms.

8.8.C, A.5.A

Example A

$2x + 6 + 3x = 21$ Since 2x and 3x are like terms, they can be combined.

$5x + 6 = 21$ x is being multiplied by 5. Then 6 is being added. Work backward. Subtract 6 from both sides.

$\quad -6 \quad -6$

$\frac{5x}{5} = \frac{15}{5}$ Since x is being multiplied by 5, divide both sides by 5.

$x = 3$

FIGURE 1.1

Example B

$-4n + 2 - 6n = 82$ Since -4n and -6n are like terms, they can be combined.

$-10n + 2 = 82$ n is being multiplied by -10. Then 2 is being added. Work backward. Subtract 2 from both sides.

$\quad -2 \quad -2$

$\frac{-10n}{-10} = \frac{80}{-10}$ Since n is being multiplied by -10, divide both sides by -10.

$n = -8$

FIGURE 1.2

Example C

$$\frac{m}{3} - 8 + 7 = 4$$

Since -8 and 7 are like terms, they can be combined.

$$\frac{m}{3} - 1 = 21$$

m is being divided by 3. Then 1 is being subtracted. Work backward.

$$\begin{array}{r} +1 \quad +1 \end{array}$$

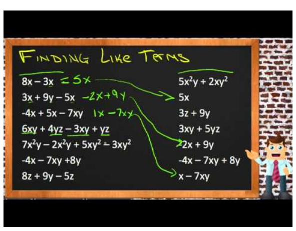
Add 1 to both sides.

$$(3)\frac{m}{3} = 22(3)$$

Since m is being divided by 3, multiply both sides by 3.

$$m = 66$$

FIGURE 1.3



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Vocabulary

Like Terms = terms whose variables and exponents are the same. In other words, terms that are "like" each other.

Combine Like Terms = a mathematical process in which like terms are added or subtracted in order to simplify the expression or equation.

Independent Practice

Solve each equation.

Table 1.2:

1. $6a+5a=-11$	2. $-6n-2n=16$
3. $-3+6-3x=-18$	4. $x+11+8x=29$
5. $0=-5n-2n$	6. $-10=-14v+14v$
7. $43=8.3m+13.2m$	8. $a-2+3=-2$
9. $4x+6+3=17$	10. $-10p+9p=12$
11. $5x+8-5x=8$	12. $\frac{3}{4}x-1+\frac{1}{2}x=11$
13. $\frac{6}{7}y+\frac{2}{5}+\frac{3}{5}+\frac{1}{7}y=10$	14. $7q+4-3q-7+5q=15$
15. $6.25x-\frac{1}{4}x+2.25+7.3x=0$	16. $0.5y+5-5y+7+4.5y=0.25$

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Multi-Step Equations with Like Terms: Shipments

You may also like...

Solving Combining Like Terms

- Equation
- Expression
- Variable
- Like terms
- Inverse operations

1.3 Solving Equations Using the Distributive Property

You will solve an equation using the distributive property.

Example A

$4(y-2)=7$ Distribute 4 to the expression in parenthesis.

$4y-8=7$ y is being multiplied by 4. Then 8 is being subtracted. Work backward.

$+8 \quad +8$ Add 8 to both sides.

$\frac{4y}{4}=\frac{15}{4}$ Since y is being multiplied by 4, divide both sides by 4.

$y = 3.75 \text{ or } \frac{15}{4}$

FIGURE 1.4

Example B

$11=-(c+9)$ Distribute -1 to the expression in parenthesis.

$11=-c-9$ c is being multiplied by -1. Then 9 is being subtracted. Work backward.

$+9 \quad +9$ Add 9 to both sides.

$\frac{20}{-1}=\frac{-c}{-1}$ Since c is being multiplied by -1, divided both sides by -1.

$-20=c$

FIGURE 1.5

Example C

$-3(x - 1) + 8x = -9$ Distribute -3 to the expression in parenthesis.

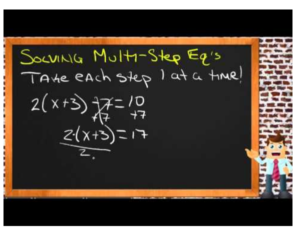
$-3x + 1 + 8x = -9$ Since -3x and 8x are like terms, they can be combined.

$5x + 1 = -9$ x is being multiplied by 5. Then 1 is being added. Work backward.
$\quad -1 \quad -1$ Subtract 1 from both sides.

$\frac{5x}{5} = \frac{-10}{5}$ Since x is being multiplied by 5, divide both sides by 5.

$x = -2$

FIGURE 1.6



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Vocabulary

Distributive Property = the mathematical law which states that a(b + c) = ab + ac.

Independent Practice

Solve each equation.

TABLE 1.3:

1. $5(3x+10)=215$	2. $6(5y-2)=18$
3. $68=4(3b-1)$	4. $-7(x-3.5)=0$
5. $-185=5(3x-4)$	6. $-\frac{1}{3}(x+6)=21$
7. $22=2(4h-9)$	8. $-6(2w+4)=-84$
9. $5(-10-6f)=-290$	10. $42=7(6-7x)$
11. $4(y-5)=-15$	12. $\frac{2}{3}(3x-12)=10$
13. $5(x+6)-3=37$	14. $5(1+4m)-2m=-13$
15. $-(n-8)+10=-2$	16. $8=8v-4(v+8)$
17. $10(1+3b)+15b=-20$	18. $-5-8(1+7n)=-8$
19. $\frac{2}{3}(6x+9)-x-2=-17$	20. $-7.2+2(2.5x-4)=12$

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Multi-Step Equations

You May Also Like...

Multi-Step Equations

- Equation
- Expression
- Solve
- Simplify
- Inverse operations
- Variable
- Like Terms

1.4 Solving Equations with a Variable on Both Sides

You will use distribution to solve equations with variables on both sides.

Example A

$9b + 1 = 2b + 29$

$-2b \qquad -2b$ To collect the variables on one side, subtract 2b from both sides.

$7b + 1 = 29$ b is being multiplied by 7. Then 1 is being added. Work backward.

$-1 \quad -1$ Subtract 1 from both sides.

$7b = 28$ Since b is being multiplied by 7, divide both sides by 7.

FIGURE 1.8

Example B

$-2(n + 3) = 4n - 3$ Distribute -2 to the expression in parenthesis.

$-2n - 6 = 4n - 3$

$-4n \qquad -4n$ To collect the variables on one side, subtract 4n from both sides.

$-6n - 6 = -3$ n is being multiplied by -6. Then 6 is being subtracted. Work backward.

$+6 \quad +6$ Add 6 to both sides.

$\frac{-6n}{-6} = \frac{3}{-6}$ Since n is being multiplied by -6, divide both sides by -6.

$n = -.5 \text{ or } -\frac{1}{2}$

FIGURE 1.8

Example C

$$-2x + 8 - 3x = -5x - 2$$

$$-2x + 8 - 3x = -5x - 2$$

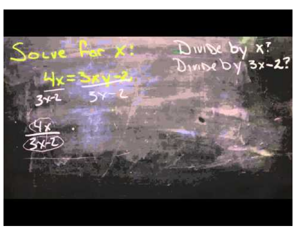
Since -2x and -3x are like terms, they can be combined.

$$\begin{array}{rcl} -5x + 8 & = & -5x - 2 \\ +5x & & +5x \\ \hline 8 & = & -2 \end{array}$$

To collect the variables on one side, add 5x to both sides.

Since this is a false statement (8 is not equal to -2), the answer is NO SOLUTION. When the answer is a true statement (for example, 3 = 3), the answer is INFINITE SOLUTIONS.

FIGURE 1.9



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Independent Practice.

Solve each equation.

TABLE 1.4:

1. $5x-17=4x+36$	2. $36+19c=24c+6$
3. $-3y+8=2y-2$	4. $4+6p=-8p+32$
5. $-2a+6=30-5a$	6. $6x-7=4x+1$
7. $6y-8=1+9y$	8. $-14g-8=-10g+40$
9. $5x+6=5x-10$	10. $6p+2=-3p-1$
11. $-3x+9=9-3x$	12. $10x=2x-16$
13. $6y=-1+6y$	14. $-5m+2+4m=-2m+11$
15. $2(3p+5)+p=13-2p+15$	16. $-3y-10=4(y+2)+2y$
17. $2(3b-4)=8b-11$	18. $-6(2x+1)=-3x+7-9x$
19. $8s-10=27-(3s-7)$	20. $3b+12=3(b-6)+4$

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Rubber Ducky Math

You May Also Like...

Equations with Variables on Both Sides

- Inverse Operations
- Solve
- Variable
- Like terms
- Distribution Property

1.5 Solving Rational Equations

You will learn how to solve an equation that involves fractions.

Example A

$$\frac{2}{n} = \frac{5}{n-3}$$

Cross multiply to eliminate the denominators. (Rewrite in distribution form)

$$5(n) = 2(n-3)$$

Distribute the 5 and the 2 to the expressions in parenthesis.

$$5n = 2n - 6$$
$$-2n \quad -2n$$

To collect variables on one side, subtract 2n from both sides.

$$\frac{3n}{3} = \frac{-6}{3}$$

Since n is being multiplied by 3, divide both sides by 3.

$$n = -2$$

FIGURE 1.10

Example B

$$\frac{12}{y-2} = 8$$

Cross multiply to eliminate the denominators. (Rewrite in distribution form)

$$8(y-2) = 12(1)$$

Distribute the 8 and the 12 to the expressions in parenthesis.

$$8y - 16 = 12$$
$$+16 \quad +16$$

y is being multiplied by 8. Then 16 is being subtracted. Work backward. Add 16 to both sides.

$$\frac{7y}{7} = \frac{28}{7}$$

Since y is being multiplied by 7, divide both sides by 7.

$$y = 4$$

FIGURE 1.11

Example C

$$\frac{4}{d+3} = \frac{6}{d-1}$$

Cross multiply to eliminate the denominators.
(Rewrite in distribution form)

$$6(d+3) = 4(d-1)$$

Distribute the 6 and the 4 to the expressions in parenthesis.

$$\begin{array}{rcl} 6d + 18 & = & 4d - 4 \\ -4d & & -4d \\ \hline \end{array}$$

To collect the variables on one side, subtract 4d from both sides.

$$\begin{array}{rcl} 2d + 18 & = & -4 \\ -18 & & -18 \\ \hline \end{array}$$

d is being multiplied by 2. Then 18 is being added. Work backward.
Subtract 18 from both sides.

$$\frac{2d}{2} = \frac{-22}{2}$$

Since d is being multiplied by 2, divide both sides by 2.

$$d = -11$$

FIGURE 1.12

Vocabulary

Rational Equation = an equation in which one or more of the terms is a fractional one.

Independent Practice

Solve the following rational equations using cross products.

TABLE 1.5:

1. $\frac{3}{c} = \frac{4}{c-3}$	2. $\frac{1}{x-1} = 3$
3. $\frac{2}{r} = \frac{2}{2-r}$	4. $\frac{5}{x+3} = \frac{2}{x}$
5. $\frac{-4}{x-1} = \frac{2}{x}$	6. $\frac{3}{c+2} = \frac{2}{c+2}$
7. $4 = \frac{8}{x+2}$	8. $\frac{2}{j+4} = \frac{4}{j-1}$
9. $\frac{5}{y-3} = \frac{-8}{y-4}$	10. $\frac{-2}{-b+5} = \frac{1}{b-2}$
11. $\frac{2}{-x-5} = \frac{3}{-2x-3}$	12. $\frac{6}{x+1} = \frac{-3}{3-x}$
13. $\frac{-2}{x+5} = \frac{-1}{2-x}$	14. $\frac{-4}{1+x} = \frac{-3}{5-3x}$

You May Also Like...

Rational Equations Using Proportions

- Rational Numbers
- Cross Products
- Distributive Property

1.6 Solving Literal Equations

You will learn how to solve for any specified variable in any given formula.

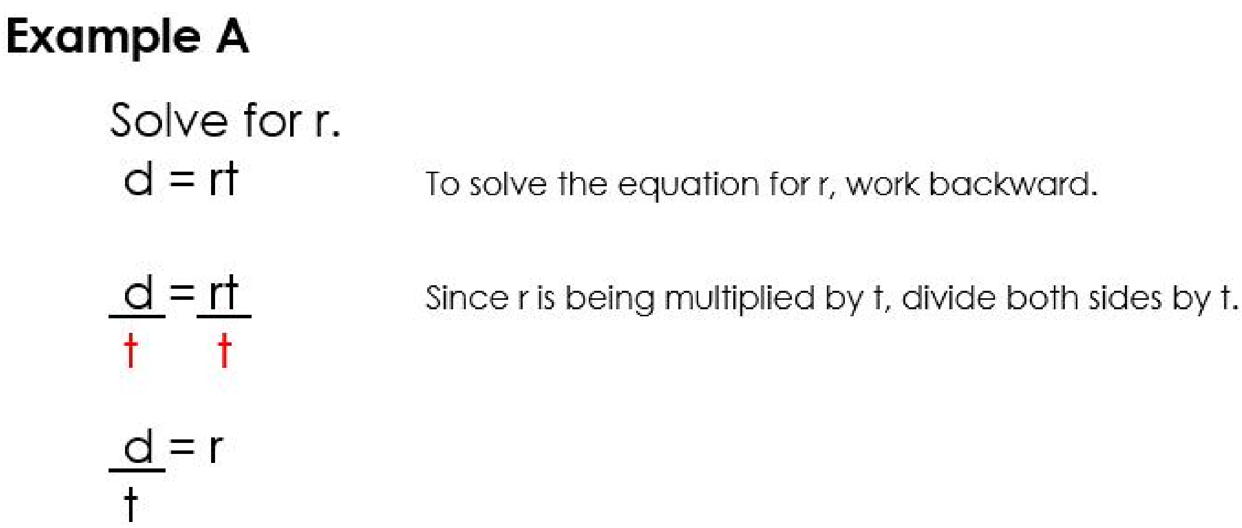

FIGURE 1.13

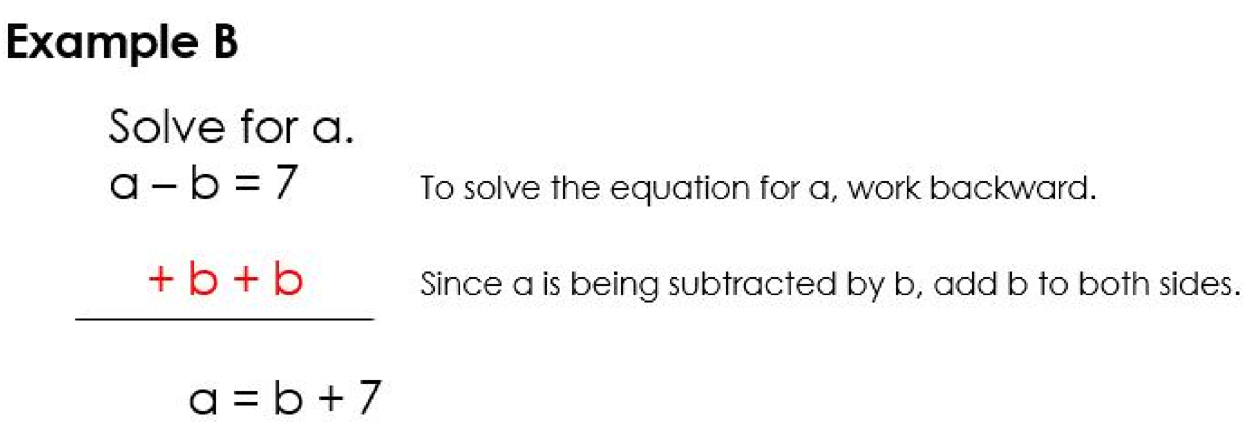

FIGURE 1.14

Example C

Solve for h.

$A = \frac{bh}{2}$ To solve for h, cross multiply to eliminate the denominator.

$1(bh) = 2(A)$ Distribute the 1 and the 2 to the expressions in parenthesis.

$\frac{bh}{b} = \frac{2A}{b}$ Since h is being multiplied by b, divide both sides by b.

$h = \frac{2A}{b}$

FIGURE 1.15

Vocabulary

Literal Equation = an equation made up of mostly letters or variables.

Independent Practice

Solve the following equations.

TABLE 1.6:

1. $5 = x + y$	Solve for x	2. $w = x + 5$	Solve for x
3. $a + b = 3$	Solve for a	4. $a + b = 3$	Solve for b
5. $p + t = q$	Solve for p	6. $a^2 + b^2 = c^2$	Solve for c^2
7. $A = lw$	Solve for w	8. $A = \prod r^2$	Solve for $\prod$
9. $d = rt$	Solve for t	10. $r = \frac{m}{2p}$	Solve for m
11. $\frac{c}{d} = \prod$	Solve for c	12. $\frac{c}{d} = \prod$	Solve for d

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- Formula
- Multi-variable
- Literal equations

1.7 Chapter 1 Review

Mixed Review

Solve the following equations for the unknown variable.

TABLE 1.7:

1. $6x-1.3=3.2$	2. $4(x+3)=1$
3. $\frac{3}{5}x+\frac{5}{2}=\frac{2}{3}$	4. $10y+5=10$
5. $1.3x-0.7x=12$	6. $-10a-2(a+5)=14$
7. $3(x-1)-2(x+3)=0$	8. $3x+6=x+15$
9. $A=\frac{(b1+b2)}{2}h$ Solve for h	10. $\frac{x-3}{5}=7$
11. $42x+12=5x-3$	12. $\frac{3}{x+1}=\frac{2}{x-2}$
13. $\frac{3}{x+1}=\frac{2}{x}$	14. C=∏d Solve for d
15. $2.3x+2(0.75x-3.5)=7.5$	16. $9(x-2)-3x=3$
17. $\frac{5(q-7)}{12}=\frac{2}{3}$	18. $12x-16-14x-21=3$
19. $0.1(3.2+2x)+0.5(3-0.2x)=0$	20. $\frac{-4}{x-4}=\frac{3}{x+1}$
21. $3(x+3)-2(x-1)=0$	22. $p=2l+2w$ Solve for w
23. $2\left(a-\frac{1}{3}\right)=\frac{2}{5}\left(a+\frac{2}{3}\right)$	24. $2\left(5a-\frac{1}{3}\right)=\frac{2}{7}$
25. $\frac{2}{7}\left(t+\frac{2}{3}\right)=\frac{1}{5}\left(t-\frac{2}{3}\right)$	26. $P=4s$ Solve for s

Ch. 2: Relations and Functions

SFDR Algebra I

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CHAPTER 1 Relations and Functions

CHAPTER OUTLINE

1.1 Identifying Attributes of Relations and Functions

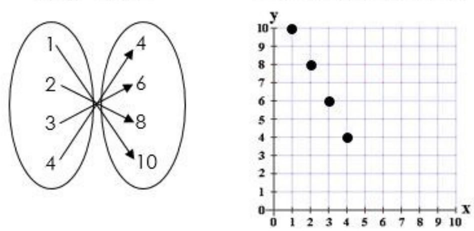
You will be able to identify attributes of a function.

Example A

Relations can be represented in various ways.

TABLE **MAPPING** **DISCRETE GRAPH**

x	y
1	10
2	8
3	6
4	4



Domain {x|x = 1, 2, 3, 4}

Range {y|y = 4, 6, 8, 10}

This relation is a function because the x values do not repeat

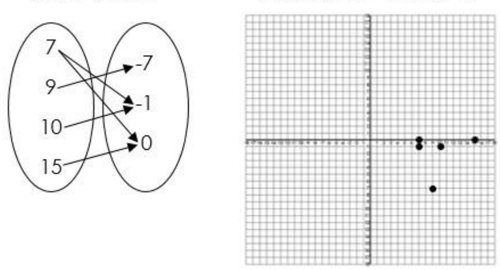
FIGURE 1.1

Example B

Relations can be represented in various ways.

TABLE **MAPPING** **DISCRETE GRAPH**

x	y
7	-1
9	-7
10	-1
7	0
15	0



Domain {x|x = 7, 9, 10, 15}

Range {y|y = -7, -1, 0}

This relation is not a function because the x value of 7 is repeating.

FIGURE 1.2

Vocabulary

Relation = a collection or set of ordered pairs.

Function = a special relationship where each input has a single output and is often written as f(x) where x is the input value.

Domain = input values, x values, independent values

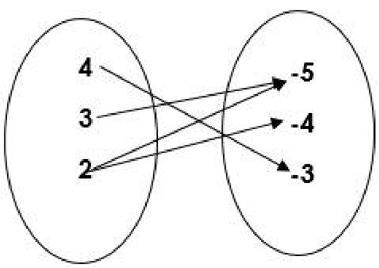
Range = output values, y values, dependent values

Independent Practice

Given the relation, identify the domain and range and determine if the relation is a function.

1. **{(8, 2), (-4, 1), (-6, 2), (1, 9)}**

2.

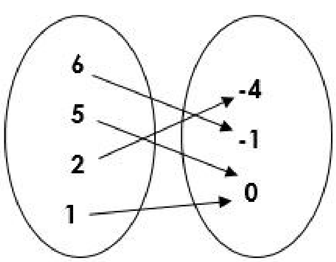


3.

x	-2	-1	0	1	2
y	1	1	1	1	1

4. **{(1. 3), (1, 0), (1, -2), (1, 8)}**

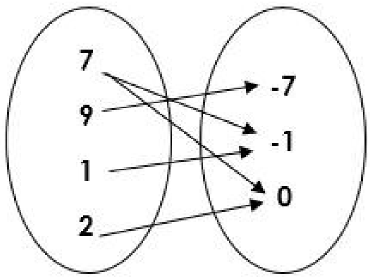
5.



6. Field Trip

x	f(x)
75	2
68	2
125	3

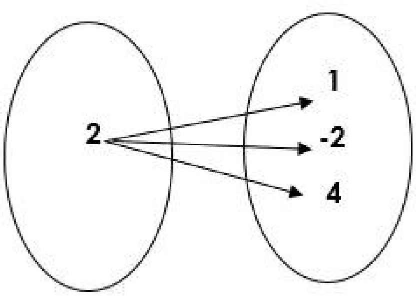
7.



8.

x	y
3	-2
5	-1
4	0
3	1

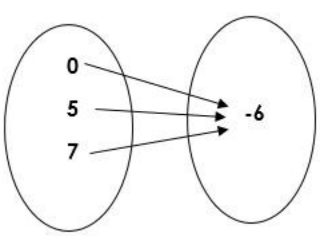
9.



10.

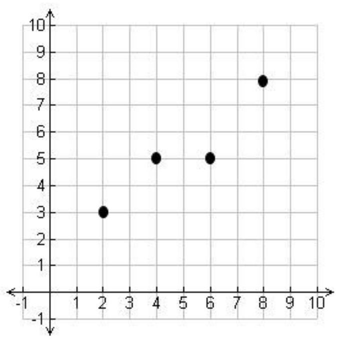
x	3	5	7
f (x)	1	15	30

11.



12. **{(2, 4), (3, 7), (6, 2), (5, 8), (6, 10)}**

13.



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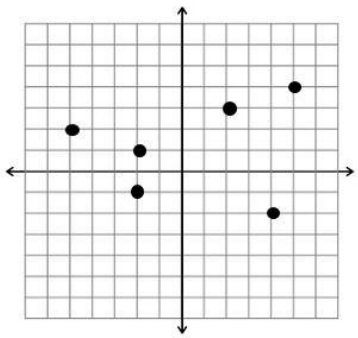


15.

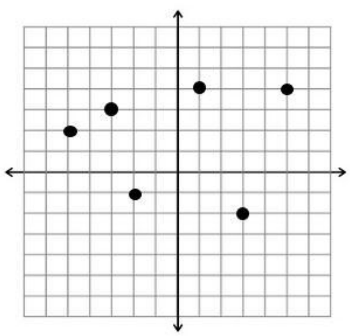


FIGURE 1.14

- Function
- Function notation
- Relation
- Mapping
- Table
- Graph

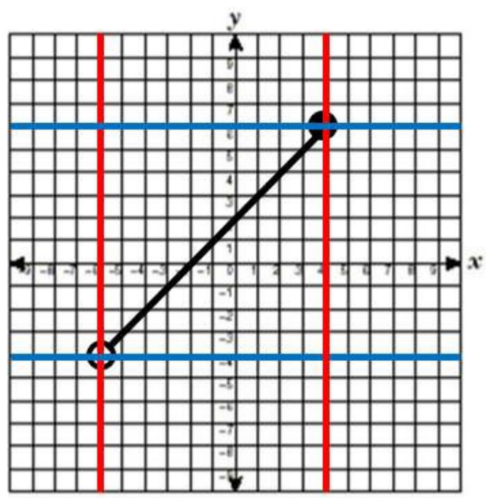
1.2 Domain and Range from Continuous Graphs

You will be able to identify the reasonable domain and the range of real-world situations and represent them with a graph or an inequality.

Domain and Range can be found from a continuous graph by creating a hashtag (#) around the graph to identify its boundaries.

Open circle or Dashed Line - Use <
Closed circle or Solid Line - Use ≤

Example A



To find the domain, look at the width of the graph (left to right). Where does the hashtag touch the x axis? Notice that the hashtag touches the x axis at -6 and 4.

Since the left boundary has an open circle, use <.

Since the right boundary has a closed circle, use ≤.

Domain: $\{x \mid -6 < x \leq 4\}$

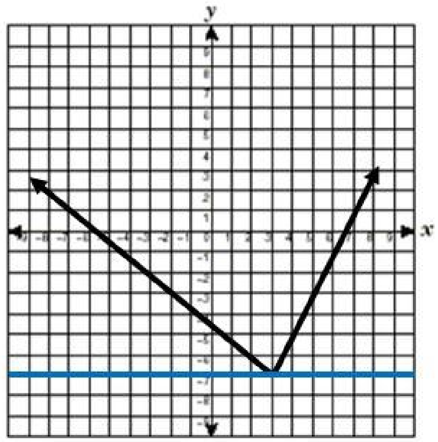
To find the range, look at the height of the graph (bottom to top). Where does the hashtag touch the y axis? Notice that the hashtag touches the y axis at -4 and 6.

Since the bottom boundary has an open circle, use <.

Since the top boundary has a closed circle, use ≤.

Range: $\{y \mid -4 < y \leq 6\}$

Example B



The domain, the width of the graph (left to right), could not be hashtagged because the arrows represent that the graph continues infinitely.

Since the left and right sides of the graph have no boundary, the answer is ALL REAL NUMBERS.

Domain: {x | all real numbers}

To find the range, look at the height of the graph (bottom to top). Where does the hashtag touch the y axis? Notice that the hashtag touches the y axis at -7. Also notice that the top of the graph could not be hashtagged because the arrows represent that the graph continues infinitely.

Since the bottom boundary has a solid line, use ≤.

Since the top could not be hashtagged, there is only one boundary.

Range: $\{y \mid -7 \leq y\}$ or $\{y \mid y \geq -7\}$

Vocabulary

Continuous Graph = a graph which allows x-values to be ANY points including fractions and decimals and is not restricted to defined separate values.

Independent Practice

Identify if the following graphs are functions or not, also determine their domain and range.

Table 1.1:

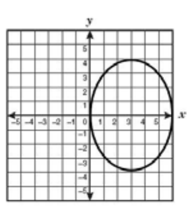
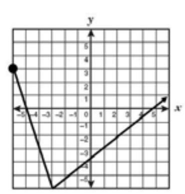
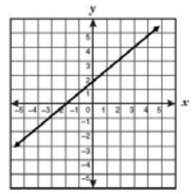
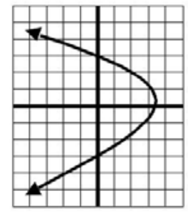
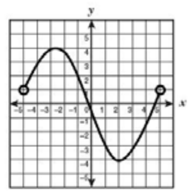
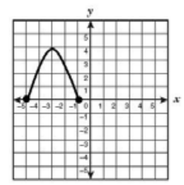
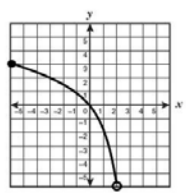
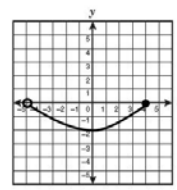
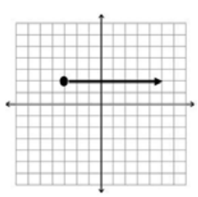
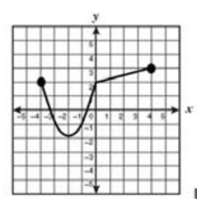
1. 	2. 

TABLE 1.1: (continued)

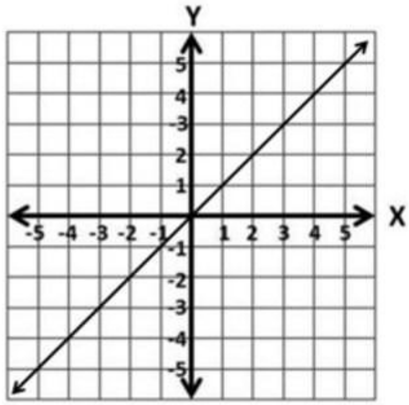
3. 	4. 
5. 	6. 
7. 	8. 
9. 	10. 

- Domain
- Range
- Continuous
- Discrete
- Open circle
- Closed circle
- Set notation
- Real numbers

1.3 Parent Functions

You will be able to determine the effect on the graphs of the linear and quadratic parent functions when specific values are changed.

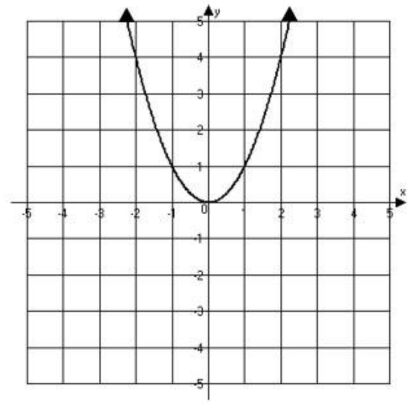
Linear Parent Function: y = x



Characteristics:

- a line
- intersects the origin
- has a positive slope
- the coefficient of x is 1

Quadratic Parent Function: $y = x^2$



Characteristics:

- parabola (u-shape)
- intersects the origin
- opens upward
- the coefficient of x^2 is 1

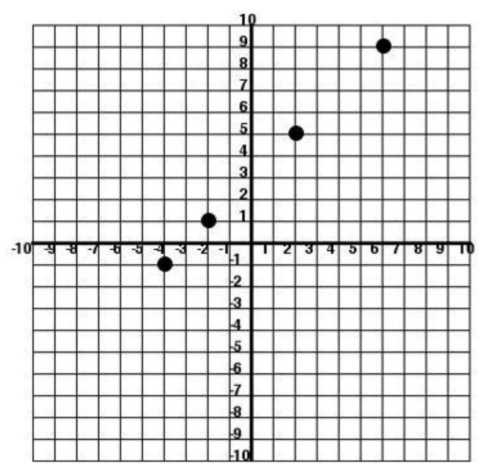
To identify the parent function of a table, set of ordered pairs or a mapping diagram, plot the points to determine if the graph is a line or a parabola.

Example A

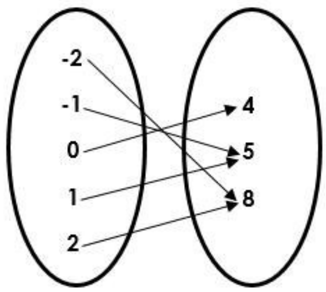
{(-4, -1), (-2, 1), (2, 5), (6, 9)}

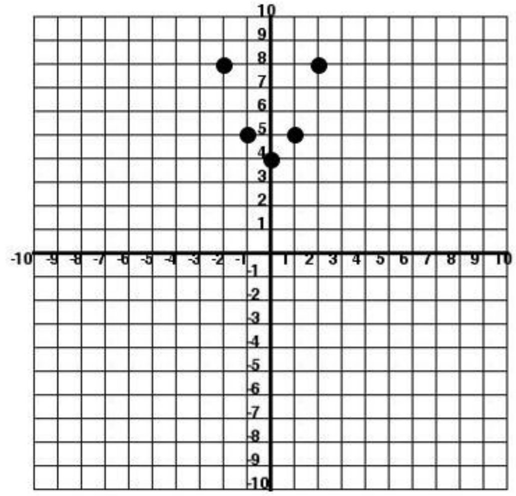
Linear Function

Parent Function y = x



Example B





Quadratic Function

Parent Function $y = x^2$

Vocabulary

Parent Function = the simplest function of a family of functions.

Independent Practice

Graph the points, then name and describe the parent function of the tables below.

TABLE 1.2:

1.

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

2.

x	y
-3	-3
-2	-2
-1	-1
0	0
1	1
2	2
3	3

3.

x	y
-2	4
-1	1
0	0
1	1
2	4

4.

x	y
-2	-8
-1	-1
0	0
1	1
2	8

5.

x	y
-2	7
-1	5
0	3
1	1
2	-1
3	-3
4	-5

6.

x	y
-4	6
-2	0
0	-2
2	0
4	6

Name the parent function of the table and set of ordered pairs shown below:

7.

x	-2	-1	0	1	2
y	1	-2	-3	-2	1

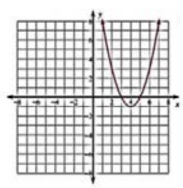
8.

{(-1. -5), (0, -2), (1, 1), (2, 4), (3, 7)}

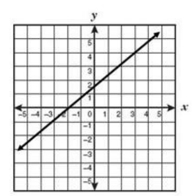
Name the parent function of the graphs shown below:

TABLE 1.3:

9.



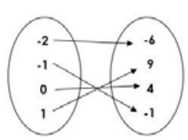
10.



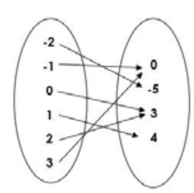
Name the parent function of the given mappings shown below:

TABLE 1.4:

11.



12.



Name the parent function:

13. Hi, I'm the parent of every graph that is a line.
14. Hi, all my kids have a graph that looks like a U.
15. Hi, my graph consist of the points (-1,-1), (0,0) and (1,1).
16. Hi, my graph contains the points (-2,4), (0,0) and (2,4).
17. Hi, I have a domain and range of all real numbers.
18. Hi, I have a domain of all real numbers and a range of all real numbers greater or equal to 0.

- Parent Function
- Linear
- Quadratic
- Parabola
- Origin

1.4 Evaluating Functions

You will use substitution to evaluate an equation.

Example A

If y = 3x – 4, what is the value of x when y = 11?

Since y = 11, substitute 11 for y, then solve.

$$11 = 3x - 4$$

x is being multiplied by 3. Then 4 is being subtracted. Work backward.

$$+4 \qquad +4$$

Add 4 to both sides.

$$\frac{15}{3} = \frac{3x}{3}$$

Since x is being multiplied by 3, divide both sides by 3.

$$5 = x$$

FIGURE 1.24

Example B

If f(x) = -3x + 1, what is the value of f(x) when x = 2?

Since x = 2, substitute 2 for x, then solve.

$$f(x) = -3(2) + 2$$

$$f(x) = -4$$

FIGURE 1.25

Example C

If (2, 7) is a point on the line whose equation is $y = x + \frac{n}{3}$, determine the value of n.

The ordered pair (2, 7), states that the value of x is 2 and the value of y is 7. Substitute 2 for x and 7 for y, then solve.

$$7 = 2 + \frac{n}{3}$$

n is being divided by 3. Then 2 is being added. Work backward.

$$-2 \quad -2$$

Subtract 2 from both sides.

$$(3)5 = \frac{n}{3}(3)$$

Since n is being divided by 3, multiply both sides by 3.

$$15 = n$$

FIGURE 1.26

Independent Practice

1. If y = 5x + 7, what is the value of y when x = 2?
2. If f(x) = -4x - 12 + x, what is the value of f(x) when x = 3?
3. If y = 6 - 2x, what is the value of x when y = 10?
4. If f(t) = 3(-2t + 7), what is the value of t when f(t) = 39?
5. If f(s) = 2(3s - 4) - 2s + 1, what is the value of f(s) when s = 5?
6. If y = -(5x + 1) + 8x - 4, what is the value of x when y = 22?
7. If (3, 8) is a point on the line whose equation is y = 2x + n, determine the value of n.
8. If (2, 7) is a point on the line whose equation is $y = \frac{w}{5} + x$ determine the value of w.
9. If (4, -10) is a point on the line whose equation is y = h + 7x, determine the value of h.
10. If (-5, 2) is a point on the line whose equation is $y = x - \frac{m}{8}$, determine the value of m.
11. If (-1, 9) is a point on the line whose equation is f(x) = -2x + p, determine the value of p.
12. If (3, 5) is a point on the line whose equation is f(x) = 5x - c, determine the value of c.

- Substitution
- Evaluate
- Value

1.5 Arithmetic Sequences

You will be able to identify and use the pattern of a sequence to find the nth term.

Example A

Determine whether the sequence is an arithmetic sequence.

1, 3, 9, 17

+2 +6 +8

Find the difference between each term.

The difference between the terms is not the same

This sequence is not an arithmetic sequence.

If the difference between the terms is the same, then the sequence is an arithmetic sequence.

FIGURE 1.27

Example B

Determine whether the sequence is an arithmetic sequence. If so, find the next three terms.

5, 2, -1, -4

–3 –3 –3

Find difference between each term.

The difference between the terms is the same.

This sequence is an arithmetic sequence.

To find the next 3 terms, continue the pattern.

-4 – 3 = -7
-7 – 3 = -10
-10 – 3 = -13

-7, -10, and -13 are the next three terms in the arithmetic sequence.

FIGURE 1.28

Example C

Find the 14^{th} term in the following arithmetic sequence.

$$2.6, \quad 3.1, \quad 3.6, \quad 4.1$$

$$+0.5 \quad +0.5 \quad +0.5$$

Find the common difference between each term.

The common difference is +0.5

$$a_n = a_1 + (n - 1)d$$

Write the formula to find the nth term.

n represents the term you are looking for (14)

a_1 represents the first term (2.6)

d represents the common difference (0.5)

Substitute the values and solve.

$$a_{14} = 2.6 + (14 - 1)0.5$$

$$a_{14} = 2.6 + (13)0.5$$

$$a^{14} = 9.1$$

The 14^{th} term in the arithmetic sequence is 9.1

FIGURE 1.29

Vocabulary

Arithmetic Sequence = A pattern in which each term is equal to the previous term, plus or minus a constant. The constant is called the common difference (d).

Independent Practice

Determine if the following sequences are arithmetic sequences. Explain.

Table 1.5:

1. 7, 10, 13, 16	2. -19, -15, -11, -7
3. 3, 7, 9, 12	4. -4, -3, 0, 3, 4

Determine if the following sequences are arithmetic. If so, find the next three terms.

Table 1.6:

5. -4, -2, 0, 2	6. 1.5, 2, 2.3, 3.5
7. 8.7, 10.2, 11.7, 13.2	8. 13.25, 13.5, 13.75, 14

Find the rule for each one of the following arithmetic sequences.

Table 1.7:

9. 3, 6, 9, 12	10. 39, 32, 25, 18
11. -20, -13, -6, 1	12. -34, -64, -94, -124

Find the indicated term for each of the following arithmetic sequences.

Table 1.8:

13. -4, -1.5, 1, 3.5	Find the 25^{th} term.
14. -3, 0, 3, 6	Find the 56^{th} term.
15. 18, 15, 12, 9	Find the 13^{th} term.
16. 23, 28, 33, 38	Find the 100^{th} term.

Table 1.9:

17. $a_1 = 14$	$d = 3$	Find the 24^{th} term.
18. $a_1 = 12$	$d = -2$	Find the 30^{th} term.
19. $a_1 = -6$	$d = -8$	Find the 11^{th} term.
20. $a_1 = 80$	$d = 13$	Find the 20^{th} term.

- Sequence
- Arithmetic sequence
- Geometric sequence

1.6 Chapter 2 Review

Chapter 2 Review

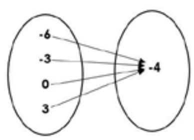
Give the domain and range and determine if the relation is a function.

1. {(3, 2), (4, 6), (-5, 7), (-4, 8)}
2. {(1, 12), (1, 14), (1, 16), (1, 18)}
3. {(-2, -12), (0, 6), (1, -3), (4, 6)}

Give the relation, domain, and range. Explain if the relation is a function.

TABLE 1.10:

4.



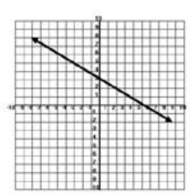
5.

x	-3	0	6	9	12
y	-4	-2	2	4	6

Determine if the following graphs are functions. Give the domain and range and determine if the graphs are discrete or continuous.

TABLE 1.11:

6.



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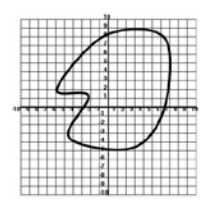

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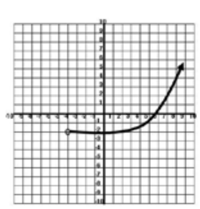

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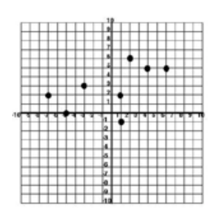
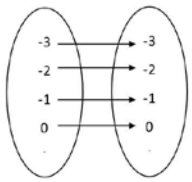


Name the parent function of the following functions

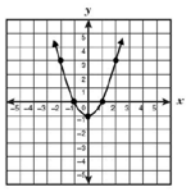
TABLE 1.12:

10.

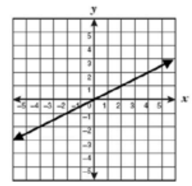


11. $y = x^2 + 3x - 2$
12. $y = x^2 - 3$
13. $y = \frac{1}{5}x + 4$
14. $y = 2x + 5$
15. $y = 2x^2$

16.



17.



18.

x	y
-4	2
-2	2
0	10
2	26

19.

x	y
0	-3
5	2
10	7
15	12

TABLE 1.13:

Evaluate the following equations.

20. Find the domain of the equation $y = 2x + 3$ when the range is {2}.

21. For $f(x) = 4x - 9$, find $f(x)$ when $x = 4$.

22. Consider the function $f(x) = \frac{1}{4}x^2 + 3$. Find the value of $f(x)$ when $x = 10$.

23. If (2,y) is a solution to $y = -3x + 1$, find the value of y.

24. If (x, 0) is a solution to the function $f(x) = -x - 3$, find the value of x.

25. Find the range of the function $f(x) = \frac{1}{2}x + 3$, when the domain is {-2}.

TABLE 1.14:

Determine if the following sequences are arithmetic by finding the common difference. If so, find the next three terms. 26. -1, 0, 2, 3 27. 2, 4, 6, 8 28. -10, -2, 6, 14 29. 10, 12, -12, 10
Find the indicated term given the first term and the common difference. 30. $a_1 = 2$, d=-2, find the 11th term. 31. $a_1 = 30$, d=0.5, find the 32th term. 32. $a_1 = -5$, d=5, find the 14th term. 33. $a_1 = 8$, d=0.2, find the 50th term.

Linear Functions

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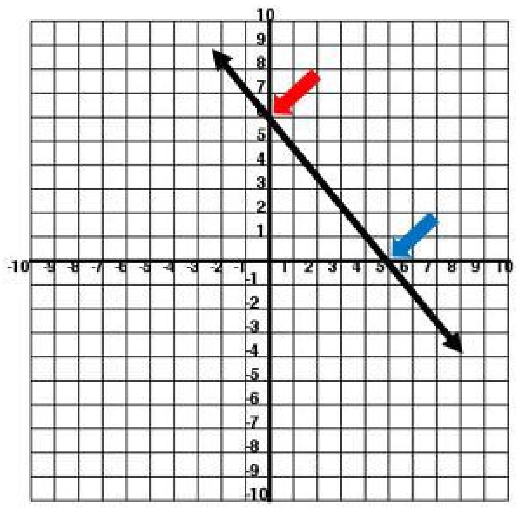
CHAPTER 3 Linear Functions

CHAPTER OUTLINE

3.1 X and Y Intercepts of Linear Functions

You will learn how to identify and/or solve for the x-intercept and the y-intercept of a function.

Example A



To find the x-intercept from a graph, determine the point at which the graph intersects the x-axis.

The x-intercept is (5, 0).

To find the y-intercept from a graph, determine the point at which the graph intersects the y-axis.

The y-intercept is (0, 6).

FIGURE 1.1

Example B

To find the x and y intercepts from a table, look for the number zero.

x	y
-4	0
-3	1
-2	2
-1	3
0	4
1	5

(-4, 0) is the x-intercept because the y value is 0

(0, 4) is the y-intercept because the x value is 0

FIGURE 1.2

x	y
-5	8
0	7
5	6
10	5
15	4
20	3
25	2
30	1
35	0

(0, 7) is the y-intercept because the x value is 0

The x-intercept is not visible in the graph. You must continue the pattern to find the other number zero.

(35, 0) is the x-intercept because the y value is 0

FIGURE 1.3

Example C

$-2x + 3y = 12$

To find the x-intercept from an equation, substitute 0 for y and solve for x.

$-2x + 3(0) = 12$

$-2x + 0 = 12$

$$\frac{-2x}{-2} = \frac{12}{-2}$$

$x = -6$

The x-intercept is (-6, 0)

FIGURE 1.4

$-2x + 3y = 12$

To find the y-intercept from an equation, substitute 0 for x and solve for y.

$-2(0) + 3y = 12$

$0 + 3y = 12$

$\frac{3y}{3} = \frac{12}{3}$

$y = 4$

The y-intercept is (0, 4)

FIGURE 1.5

Example D

Manuel started with $9 in his piggy bank. Each week he takes out $1.50 for a snack. The amount of money in his piggy bank after x weeks is represented by the function f(x) = 9 – 1.5x.

The y-intercept is (0, 9) because at 0 weeks Manuel has 9 dollars in his piggy bank.

The x-intercept is (6, 0) because after 6 weeks, Manuel has 0 dollars left in his piggy bank.

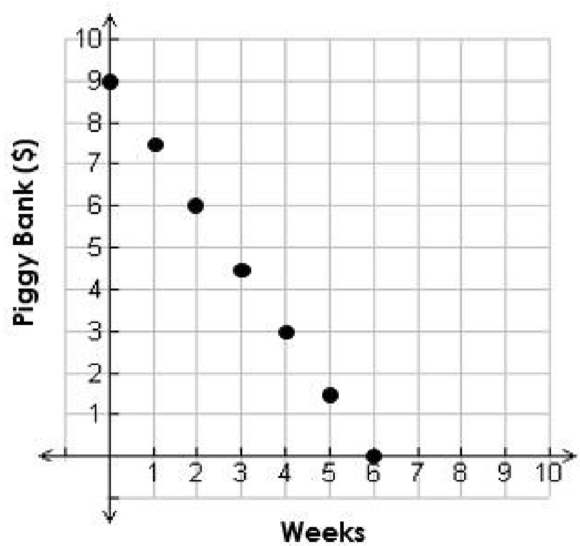


FIGURE 1.6

Vocabulary

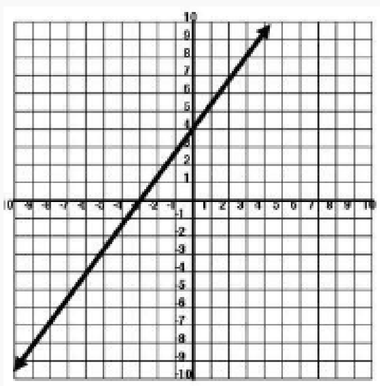
x-Intercept = the point at which a line crosses the x axis and the y value is 0. y-Intercept = the point at which a line crosses the y axis and the x value is 0.

Independent Practice

Find the x and y intercepts. Write your answers as order pairs.

TABLE 1.1:

1.



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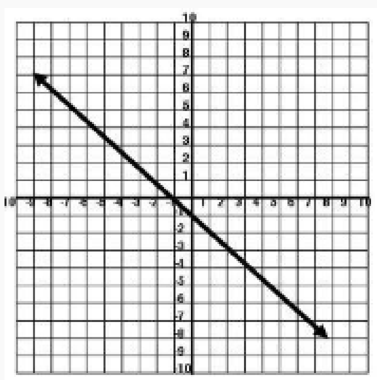


3.

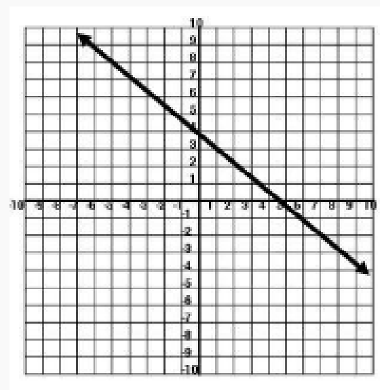


TABLE 1.2:

4.

x	y
-3	-2
-2	0
-1	2
0	4
1	6

5.

x	0	3	6	9	12
y	-4	-3	-2	-1	0

6.

x	y
0	5
2	6
4	7
6	8
8	9

7.

x	-4	-3	-2	-1
y	0	6	12	18

Find the x and y intercepts from the following equations.

TABLE 1.3:

8. $3x - y = 3$

9. $3y - 2x = 6$

10. $2x = 4y - 8$

Find the x and y intercepts from the following equations and graph the line.

TABLE 1.4:

11. 8x - 3y = 24	12. 5x - 4y = 20	13. 7x + 3y = 21	14. -7x + 2y = 14

Using intercepts in real world situations.

15.

Rosa has $30 to spend on refreshments for herself and her friends at the carnival. The equation 3x + 2y = 30 describes the number of small popcorns, x, and small drinks, y, she can buy. Graph this function, find its intercepts and describe the meaning of each intercept.

x-intercept __________

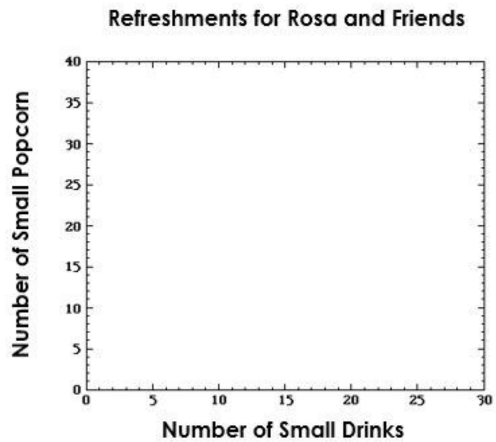
Meaning of x-intercept

____
____

y-intercept __________

Meaning of y-intercept

____
____



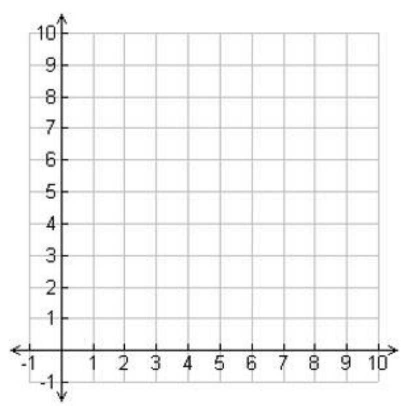
16.

At the local grocery store, strawberries cost $3.00 per pound and bananas cost $1.00 per pound.

a. If I have $10 to spend on strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly $10.

b. Plot the point representing 3 pounds of strawberries and 2 pounds of bananas. Will that cost more or less than $10?

c. Do the same for the point representing 1 pound of strawberries and 5 pounds of bananas.



The graph shows the distance of an elevator at the First National Bank, from its destination as a function of time. Use the graph to answer the following questions.

17. What is the x-intercept?

a. 20 b. 0 c. 500 d. 300

18. What does the x-intercept represent?

a. The time it takes to reach the first floor.

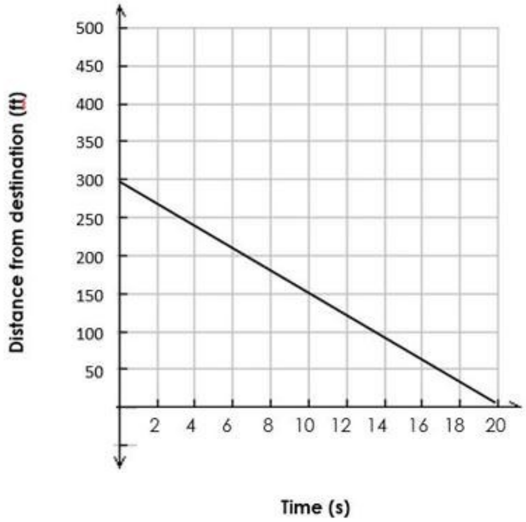
b. The total distance the elevator travels.

c. The distance that the elevator has traveled at any time.

d. The number of seconds that have passed for any given distance.

19. What is the y-intercept?

a. 300 b. 0 c. 20 d. 500



20. What does the y-intercept represent?

a. The total distance the elevator travels.

b. The number of seconds that have passed for any given distance.

c. The time it takes to reach the first floor.

d. The distance that the elevator has traveled at any time.

21.

Thomas is reading a 500 page book. He reads 5 pages every 8 minutes. The number of pages Thomas has left to read after x minutes is presented by the function $f(x) = 500 - \frac{5}{8}x$. Graph this function using the intercepts.

a. Meaning of y-intercept.

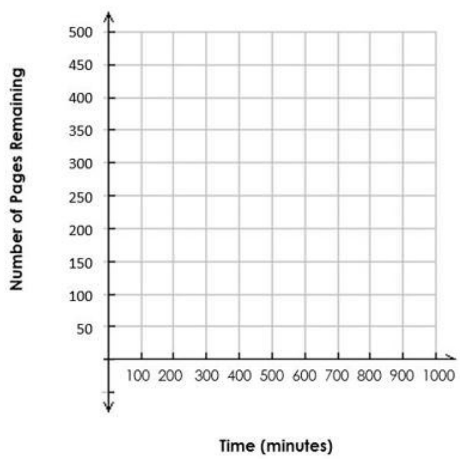
____

b. Meaning of x-intercept.

____

c. After how many minutes does Thomas have 350 pages left?

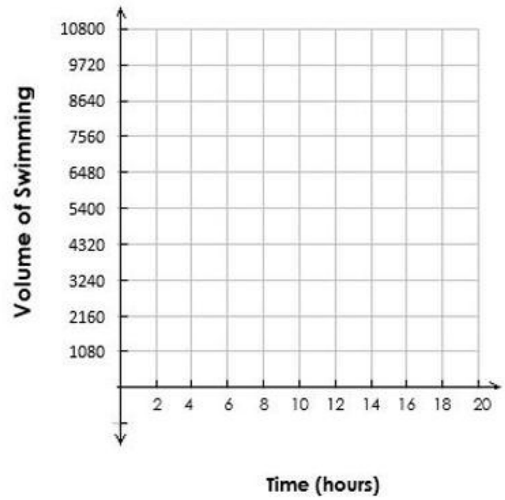
____



22.

Mathew is draining his swimming pool. The pool has a capacity of 10,800 gallons and the water level drops at a rate of 540 gal/hr.

How long will it take to empty the swimming pool? _________



- intercepts
- x-intercept
- y-intercept
- x-axis
- y-axis
- zero

3.2 Slope of Linear Functions

You will be able to identify and calculate the slope of a real world situation given a table, a set of ordered pairs, an equation, a graph or context.

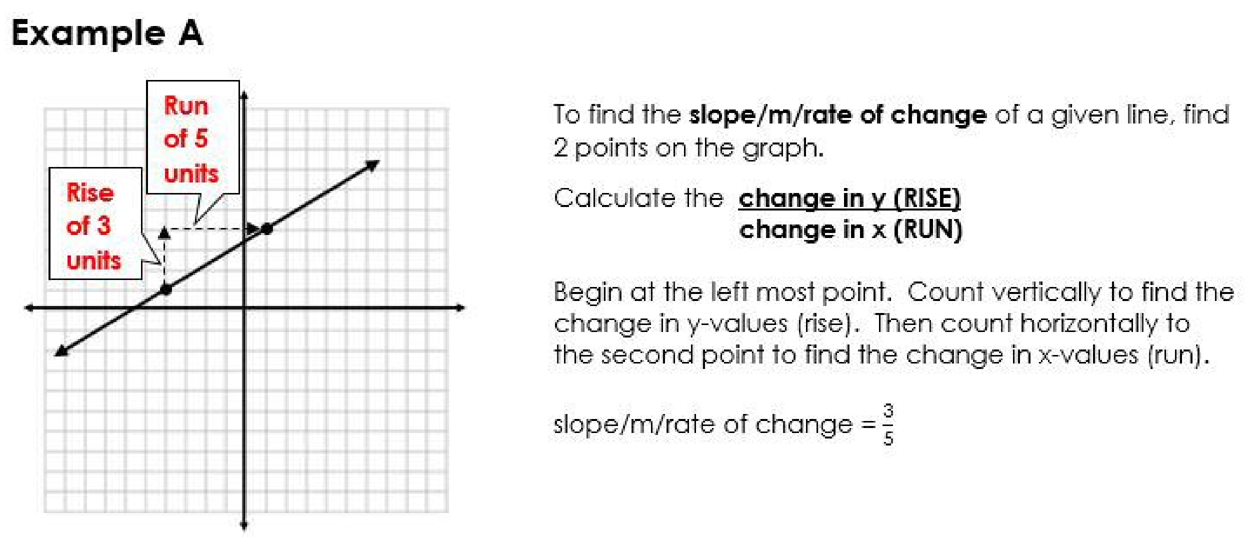


FIGURE 1.12

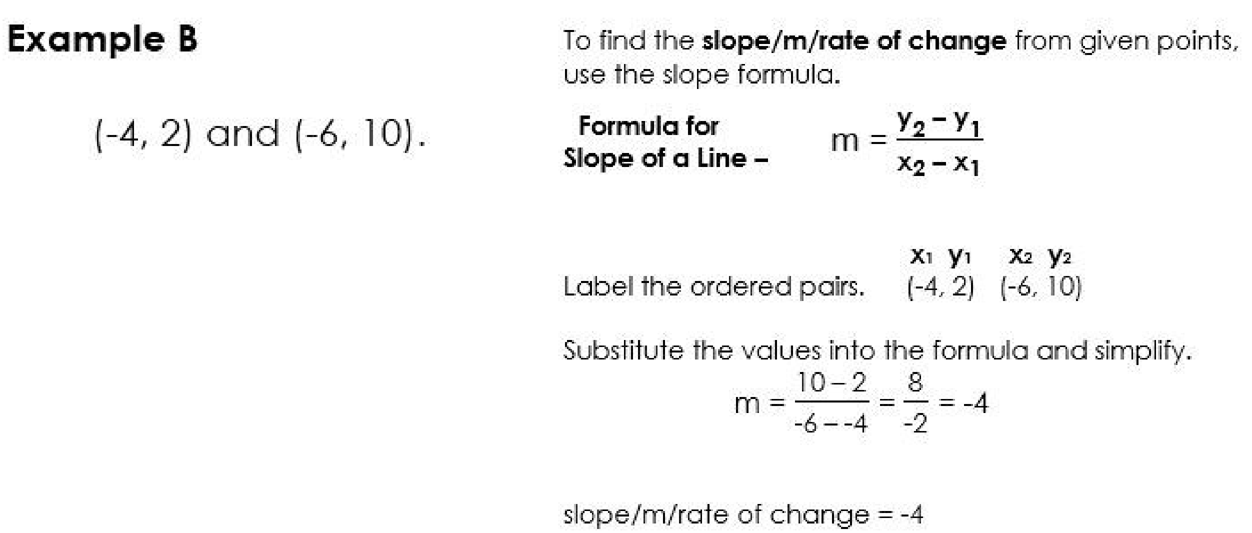


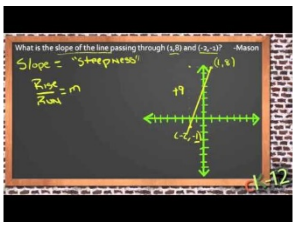
FIGURE 1.13

Example C

x	y
-3	-5
1	3
3	7
6	13

To find the slope from a table, pick any two points and use the Slope of a Line Formula. Follow the same steps as shown above.

FIGURE 1.14



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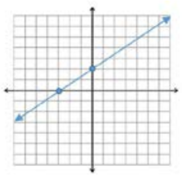
Vocabulary

Slope = the measure of the steepness of a line.

Independent Practice

Find the slope/m/rate of change of the given lines.

TABLE 1.5:

1.


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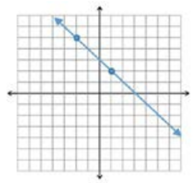

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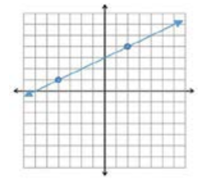

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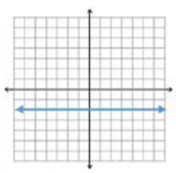

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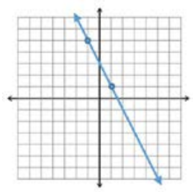

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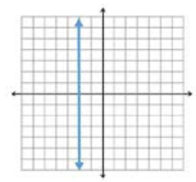

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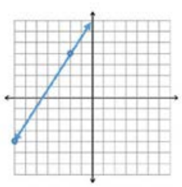

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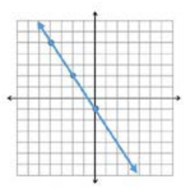

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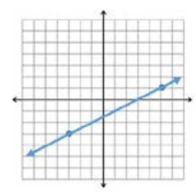

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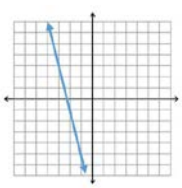

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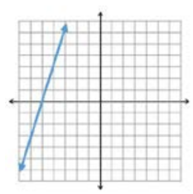

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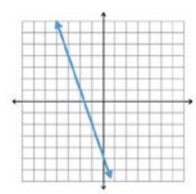


Find the slope/m/rate of change of a line containing the given points.

TABLE 1.6:

1. (2, 5) and (3, 7)	2. (3, 1) and (-1, 5)
3. (-1, 3) and (2, 7)	4. (0, 2) and (1, 5)
5. (2, -1) and (-3, 6)	6. (12, 6) and (-8, 6)
7. (-5, -2) and (0, 8)	8. (9, 7) and (9, -4)
9. (0, 2) and (-6, -2)	10. (3, 7) and (3, 10)

Each table shows a linear relationship. Find the slope.

TABLE 1.7:

11.

x	y
-4	-10
-2	-4
0	2
2	8

12.

x	y
10	-5
12	-5
14	-5
16	-5

13.

x	y
0	-2
-5	-6
-10	-10
-15	-14

14.

x	y
-3	-5
-1	1
2	10
5	19

15.

x	y
0	-2
0	-6
0	-10
0	-14

16.

x	y
0	-1
5	-5
10	-9
15	-13

- slope
- rate of change

3.3 Graphing Linear Functions

You will be able to graph the line of any given equation on a coordinate plane and identify its key features.

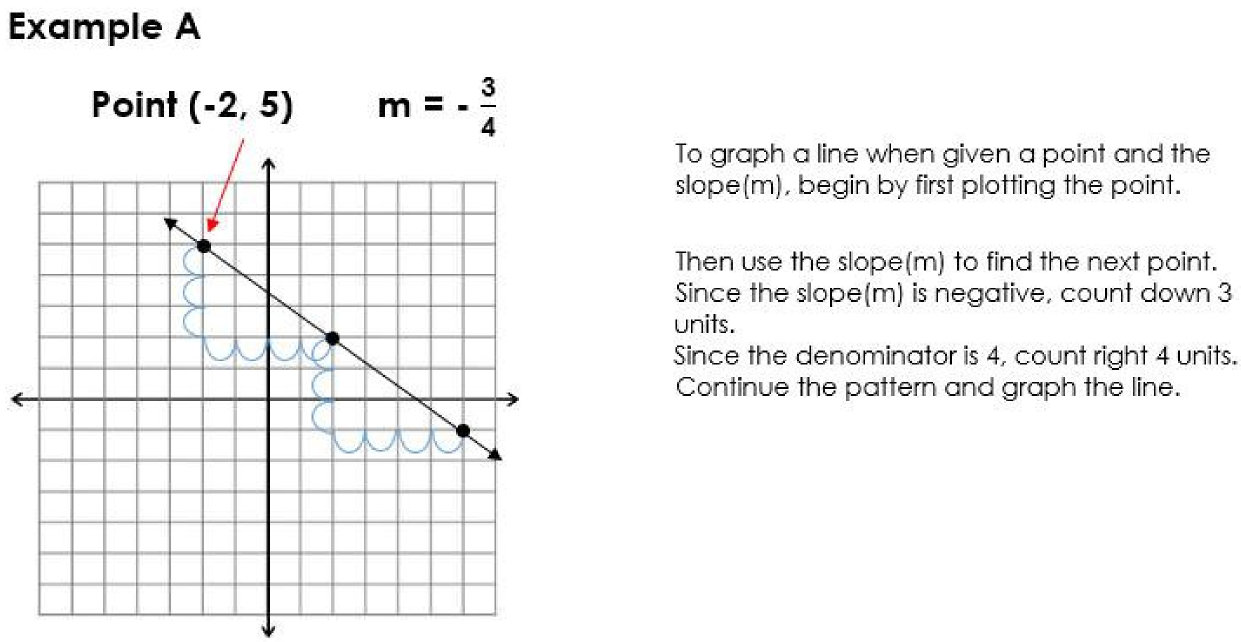


FIGURE 1.15

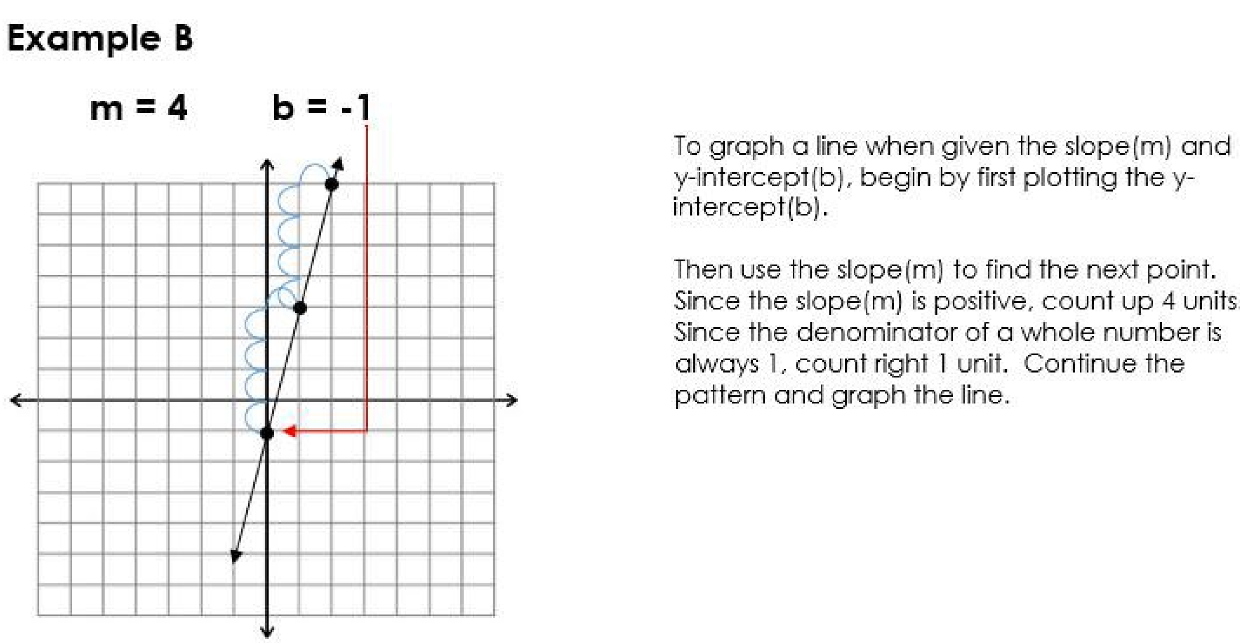


FIGURE 1.16

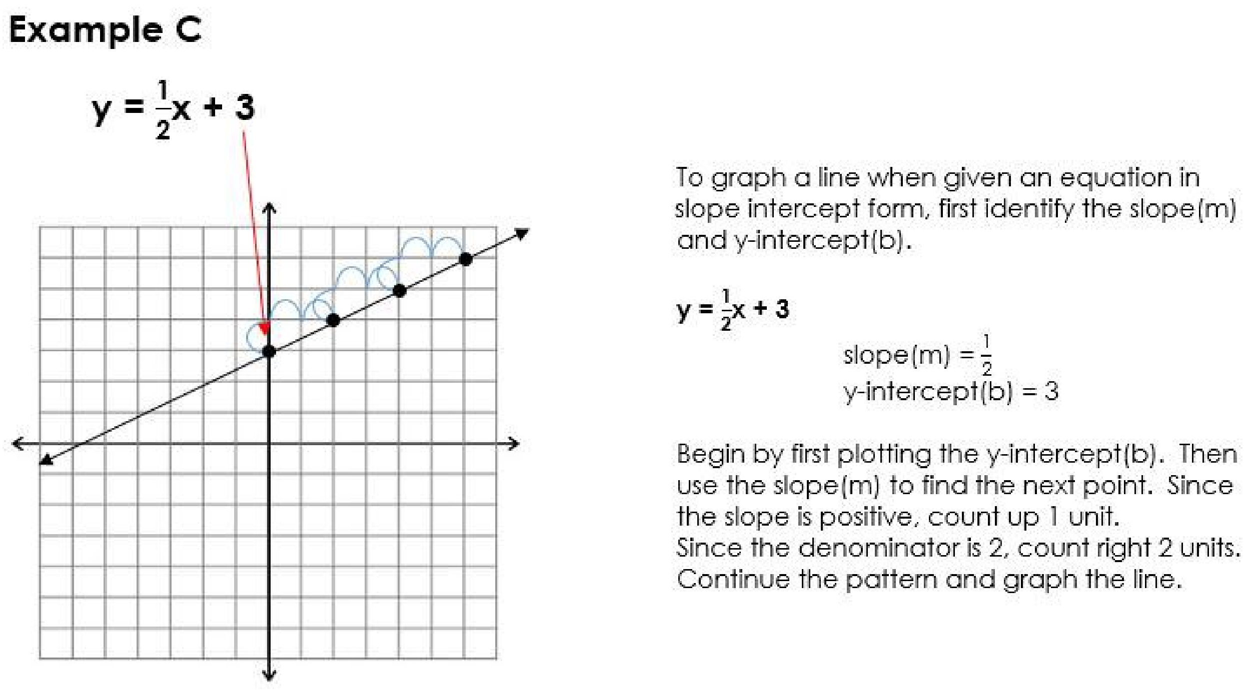


FIGURE 1.17

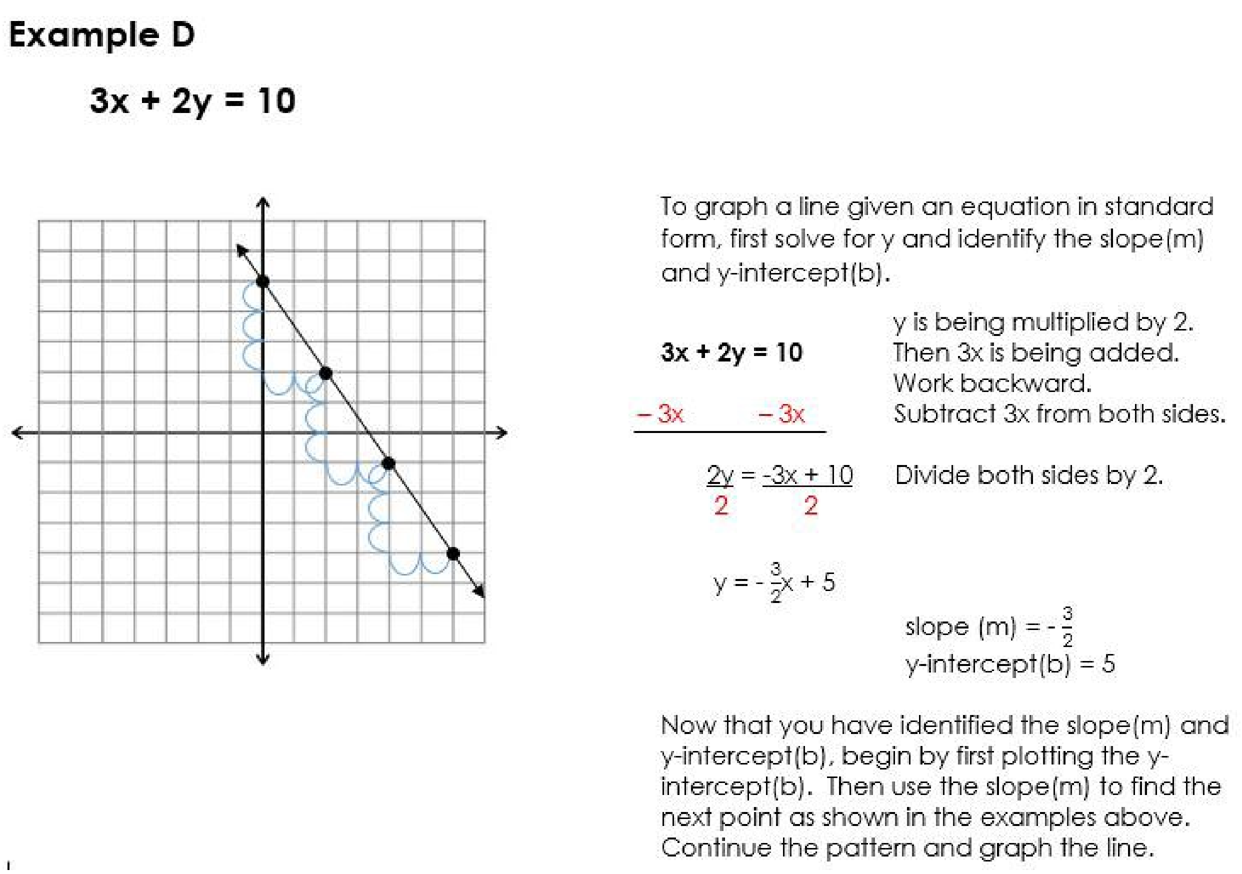


FIGURE 1.18

Independent Practice

Plot the point, count the slope and then graph the line.

1. (-2, 3) $m = -3$
2. (-3, 3) $m = \frac{2}{5}$
3. (0, -2) m = undefined
4. (0, -3) $m = 0$
5. (-4, 2) $m = -\frac{1}{5}$
6. (-1, -4) $m = 4$
7. (0, 0) $m = \frac{3}{2}$
8. (3, 0) $m = \frac{1}{2}$
9. (0, 0) $m = 1$
10. (-5, 0) $m = \frac{1}{4}$

FIGURE 1.19

Graph the line when given the slope (m) and y-intercept (b).

11. $m = 2$ $b = 0$
12. $m = \frac{1}{3}$ $b = 3$
13. $m = \frac{3}{4}$ $b = -4$
14. $m = 0$ $b = 3$
15. $m = \frac{5}{2}$ $b = -2$
16. $m = -3$ $b = 4$
17. $m = -3$ $b = -1$
18. $m = -1$ $b = -1$
19. $m = -\frac{2}{3}$ $b = 1$
20. $m = 0$ $b = -4$

FIGURE 1.20

Graph the line given the equation in slope-intercept form y=mx+b. Identify the slope (m) and y-intercept (b).

21. $y = 3x + 1$
22. $y = \frac{-1}{2}x + 4$
23. $y = -3x - 1$
24. $y = \frac{2}{3}x$
25. $y = \frac{1}{3}x + 4$
26. $y = 5$
27. $y = \frac{4}{3}x + 2$
28. $y = x + 4$
29. $y = -\frac{2}{5}x - 1$
30. $y = x$

FIGURE 1.21

Graph the equation given in standard form (Ax + By = C).

31. $2x + 2y = 10$
32. $x + y = 1$
33. $x + 3y = 12$
34. $2x + y = -3$
35. $2x + 3y = 6$
36. $3x + 4y = 12$
37. $3x - y = 1$
38. $6x - 5y = 20$
39. $x - 4y = 8$
40. $2x - 2y = -6$

FIGURE 1.22

- Graph
- Linear
- Slope-intercept form
- Y-intercept

3.4 Applications Using Linear Functions

You will be able to write an equation that represents a real-world situation. You will be able to identify and interpret the slope and intercepts.

Example A

Pizza Pro charges $8 a pizza plus $2 per topping.

Write an equation to represent the relationship between the total charge, c, and the number of pizza toppings, t, ordered.

$$\underbrace{c}_{\text{total charge}} = \underbrace{2}_{\text{cost per topping}}\underbrace{t}_{\text{Number of toppings}} + \underbrace{8}_{\text{initial cost for 1 pizza}}$$

Write the equation as a function of x.

$f(x) = 2x + 8$

What is the y-intercept?

The y-intercept is the initial cost without any toppings. The y-intercept is 8

What is the rate of change/slope for this situation?

The rate of change/slope is the cost per topping. The slope/rate of change is 2.

What would be the amount charged if you ordered a pizza with 3 toppings?

$c = 2t + 8$ Substitute 3 for t.

$c = 2(3) + 8$

$c = \$12$

FIGURE 1.23

Independent Practice

1. Castle Bounce Fun charges a fee of $30 plus $2 per hour to rent a castle bounce. Write an equation to determine *c*, the total cost to rent the castle bounce if *h* represents the number of hours the castle bounce has been rented?
2. Juanita charges a $20 initial fee plus $5 per hour to clean a house. What is the equation that describes the relationship between the number of hours she works, *h*, and the amount of money she earns, *m*.?

3. A submarine is 12 feet below the surface of the ocean. It is descending at a rate of 4 feet per minute. What is the equation that represents the distance in feet, *f*, at any given time, *t*.?

4. Mike's grandfather gave him $50 for his birthday and told him to put it in a savings account. Mike's grandfather also told him that he would give him $10 a month to add to the account. Write an equation to determine b, the total balance in his account, after a certain number of months, *m*.

5. For her cellular phone service, Brianna pays $32 a month, plus $0.75 for each minute over the allowed minutes in her plan. Write an equation to represent, *c*, the monthly cost Brianna will pay for number of minutes used after the allowed number of minutes in her plan, *m*.
   a. Rewrite the equation in terms of x and y.
   b. Write the equation as a function of x.
   c. What is the y-intercept?
   d. What is the rate of change/slope for this situation?

6. A local bowling alley charges a fee of $3 to rent bowling shoes and a fee of $5 per game bowled.
   a. Write an equation to represent the relationship between the amount charged, *c*, and the number of games, *g*, bowled.
   b. Rewrite the equation as a function of *x*.
   c. What would be amount charged after 3 games?
   d. What would be the amount charged after 7 games?
   e. What is the rate of change for this situation?

7. Alfred weighs 165 pounds, but is on a diet that allows him to lose 1.5 pounds per week.
   a. Write an equation representing Alfred's weight after 'x' weeks.
   b. Rewrite the equation as a function of *x*.
   c. How many weeks will take Alfred to weigh 150 pounds?
   d. What is the rate of change for this situation?

8. At the beginning of the school year, teachers had 240,000 sheets of copier paper to use. If 2000 sheets of paper are used each day during a school year, write an equation to describe *s*, the number of sheets that are left after *d*, days of school?
   a. Write an equation to describe *s*, the number of sheets that are left after *d* days of school.
   b. How many sheets of paper will be left after 30 days of school?
   c. How many sheets of paper will be left after 60 and 90 days of school?
   d. What is the rate of change?

e. If you are graphing this relationship, what would be the intersection with the y-axis?
f. How many days would it take to run out of paper? Does that number correspond to the x-intercept?

9. A caterer charges a $75 fee to provide the equipment for a party and $7.50 per person for the food.

a. Write a function describing the relationship between the caterer's cost, c, and the number of people who will attend the party, p.
b. Rewrite the equation as a function of x.
c. What is the rate of change for this problem?
d. What is the y-intercept?
e. Sketch a graph illustrating the y-intercept and the slope/rate of change.
f. What would be a possible domain for this situation?
g. What would be a possible range for this situation?

- interpretation
- slope
- zeros
- x-intercept
- y-intercept
- slope
- rate of change

3.5 Writing Linear Functions

You will be able to write linear equations in various forms given different constraints.

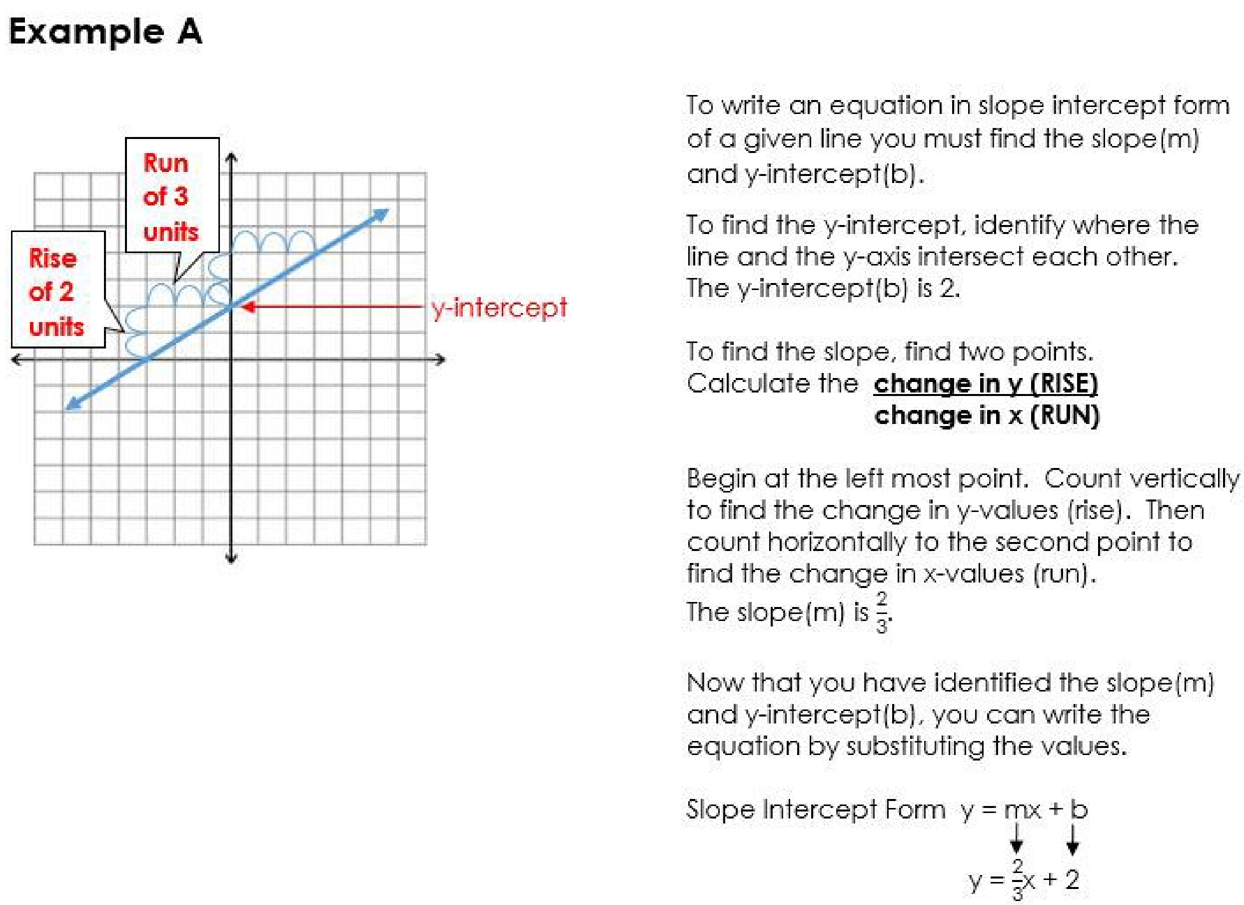
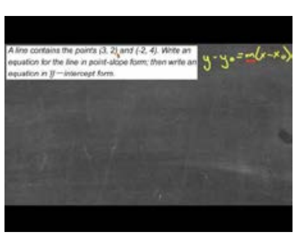


FIGURE 1.24



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Example B

Slope of 5 and passes through the point (2, 4)

To write an equation of a line in point-slope form when given a point and the slope, use the point-slope formula.

$$y - y_1 = m(x - x_1)$$

The slope(m) is 5 The point is (2, 4), with $x_1 = 2$ and $y_1 = 4$

Substitute the values into the formula.

Point Slope Form $\longrightarrow y - 4 = 5(x - 2)$

To write the equation in slope intercept form, solve the equation for y.

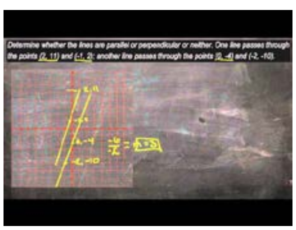
$y - 4 = 5(x - 2)$ Distribute the 5

$y - 4 = 5x - 10$ Add 4 to both sides.

$\quad + 4 \qquad + 4$

$y = 5x - 6 \longleftarrow$ Slope Intercept Form

FIGURE 1.25



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Example C

A line containing the points (3, -7) and (0, -1).

To write an equation of a line when given 2 points, first use the slope of a line formula to find the slope(m).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Label the ordered pairs. $\begin{matrix} x_1 & y_1 \\ (3, & -7) \end{matrix}$ $\begin{matrix} x_2 & y_2 \\ (0, & -1) \end{matrix}$

Substitute the values into the formula and simplify.

$$m = \frac{-1 - -7}{0 - 3} = \frac{6}{-3} = -2$$

The slope(m) is -2.

Now that you have identified the slope(m), you can use the slope and one of the points and substitute the values into the point-slope formula.

$$\mathbf{y - y_1 = m(x - x_1)}$$

The slope(m) is -2 A point is $\begin{matrix} x_1 & y_1 \\ (3, & -7) \end{matrix}$

Substitute the values into the formula.

$$y - -7 = -2(x - 3)$$

Point Slope Form $\rightarrow y + 7 = -2(x - 3)$

To write the equation in slope intercept form, solve the equation for y.

$y + 7 = -2(x - 3)$ Distribute the -2

$y + 7 = -2x + 6$ Subtract 7 from both sides.

$\underline{\;-7 \qquad -7\;}$

$y = -2x - 1 \leftarrow$ Slope Intercept Form

FIGURE 1.26

Example D

$$y = 9x - 2$$
$$y = 9x + 1$$

To determine if 2 lines are parallel or perpendicular, you must first identify their slopes(m).

Lines that are parallel have the same slopes(m).

$$y = 9x - 2$$
$$y = 9x + 1$$

The slope(m) in both equations is 9, which means they are parallel.

$$y = \frac{4}{5}x + 3$$
$$y = -\frac{5}{4}x + 4$$

Lines that are perpendicular have opposite reciprocal slopes.

$$y = \frac{4}{5}x + 3$$
$$y = -\frac{5}{4}x + 4$$

One slope is positive and one slope is negative, which means they are opposites. The slopes are "flipped", which means they are reciprocals. The slopes are opposite reciprocals, which means they are perpendicular.

FIGURE 1.27

Independent Practice

Write the equation of the following lines in slope-intercept form.

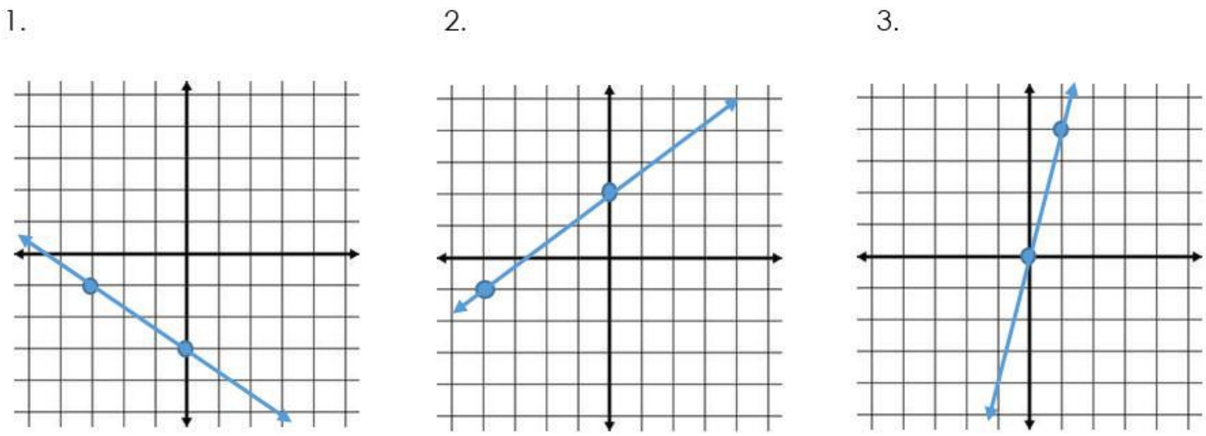


FIGURE 1.28

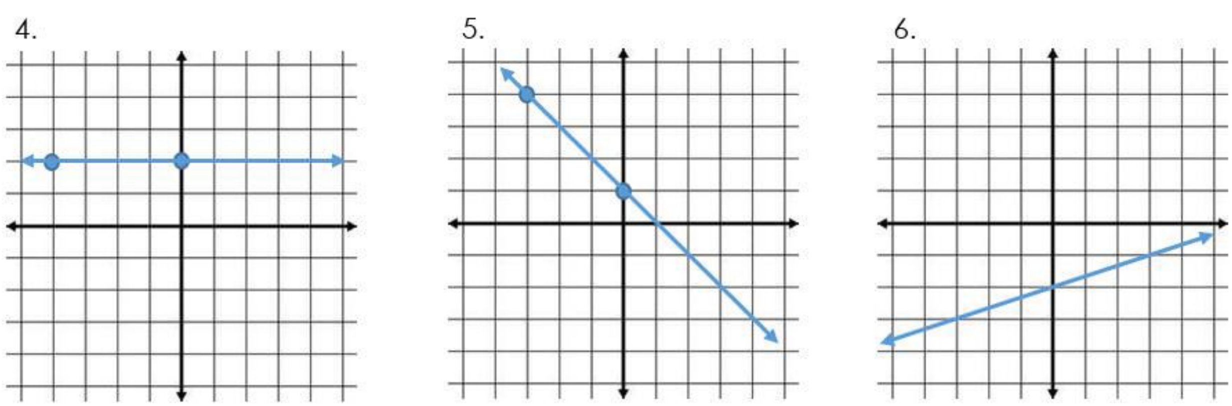
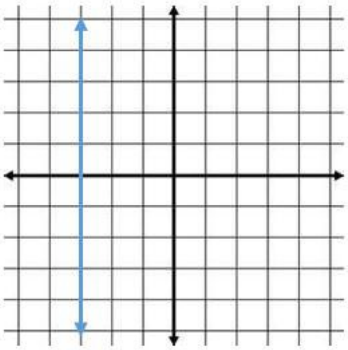


FIGURE 1.29

TABLE 1.8: (continued)

7.



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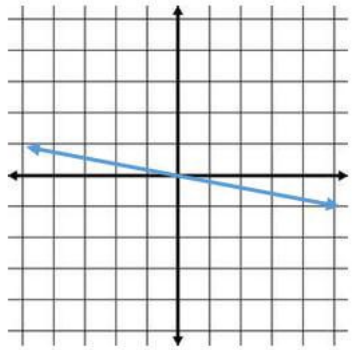

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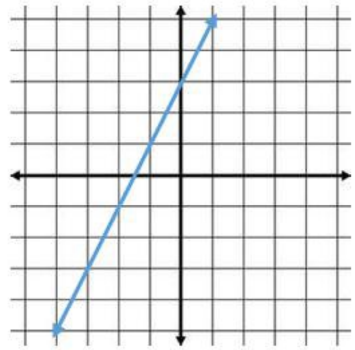

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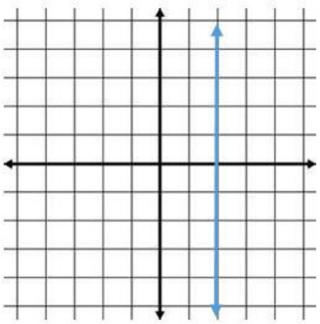


FIGURE 1.30

Point-Slope

Write the equation of the line in point-slope form first and then in slope-intecept form.

11. Slope of 1 and passes through the point (-2, 4).
12. Slope of $\frac{1}{3}$ and passes through the point (0, 0).
13. Slope of - $\frac{1}{3}$ and passes through the point (3, 4).
14. Slope of $\frac{1}{2}$ and passes through the point (2, -2).
15. Slope of 5 and a y-intercept of 3.
16. Slope of $\frac{3}{4}$ and passes through (-4, 1)
17. Slope of -$\frac{1}{10}$ and passes through the point (5, -1)
18. Slope of -1 and x-intercept of -1.
19. The line has a slope of 7 and a y-intercept of -2.
20. The line has a slope of -5 and a y-intercept of 6.
21. The line has a slope of -$\frac{1}{4}$ and contains the point (4, -1).

22. The line has a slope of 5 and f(0)= -3
23. m=5, f(0) = -3.
24. m=-7, f(2) = -1
25. m= $\frac{1}{3}$ f(-1) = $\frac{2}{3}$
26. m=4.2, f(-3) = 7.1

Write the equation of the line given 2 points.

Write the equation of the line in point-slope form $y - y_1 = m(x - x_1)$ first and then in slope intercept form $y - y_1 = mx+b$.

27. The line contains the points (3, 6) and (-3, 0).
28. The line contains the points (-1, 5) and (2, 2).
29. The line goes through the points (-2, 3) and (-1, -2).
30. The line contains the points (10, 12) and (5, 25).
31. The line goes through the points (2, 3) and (2, -3).
32. The line contains the points (3, 5) and (-3, 3).
33. The line contains the points (10, 15) and (12, 20).
34. The line goes through the points (-2, 3) and (-1, -2).
35. The line contains the points (1, 1) and (5, 5).
36. The line goes through the points (2, 3) and (0, 3).
37. A horizontal line passing through (5, 4).
38. A vertical line passing through (-1, 3).
39. x-intercept of 4 and y-intercept of 4.
40. x-intercept of -2 and y-intercept of 5.
41. x-intercept of 3 and y-intercept of 1.

Parallel and Perpendicular LInes

Determine whether the following lines are parallel, perpendicular or neither.

42. 43. 44.

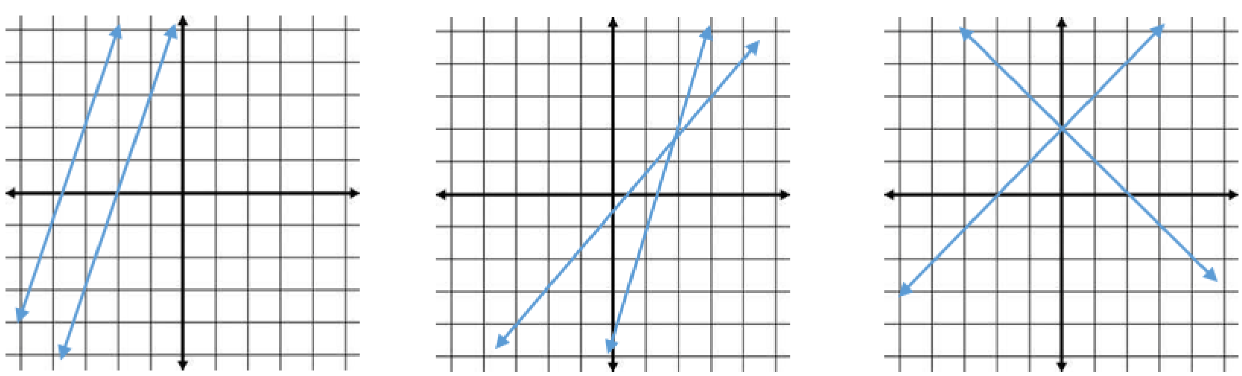
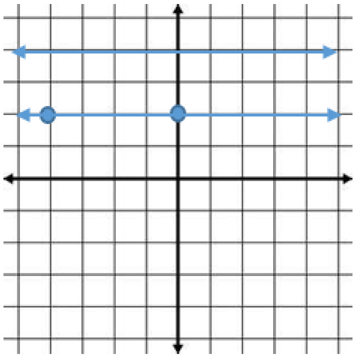
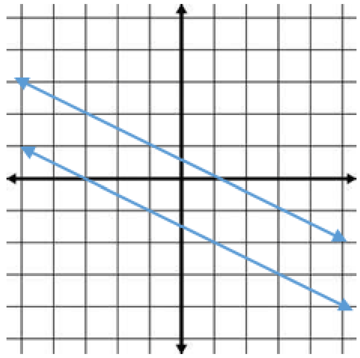


FIGURE 1.31

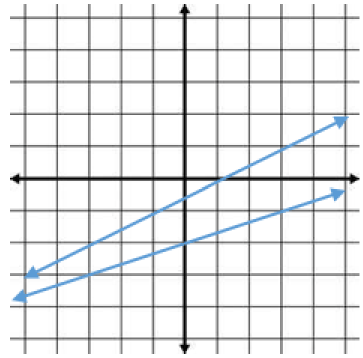
45.



46.



47.



48. $y = 3x + 5$ and $y = -\frac{1}{3}x + 4$

49. $y = 5x - 3$ and $y = -5x - 8$

50. $y = \frac{1}{2}x + 2$ and $y = \frac{x}{2} - 2$

51. $y = x$ and $y = x - 2$

52. $y = \frac{4}{3}x + 5$ and $y = -\frac{4}{3}x + 1$
53. $y = 5$ and $x = 2$
54. $y = -\frac{1}{3}x + 7$ and $y = -3x - 5$
55. $y = -\frac{3}{4}x + 2$ and $y = \frac{1}{4}x + 1$
56. $y = 2x - 4$ and $y = 2x - 7$
57. $y = 4$ and $y = -7$

FIGURE 1.32

58. $3y = 12x + 6$ and $10 + y = 4x$
59. $5x + 10y = 20$ and $y = 2x - 7$
60. $y = 3x - 3$ and $y + 7 = -9$
61. $8x + 14 = 2y$ and $7x = 2y + 16$
62. $y = -\frac{1}{5}x$ and $3y = 15x + 3$
63. $18 = 2x - 3y$ and $-5 + y = -\frac{3}{2}x$

PLIX (Play Learn Interact eXplore)

Trip Functions

You May Also Like...

Determining the Equation of a Line

- Point-slope formula
- Point-slope form
- Perpendicular lines
- Substituition

3.6 Transformations of Linear Functions

You will be able to graph new functions when specific values of a function are changed. You will also be able to determine the effect of the change.

Example A

$y = 3x - 2$
$y = \frac{1}{4}x + 1$

To determine if given equations become steeper or flatter you must compare their slopes.
The slope of the first equation is 3.
The slope of the second equation is $\frac{1}{4}$.
Because the slope has decreased from 3 to $\frac{1}{4}$, this means the line has become flatter.

To determine if the line has translated up or down you must compare their y-intercepts.
The y-intercept of the first equation is -2.
The y-intercept of the second equation is 1.
Because the y-intercept has increased from -2 to 1, this means the line has translated up.

FIGURE 1.33

Example B

$y = -2x + 6$
$y = -4x + 2$

To determine if given equations become steeper or flatter you must compare their slopes.
The slope of the first equation is -2.
The slope of the second equation is -4.
When checking for steepness, do not take into account the negative signs.
The negative only changes the direction of the line. Because the slope has increased from 2 to 4, this means the line has become steeper.

To determine if the line has translated up or down you must compare their y-intercepts.
The y-intercept of the first equation is 6.
The y-intercept of the second equation is 2.
Because the y-intercept has decreased from 6 to 2, this means the line has translated down.

FIGURE 1.34

Example C

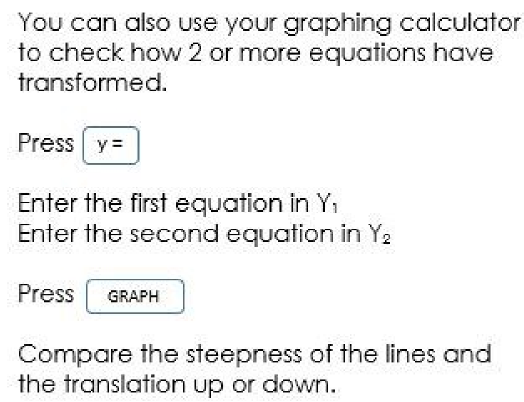


FIGURE 1.35

Independent Practice.

Graph Line 1 and Line 2 in your calculator and compare. Tell whether Line 2 is steeper or flatter, then determine if Line 2 translated up or down.

TABLE 1.9:

Line 1	Line 2
1. $y = \frac{1}{2} + 2$	$y = 2x + 1$
2. $y = 10x - 2$	$y = 3x - 3$
3. $y = \frac{5}{11}x - 5$	$y = 6x + 1$
4. $y = 14x - 7$	$y = 7x - 1$
5. $y = \frac{3}{2}x + 1$	$y = \frac{5}{2}x + 1$
6. $y = 5x - 5$	$y = 7x - 2$
7. $y = -\frac{5}{2}x + 4$	$y = 15x - 6$
8. $y = -10x + 2$	$y = 2x + 3$

TABLE 1.9: (continued)

9. $y = -5x - 5$	$y = -2x + 1$
10. $y = 10x + 2$	$y = -5x + 2$
11. $y = 14x - 4$	$y = -7x + 2$
12. $y = x$	$y = 4x - 2$
13. $y = 2x - 4$	$y = -2x + 1$

- Transformation
- Slide
- Rotation
- Steeper
- Flatter

3.7 Scatter Plots

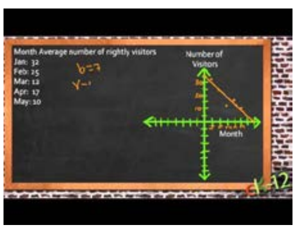
A1 4(C) Write, with and without technology, linear functions that provide a reasonable fit to data to estimate solutions and make predictions for real-world problems. TEKS: A1 4(C)

Learning Objective

Here you'll learn how to make a scatter plot of a set of data. You'll also learn how to find the line that best fits that data.

What if you had a graph with many random ordered pairs plotted on it? How could you find the line that best describes those plotted points? After completing this concept, you'll be able to find the line of best fit for scattered data.

Watch This



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/133186

CK-12 Foundation: 0506S Fitting a Line to Data (H264)

Guided Practice

In real-world problems, the relationship between our dependent and independent variables is linear, but not perfectly so. We may have a number of data points that don't quite fit on a straight line, but we may still want to find an equation representing those points. In this lesson, we'll learn how to find linear equations to fit real-world data.

Make a Scatter Plot

A **scatter plot** is a plot of all the ordered pairs in a table. Even when we expect the relationship we're analyzing to be linear, we usually can't expect that all the points will fit perfectly on a straight line. Instead, the points will be "scattered" about a straight line.

There are many reasons why the data might not fall perfectly on a line. Small errors in measurement are one reason; another reason is that the real world isn't always as simple as a mathematical abstraction, and sometimes math can only describe it approximately.

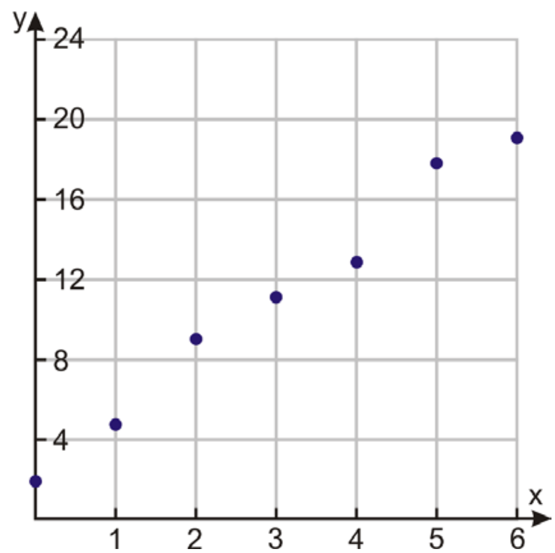
Example A

Make a scatter plot of the following ordered pairs:

(0, 2); (1, 4.5); (2, 9); (3, 11); (4, 13); (5, 18); (6, 19.5)

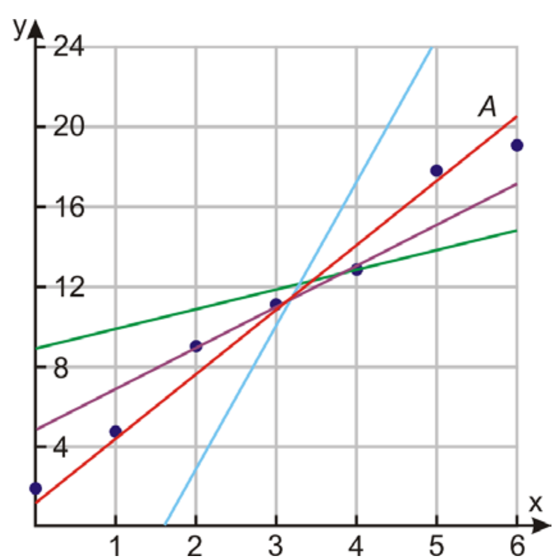
Solution

We make a scatter plot by graphing all the ordered pairs on the coordinate axis:

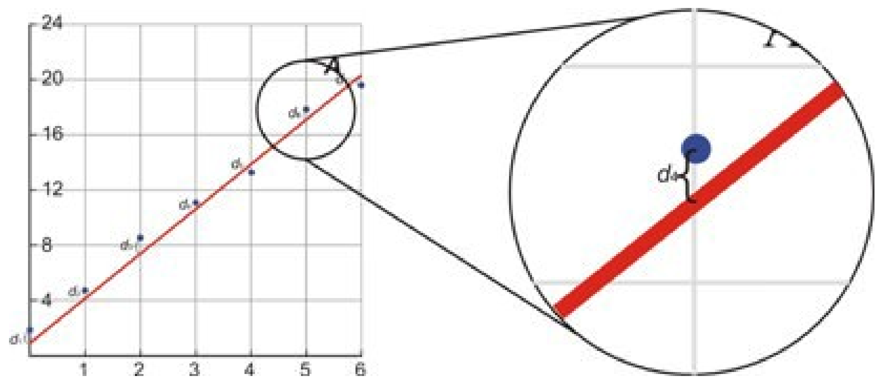


Fit a Line to Data

Notice that the points look like they might be part of a straight line, although they wouldn't fit perfectly on a straight line. If the points were perfectly lined up, we could just draw a line through any two of them, and that line would go right through all the other points as well. When the points aren't lined up perfectly, we just have to find a line that is as close to all the points as possible.



Here you can see that we could draw many lines through the points in our data set. However, the red line *A* is the line that best fits the points. To prove this mathematically, we would measure all the distances from each data point to line *A* and then we would show that the sum of all those distances—or rather, the square root of the sum of the squares of the distances—is less than it would be for any other line.



Actually proving this is a lesson for a much more advanced course, so we won't do it here. And finding the best fit line in the first place is even more complex; instead of doing it by hand, we'll use a graphing calculator or just "eyeball" the line, as we did above—using our visual sense to guess what line fits best.

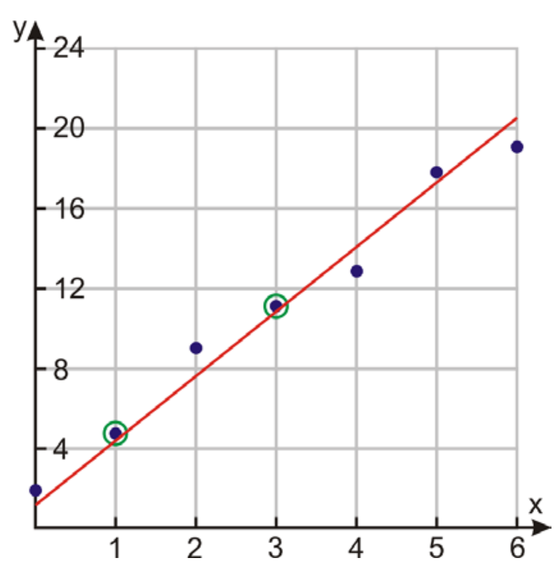
For more practice eyeballing lines of best fit, try the Java applet at http://mste.illinois.edu/activity/regression/ . Click on the green field to place up to 50 points on it, then use the slider to adjust the slope of the red line to try and make it fit the points. (The thermometer shows how far away the line is from the points, so you want to try to make the thermometer reading as *low* as possible.) Then click "Show Best Fit" to show the actual best fit line in blue. Refresh the page or click "Reset" if you want to try again. For more of a challenge, try scattering the points in a less obvious pattern.

Write an Equation For a Line of Best Fit

Once you draw the line of best fit, you can find its equation by using two points on the line. Finding the equation of the line of best fit is also called **linear regression.**

Caution: Make sure you don't get caught making a common mistake. Sometimes the line of best fit won't pass straight through any of the points in the original data set. This means that you can't just use two points from the data set **- you need to use two points that are on the line**, which might not be in the data set at all.

In Example 1, it happens that two of the data points *are* very close to the line of best fit, so we can just use these points to find the equation of the line: (1, 4.5) and (3, 11).



Start with the slope-intercept form of a line: $y = mx + b$
Find the slope: $m = \frac{11-4.5}{3-1} = \frac{6.5}{2} = 3.25$.

So $y = 3.25x + b$.

Plug (3, 11) into the equation: $11 = 3.25(3) + b$
$b = 1.25$

So **the equation for the line that fits the data best is** $y = 3.25x + 1.25$.

Perform Linear Regression With a Graphing Calculator

The problem with eyeballing a line of best fit, of course, is that you can't be sure how accurate your guess is. To get the most accurate equation for the line, we can use a graphing calculator instead. The calculator uses a mathematical algorithm to find the line that minimizes the sum of the squares.

Example B

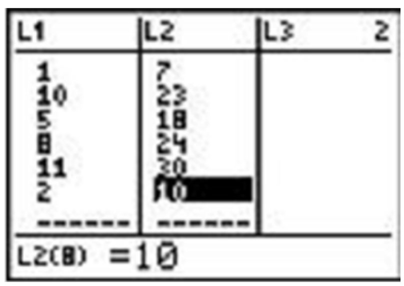
Use a graphing calculator to find the equation of the line of best fit for the following data:

(3, 12), (8, 20), (1, 7), (10, 23), (5, 18), (8, 24), (11, 30), (2, 10)

Solution

Step 1: *Input the data in your calculator.*

Press **[STAT]** and choose the **[EDIT]** option. Input the data into the table by entering the $x-$values in the first column and the $y-$values in the second column.



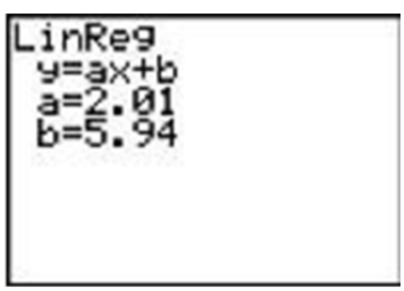
Step 2: *Find the equation of the line of best fit.*

Press **[STAT]** again use right arrow to select **[CALC]** at the top of the screen.

Chose option number 4, *LinReg*$(ax+b)$, and press **[ENTER]**

The calculator will display *LinReg*$(ax+b)$.

Press **[ENTER]** and you will be given the $a-$ and $b-$values.



Here a represents the slope and b represents the $y-$intercept of the equation. The linear regression line is $y = 2.01x + 5.94$.

Step 3. *Draw the scatter plot.*

To draw the scatter plot press **[STATPLOT] [2nd] [Y=].**

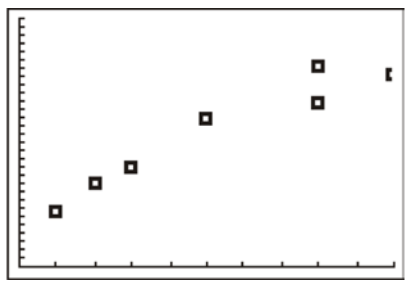
Choose Plot 1 and press **[ENTER]**.

Press the On option and set the Type as scatter plot (the one highlighted in black).

Make sure that the X list and Y list names match the names of the columns of the table in Step 1.

Choose the box or plus as the mark, since the simple dot may make it difficult to see the points.

Press **[GRAPH]** and adjust the window size so you can see all the points in the scatter plot.

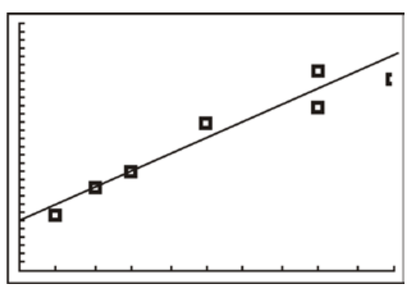


Step 4. *Draw the line of best fit through the scatter plot.*

Press **[Y=]**

Enter the equation of the line of best fit that you just found: $y = 2.01x + 5.94$.

Press **[GRAPH]**.



Solve Real-World Problems Using Linear Models of Scattered Data

Once we've found the line of best fit for a data set, we can use the equation of that line to predict other data points.

Example C

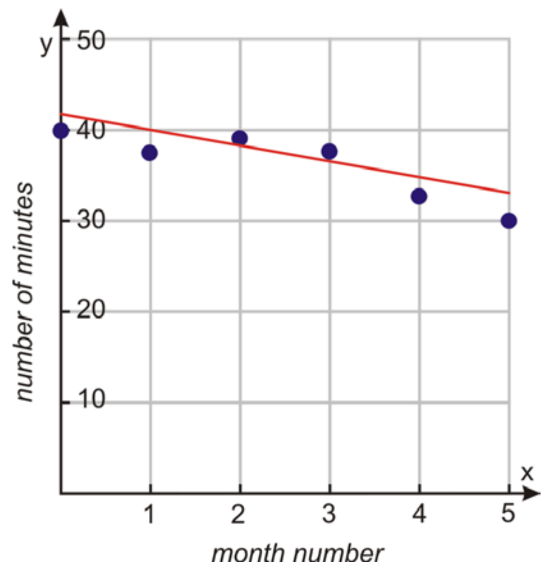
Nadia is training for a 5K race. The following table shows her times for each month of her training program. Find an equation of a line of fit. Predict her running time if her race is in August.

TABLE 1.10:

Month	Month number	Average time (minutes)
January	0	40
February	1	38
March	2	39
April	3	38
May	4	33
June	5	30

Solution

Let's make a scatter plot of Nadia's running times. The independent variable, x, is the month number and the dependent variable, y, is the running time. We plot all the points in the table on the coordinate plane, and then sketch a line of fit.



Two points on the line are (0, 42) and (4, 34). We'll use them to find the equation of the line:

$$m = \frac{34-42}{4-0} = -\frac{8}{4} = -2$$
$$y = -2x + b$$
$$42 = -2(0) + b \Rightarrow b = 42$$
$$y = -2x + 42$$

In a real-world problem, the slope and $y-$intercept have a physical significance. In this case, the slope tells us how Nadia's running time changes each month she trains. Specifically, it decreases by 2 minutes per month. Meanwhile, the $y-$intercept tells us that when Nadia started training, she ran a distance of 5K in 42 minutes.

The problem asks us to predict Nadia's running time in August. Since June is defined as month number 5, August will be month number 7. We plug $x = 7$ into the equation of the line of best fit:

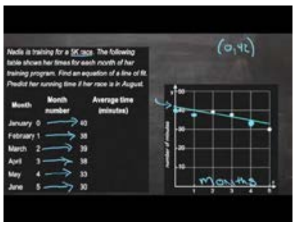
$$y = -2(7) + 42 = -14 + 42 = 28$$

The equation predicts that **Nadia will run the 5K race in 28 minutes.**

In this solution, we eyeballed a line of fit. Using a graphing calculator, we can find this equation for a line of fit instead: $y = -2.2x + 43.7$

If we plug $x = 7$ into this equation, we get $y = -2.2(7) + 43.7 = 28.3$. This means that **Nadia will run her race in 28.3 minutes.** You see that the graphing calculator gives a different equation and a different answer to the question. The graphing calculator result is more accurate, but the line we drew by hand still gives a good approximation to the result. And of course, there's no guarantee that Nadia will actually finish the race in that exact time; both answers are estimates, it's just that the calculator's estimate is slightly more likely to be right.

Watch this video for help with the Examples above.



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CK-12 Foundation: Fitting a Line to Data

Vocabulary

- A **scatter plot** is a plot of all the ordered pairs in a table. Even when we expect the relationship we're analyzing to be linear, we usually can't expect that all the points will fit perfectly on a straight line. Instead, the points will be "scattered" about a straight line.
- Once you draw the line of best fit, you can find its equation by using two points on the line. Finding the equation of the line of best fit is also called **linear regression.**

Guided Practice

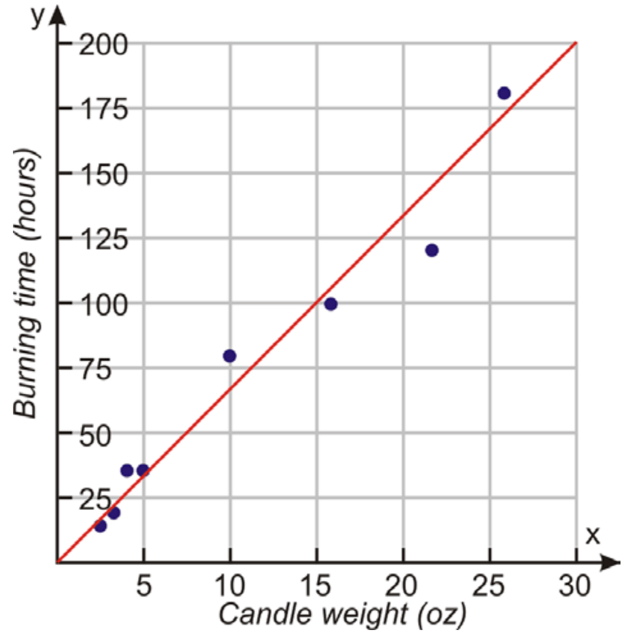
Peter is testing the burning time of "BriteGlo" candles. The following table shows how long it takes to burn candles of different weights. Assume it's a linear relation, so we can use a line to fit the data. If a candle burns for 95 hours, what must be its weight in ounces?

TABLE 1.11:

Candle weight (oz)	Time (hours)
2	15
3	20
4	35
5	36
10	80
16	100
22	120
26	180

Solution

Let's make a scatter plot of the data. The independent variable, x, is the candle weight and the dependent variable, y, is the time it takes the candle to burn. We plot all the points in the table on the coordinate plane, and draw a line of fit.



Two convenient points on the line are (0,0) and (30, 200). Find the equation of the line:

$$m = \frac{200}{30} = \frac{20}{3}$$
$$y = \frac{20}{3}x + b$$
$$0 = \frac{20}{3}(0) + b \Rightarrow b = 0$$
$$y = \frac{20}{3}x$$

A slope of $\frac{20}{3} = 6\frac{2}{3}$ tells us that for each extra ounce of candle weight, the burning time increases by $6\frac{2}{3}$ hours. A $y-$intercept of zero tells us that a candle of weight 0 oz will burn for 0 hours.
The problem asks for the weight of a candle that burns 95 hours; in other words, what's the $x-$value that gives a $y-$value of 95? Plugging in $y = 95$:

$$y = \frac{20}{3}x \Rightarrow 95 = \frac{20}{3}x \Rightarrow x = \frac{285}{20} = \frac{57}{4} = 14\frac{1}{4}$$

A candle that burns 95 hours weighs 14.25 oz.

A graphing calculator gives the linear regression equation as $y = 6.1x + 5.9$ and a result of **14.6 oz.**

Explore More

For problems 1-4, draw the scatter plot and find an equation that fits the data set by hand.

1. (57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)

2. (32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (39, 48)
3. (12, 18); (5, 24); (15, 16); (11, 19); (9, 12); (7, 13); (6, 17); (12, 14)
4. (3, 12); (8, 20); (1, 7); (10, 23); (5, 18); (8, 24); (2, 10)
5. Use the graph from problem 1 to predict the $y-$values for two $x-$values of your choice that are not in the data set.
6. Use the graph from problem 2 to predict the $x-$values for two $y-$values of your choice that are not in the data set.
7. Use the equation from problem 3 to predict the $y-$values for two $x-$values of your choice that are not in the data set.
8. Use the equation from problem 4 to predict the $x-$values for two $y-$values of your choice that are not in the data set.

For problems 9-11, use a graphing calculator to find the equation of the line of best fit for the data set.

9. (57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)
10. (32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (95, 105); (39, 48)
11. (12, 18); (3, 26); (5, 24); (15, 16); (11, 19); (0, 27); (9, 12); (7, 13); (6, 17); (12, 14)
12. Graph the best fit line on top of the scatter plot for problem 10. Then pick a data point that's close to the line, and change its $y-$value to move it much farther from the line.
    a. Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
    b. How much did the slope of the best fit line change when you changed that point?
13. Graph the scatter plot from problem 11 and change one point as you did in the previous problem.
    a. Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
    b. Did changing that one point seem to affect the slope of the best fit line more or less than it did in the previous problem? What might account for this difference?
14. Shiva is trying to beat the samosa-eating record. The current record is 53.5 samosas in 12 minutes. Each day he practices and the following table shows how many samosas he eats each day for the first week of his training.

TABLE 1.12:

Day	No. of samosas
1	30
2	34
3	36
4	36
5	40
6	43
7	45

(a) Draw a scatter plot and find an equation to fit the data.

(b) Will he be ready for the contest if it occurs two weeks from the day he started training?

(c) What are the meanings of the slope and the $y-$intercept in this problem?

15. Anne is trying to find the elasticity coefficient of a Superball. She drops the ball from different heights and measures the maximum height of the ball after the bounce. The table below shows the data she collected.

TABLE 1.13:

Initial height (cm)	**Bounce height (cm)**
30	22
35	26
40	29
45	34
50	38
55	40
60	45
65	50
70	52

(a) Draw a scatter plot and find the equation.

(b) What height would she have to drop the ball from for it to bounce 65 cm?

(c) What are the meanings of the slope and the $y-$intercept in this problem?

(d) Does the $y-$intercept make sense? Why isn't it (0, 0)?

16. The following table shows the median California family income from 1995 to 2002 as reported by the US Census Bureau.

TABLE 1.14:

Year	**Income**
1995	53,807
1996	55,217
1997	55,209
1998	55,415
1999	63,100
2000	63,206
2001	63,761
2002	65,766

(a) Draw a scatter plot and find the equation.

(b) What would you expect the median annual income of a Californian family to be in year 2010?

(c) What are the meanings of the slope and the $y-$intercept in this problem?

(d) Inflation in the U.S. is measured by the Consumer Price Index, which increased by 20% between 1995 and 2002. Did the median income of California families keep up with inflation over that time period? (In other words, did it increase by at least 20%?)

3.8 Direct Variation.

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a "per pound" basis, and that if you buy two-fifths the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries, or \$5.00.

Similarly, if you bought 10 pounds of strawberries (twice the amount) you would pay twice \$12.50, and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x, then we write the relationship as

$$y = kx$$

k is called the **constant of variation**

If we were to graph this function, you can see that it would pass through the origin, because $y = 0$ when $x = 0$,whatever the value of k. So we know that a direct variation, when graphed, has a single intercept at (0, 0).

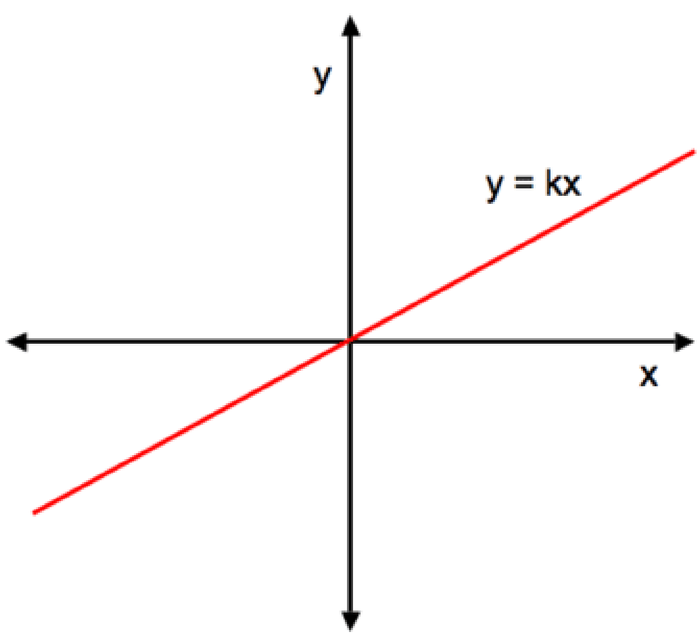


FIGURE 1.36

For examples of how to plot and identify direct variation functions, see the video http://neaportal.k12.ar.us/index.php/2010/06/slope-and-direct-variation/ .

Example A

If y varies directly with x according to the relationship $y = kx$, *and* $y = 7.5$ *when* $x = 2.5$, *determine the constant of variation,* k.

Solution

We can solve for the constant of proportionality using substitution. Substitute $x = 2.5$ and $y = 7.5$ into the equation

$$y = kx$$

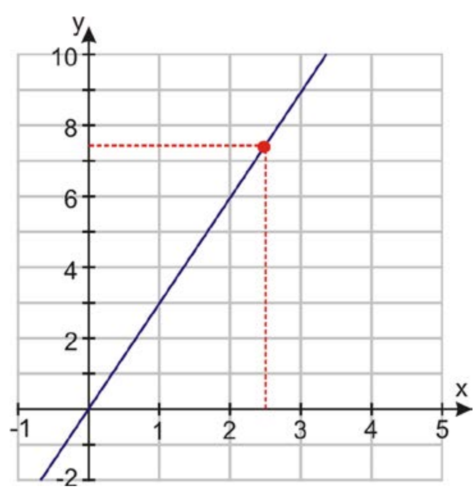
$$7.5 = k(2.5)$$

Divide both sides by 2.5

$$k = \frac{7.5}{2.5} = 3$$

The constant of variation, k**, is 3.**

We can graph the relationship quickly, using the intercept (0, 0) and the point (2.5, 7.5). The graph is shown below. It is a straight line with slope 3.



The graph of a direct variation always passes through the origin, and always has a slope that is equal to the constant of variation, k.

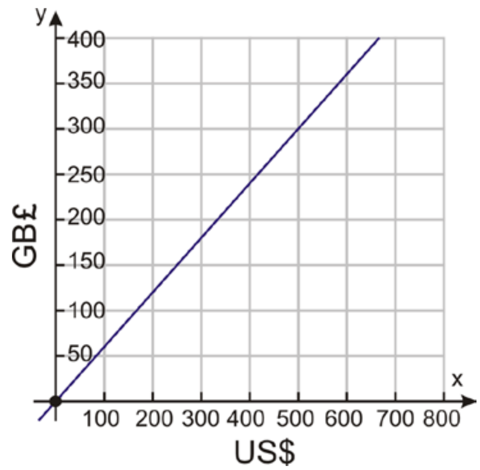
Example B

The graph shown below is a conversion chart used to convert U.S. dollars (US$) to British pounds (GB£) in a bank on a particular day. Use the chart to determine:

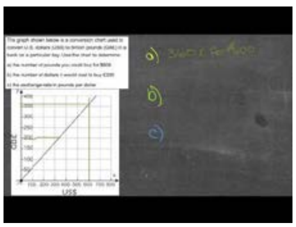
a) the number of pounds you could buy for $600

b) the number of dollars it would cost to buy £200

c) the exchange rate in pounds per dollar



Watch this video for help with the Example above.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/133285

CK-12 Foundation: Direct Variation Models

Solution

We can read the answers to a) and b) right off the graph. It looks as if at $x = 600$ the graph is about one fifth of the way between £350 and £400. So $600 would buy £360.

Similarly, the line $y = 200$ appears to intersect the graph about a third of the way between $300 and $400. We can round this to $330, so it would cost approximately $330 to buy £200.

To solve for the exchange rate, we should note that as this is a direct variation - the graph is a straight line passing through the origin. The slope of the line gives the constant of variation (in this case the **exchange rate**) and it is equal to the ratio of the $y-$value to $x-$value at any point. Looking closely at the graph, we can see that the line passes through one convenient lattice point: (500, 300). This will give us the most accurate value for the slope and so the exchange rate.

$$y = kx \Rightarrow k = \frac{y}{x}$$

$$\text{k} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar.}$$

Graph Direct Variation Equations

We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of variation, k.

Example C

Plot the following direct variations on the same graph.

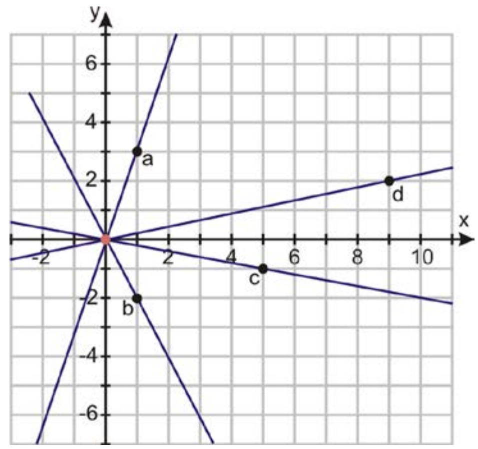
a) $y = 3x$

b) $y = -2x$

c) $y = -0.2x$

d) $y = \frac{2}{9}x$

Solution



All lines pass through the origin (0,0), so this will be the initial point. Apply the slope to determine additional points for each of the equations.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time one quantity increases at the same rate another quantity increases (for example, doubling when it doubles and tripling when it triples), we say that they follow a direct variation.

Newton's Second Law

In 1687 Sir Isaac Newton published the famous *Principia Mathematica*. It contained, among other things, his second law of motion. This law is often written as $F = m \cdot a$, where a force of F Newtons applied to a mass of m kilograms results in acceleration of a meters per second^2. Notice that if the mass stays constant, then this formula is basically the same as the direct variation equation, just with different variables—and m is the constant of variation.

Example D

If a 175 Newton force causes a shopping cart to accelerate down the aisle with an acceleration of $2.5\ m/s^2$, *calculate:*

a) *The mass of the shopping cart.*

b) *The force needed to accelerate the same cart at* $6\ m/s^2$.

Solution

a) We can solve for m (the mass) by plugging in our given values for force and acceleration. $F = m \cdot a$ becomes $175 = m(2.5)$, and then we divide both sides by 2.5 to get $70 = m$.

So **the mass of the shopping cart is 70 kg.**

b) Once we have solved for the mass, we simply substitute that value, plus our required acceleration, back into the formula $F = m \cdot a$ and solve for F. We get $F = 70 \times 6 = 420$.

So **the force needed to accelerate the cart at** $6\ m/s^2$ **is 420 Newtons.**

Vocabulary

- If a variable y varies *directly* with variable x, then we write the relationship as $y = kx$, where k is a constant called the **constant of variation.**

Guided Practice

The volume of water in a fish-tank, V, varies directly with depth, d. If there are 15 gallons in the tank when the depth is 8 inches, calculate how much water is in the tank when the depth is 20 inches.

Solution

Since the volume, V, depends on depth, d, we'll use an equation of the form $y = kx$, but in place of y we'll use V and in place of x we'll use d:

$$V = kd$$

We know that when the depth is 8 inches the volume is 15 gallons, so to solve for k, we plug in 15 for V and 8 for d

$$15 = k(8)$$

Divide both sides by 8

$$k = \frac{15}{8} = 1.875$$

Now to find the volume of water at the final depth, we use $V = kd$ again, but this time we can plug in our new d and the value we found for k:

$$\begin{aligned} V &= 1.875(20) \\ V &= 37.5 \end{aligned}$$

At a depth of 20 inches, the volume of water in the tank is 37.5 gallons.

Independent Practice.

Explore More

For 1-4, plot the following direct variations on the same graph.

1. $y = \frac{4}{3}x$
2. $y = -\frac{2}{3}x$
3. $y = -\frac{1}{6}x$
4. $y = 1.75x$
5. Dasan's mom takes him to the video arcade for his birthday.

For each equation tell whether y varies directly as x.

1. $y = \frac{1}{3}x - 10$
2. $y = 2x$

From the table below tell whether y varies directly as x. If so, name the constant of variation and the equation that shows the relationship.

3.

x	.5	8	12	16
y	2	4	6	8

4.

x	2	3	4	5
y	4	3	8	4

FIGURE 1.37

Solve. Write the answer to the nearest hundredth.

5. y varies directly with x. $y = 6$ when $x = 1$. Find y when $x = 7$.
6. y varies directly with x. $y = 4$ when $x = 2$. Find y when $x = 6$.
7. A is proportional to B. $A = 6$ when $B = 2$. Find A when $B = 1$.
8. P is proportional to Q. $P = 4$ when Q is 1. Find Q when $P = 6$.
9. S is proportional to R. $R = 2.5$ when S is 1. Find R when $S = 10$.
10. D is proportional to C. $D = 18$ when $C = 2$. Find D when $C = 6$.
11. x is proportional to w. $x = 2$ when $w = 0.4$. Find x when $w = 1$.

FIGURE 1.38

a. In the first 10 minutes, he spends $3.50 playing games. If his allowance for the day is $20, how long can he keep playing games before his money is gone?
b. He spends the next 15 minutes playing Alien Invaders. In the first two minutes, he shoots 130 aliens. If he keeps going at this rate, how many aliens will he shoot in fifteen minutes?

c. The high score on this machine is 120000 points. If each alien is worth 100 points, will Dasan beat the high score? What if he keeps playing for five more minutes?

6. The current standard for low-flow showerheads is 2.5 gallons per minute.
   a. How long would it take to fill a 30-gallon bathtub using such a showerhead to supply the water?
   b. If the bathtub drain were not plugged all the way, so that every minute 0.5 gallons ran out as 2.5 gallons ran in, how long would it take to fill the tub?
   c. After the tub was full and the showerhead was turned off, how long would it take the tub to empty through the partly unplugged drain?
   d. If the drain were immediately unplugged all the way when the showerhead was turned off, so that it drained at a rate of 1.5 gallons per minute, how long would it take to empty?
7. Amin is using a hose to fill his new swimming pool for the first time. He starts the hose at 10 PM and leaves it running all night.
   a. At 6 AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
   b. At 10 AM he measures again and realizes his earlier calculations were wrong. The pool is still only three quarters full. When will it actually be full?
   c. After filling the pool, he needs to chlorinate it to a level of 2.0 ppm (parts per million). He adds two gallons of chlorine solution and finds that the chlorine level is now 0.7 ppm. How many more gallons does he need to add?
   d. If the chlorine level in the pool decreases by 0.05 ppm per day, how much solution will he need to add each week?
8. Land in Wisconsin is for sale to property investors. A 232-acre lot is listed for sale for $200,500.
   a. Assuming the same price per acre, how much would a 60-acre lot sell for?
   b. Again assuming the same price, what size lot could you purchase for $100,000?
9. The force (F) needed to stretch a spring by a distance x is given by the equation $F = k \cdot x$, where $k$is the spring constant (measured in Newtons per centimeter, or N/cm). If a 12 Newton force stretches a certain spring by 10 cm, calculate:
   a. The spring constant, k
   b. The force needed to stretch the spring by 7 cm.
   c. The distance the spring would stretch with a 23 Newton force.
10. Angela's cell phone is completely out of power when she puts it on the charger at 3 PM. An hour later, it is 30% charged. When will it be completely charged?
11. It costs $100 to rent a recreation hall for three hours and $150 to rent it for five hours.
    a. Is this a direct variation?
    b. Based on the cost to rent the hall for three hours, what would it cost to rent it for six hours, assuming it is a direct variation?
    c. Based on the cost to rent the hall for five hours, what would it cost to rent it for six hours, assuming it is a direct variation?
    d. Plot the costs given for three and five hours and graph the line through those points. Based on that graph, what would you expect the cost to be for a six-hour rental?

3.9 Chapter 3 Review

Find the x and y intercept of the following graphs.

1. 2. 3.

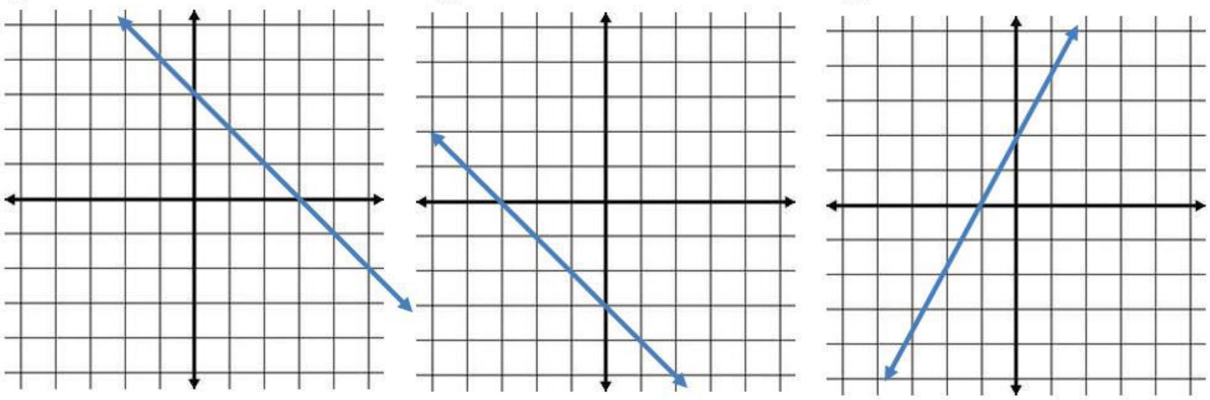


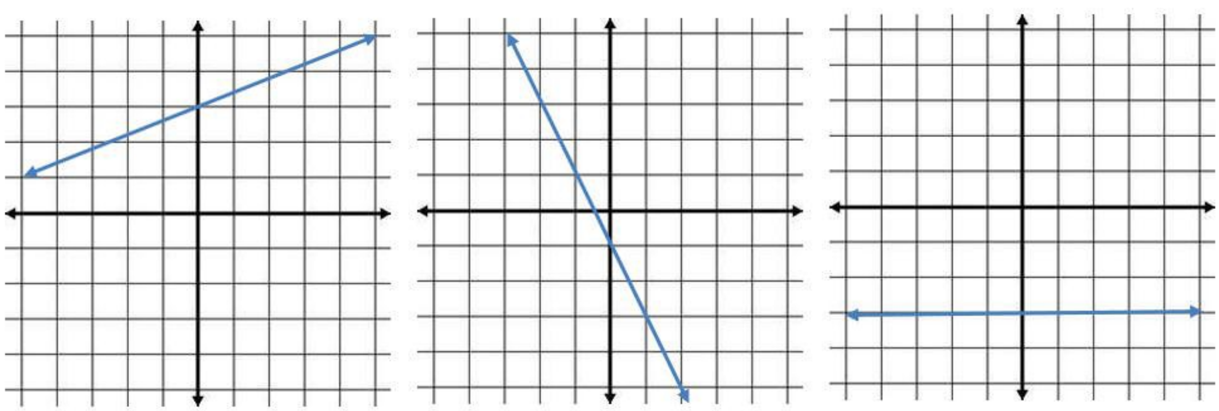
FIGURE 1.39

TABLE 1.15:

4. $x - y = -2$
5. $6x + 2y = 12$
6. $y = \frac{1}{3}x - 4$
7. $y = x$
8. $y = \frac{2}{3}x + 5$
9. $y = 14x + 7$

Find the slope of the following graphs.

10. 11. 12.



Find the slope of the line passing through the given points.

13. (-5, 7) and (0, 0)
14. (-3, -5) and (3, 11)
15. (3, -5) and (-2, 9)
16. (-5, 7) and (-5, 11)
17. (9, 9) and (-9, -9)
18. (3, 5) and (-2, 7)
19. (2.5, 3) and (8, 3.5)

20.

x	y
0	0
30	-20
60	-40
90	-60

21.

x	y
-2	-12
0	-2
2	8
4	18

22.

x	y
-5	-6
0	-5
5	-4
10	-3

Plot the point, count the slope and then graph the line.

23. (3, 3) $m = -1$
24. (-4, 3) $m = \frac{1}{5}$
25. (0, -2) $m = 0$

Graph the line when given the slope (m) and y-intercept (b).

26. $m = -3$ $b = -1$
27. $m = -\frac{1}{4}$ $b = 1$
28. $m = \frac{2}{7}$ $b = -5$

Graph the line given the equation in slope-intercept form y=mx+b. Identify the slope (m) and y-intercept (b).

29. $y = 0.5x + 3$
30. $y = \frac{3}{5}x - 3$
31. $y = -3x + 1$

Graph the equation given in standard form (Ax + By = C).

32. $2x - 2y = -10$
33. $2x + y = 4$
34. $-x - 3y = -12$

Systems of Linear Equations

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CHAPTER 4 Systems of Linear Equations

CHAPTER OUTLINE

4.1 Identifying Solutions of Systems of Linear Equations

System of Equations

Example A

Determine which of the points (1, 3), (0, 2), or (2, 7) is a solution to the following system of equations:

$$y = 4x - 1$$
$$y = 2x + 3$$

Solution To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.Point (1, 3):

$$\begin{aligned} y &= 4x - 1 \\ 3 &= 4(1) - 1 \\ 3 &= 3 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 3 &= 2(1) + 3 \\ 3 &\neq 5 \text{ solution does not check} \end{aligned}$$

Point (1, 3) is on the line $y = 4x - 1$, but it is not on the line $y = 2x + 3$, so it is not a solution to the system.Point (0, 2):

$$\begin{aligned} y &= 4x - 1 \\ 2 &= 4(0) - 1 \\ 2 &\neq -1 \text{ solution does not check} \end{aligned}$$

Point (0, 2) is not on the line $y = 4x - 1$, so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.Point (2, 7):

$$\begin{aligned} y &= 4x - 1 \\ 7 &= 4(2) - 1 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 7 &= 2(2) + 3 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

Point (2, 7) is the solution to the system since it lies on both lines.

Independent Practice

Determine which ordered pair satisfies the system of linear equations.

1.

$$y = 3x - 2$$
$$y = -x$$

1. (1, 4)
2. (2, 9)
3. $\left(\frac{1}{2}, \frac{-1}{2}\right)$

2.

$$\begin{aligned} y &= 2x - 3 \\ y &= x + 5 \end{aligned}$$

a. (8, 13)
b. (-7, 6)
c. (0, 4)

3.

$$\begin{aligned} 2x + y &= 8 \\ 5x + 2y &= 10 \end{aligned}$$

a. (-9, 1)
b. (-6, 20)
c. (14, 2)

4.

$$\begin{aligned} 3x + 2y &= 6 \\ y &= \frac{1}{2}x - 3 \end{aligned}$$

a. $\left(3, \frac{-3}{2}\right)$
b. (-4, 3)
c. $\left(\frac{1}{2}, 4\right)$

5.

$$\begin{aligned} 2x - y &= 10 \\ 3x + y &= -5 \end{aligned}$$

a. (4, -2)
b. (1, -8)
c. (-2, 5)

of

4.2 Solving Linear Systems by Graphing

A1 2(B) Write linear equations in two variables in various forms, including y = mx + b, Ax + By = C, and y - y_1 = m(x - x_1), given one point and the slope and given two points.

A1 3(F) Graph systems of two linear equations in two variables on the coordinate plane and determine the solutions if they exist.

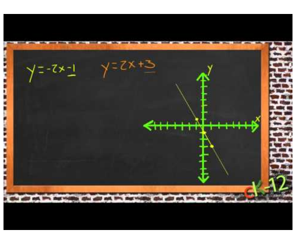
A1 5(C) Solve systems of two linear equations with two variables for mathematical and real-world problems. AI 5(C) Solve systems of two linear equations with two variables for mathematical and real-world problems. TEKS: A1 2(B); A1 5(C); A1 3(F)

Learning Objective

Here you'll learn how to determine whether an ordered pair is a solution to a system of equations. You'll also learn how to solve a system of equations by graphing. Finally, you'll solve word problems involving systems of equations.

What if you were given a set of linear equations like $y = -3x + 4$ and $y = 6x - 1$? How could you determine the solution(s) that both equations have in common? After completing this concept, you'll be able to determine if an ordered pair is a solution to a system of equations and you'll find such solutions by graphing.

Watch This



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CK-12 Foundation: 0701S Linear Systems by Graphing (H264)

Guided Instruction

In this concept, first we'll discover methods to determine if an ordered pair is a solution to a system of two equations. Then we'll learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, we'll look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

A linear system of equations is a set of equations that must be solved together to find the one solution that fits them both.

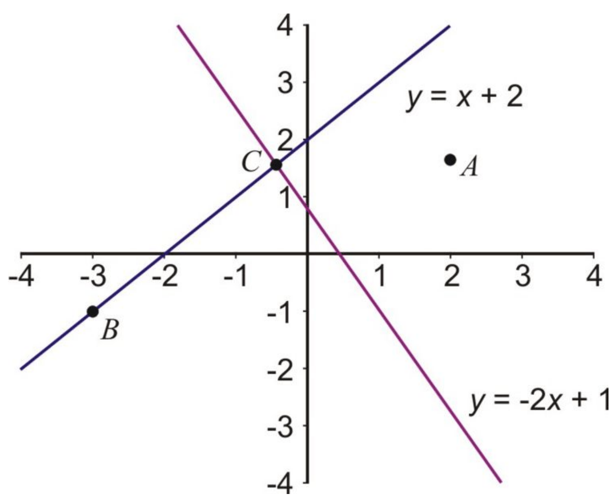
Consider this system of equations:

$$y = x + 2$$
$$y = -2x + 1$$

Since the two lines are in a system, we deal with them together by graphing them on the same coordinate plane.

We can use the following methods to graph:

- Graph using your y-intercept and slope
- Graph using your x- and y-intercepts
- Graph using points from a table
- Graph using your calculator



We already know that any point that lies on a line is a solution to the equation for that line. That means that any point that lies on *both* lines in a system is a solution to both equations.

So in this system:

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point B is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In fact, point C is the only solution to the system, because it is the only point that lies on both lines. For a system of equations, the solution is the intersection of the two lines, which are the coordinates of that intersection point.

You can confirm the solution by plugging it into the system of equations, and checking that the solution works in each equation.

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point, (if there is one) that lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

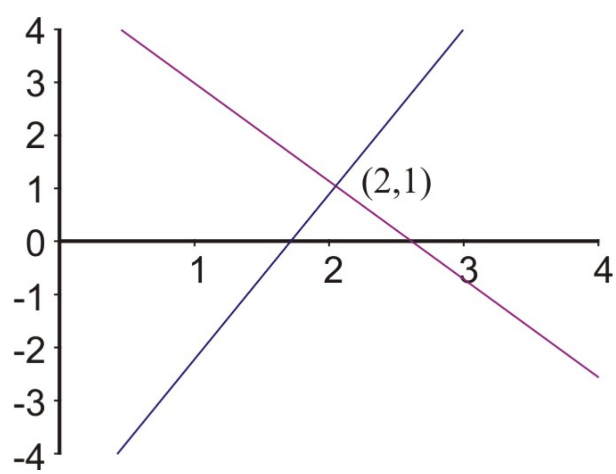
Example A

Solve the following system of equations by graphing:

$$\begin{aligned} y &= 3x - 5 \\ y &= -2x + 5 \end{aligned}$$

Solution

Graph both lines on the same coordinate plane using any method you like.



The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point (2, 1). So **the solution is** $x = 2, y = 1$ **or (2, 1).**

Solving a System of Equations Using a Graphing Calculator

As an alternative to graphing by hand, you can use a graphing calculator to find or check solutions to a system of equations.

Example B

Solve the following system of equations using a graphing calculator.

$$x-3y=4$$
$$2x+5y=8$$

To input the equations into the calculator, you need to rewrite them in slope-intercept form, $y=mx+b$.

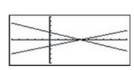
$$x-3y=4$$
$$y=\frac{1}{3}x-\frac{4}{3}$$

$$2x+5y=8$$
$$y=-\frac{2}{5}x+\frac{8}{5}$$

Press the **[y=]** button on the graphing calculator and enter the two functions as:

$$Y_1=(\frac{1}{3})x-(\frac{4}{3})$$
$$Y_2=(\frac{-2}{5})x+(\frac{8}{5})$$

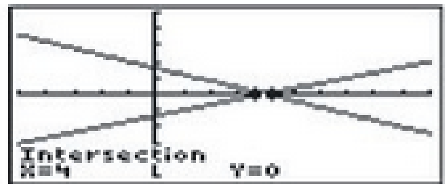
Now press **[GRAPH]**. Here's what the graph should look like on a TI-83 graphing calculator.



There are a few different ways to find the intersection point.

Option 1:

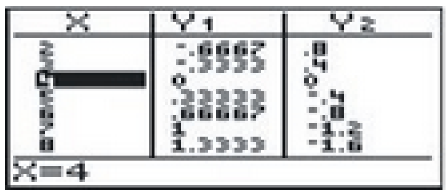
- Using the **[2nd] [TRACE]** function (see the second screen below)
- Scroll down and select **[intersect]**
- The calculator will display the graph with the question **[FIRSTCURVE?]** then press **[ENTER]**
- The calculator now shows **[SECONDCURVE?]** then press **[ENTER]**
- The calculator displays **[GUESS?]** then press **[ENTER]**
- The calculator displays the solution at the bottom of the screen (see the right screen below).



The point of intersection is $x = 4$ and $y = 0$, which is the ordered pair (4,0).

***Option 2*:**

- Look at the table of values by pressing **[2nd] [GRAPH]** shows a table of values for this system of equations (see screen below)
- Scroll down until the $y-$values for the two functions are the same. In this case this occurs at $x = 4$ and $y = 0$.



Vocabulary

- A **linear system of equations** is two or more equations.
- A **solution** to a system is an ordered pair that makes all equations true. In other words, the **intersention** point.

Guided Practice

Solve the following system of equations by graphing:

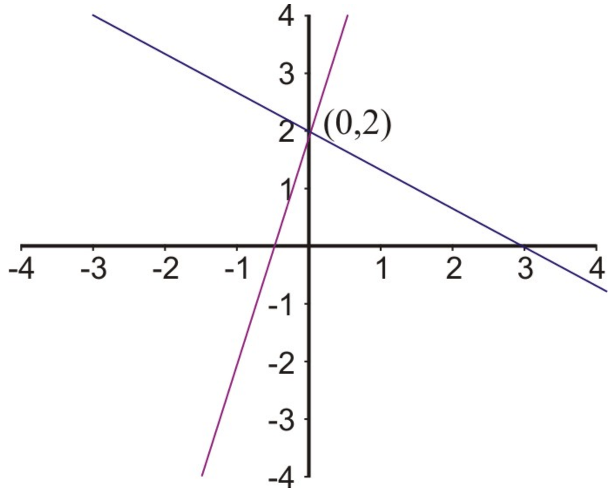
$$2x + 3y = 6$$
$$4x - y = -2$$

Solution

Since the equations are in standard form, we need to rewrite them in slope intercept form in order to graph.

$$2x + 3y = 6$$
$$y = -\frac{2}{3}x + 2$$

$$-4x + y = 2$$
$$y = 4x + 2$$



The graph shows that the lines intersect at (0, 2). Therefore, **the solution to the system of equations is** $x = 0, y = 2$.

Explore More

Solve the following systems using the graphing method.

1. $y = x + 3$ and $y = -x + 3$
2. $y = 3x - 6$ and $y = -x + 6$
3. $2x = 4$ and $y = -3$
4. $y = -x + 5$ and $-x + y = 1$
5. $x + 2y = 8$ and $5x + 2y = 0$
6. $3x + 2y = 12$ and $4x - y = 5$
7. $5x + 2y = -4$ and $x - y = 2$
8. $2x + 4 = 3y$ and $x - 2y + 4 = 0$
9. $y = \frac{1}{2}x - 3$ and $2x - 5y = 5$
10. $y = 4$ and $x = 8 - 3y$
11. Solve the following system using the graphing method:

$$y = \frac{3}{5}x + 5$$
$$y = -2x - \frac{1}{2}$$

Why is it difficult to find the real solution to this system?

12. Solve the following system using the graphing method:

$$y = 4x + 8$$
$$y = 5x + 1$$

1. Do these lines appear to intersect?
2. Based on their equations, are they parallel?

Solve each system of equations by graphing

13. $y = 2x + 6$
    $y = -x - 3$

14. $4x - y = 3$
    $2x + y = 9$

15. $y = \frac{5}{2}x - 4$
    $y = -x + 3$

16. $y = -7x - 3$
    $y = 4$

17. $y = \frac{3}{4}x - 5$
    $y = \frac{3}{4}x$

18. $y = 2x + 3$
    $-10x + 5y = 15$

FIGURE 1.1

4.3 Solving Linear Systems Using Substitution

A1 5(C) Solve systems of two linear equations with two variables for mathematical and real-world problems.

A1 2(B) Write linear equations in two variables in various forms, including y = mx + b, Ax + By = C, and y - y1 = m(x - x1), given one point and the slope and given two points.

A1 1(B). Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution.

TEKS: A1 5(C); A1 2(B); A1 1B

Learning Objective

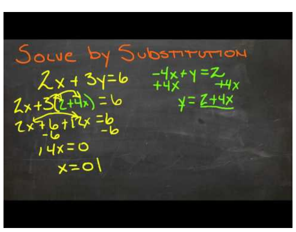
Here you'll learn how to use substitution to solve systems of linear equations in two variables. You'll then solve real-world problems involving such systems.

What if you were given a system of linear equations like $x-y=7$ and $3x-4y=-3$? How could you substitute one equation into the other to solve for the variables? After completing this concept, you'll be able to solve a system of linear equations by substitution.

Guided Instruction

In this lesson, we'll learn to solve a system of two equations using the method of substitution.

Watch this video for help with the Examples below.



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CK-12 Foundation: Linear Systems by Substitution

Example A

Let's look at an example where the equations are written in **standard form.**

Solve the system

$$\begin{aligned} 2x+3y&=6 \\ -4x+y&=2 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. So the easiest way to start is to use this equation to solve for y.

Solve the second equation for y:

$$\begin{aligned} -4x+y&=2 && \textit{add } 4x \textit{ to both sides}:\\ y&=2+4x \end{aligned}$$

Substitute this expression into the first equation:

$$\begin{aligned} 2x+3(2+4x)&=6 && \textit{distribute the } 3:\\ 2x+6+12x&=6 && \textit{collect like terms}:\\ 14x+6&=6 && \textit{subtract } 6 \textit{ from both sides}:\\ 14x&=0 && \textit{and hence}:\\ x&=0 \end{aligned}$$

Substitute $x=0$ back into our expression and solve for y:

$$\begin{aligned} y&=2+2x\\ y&=2+4(0)\\ y&=2 \end{aligned}$$

As you can see, we end up with the same solution $(x=0, y=2)$ that we found when we graphed these functions in the previous lesson.

Example B

Solve the system

$$\begin{aligned} 2x+3y&=3\\ 2x-3y&=-1 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x:

$$\begin{aligned} 2x+3y&=3 && \textit{subtract } 3y \textit{ from both sides}:\\ 2x&=3-3y && \textit{divide both sides by } 2:\\ x&=\frac{1}{2}(3-3y) \end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned}
\cancel{2}\cdot\frac{1}{2}(3-3y)-3y &= -1 && \textit{cancel the fraction and re-write terms:}\\
3-3y-3y &= -1 && \textit{collect like terms:}\\
3-6y &= -1 && \textit{subtract 3 from both sides:}\\
-6y &= -4 && \textit{divide by } -6:\\
y &= \frac{2}{3}
\end{aligned}$$

Substitute $y=\frac{2}{3}$into the expression we isolated and solve for x:

$$\begin{aligned}
x &= \frac{1}{2}(3-3y)\\
x &= \frac{1}{2}\left(3-3\left(\frac{2}{3}\right)\right)\\
x &= \frac{1}{2}
\end{aligned}$$

So our solution is $x=\frac{1}{2}, y=\frac{2}{3}$.

Vocabulary

- Solving linear systems **by substitution** means to replace a variable with an equivalent expression.

Guided Practice

Solve the system

$$\begin{aligned}
8x+10y &= 2\\
4x-15y &= -19
\end{aligned}$$

Solution:

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x:

$$\begin{aligned}
8x+10y &= 2 && \textit{subtract } 10y \textit{ from both sides:}\\
8x &= 2-10y && \textit{divide both sides by } 8:\\
x &= \frac{1}{8}(2-10y)
\end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned}
4 \cdot \frac{1}{8}(2-10y)-15y &= -19 && \mathit{simplify\ the\ fraction:}\\
\frac{1}{2}(2-10y)-15y &= -19 && \mathit{distribute\ the\ fraction\ and\ re-write\ terms:}\\
1-5y-15y &= -19 && \mathit{collect\ like\ terms:}\\
1-20y &= -19 && \mathit{subtract\ 1\ from\ both\ sides:}\\
-20y &= -20 && \mathit{divide\ by\ -20:}\\
y &= 1
\end{aligned}$$

Substitute into the expression we got for x:

$$\begin{aligned}
x &= \frac{1}{8}(2-10y) && \mathit{Substitute\ the\ y-value\ into\ the\ x\ equation:}\\
x &= \frac{1}{8}(2-10(1)) && \mathit{Simplify}\\
x &= \frac{1}{8}(2-10)\\
x &= \frac{1}{8}(-8)\\
x &= -1
\end{aligned}$$

So our solution is $x=-1, y=1$.

Try This

For lots more practice solving linear systems, check out this web page: http://www.algebra.com/algebra/homework/coordinate/practice-linear-system.epl

After clicking to see the solution to a problem, you can click the back button and then click Try Another Practice Linear System to see another problem.

Explore More

1. Solve the system:

$$\begin{aligned}
x+2y &= 9\\
3x+5y &= 20
\end{aligned}$$

2. Solve the system:

$$\begin{aligned}
x-3y &= 10\\
2x+y &= 13
\end{aligned}$$

3. Solve the system:

$$\begin{aligned}
2x+0.5y &= -10\\
x-y &= -10
\end{aligned}$$

4. Solve the system:

$$\begin{aligned} 2x+0.5y &= 3 \\ x+2y &= 8.5 \end{aligned}$$

5. Solve the system:

$$\begin{aligned} 3x+5y &= -1 \\ x+2y &= -1 \end{aligned}$$

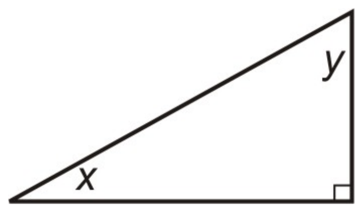
6. Solve the system:

$$\begin{aligned} 3x+5y &= -3 \\ x+2y &= -\frac{4}{3} \end{aligned}$$

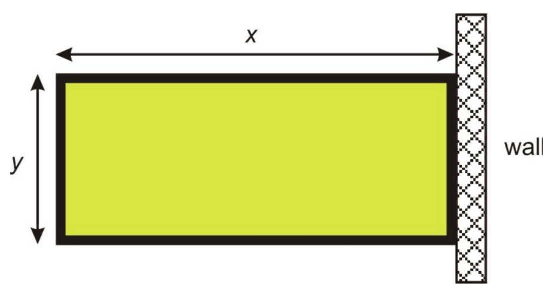
7. Solve the system:

$$\begin{aligned} x-y &= -\frac{12}{5} \\ 2x+5y &= -2 \end{aligned}$$

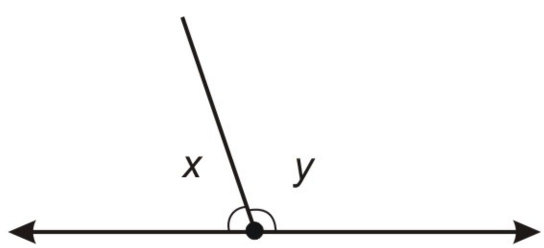
8. Of the two non-right angles in a right angled triangle, one measures twice as many degrees as the other. What are the angles?



9. The sum of two numbers is 70. They differ by 11. What are the numbers?
10. A number plus half of another number equals 6; twice the first number minus three times the second number equals 4. What are the numbers?
11. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



12. A ray cuts a line forming two angles. The difference between the two angles is 18°. What does each angle measure?



13. Jason is five years older than Becky, and the sum of their ages is 23. What are their ages?

Learning Objective Learning Objective Learning Objective

4.4 Solving Linear Systems Using Elimination

A1 5(C) Solve systems of two linear equations with two variables for mathematical and real-world problems.

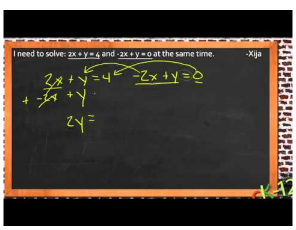
A1 1(B) Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution. TEKS: A1 5(C); A1 1(B)

Learning Objective

Here you'll learn how to solve systems of linear equations in two variables by eliminating one of the variables. You'll then solve real-world problems involving such systems.

What if you were given a system of linear equations like $x+4y=7$ and $3x-4y=-3$? How could you solve for one of the variables by eliminating the other? After completing this concept, you'll be able to solve a system of linear equations by elimination.

Watch This



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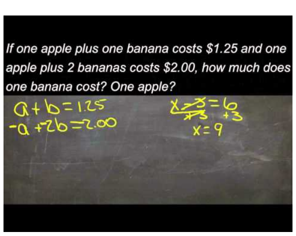
CK-12 Foundation: 0704S Solving Linear Systems by Elimination (H264)

Guided Instruction

In this lesson, we'll see how to use simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns (x and y) to a single unknown (either x or y), this method is often referred to by ***solving by elimination***. We eliminate one variable in order to make our equations solvable! To illustrate this idea, let's look at the simple example of buying apples and bananas.

Watch this video for help with the Examples below.



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CK-12 Foundation: Linear Systems by Elimination

Example A

If one apple plus one banana costs $1.25 and one apple plus 2 bananas costs $2.00, how much does one banana cost? One apple?

It shouldn't take too long to discover that each banana costs $0.75. After all, the second purchase just contains 1 more banana than the first, and costs $0.75 more, so that one banana must cost $0.75.

Here's what we get when we describe this situation with algebra:

$$\begin{aligned} a+b &= 1.25 \\ a+2b &= 2.00 \end{aligned}$$

$$\begin{aligned} -1(a+b=1.25) \quad &\rightarrow \qquad -a-b=-1.25 \\ a+2b=2.00 \\ a+2b &= 2.00 \\ b &= .75 \end{aligned}$$

That gives us the cost of one banana. To find out how much one apple costs, we substitute $0.75 for b.

$$\begin{aligned} a+b &= 1.23 \\ a+0.75 &= 1.25 \\ a &= 0.50 \end{aligned}$$

So an apple costs 50 cents.

Solving Linear Systems Using Addition of Equations

Often considered the easiest and most powerful method of solving systems of equations, the elimination method lets us combine two equations in such a way that the resulting equation has only one variable. We can then use simple algebra to solve for that variable. Then, if we need to, we can substitute the value we get for that variable back into either one of the original equations to solve for the other variable.

Example B

Solve this system using elimination:

$$\begin{aligned} 3x+2y &= 11 \\ 5x-2y &= 13 \end{aligned}$$

Solution

A simpler way to visualize this is to keep the equations as they appear above, and to add them together vertically, going down the columns. However, just like when you add units, tens and hundreds, you MUST be sure to keep the x's and y's in their own columns.

$$\begin{aligned} 3x+2y &= 11 \\ \underline{5x-2y} &\underline{= 13} \\ 8x &= 24 \\ x &= 3 \end{aligned}$$

To find a value for y, we simply substitute our value for x back in.

Substitute $x = 3$ into the second equation:

$$\begin{aligned} 5x-2y &= 13 \\ 5(3)-2y &= 13 \\ 15-2y &= 13 \\ -2y &= -2 \\ y &= 1 \end{aligned}$$

Therefore, the solution is (3,1).

Example C

Peter examines the coins in the fountain at the mall. He counts 107 coins, all of which are either pennies or nickels. The total value of the coins is $3.47. How many of each coin did he see?

Solution

We have 2 types of coins, so let's call the number of pennies x and the number of nickels y. The total value of all the pennies is just x, since they are worth 1¢ each. The total value of the nickels is $5y$. We are given two key pieces of information to make our equations: the number of coins and their value in cents.

$$\begin{aligned} \text{\# of coins equation} : x+y &= 107 \\ \text{value equation} : x+5y &= 347 \end{aligned}$$

We'll jump straight to subtracting the two equations:

$$\begin{aligned} x+y = 107 \qquad & \qquad x+y = 107 \\ -(x+5y = 347) \quad \rightarrow & \quad \underline{-x-5y = -347} \\ & \qquad -4y = -240 \\ & \qquad y = 60 \end{aligned}$$

Substituting this value back into the first equation:

$$\begin{aligned} x+60 &= 107 \\ x &= 47 \end{aligned}$$

So Peter saw 47 pennies (worth 47 cents) and 60 nickels (worth $3.00) making a total of $3.47.

Vocabulary

- The purpose of using the **elimination method** is to cancel, or eliminate, a variable by either adding the two equations. Sometimes the equations must be multiplied by a number first, in order to cancel out a variable.

Guided Practice

Solve this systems using elimination.

$$\begin{aligned} x+y&=7 \\ x-y&=1.5 \end{aligned}$$

Solution

$$\begin{aligned} x+y&=7 \\ \underline{\quad x-y}&\underline{=1.5} \\ 2x&=8.5 \\ x&=4.25 \end{aligned}$$

To find a corresponding value for y, we plug our value for x into either equation and isolate our unknown. In this example, we'll plug it into the first equation:

$$\begin{aligned} x+y&=7 \\ 4.25+y&=7 \\ y&=2.75 \end{aligned}$$

Therefore, the solution is (4.25, 2.75).

Explore More

1. Solve the system:

$$\begin{aligned} 3x+4y&=2.5 \\ 5x-4y&=25.5 \end{aligned}$$

2. Solve the system:

$$\begin{aligned} 2x+-y&=10 \\ 3x+y&=-5 \end{aligned}$$

3. Solve the system:

$$\begin{aligned} 5x+7y&=-31 \\ 5x-9y&=17 \end{aligned}$$

4. Solve the system:

$$3y-4x=-33$$
$$5x-3y=40.5$$

5. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for $2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs $1.79. What is the cost of a candy bar and a fruit roll-up individually?
6. A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane) and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at 275 miles per hour. Another, identical plane, moving in the opposite direction has a ground-speed of 227 miles per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.
7. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a 12-mile journey costs $14.29 and a 17-mile journey costs $19.91, calculate:
   a. the pick-up fee
   b. the per-mile rate
   c. the cost of a seven mile trip
8. Calls from a call-box are charged per minute at one rate for the first five minutes, then a different rate for each additional minute. If a 7-minute call costs $4.25 and a 12-minute call costs $5.50, find each rate.
9. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns $35 per hour, and the builder earns $28 per hour. Together they were paid $330.75, but the plumber earned $106.75 more than the builder. How many hours did each work?
10. Paul has a part time job selling computers at a local electronics store. He earns a fixed hourly wage, but can earn a bonus by selling warranties for the computers he sells. He works 20 hours per week. In his first week, he sold eight warranties and earned $220. In his second week, he managed to sell 13 warranties and earned $280. What is Paul's hourly rate, and how much extra does he get for selling each warranty?

4.5 Solving Linear Systems Using Elimination Continued

A1 5(C) Solve systems of two linear equations with two variables for mathematical and real-world problems.

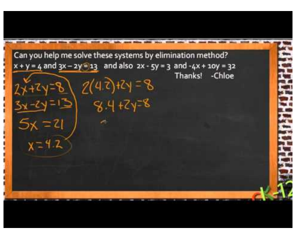
A1 1(B). Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution. TEKS: A1 5(C); A1 1(B)

Learning Objective

Here you'll learn how to solve systems of linear equations in two variables by first multiplying and then eliminating one of the variables. You'll then solve real-world problems involving such systems.

What if you were given a system of linear equations like $x-2y=7$ and $3x-4y=-3$? How could you solve for one of the variables by eliminating the other? After completing this concept, you'll be able to solve a system of linear equations by multiplication and then elimination.

Watch This



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CK-12 Foundation: 0705S Solving Linear Systems by Elimination and Multiplication (H264)

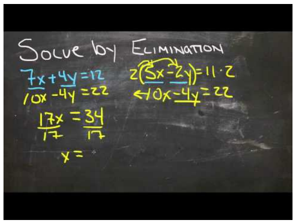
Guided Instruction

Solving a Linear System by Multiplying One Equation

If we can multiply every term in an equation by a fixed number (a **scalar**), that means we can use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match.

This is easiest to do when the coefficient as a variable in one equation is a multiple of the coefficient in the other equation.

Watch this video for help with the Examples below.



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CK-12 Foundation: Linear Systems by Elimination and Multiplication

Example A

Solve the system:

$$\begin{aligned}7x+4y&=17\\5x-2y&=11\end{aligned}$$

Solution

You can easily see that if we multiply the second equation by 2, the coefficients of y will be +4 and -4, allowing us to solve the system by addition:

$$\begin{array}{rcr}7x+4y=17 & & 7x+4y=17\\ \underline{2(5x-2y=11)} & \rightarrow & \underline{10x-4y=22}\\ & & 17x=34\\ & & x=2\end{array}$$

Now simply substitute this value for x back into equation 1:

$$\begin{aligned}7x+4y&=17\\7(2)+4y&=17\\14+4y&=17\\4y&=3\\y&=0.75\end{aligned}$$

Example B

Solve the following system

$$\begin{aligned}2x+2y&=400\\8x-8y&=400\end{aligned}$$

Solution

$$\begin{array}{rcr} 4(2x+2y=400) & \rightarrow & 8x+8y=1600 \\ \underline{8x-8y=400} & & \underline{8x-8y=400} \\ & & 16x=2000 \\ & & x=125 \end{array}$$

Substitute this value back into the first equation:

$$\begin{aligned} 2x+2y&=400 \\ 2(125)+2y&=400 \\ 250+2y&=400 \\ 2y&=150 \\ y&=75 \end{aligned}$$

Vocabulary

- **Elimination method:** The purpose of the **elimination method** to solve a system is to cancel, or eliminate, a variable by either adding or subtracting the two equations. Sometimes the equations must be multiplied by scalars first, in order to cancel out a variable.

- **Least common multiple:** The ***least common multiple*** is the smallest value that is divisible by two or more quantities **without a remainder**.

Guided Practice

Solve the system $\begin{cases} 12x+21y=18 \\ 6x+\ \ 5y=20 \end{cases}$.

Solution:

Neither x nor y have additive inverse coefficients, but the x-variables do share a common factor of 2. Thus we can most easily eliminate x.

We need to multiply the second equation by -2.

$$\begin{cases} 12x\ \ +21y=18 \\ -2(6x+\ \ 5y=20) \end{cases} \rightarrow \begin{cases} 12x+21y=18 \\ -12x-10y=-40 \end{cases}$$

$$\begin{array}{lr} \text{Add the two equations} & 11y=-22 \\ \text{Divide by} & y=-2 \end{array}$$

To find the $x-$value, use the Substitution Property in either equation.

$$\begin{aligned} 12x+21(-2) &= 18 \\ 12x-42 &= 18 \\ 12x &= 60 \\ x &= 5 \end{aligned}$$

The solution to this system is $(5,-2)$.

Explore More

Solve the following systems using multiplication.

1. $5x-10y=15$
   $3x-2y=3$
2. $5x-y=10$
   $3x-2y=-1$
3. $5x+7y=15$
   $7x-3y=5$
4. $9x+5y=9$
   $12x+8y=12.8$
5. $4x-3y=1$
   $3x-4y=4$
6. $7x-3y=-3$
   $6x+4y=3$
7. $x=3y$
   $x-2y=-3$
8. $y=3x+2$
   $y=-2x+7$
9. $5x-5y=5$
   $5x+5y=35$
10. $y=-3x-3$
    $3x-2y+12=0$
11. $3x-4y=3$
    $4y+5x=10$
12. $9x-2y=-4$
    $2x-6y=1$

Inequalities

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CHAPTER 5 Inequalities

CHAPTER OUTLINE

5.1 Solving One-Step Inequalities

A1 5(B) Solve linear inequalities in one variable, including those for which the application of the distributive property is necessary and for which variables are included on both sides.

TEKS: A1 5(B);

Learning Objective

Here you'll learn how to solve inequalities by isolating the variable on one side of the inequality sign. You'll also learn how to graph their solution set.

What if you had an inequality with an unknown variable like $x-12>-5$? How could you isolate the variable to find its value? After completing this concept, you'll be able to solve one-step inequalities like this one.

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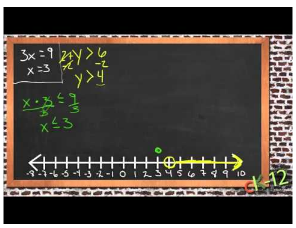

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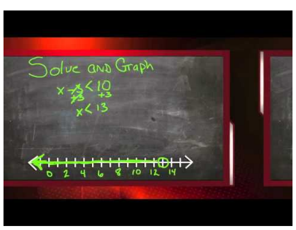
CK-12 Foundation: 0602S Solving One-Step Inequalities (H264)

Guided Instruction

To solve an inequality we must isolate the variable on one side of the inequality sign. To isolate the variable, we use the same basic techniques used in solving equations.

We can solve some inequalities by adding or subtracting a constant from one side of the inequality.

Watch this video for help with the Examples below.



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CK-12 Foundation: Solving One-Step Inequalities

Example A

Solve the inequality and graph the solution set.

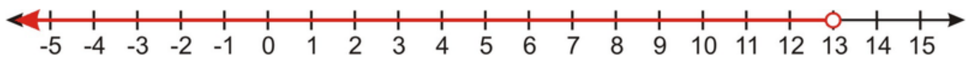
$x-3<10$

Solution

Starting inequality: $x-3<10$

Add **3** to both sides of the inequality: $x-3+3<10+3$

Simplify: $x<13$



Example B

Solve the inequality and graph the solution set.

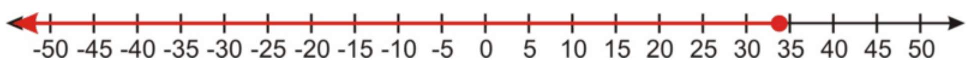
$x-20 \leq 14$

Solution:

Starting inequality: $x-20 \leq 14$

Add **20** to both sides of the inequality: $x-20+20 \leq 14+20$

Simplify: $x \leq 34$



Solving Inequalities Using Multiplication and Division

We can also solve inequalities by multiplying or dividing both sides by a constant. For example, to solve the inequality $5x<3$, we would divide both sides by 5 to get $x<\frac{3}{5}$.

However, something different happens when we multiply or divide by a negative number. We know, for example, that 5 is greater than 3. But if we multiply both sides of the inequality

$$5>3$$

by -2, we get

$$-10>-6$$

And we know that's **not true**; -10 is less than -6.

This happens whenever we multiply or divide an inequality by a negative number, and so we have to flip the sign around to make the inequality true. For example, to multiply $2<4$ by -3, first we multiply the 2 and the 4 each by -3, and then we change the <sign to a >sign, so we end up with $-6>-12$.

The same principle applies when the inequality contains variables.

Example C

Solve the inequality.

$4x<24$

Solution:

Original problem: $4x<24$

Divide both sides by 4: $\frac{4x}{4}<\frac{24}{4}$

Simplify: $x<6$

Example D

Solve the inequality.

$-5x \leq 21$

Solution:

Original problem: $-5x \leq 21$

Divide both sides by -5 : $\frac{-5x}{-5} \geq \frac{21}{-5}$***Flip the inequality sign.***

Simplify: $x \geq -\frac{21}{5}$

Vocabulary

- The answer to an **inequality** is usually an **interval of values**.
- Solving inequalities works just like solving an equation. To solve, we isolate the variable on one side of the equation.
- When **multiplying or dividing both sides of an inequality by a negative number**, you need to ***reverse the inequality***.

Guided Practice

Solve each inequality.

a) $x+8 \leq -7$

b) $x+4 > 13$

c) $\frac{x}{25} < \frac{3}{2}$

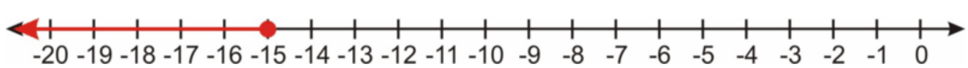
d) $\frac{x}{-7} \geq 9$

Solutions:

a) Starting inequality: $x+8 \leq -7$

Subtract **8** from both sides of the inequality: $x+8-8 \leq -7-8$

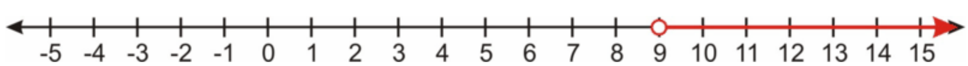
Simplify: $x \leq -15$



b) Starting inequality: $x+4 > 13$

Subtract **4** from both sides of the inequality: $x+4-4 > 13-4$

Simplify: $x > 9$



c) Original problem: $\frac{x}{25} < \frac{3}{2}$

Multiply both sides by 25: $25 \cdot \frac{x}{25} < \frac{3}{2} \cdot 25$

Simplify: $x < \frac{75}{2}$ or $x < 37.5$

d) Original problem: $\frac{x}{-7} \geq 9$

Multiply both sides by -7: $-7 \cdot \frac{x}{-7} \leq 9 \cdot (-7)$***Flip the inequality sign.***

Simplify: $x \leq -63$

Explore More

For 1-8, solve each inequality and graph the solution on the number line.

1. $x-5<35$
2. $x+15\geq -60$
3. $x-2\leq 1$
4. $x-8>-20$
5. $x+11>13$
6. $x+65<100$
7. $x-32\leq 0$
8. $x+68\geq 75$

For 9-12, solve each inequality. Write the solution as an inequality and graph it.

9. $3x\leq 6$
10. $\frac{x}{5}>-\frac{3}{10}$
11. $-10x>250$
12. $\frac{x}{-7}\geq -5$

[U+EFFE] AI 5(B) Solve linear inequalities in one variable, including those for which the application of the distributive property is necessary and for which variables are included on both sides. Here you'll learn how to solve inequalities by isolating the variable on one side of the inequality sign. You'll also learn how to graph their solution set.

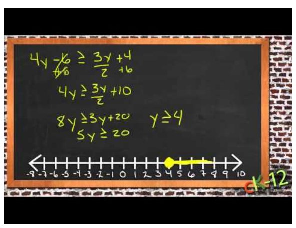
5.2 Solving Multi-Step Inequalities

Learning Objective

Here you'll learn how to solve inequalities that require several steps to arrive at the solution. You'll also graph their solution set.

What if you had an inequality with an unknown variable on both sides like $2(x-2) > 3x-5$? How could you isolate the variable to find its value? After completing this concept, you'll be able to solve multi-step inequalities like this one.

Watch This



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CK-12 Foundation: 0603S Solving Multi-Step Inequalities (H264)

Try This

For additional practice solving inequalities, try the online game at http://www.aaamath.com/equ725x7.htm#section2 . If you're having a hard time with multi-step inequalities, the video at http://www.schooltube.com/video/aa66df49e0af4f85a5e9/MultiStep-Inequalities will walk you through a few.

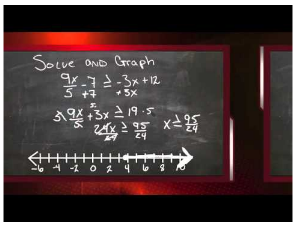
Guided Instruction

The general procedure for solving multi-step inequalities is almost exactly like the procedure for solving multi-step equations:

1. Clear parentheses on both sides of the inequality and collect like terms.

2. Add or subtract terms so the variable is on one side and the constant is on the other side of the inequality sign.
3. Multiply and divide by whatever constants are attached to the variable. **Remember to change the direction of the inequality if you multiply or divide by a negative number.**

Watch this video for help with the Examples below.



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CK-12 Foundation: Solving Multi-Step Inequalities

Example A

Solve the inequality $\frac{9x}{5} - 7 \geq -3x + 12$ *and graph the solution set.*

Solution

Original problem:

$$\frac{9x}{5} - 7 \geq -3x + 12$$

Add $3x$ to both sides:

$$\frac{9x}{5} + 3x - 7 \geq -3x + 3x + 12$$

Simplify:

$$\frac{24x}{5} - 7 \geq 12$$

Add 7 to both sides:

$$\frac{24x}{5} - 7 + 7 \geq 12 + 7$$

Simplify:

$$\frac{24x}{5} \geq 19$$

Multiply 5 to both sides:

$$5 \cdot \frac{24x}{5} \geq 5 \cdot 19$$

Simplify:

$$24x \geq 95$$

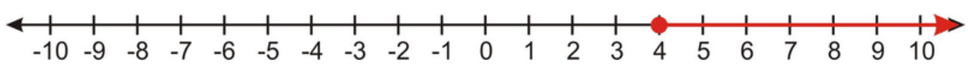
Divide both sides by 24:

$$\frac{24x}{24} \geq \frac{95}{24}$$

Simplify:

$$x \geq \frac{95}{24}$$

Graph:



Example B

Solve the inequality $-25x+12 \leq -10x-12$ *and graph the solution set.*

Solution:

Original problem:

$$-25x+12 \leq -10x-12$$

Add $10x$ to both sides:

$$-25x+10x+12 \leq -10x+10x-12$$

Simplify:

$$-15x+12 \leq -12$$

Subtract 12:

$$-15x+12-12 \leq -12-12$$

Simplify:

$$-15x \leq -24$$

Divide both sides by -15:

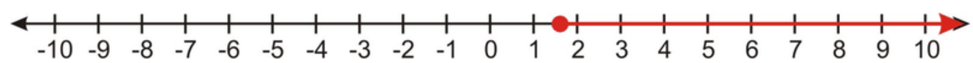
$$\frac{-15x}{-15} \geq \frac{-24}{-15}$$

flip the inequality sign

Simplify:

$$x \geq \frac{8}{5}$$

Graph:



Example C

Solve the inequality $4x-2(3x-9) \leq -4(2x-9)$.

Solution:

Original problem:

$$4x-2(3x-9) \leq -4(2x-9)$$

Simplify parentheses:

$$4x-6x+18 \leq -8x+36$$

Collect like terms:

$$-2x+18 \leq -8x+36$$

Add $8x$ to both sides:

$$-2x+8x+18 \leq -8x+8x+36$$

Simplify:

$$6x+18 \leq 36$$

Subtract 18:

$$6x+18-18 \leq 36-18$$

Simplify:

$$6x \leq 18$$

Divide both sides by 6:

$$\frac{6x}{6} \leq \frac{18}{6}$$

Simplify:

$$x \leq 3$$

Vocabulary

- The answer to an **inequality** is usually an **interval of values**.
- Solving inequalities works just like solving an equation. To solve, we isolate the variable on one side of the equation.
- When multiplying or dividing both sides of an inequality by a negative number, you need to ***reverse the inequality***.

Guided Practice

Solve the inequality $\frac{5x-1}{4} > -2(x+5)$.

Solution:

Original problem:

$$\frac{5x-1}{4} > -2(x+5)$$

Simplify parenthesis:

$$\frac{5x-1}{4} > -2x-10$$

Multiply both sides by 4:

$$4 \cdot \frac{5x-1}{4} > 4(-2x-10)$$

Simplify:

$$5x-1 > -8x-40$$

Add $8x$ to both sides:

$$5x+8x-1 > -8x+8x-40$$

Simplify:

$$13x-1 > -40$$

Add 1 to both sides:

$$13x-1+1 > -40+1$$

Simplify:

$$13x > -39$$

Divide both sides by 13:

$$\frac{13x}{13} > -\frac{39}{13}$$

Simplify:

$$x > -3$$

Explore More

Solve each multi-step inequality.

1. $n+6<15$
2. $2m \leq 18$
3. $-\frac{n}{4} \leq -5$
4. $8+2c<-6$
5. -2$-2f-3<3$
6. $y+8+3y \leq -4$
7. $-5g+2+4g>-2g+11$
8. $2(3w-4) \leq 8w-10$
9. $x-5>2x+3$
10. $2(x-3) \leq 3x-2$
11. $3(x+1) \geq 2x+5$
12. $2(x-9) \geq -1(4x+7)$
13. $\frac{x}{3}<x+7$
14. $\frac{x}{4}<2x-21$
15. $\frac{3(x-4)}{12} \leq \frac{2x}{3}$
16. $2\left(\frac{x}{4}+3\right)>6(x-1)$
17. $9x+4 \leq -2\left(x+\frac{1}{2}\right)$

5.3 Graphing Linear Inequalities

A1 5(B) Solve linear inequalities in one variable, including those for which the application of the distributive property is necessary and for which variables are included on both sides.

A1 3(D) Graph the solution set of linear inequalities in two variables on the coordinate plane.

TEKS: A1 5(B); A1 3(D)

Learning Objective

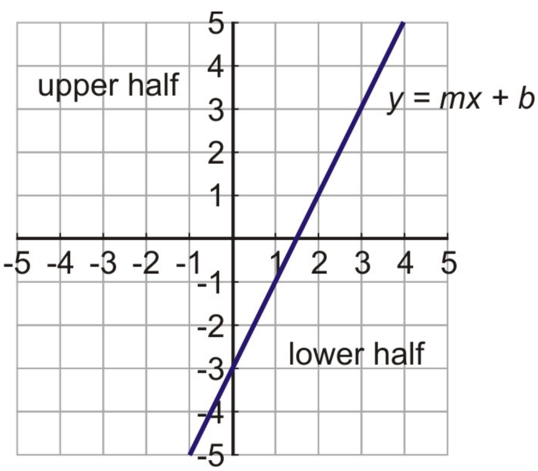
Here you'll learn how to graph linear inequalities in two variables of the form $y > mx + b$or $y < mx + b$.You'll also solve real-world problems involving such inequalities.

What if you were given a linear inequality like $2x - 3y \leq 5$? How could you graph that inequality in the coordinate plane? After completing this concept, you'll be able to graph linear inequalities in two variables like this one.

Guided Instruction

A **linear inequality** in two variables takes the form $y > mx + b$ or $y < mx + b$. Linear inequalities are closely related to graphs of straight lines; recall that a straight line has the equation $y = mx + b$.

When we graph a line in the coordinate plane, we can see that it divides the plane in half:



The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

$>$ The solution set is the half plane above the line.

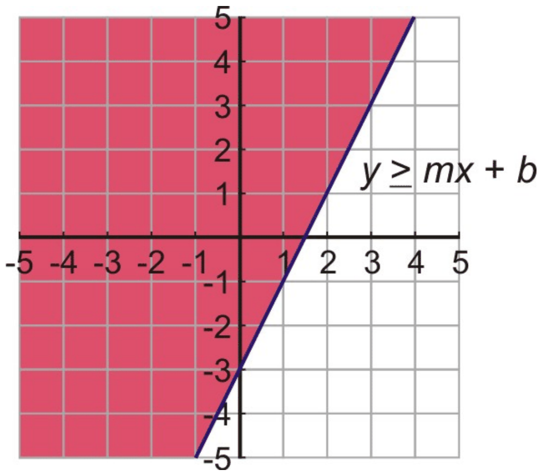
$\geq$ The solution set is the half plane above the line and also all the points on the line.

The solution set is the half plane below the line.

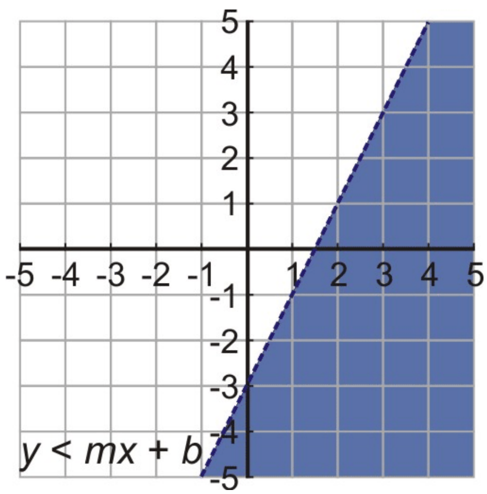
$\leq$ The solution set is the half plane below the line and also all the points on the line.

For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

This is a graph of $y \geq mx+b$; the solution set is the line and the half plane above the line.



This is a graph of $y < mx+b$; the solution set is the half plane above the line, not including the line itself.



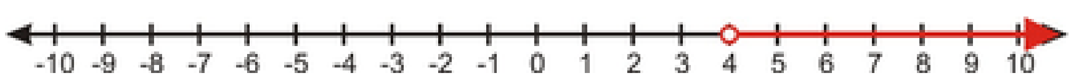
Special Case: Graph Linear Inequalities in One Variable in the Coordinate Plane

In the last few sections we graphed inequalities in one variable on the number line. We can also graph inequalities in one variable on the coordinate plane. We just need to remember that when we graph an equation of the type $x = a$ we get a vertical line, and when we graph an equation of the type $y = b$ we get a horizontal line.

Graph the inequality $x > 4$ *on the coordinate plane.*

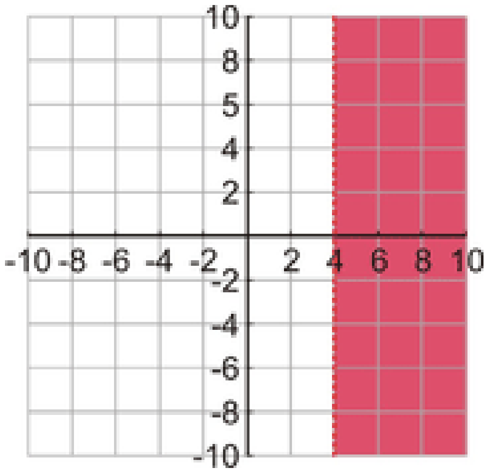
Solution

First let's remember what the solution to $x > 4$ looks like on the number line.



The solution to this inequality is the set of all real numbers x that are bigger than 4, not including 4. The solution is represented by a line.

In two dimensions, the solution still consists of all the points to the right of $x = 4$,, but for all possible $y-$values as well. This solution is represented by the half plane to the right of $x = 4$.. (You can think of it as being like the solution graphed on the number line, only stretched out vertically.)



The line $x = 4$ is dashed because the equals sign is not included in the inequality, meaning that points on the line are not included in the solution.

The general procedure for graphing inequalities in two variables is as follows:

1. Re-write the inequality in slope-intercept form: $y = mx + b$. Writing the inequality in this form lets you know the direction of the inequality.
2. Graph the line of the equation $y = mx + b$ using your favorite method (plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
3. Shade the half plane above the line if the inequality is "greater than." Shade the half plane under the line if the inequality is "less than."

Watch this video for help with the Examples below.

MEDIA

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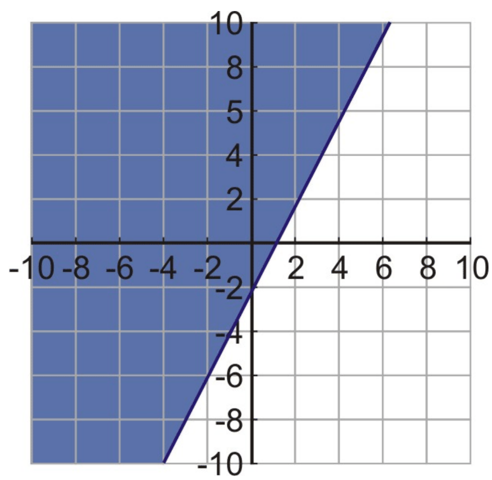
CK-12 Foundation: Linear Inequalities in Two Variables

Example A

Graph the inequality $y \geq 2x - 3$.

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x - 3$;then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.



Example B

Graph the inequality $5x - 2y > 4$.

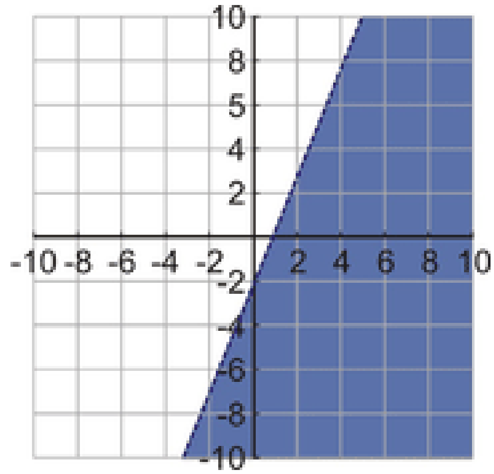
Solution

First we need to rewrite the inequality in slope-intercept form:

$$-2y > -5x + 4$$
$$y < \frac{5}{2}x - 2$$

Notice that the inequality sign changed direction because we divided by a negative number.

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example C

A retailer sells two types of coffee beans. One type costs \$9 per pound and the other type costs \$7 per pound. Find all the possible amounts of the two different coffee beans that can be mixed together to get a quantity of coffee beans costing \$8.50 or less.

Solution

Let $x =$ weight of \$9 per pound coffee beans in pounds.

Let $y =$ weight of \$7 per pound coffee beans in pounds.

The cost of a pound of coffee blend is given by $9x+7y$.

We are looking for the mixtures that cost \$8.50 or less. We write the inequality

$$9x+7y \le 8.50$$

Since this inequality is in standard form, it's easiest to graph it by finding the $x-$ and $y-$intercepts.

When $x = 0$, we have

$$7y = 8.50$$

$$y = \frac{8.50}{7} \approx 1.21$$

When $y = 0$, we have

$$9x = 8.50$$

$$x = \frac{8.50}{9} \approx 0.94$$

We can then graph the line that includes those two points.

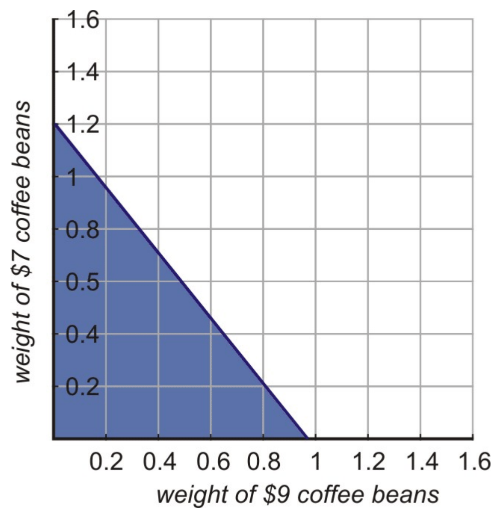
The other method, which works for any linear inequality in any form, is to plug a random point into the inequality and see if it makes the inequality true. Any point that's not on the line will do; the point (0, 0) is usually the most convenient.

In this case, plugging in 0 for x and y would give us

$$9(0)+7(0) \leq 8.50$$

$$0 \leq 8.50$$

which is true. That means we should shade the half of the plane that includes (0, 0). If plugging in (0, 0) gave us a false inequality, that would mean that the solution set is the part of the plane that does *not* contain (0, 0).



Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always non-negative, so points outside the first quadrant don't represent real-world solutions to this problem.

Vocabulary

- For an inequality that does **not** include the equal sign, we draw a **dashed line** to show that the points in the line *are not* part of the solution.
- For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.
- The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

$>$The solution set is the half plane above the dashed line.

$\geq$The solution set is the half plane above the solid line and also all the points on the line.

The solution set is the half plane below the dashed line.

$\leq$The solution set is the half plane below the solid line and also all the points on the line.

Guided Practice

Julius has a job as an appliance salesman. He earns a commission of $60 for each washing machine he sells and $130 for each refrigerator he sells. How many washing machines and refrigerators must Julius sell in order to make $1000 or more in commissions?

Solution

Let $x =$ number of washing machines Julius sells.

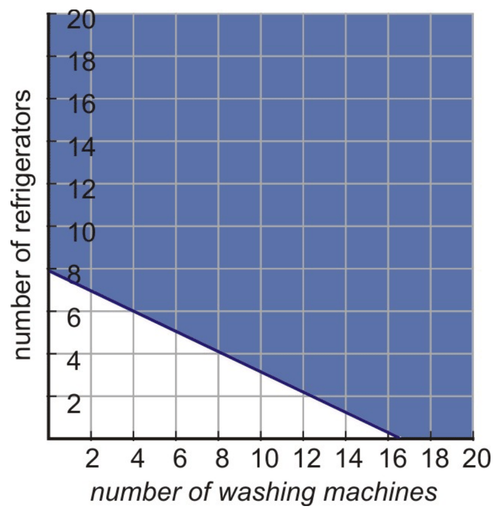
Let $y =$ number of refrigerators Julius sells.

The total commission is $60x + 130y$.

We're looking for a total commission of $1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Once again, we can do this most easily by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $130y = 1000$, or $y = \frac{1000}{30} \approx 7.69$. When $y = 0$, we have $60x = 1000$, or $x = \frac{1000}{60} \approx 16.67$.

We draw a solid line connecting those points, and shade above the line because the inequality is "greater than." We can check this by plugging in the point (0, 0): selling 0 washing machines and 0 refrigerators would give Julius a commission of $0, which is *not* greater than or equal to $1000, so the point (0, 0) is *not* part of the solution; instead, we want to shade the side of the line that does *not* include it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius's commission should be non-negative.

Independent Practice

Graph the following inequalities and determine if the given point is a solution to the given inequality.

1. $y \geq -3x + 4$, (3, -4)
2. $y \leq \frac{3}{5}x - 5$, (2, 4)
3. $y > -x - 5$, (-1, -4)
4. $y > -4$, (-6, -2)
5. $y > 2x - 5$, (0, 0)
6. $y \geq \frac{7}{4}x + 2$, (4, -2)
7. $x < -5$, (5, 5)

8. $y \leq \frac{4}{3}x - 4$, (6, 4)

9. $3x - 2y < 10$, (3, -1)

10. $5x - 3y \leq -15$, (-4, -1)

11. $y \leq 4$, (-3, 6)

12. $x - y > 2$, (2, 0)

Graph the Inequalities

13. $y > -\frac{x}{2} - 6$

14. $3x - 4y \geq 12$

15. $y \geq -3x + 4$

16. $y \leq \frac{3}{5}x - 5$

17. $y + 5 \leq -4x + 10$

18. $x - \frac{1}{2}y \geq 5$

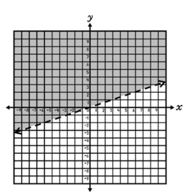
19. $6x + y < 20$

20. $x - 2y > 10$

Choose the inequality that correctly matches the graph.

TABLE 1.1:

21. figure*



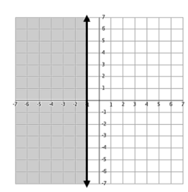
a. $y > \frac{1}{3}x$

b. $y \geq \frac{1}{3}x$

c. $y > -\frac{1}{3}x$

d. $y < -\frac{1}{3}x$

22. figure*



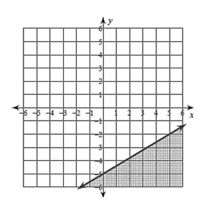
a. $y < -1$

b. $y \leq -1$

c. $x < -1$

d. $x \leq -1$

23. figure*



a. $y < \frac{3}{5}x - 5$

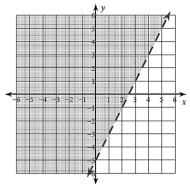
b. $y \geq \frac{3}{5}x - 5$

c. $y > \frac{3}{5}x - 5$

d. $y \leq \frac{3}{5}x - 5$

TABLE 1.1: (continued)

24. figure*



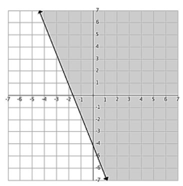
a. $-10x + 5y > -25$

b. $y < 2x + 5$

c. $-16x + 8y > 40$

d. $y \geq 2x - 5$

25. figure*



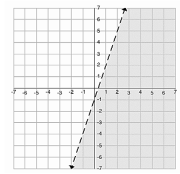
a. $y \geq -\frac{5}{2}x + 4$

b. $y \geq -\frac{5}{2}x - 4$

c. $y > \frac{5}{2}x - 4$

d. $y \leq \frac{5}{2}x + 4$

26. figure*



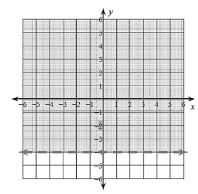
a. $y > 3x + 1$

b. $y < -3x - 1$

c. $y < 3x - 1$

d. $y > 3x - 1$

27. figure*



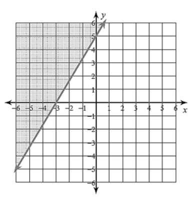
a. $y > 4$

b. $y \geq -4$

c. $x > -4$

d. $y > -4$

28. figure*



a. $5x + 3y \leq -15$

b. $y < \frac{5}{3}x + 5$

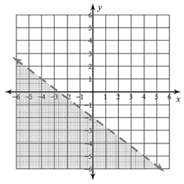
c. $5x - 3y \leq -15$

d. $y \geq 3x + 5$

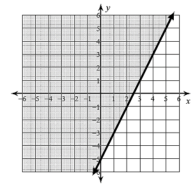
Write the inequality of the graph shown below.

TABLE 1.2:

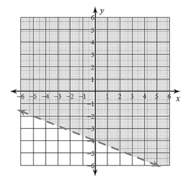
29. figure*



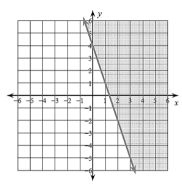
30. figure*



31. figure*



32. figure*



Determine if the given point is a solution to the inequality.

33. Is the point (7, -3) a solution to the inequality y ?

34. Is the point (-2, -4) a solution to the inequality $2y \geq 6x+4$?

35. Is the point (1, 3) a solution to the inequality $3x+4y > 10$?

36. Is the point (3, 4) a solution to the inequality $y \leq \frac{1}{3}x+3$?

37. Is the point (0, 7) a solution to the inequality $2x-2y > 9$?

38. Is the point (-1, 5) a solution to the inequality $y < 8x+13$?

Explore More

39. Remember what you learned in the last chapter about families of lines.

1. What do the graphs of $y > x+2$ and $y < x+5$ have in common?
2. What do you think the graph of $x+2 < y < x+5$ would look like?

40. How would the answer to problem 6 change if you subtracted 2 from the right-hand side of the inequality?

41. How would the answer to problem 7 change if you added 12 to the right-hand side?

42. How would the answer to problem 8 change if you flipped the inequality sign?

43. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and nighttime minutes could you use in one week if you wanted to pay less than \$20?

44. Suppose you are graphing the inequality $y > 5x$.

1. Why can't you plug in the point (0, 0) to tell you which side of the line to shade?
2. What happens if you do plug it in?
3. Try plugging in the point (0, 1) instead. Now which side of the line should you shade?

45. A theater wants to take in at least \$2000 for a certain matinee. Children's tickets cost \$5 each and adult tickets cost \$10 each.

1. If x represents the number of adult tickets sold and y represents the number of children's tickets, write an inequality describing the number of tickets that will allow the theater to meet their minimum take.
2. If 100 children's tickets and 100 adult tickets have already been sold, what inequality describes how many *more* tickets of both types the theater needs to sell?
3. If the theater has only 300 seats (so only 100 are still available), what inequality describes the *maximum* number of additional tickets of both types the theater can sell?

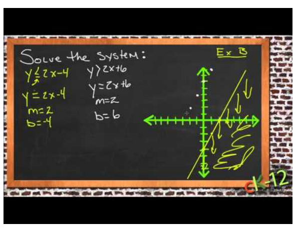
TABLE 1.3:

5.4 Systems of Linear Inequalities -

Here you'll learn how to graph and solve a system of two or more linear inequalities. You'll also determine if such systems are consistent or inconsistent.

What if you were given a system of linear inequalities like $6x - 2y \geq 3$ and $2y - 3x \leq 7$? How could you determine its solution? After completing this Concept, you'll be able to find the solution region of systems of linear inequalities like this one.

Watch This



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CK-12 Foundation: 0710S Systems of Linear Inequalities

Guidance

In the last chapter you learned how to graph a linear inequality in two variables. To do that, you graphed the equation of the straight line on the coordinate plane. The line was solid for $\leq$ or $\geq$ signs (where the equals sign is included), and the line was dashed for <or >signs (where the equals sign is not included). Then you shaded above the line (if the inequality began with $y >$ or $y \geq$) or below the line (if it began with $y <$ or $y \leq$).

In this section, we'll see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes.

Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example A

Solve the following system:

$$\begin{aligned} 2x+3y &\leq 18 \\ x-4y &\leq 12 \end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

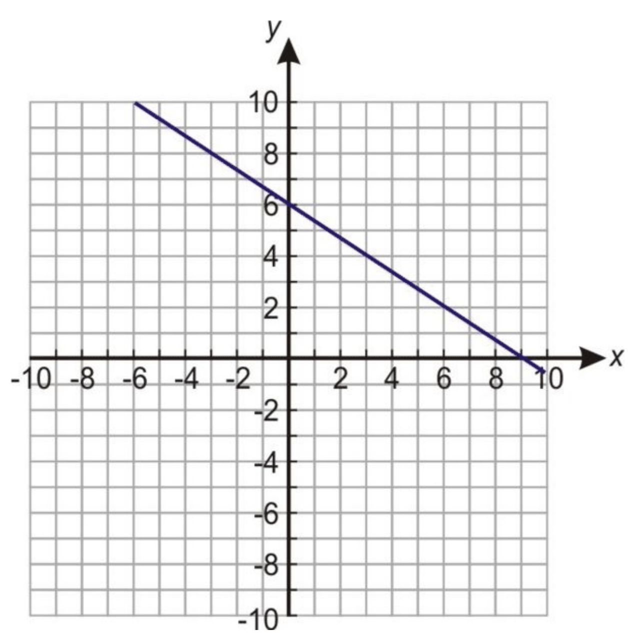
First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

$$\begin{aligned} 3y &\leq -2x+18 & \qquad y &\leq -\frac{2}{3}x+6 \\ & & &\Rightarrow \\ -4y &\leq -x+12 & \qquad y &\geq \frac{x}{4}-3 \end{aligned}$$

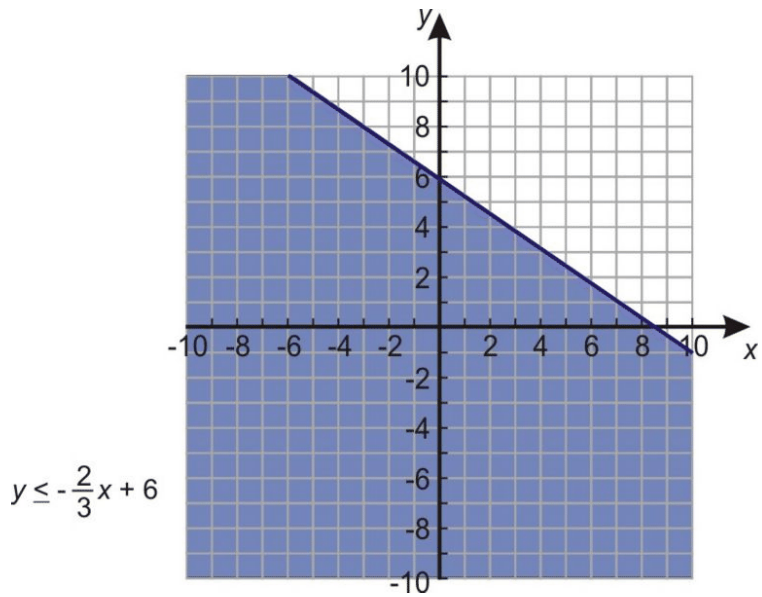
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

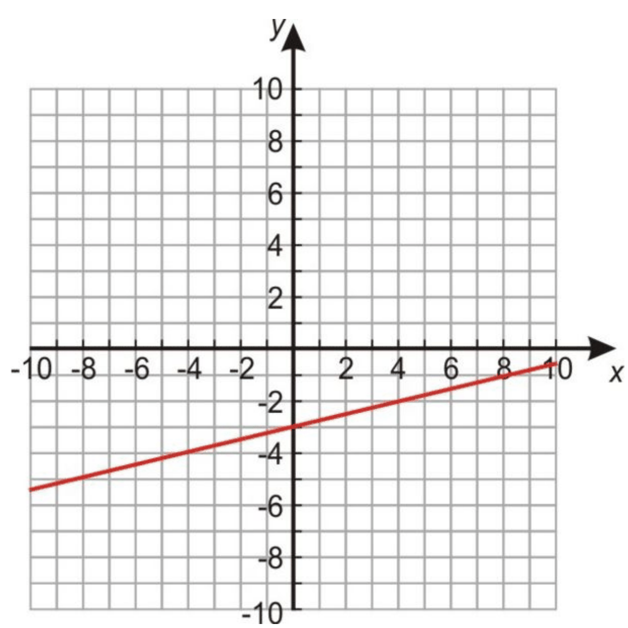
Here's the graph of the first inequality:



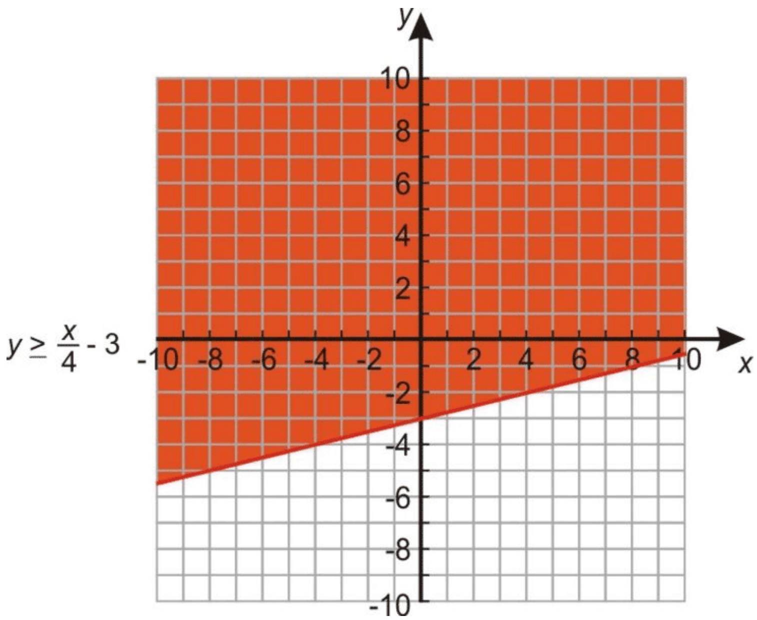
The line is solid because the equals sign is included in the inequality. Since the inequality is **less** than or equal to, we shade **below** the line.



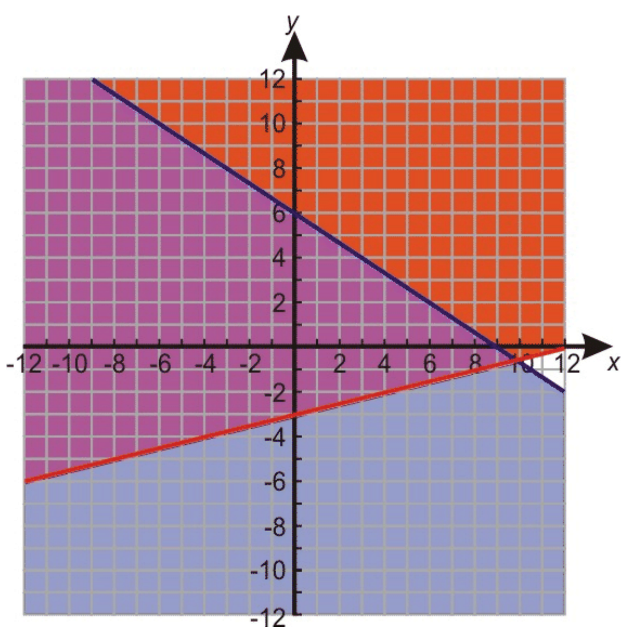
And here's the graph of the second inequality:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because y is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

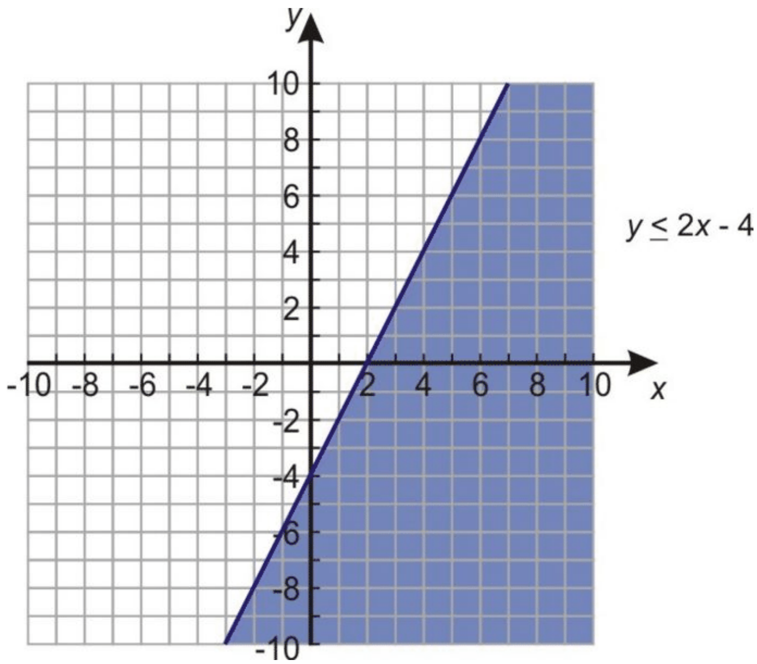
Example B

There are also situations where a system of inequalities has no solution. For example, let's solve this system.

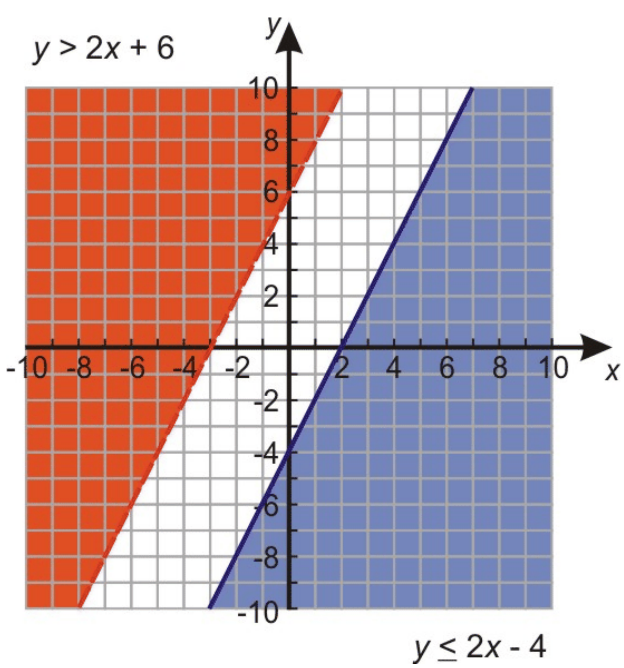
$$y \leq 2x - 4$$
$$y > 2x + 6$$

Solution

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because y is greater than.



It doesn't look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won't ever overlap since the lines won't ever cross. So this system of inequalities has no solution.

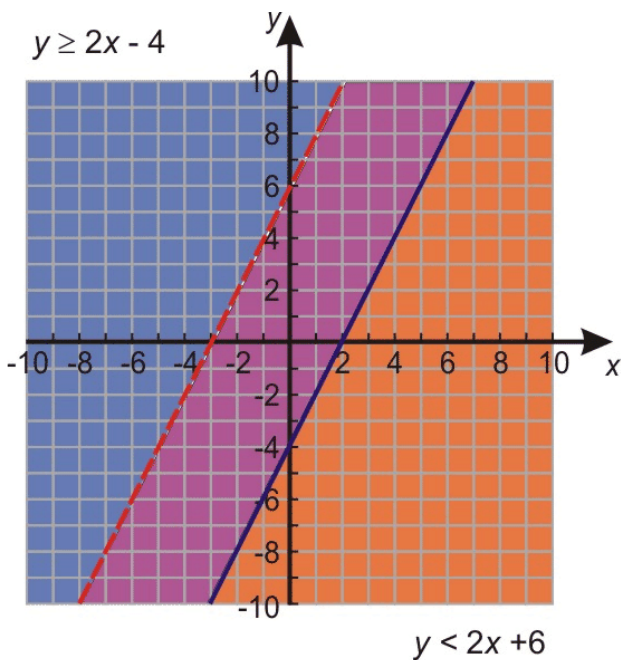
But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

To graph the system

$$\begin{aligned} y &\geq 2x - 4 \\ y &< 2x + 6 \end{aligned}$$

,

we draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region—one that continues infinitely in at least one direction. But if we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded**—a finite region with three or more sides.

Let's look at a simple example.

Example C

Find the solution to the following system of inequalities.

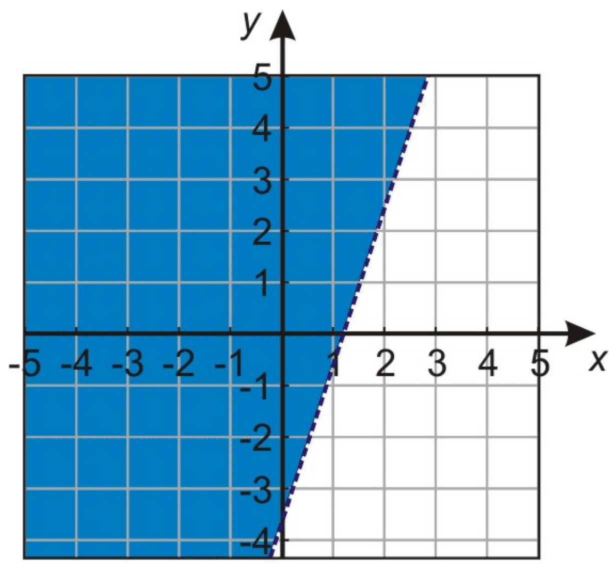
$$\begin{aligned} 3x - y &< 4 \\ 4y + 9x &< 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution

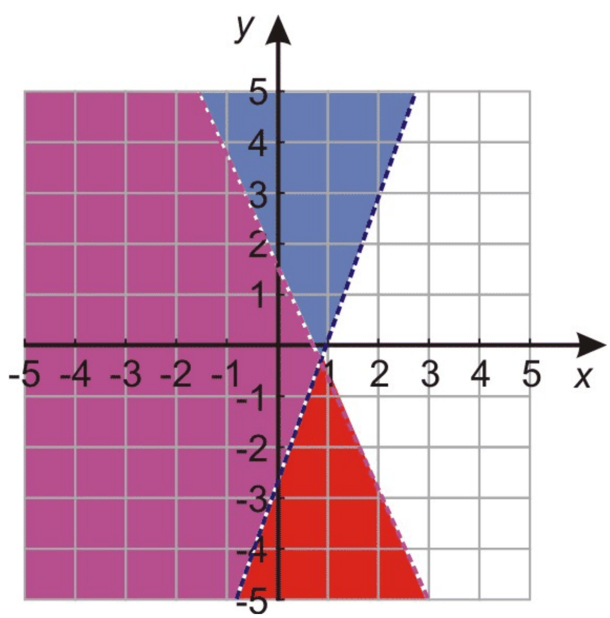
Let's start by writing our inequalities in slope-intercept form.

$$y > 3x - 4$$
$$y < -\frac{9}{4}x + 2$$
$$x \geq 0$$
$$y \geq 0$$

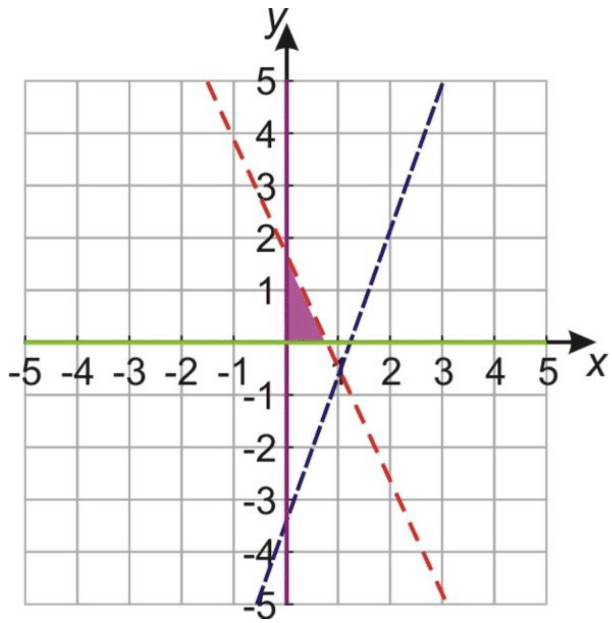
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$:



Next we graph $y < -\frac{9}{4}x + 2$:



Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.

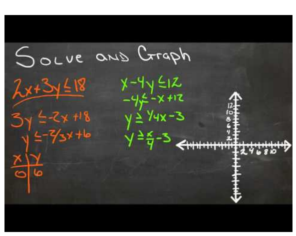


The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x - 4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

But that wasn't obvious until we actually drew the graph!

Watch this video for help with the Examples above.



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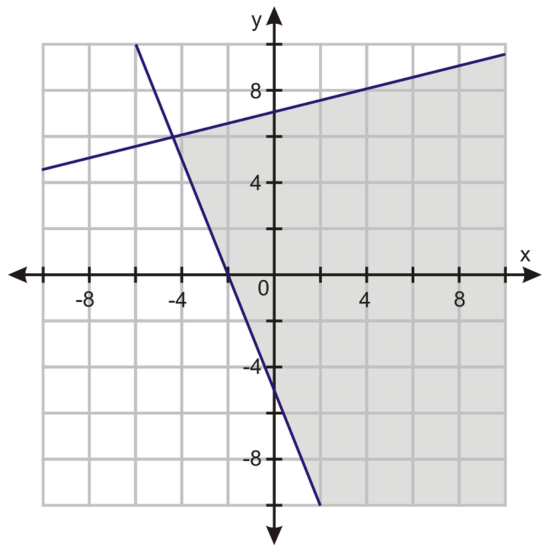
CK-12 Foundation: Systems of Linear Inequalities

Vocabulary

- **Solution for the system of inequalities:** The ***solution for the system of inequalities*** is the common shaded region between all the inequalities in the system.
- **Feasible region:** The common shaded region of the system of inequalities is called the ***feasible region***.
- **Optimization:** The goal is to locate the feasible region of the system and use it to answer a profitability, or ***optimization***, question.

Guided Practice

Write the system of inequalities shown below.



Solution:

There are two boundary lines, so there are two inequalities. Write each one in slope-intercept form.

$$y \leq \frac{1}{4}x + 7$$
$$y \geq -\frac{5}{2}x - 5$$

Independent Practice

Graph the system of linear inequalities and determine if the given ordered pairs are a solution.

(-2, 0) and (0, 3)
2. figure*

$$\begin{cases} y \leq \frac{1}{2}x + 1 \\ x + y < 3 \end{cases}$$

(0, 0) and (2, 3)

3. figure*

$$\begin{cases} y > x - 4 \\ y < x + 2 \end{cases}$$

(0, 0) and (4, -4)

4. figure*

$$\begin{cases} y \leq 3x + 2 \\ y \geq -x \end{cases}$$

(0, 2) and (-4, 0)

5. figure*

$$\begin{cases} y > \frac{1}{3} - 2 \\ x + y > -3 \end{cases}$$

(-2, 0) and (-4, -2)

Tell whether the ordered pairs is a solution of the given system of inequalities.

(2, -2)

7. figure*

$$\begin{cases} y > 2x \\ y \geq x + 2 \end{cases}$$

(2, 5)

8. figure*

$$\begin{cases} y \leq x + 2 \\ y > 4x - 1 \end{cases}$$

(1, 3)

Explore More

9. Consider the system

$$y < 3x - 5$$
$$y > 3x - 5$$

Is it consistent or inconsistent? Why?

10. Consider the system

$$y \leq 2x + 3$$
$$y \geq 2x + 3$$

Is it consistent or inconsistent? Why?

11. Consider the system

$$y \leq -x + 1$$
$$y > -x + 1$$

Is it consistent or inconsistent? Why?

12. In example 3 in this lesson, we solved a system of four inequalities and saw that one of the inequalities, $y > 3x - 4$, didn't affect the solution set of the system.

a. What would happen if we changed that inequality to $y < 3x - 4$?
b. What's another inequality that we could add to the original system without changing it? Show how by sketching a graph of that inequality along with the rest of the system.
c. What's another inequality that we could add to the original system to make it inconsistent? Show how by sketching a graph of that inequality along with the rest of the system.

13. Recall the compound inequalities in one variable that we worked with back in chapter 6. Compound inequalities with "and" are simply systems like the ones we are working with here, except with one variable instead of two.

a. Graph the inequality $x > 3$ in two dimensions. What's another inequality that could be combined with it to make an inconsistent system?
b. Graph the inequality $x \leq 4$ on a number line. What two-dimensional system would have a graph that looks just like this one?

Find the solution region of the following systems of inequalities.

14.

$$x - y < -6$$
$$2y \geq 3x + 17$$

15.

$$4y - 5x < 8$$
$$-5x \geq 16 - 8y$$

16.

$$
\begin{aligned}
5x - y &\geq 5 \\
2y - x &\geq -10
\end{aligned}
$$

17.

$$
\begin{aligned}
5x + 2y &\geq -25 \\
3x - 2y &\leq 17 \\
x - 6y &\geq 27
\end{aligned}
$$

18.

$$
\begin{aligned}
2x - 3y &\leq 21 \\
x + 4y &\leq 6 \\
3x + y &\geq -4
\end{aligned}
$$

19.

$$
\begin{aligned}
12x - 7y &< 120 \\
7x - 8y &\geq 36 \\
5x + y &\geq 12
\end{aligned}
$$

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9617 .

Lin

Exponential Functions

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Printed: September 23, 2016

CHAPTER 6 Exponential Functions

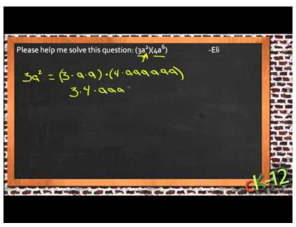
CHAPTER OUTLINE

6.1 Exponential Properties Involving Products

Here you'll learn how to write repeated multiplication in exponential form. You'll also learn how to multiply and simplify exponential expressions.

What if you wanted to simplify a mathematical expression containing exponents, like $4^3 \cdot 4^2$? How would you do so? After completing this Concept, you'll be able to use the product of powers property to simplify exponential expressions like this one.

Watch This



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CK-12 Foundation: 0801S Product of Powers

Guidance

Back in chapter 1, we briefly covered expressions involving exponents, like 3^5 or x^3. In these expressions, the number on the bottom is called the **base** and the number on top is the **power** or **exponent**. The whole expression is equal to the base multiplied by itself a number of times equal to the exponent; in other words, the exponent tells us how many copies of the base number to multiply together.

Example A

Write in exponential form.

a) $2 \cdot 2$

b) $(-3)(-3)(-3)$

c) $y \cdot y \cdot y \cdot y \cdot y$

d) $(3a)(3a)(3a)(3a)$

Solution

a) $2 \cdot 2 = 2^2$ because we have 2 factors of 2

b) $(-3)(-3)(-3) = (-3)^3$ because we have 3 factors of (-3)

c) $y \cdot y \cdot y \cdot y \cdot y = y^5$ because we have 5 factors of y

d) $(3a)(3a)(3a)(3a) = (3a)^4$ because we have 4 factors of $3a$

When the base is a variable, it's convenient to leave the expression in exponential form; if we didn't write x^7, we'd have to write $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ instead. But when the base is a number, we can simplify the expression further than that; for example, 2^7 equals $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, but we can multiply all those 2's to get 128.

Let's simplify the expressions from Example A.

Example B

Simplify.

a) 2^2

b) $(-3)^3$

c) y^5

d) $(3a)^4$

Solution

a) $2^2 = 2 \cdot 2 = 4$

b) $(-3)^3 = (-3)(-3)(-3) = -27$

c) y^5 is already simplified

d) $(3a)^4 = (3a)(3a)(3a)(3a) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot a \cdot a \cdot a \cdot a = 81a^4$

Be careful when taking powers of negative numbers. Remember these rules:

$$\begin{aligned}(\textit{negative number}) \cdot (\textit{positive number}) &= \textit{negative number}\\ (\textit{negative number}) \cdot (\textit{negative number}) &= \textit{positive number}\end{aligned}$$

So **even powers of negative numbers** are always positive. Since there are an even number of factors, we pair up the negative numbers and all the negatives cancel out.

$$\begin{aligned}(-2)^6 &= (-2)(-2)(-2)(-2)(-2)(-2)\\ &= \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4}\\ &= +64\end{aligned}$$

And **odd powers of negative numbers** are always negative. Since there are an odd number of factors, we can still pair up negative numbers to get positive numbers, but there will always be one negative factor left over, so the answer is negative:

$$\begin{aligned}(-2)^5 &= (-2)(-2)(-2)(-2)(-2)\\ &= \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)}_{-2}\\ &= -32\end{aligned}$$

Use the Product of Powers Property

So what happens when we multiply one power of x by another? Let's see what happens when we multiply x ***to the power of 5*** by x ***cubed***. To illustrate better, we'll use the full factored form for each:

$$\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{x^5} \cdot \underbrace{(x \cdot x \cdot x)}_{x^3} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^8}$$

So $x^5 \times x^3 = x^8$. You may already see the pattern to multiplying powers, but let's confirm it with another example. We'll multiply x ***squared*** by x ***to the power of 4***:

$$\underbrace{(x \cdot x)}_{x^2} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^6}$$

So $x^2 \times x^4 = x^6$. Look carefully at the powers and how many factors there are in each calculation. 5 x's times 3 x's equals $(5+3) = 8$ x's. 2 x's times 4 x's equals $(2+4) = 6$ x's.

You should see that when we take the product of two powers of x, the number of x's in the answer is the total number of x's in all the terms you are multiplying. In other words, the exponent in the answer is the sum of the exponents in the product.

Product Rule for Exponents: $x^n \cdot x^m = x^{(n+m)}$

There are some easy mistakes you can make with this rule, however. Let's see how to avoid them.

Example C

Multiply $2^2 \cdot 2^3$.

Solution

$2^2 \cdot 2^3 = 2^5 = 32$

Note that when you use the product rule you **don't multiply the bases**. In other words, you must avoid the common error of writing $2^2 \cdot 2^3 = 4^5$. You can see this is true if you multiply out each expression: 4 times 8 is definitely 32, not 1024.

Example D

Multiply $2^2 \cdot 3^3$.

Solution

$2^2 \cdot 3^3 = 4 \cdot 27 = 108$

In this case, we can't actually use the product rule at all, because it only applies to terms that have the *same base*. In a case like this, where the bases are different, we just have to multiply out the numbers by hand—the answer is *not* 2^5 or 3^5 or 6^5 or anything simple like that.

Watch this video for help with the Examples above.

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CK-12 Foundation: Products of Powers

Vocabulary

- An ***exponent*** is a power of a number that shows how many times that number is multiplied by itself. An example would be 2^3. You would multiply 2 by itself 3 times: $2 \times 2 \times 2$. The number 2 is the ***base*** and the number 3 is the ***exponent.*** The value 2^3 is called the ***power.***

- **Product Rule for Exponents:** $x^n \cdot x^m = x^{(n+m)}$

Guided Practice

Simplify the following exponents:

a. $(-2)^5$

b. $(10x)^2$

Solutions:

a. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$

b. $(10x)^2 = 10^2 \cdot x^2 = 100x^2$

Explore More

Write in exponential notation:

1. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
2. $3x \cdot 3x \cdot 3x$
3. $(-2a)(-2a)(-2a)(-2a)$
4. $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$
5. $2 \cdot x \cdot y \cdot 2 \cdot 2 \cdot y \cdot x$

Find each number.

6. 5^4
7. $(-2)^6$
8. $(0.1)^5$
9. $(-0.6)^3$
10. $(1.2)^2 + 5^3$
11. $3^2 \cdot (0.2)^3$

Multiply and simplify:

12. $6^3 \cdot 6^6$
13. $2^2 \cdot 2^4 \cdot 2^6$
14. $3^2 \cdot 4^3$
15. $x^2 \cdot x^4$
16. $(-2y^4)(-3y)$
17. $(4a^2)(-3a)(-5a^4)$

Independent Practice

18. $x \bullet x$
19. $(n^5)(n^7)$
20. $(3x)(7x^2)$
21. $y^3 \bullet y^4 \bullet y^5$
22. $-3m^2n^3 \bullet -mn^2$
23. $13y^5 \bullet -3y^{10}$
24. $(7a^3b^4)(8ab^5)$
25. $(-3xy)(-3xy^2)(-3x^2y^3)$
26. $c^5d^3 \bullet cd^5$
27. $w^{1/4} \bullet w^{2/3}$
28. $(5g^{1/2})(3g^{1/3})$
29. $x^m \bullet x^n$
30. $h^p \bullet h^p$
31. $(8y^a)(3y^a)$
32. $-4rt^2 \bullet 2rt \bullet -t^2$
33. $k^6 \bullet k^{1/5}$

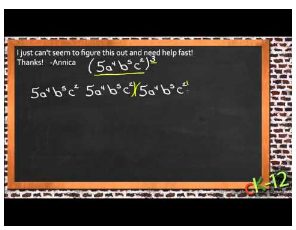
FIGURE 1.1

6.2 Exponential Terms Raised to an Exponent

Here you'll learn how to simplify exponential expressions that are raised to another secondary power.

What if you had an exponential expression that was raised to a secondary power, like $(2^3)^2$? How could you simplify it? After completing this Concept, you'll be able to use the power of a product property to simplify exponential expressions like this one.

Watch This



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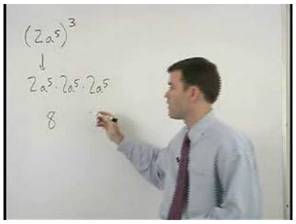
CK-12 Foundation: 0802S Power of a Product

The following video from YourTeacher.com may make it clearer how the power rule works for a variety of exponential expressions:



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YourTeacher: Power Rule- Multiplying Exponents

Guidance

What happens when we raise a whole expression to a power? Let's take x ***to the power of 4*** and ***cube it***. Again we'll use the full factored form for each expression:

$$(x^4)^3 = x^4 \times x^4 \times x^4 \qquad 3\ factors\ of\ \{x\ to\ the\ power\ 4\}$$
$$(x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^{12}$$

So $(x^4)^3 = x^{12}$. You can see that when we raise a power of x to a new power, the powers multiply.

Power Rule for Exponents: $(x^n)^m = x^{(n \cdot m)}$

If we have a product of more than one term inside the parentheses, then we have to distribute the exponent over all the factors, like distributing multiplication over addition. For example:

$$(x^2y)^4 = (x^2)^4 \cdot (y)^4 = x^8y^4.$$

Or, writing it out the long way:

$$\begin{aligned}(x^2y)^4 = (x^2y)(x^2y)(x^2y)(x^2y) &= (x \cdot x \cdot y)(x \cdot x \cdot y)(x \cdot x \cdot y)(x \cdot x \cdot y)\\ &= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y = x^8y^4\end{aligned}$$

Note that this does NOT work if you have a sum or difference inside the parentheses! For example, $(x+y)^2 \neq x^2+y^2$. This is an easy mistake to make, but you can avoid it if you remember what an exponent means: if you multiply out $(x+y)^2$ it becomes $(x+y)(x+y)$, and that's not the same as x^2+y^2. We'll learn how we can simplify this expression in a later chapter.

Example A

Simplify the following expressions.

a) $3^5 \cdot 3^7$

b) $2^6 \cdot 2$

c) $(4^2)^3$

Solution

When we're just working with numbers instead of variables, we can use the product rule and the power rule, or we can just do the multiplication and then simplify.

a) We can use the product rule first and then evaluate the result: $3^5 \cdot 3^7 = 3^{12} = 531441$.

OR we can evaluate each part separately and then multiply them: $3^5 \cdot 3^7 = 243 \cdot 2187 = 531441$.

b) We can use the product rule first and then evaluate the result: $2^6 \cdot 2 = 2^7 = 128$.

OR we can evaluate each part separately and then multiply them: $2^6 \cdot 2 = 64 \cdot 2 = 128$.

c) We can use the power rule first and then evaluate the result: $(4^2)^3 = 4^6 = 4096$.

OR we can evaluate the expression inside the parentheses first, and then apply the exponent outside the parentheses: $(4^2)^3 = (16)^3 = 4096$.

Example B

Simplify the following expressions.

a) $x^2 \cdot x^7$

b) $(y^3)^5$

Solution

When we're just working with variables, all we can do is simplify as much as possible using the product and power rules.

a) $x^2 \cdot x^7 = x^{2+7} = x^9$

b) $(y^3)^5 = y^{3\times 5} = y^{15}$

Example C

Simplify the following expressions.

a) $(3x^2y^3) \cdot (4xy^2)$

b) $(4xyz) \cdot (x^2y^3) \cdot (2yz^4)$

c) $(2a^3b^3)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number and variable separately.

a) First we group like terms together: $(3x^2y^3) \cdot (4xy^2) = (3 \cdot 4) \cdot (x^2 \cdot x) \cdot (y^3 \cdot y^2)$

Then we multiply the numbers or apply the product rule on each grouping: $= 12x^3y^5$

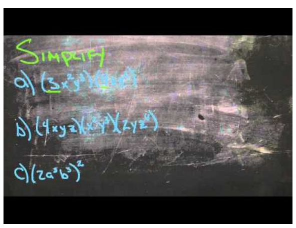
b) Group like terms together: $(4xyz) \cdot (x^2y^3) \cdot (2yz^4) = (4 \cdot 2) \cdot (x \cdot x^2) \cdot (y \cdot y^3 \cdot y) \cdot (z \cdot z^4)$

Multiply the numbers or apply the product rule on each grouping: $= 8x^3y^5z^5$

c) Apply the power rule for each separate term in the parentheses: $(2a^3b^3)^2 = 2^2 \cdot (a^3)^2 \cdot (b^3)^2$

Multiply the numbers or apply the power rule for each term $= 4a^6b^6$

Watch this video for help with the Examples above.



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CK-12 Foundation: Power of a Product

Vocabulary

- **Exponent:** An ***exponent*** is a power of a number that shows how many times that number is multiplied by itself. An example would be 2^3. You would multiply 2 by itself 3 times: $2 \times 2 \times 2$. The number 2 is the ***base*** and the number 3 is the ***exponent.*** The value 2^3 is called the ***power.***

- **Product of Powers Property:** For all real numbers χ,

$$\chi^n \cdot \chi^m = \chi^{n+m}$$

.

- **Power of a Product Property:** For all real numbers χ,

$$(\chi^n)^m = \chi^{n \cdot m}$$

.

Guided Practice

Simplify the following expressions.

a) $(x^2)^2 \cdot x^3$

b) $(2x^2y) \cdot (3xy^2)^3$

c) $(4a^2b^3)^2 \cdot (2ab^4)^3$

Solution

In problems where we need to apply the product and power rules together, we must keep in mind the order of operations. Exponent operations take precedence over multiplication.

a) We apply the power rule first: $(x^2)^2 \cdot x^3 = x^4 \cdot x^3$

Then apply the product rule to combine the two terms: $x^4 \cdot x^3 = x^7$

b) Apply the power rule first: $(2x^2y) \cdot (3xy^2)^3 = (2x^2y) \cdot (27x^3y^6)$

Then apply the product rule to combine the two terms: $(2x^2y) \cdot (27x^3y^6) = 54x^5y^7$

c) Apply the power rule on each of the terms separately: $(4a^2b^3)^2 \cdot (2ab^4)^3 = (16a^4b^6) \cdot (8a^3b^{12})$

Then apply the product rule to combine the two terms: $(16a^4b^6) \cdot (8a^3b^{12}) = 128a^7b^{18}$

Explore More

Simplify:

1. $(a^3)^4$
2. $(xy)^2$
3. $(-5y)^3$
4. $(3a^2b^3)^4$
5. $(-2xy^4z^2)^5$
6. $(-8x)^3(5x)^2$
7. $(-x)^2(xy)^3$
8. $(4a^2)(-2a^3)^4$
9. $(12xy)(12xy)^2$
10. $(2xy^2)(-x^2y)^2(3x^2y^2)$

11. $(x^3)^2$

12. $(3mn)^3$

13. $\left(\frac{m^4}{n}\right)^2$

14. $\left(\frac{x}{y^2}\right)^3$

15. $(-x)^6$

16. $(-2xy)^5$

17. $(8m^5n^5)^2$

18. $\left(\frac{x}{2y^3}\right)^4$

19. $\left(\frac{4x^2}{3y^3}\right)^2$

20. $(-y^2)^4$

FIGURE 1.2

Independent Practice

21. $(-10a^3b^2)^2$

22. $(mn^4p^3)^3$

23. $\left(\frac{c^5}{a^7}\right)^2$

24. $\left(\frac{3a^2}{5b^3}\right)^3$

25. $(2f^3g)^5$

26. $(g^c)^3$

27. $(a^xb^2)^3$

28. $(h^3)^b$

FIGURE 1.3

29. $(3y^a)^2$

30. $(m^x)^y$

31. $(y^{3/4})^4$

32. $(b^{1/3})^{3/8}$

33. $\left(\frac{a^3b^2}{a^2b}\right)^{1/3}$

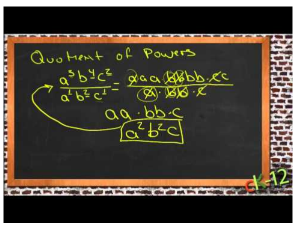
FIGURE 1.4

6.3 Exponential Properties Involving Quotients.

Here you'll learn how to simplify one exponential expression that is divided by another.

What if you had a fractional expression like $\frac{x^5}{x^2}$ in which both the numerator and denominator contained exponents? How could you simplify it? After completing this Concept, you'll be able to use the quotient of powers property to simplify exponential expressions like this one.

Watch This



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CK-12 Foundation: 0803S Quotient of Powers

Guidance

The rules for simplifying quotients of exponents are a lot like the rules for simplifying products.

Example A

Let's look at what happens when we divide x^7 by x^4:

$$\frac{x^7}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x \cdot x \cdot x}{1} = x^3$$

You can see that when we divide two powers of x, the number of x's in the solution is the number of x's in the top of the fraction minus the number of x's in the bottom. In other words, when dividing expressions with the same base, we keep the same base and simply subtract the exponent in the denominator from the exponent in the numerator.

Quotient Rule for Exponents: $\frac{x^n}{x^m} = x^{(n-m)}$

When we have expressions with more than one base, we apply the quotient rule separately for each base:

Now let's see what happens if the exponent in the denominator is bigger than the exponent in the numerator. For example, what happens when we apply the quotient rule to $\frac{x^4}{x^7}$?

The quotient rule tells us to subtract the exponents. 4 minus 7 is -3, so our answer is x^{-3}. A negative exponent! What does that mean?

Example B

$$\frac{x^5y^3}{x^3y^2} = \frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}\cdot\frac{\cancel{y}\cdot\cancel{y}\cdot y}{\cancel{y}\cdot\cancel{y}} = \frac{x\cdot x}{1}\cdot\frac{y}{1} = x^2y$$

OR

$$\frac{x^5y^3}{x^3y^2} = x^{5-3}\cdot y^{3-2} = x^2y$$

Well, let's look at what we get when we do the division longhand by writing each term in factored form:

$$\frac{x^4}{x^7} = \frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x} = \frac{1}{x\cdot x\cdot x} = \frac{1}{x^3}$$

Even when the exponent in the denominator is bigger than the exponent in the numerator, we can still subtract the powers. The x's that are left over after the others have been canceled out just end up in the denominator instead of the numerator. Just as $\frac{x^7}{x^4}$ would be equal to $\frac{x^3}{1}$ (or simply x^3), $\frac{x^4}{x^7}$ is equal to $\frac{1}{x^3}$. And you can also see that $\frac{1}{x^3}$ is equal to x^{-3}. We'll learn more about negative exponents shortly.

Example C

Simplify the following expressions, leaving all exponents positive.

a) $\frac{x^2}{x^6}$

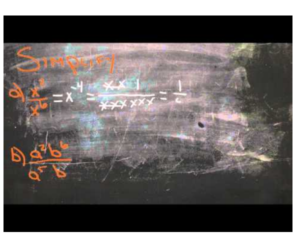
b) $\frac{a^2b^6}{a^5b}$

Solution

a) Subtract the exponent in the numerator from the exponent in the denominator and leave the x's in the denominator: $\frac{x^2}{x^6} = \frac{1}{x^{6-2}} = \frac{1}{x^4}$

b) Apply the rule to each variable separately: $\frac{a^2b^6}{a^5b} = \frac{1}{a^{5-2}}\cdot\frac{b^{6-1}}{1} = \frac{b^5}{a^3}$

Watch this video for help with the Examples above.



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CK-12 Foundation: Quotient of Powers

Vocabulary

Quotient of Powers Property: For all real numbers x,

$\frac{x^n}{x^m} = x^{n-m}$.

Guided Practice

Simplify each of the following expressions using the quotient rule.

a) $\frac{x^{10}}{x^5}$

b) $\frac{a^6}{a}$

c) $\frac{a^5b^4}{a^3b^2}$

Solution

a) $\frac{x^{10}}{x^5} = x^{10-5} = x^5$

b) $\frac{a^6}{a} = a^{6-1} = a^5$

c) $\frac{a^5b^4}{a^3b^2} = a^{5-3} \cdot b^{4-2} = a^2b^2$

Review Questions

Evaluate the following expressions.

1. $\frac{5^6}{5^2}$
2. $\frac{6^7}{6^3}$
3. $\frac{3^4}{3^{10}}$
4. $\frac{2^2 \cdot 3^2}{5^2}$
5. $\frac{3^3 \cdot 5^2}{3^7}$

Simplify the following expressions.

6. $\frac{a^3}{a^2}$
7. $\frac{x^5}{x^9}$
8. $\frac{x^6y^2}{x^2y^5}$
9. $\frac{6a^3}{2a^2}$
10. $\frac{15x^5}{5x}$
11. $\frac{25yx^6}{20y^5x^2}$

12. $\frac{x^5}{x}$

13. $\frac{6a^7}{2a^5}$

14. $\frac{9y^3}{y}$

15. $\frac{m^{10}}{m^3}$

16. $\frac{x^4y^5}{x^3y^2}$

17. $\frac{15a^4b^9}{-3ab^3}$

18. $\frac{12c^3}{-2c^9}$

19. $\frac{9x^3}{3x^3}$

20. $\frac{w^4}{w^6}$

21. $\frac{2p^8}{10p^{11}}$

22. $\frac{j^{12}k^6}{k^2}$

23. $\frac{60x^{13}y^5}{15x^4y^4}$

24. $\frac{8r^{10}t^7}{4r^8t^{10}}$

25. $\frac{w^3x^7}{w^2x^{10}}$

FIGURE 1.5

Independent Practice

26. $\dfrac{5n^8}{20n^8}$

27. $\dfrac{2n^2}{n}$

28. $\dfrac{d^{15}}{d^3}$

29. $\dfrac{x^5y^8z^2}{y^2z^6}$

30. $\dfrac{x^m}{x^n}$

31. $\dfrac{b^{5y}}{b^{3y}}$

32. $\dfrac{30x^f}{15x^g}$

33. $\dfrac{x^9}{x^c}$

34. $\dfrac{a^{1/3}}{a^{2/3}}$

35. $\dfrac{p^s}{p^{6s}}$

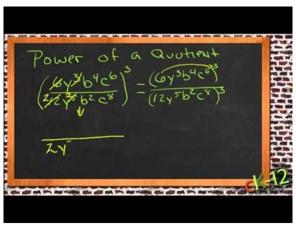
FIGURE 1.6

6.4 Exponent of a Quotient

Here you'll learn how to simplify a fraction with exponential expressions in both its numerator and denominator that is raised to another secondary power.

What if you had a fractional expression containing exponents that was raised to a secondary power, like $\left(\frac{x^8}{x^4}\right)^5$? How could you simplify it? After completing this Concept, you'll be able to use the power of a quotient property to simplify exponential expressions like this one.

Watch This



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CK-12 Foundation: 0804S Power of a Quotient

Guidance

When we raise a whole quotient to a power, another special rule applies. Here is an example:

$$\begin{aligned}\left(\frac{x^3}{y^2}\right)^4 &= \left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\\ &= \frac{(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)}{(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)}\\ &= \frac{x^{12}}{y^8}\end{aligned}$$

Notice that the exponent outside the parentheses is multiplied by the exponent in the numerator and the exponent in the denominator, separately. This is called the power of a quotient rule:

Power Rule for Quotients: $\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n\cdot p}}{y^{m\cdot p}}$

Let's apply these new rules to a few examples.

Example A

Simplify the following expressions.

a) $\frac{4^5}{4^2}$

b) $\frac{5^3}{5^7}$

c) $\left(\frac{3^4}{5^2}\right)^2$

Solution

Since there are just numbers and no variables, we can evaluate the expressions and get rid of the exponents completely.

a) We can use the quotient rule first and then evaluate the result: $\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$

OR we can evaluate each part separately and then divide: $\frac{4^5}{4^2} = \frac{1024}{16} = 64$

b) Use the quotient rule first and then evaluate the result: $\frac{5^3}{5^7} = \frac{1}{5^4} = \frac{1}{625}$

OR evaluate each part separately and then reduce: $\frac{5^3}{5^7} = \frac{125}{78125} = \frac{1}{625}$

Notice that it makes more sense to apply the quotient rule first for examples (a) and (b). Applying the exponent rules to simplify the expression *before* plugging in actual numbers means that we end up with smaller, easier numbers to work with.

c) Use the power rule for quotients first and then evaluate the result: $\left(\frac{3^4}{5^2}\right)^2 = \frac{3^8}{5^4} = \frac{6561}{625}$

OR evaluate inside the parentheses first and then apply the exponent: $\left(\frac{3^4}{5^2}\right)^2 = \left(\frac{81}{25}\right)^2 = \frac{6561}{625}$

Example B

Simplify the following expressions:

a) $\frac{x^{12}}{x^5}$

b) $\left(\frac{x^4}{x}\right)^5$

Solution

a) Use the quotient rule: $\frac{x^{12}}{x^5} = x^{12-5} = x^7$

b) Use the power rule for quotients and then the quotient rule: $\left(\frac{x^4}{x}\right)^5 = \frac{x^{20}}{x^5} = x^{15}$

OR use the quotient rule inside the parentheses first, then apply the power rule: $\left(\frac{x^4}{x}\right)^5 = (x^3)^5 = x^{15}$

Example C

Simplify the following expressions.

a) $\frac{6x^2y^3}{2xy^2}$

b) $\left(\frac{2a^3b^3}{8a^7b}\right)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number or each variable separately.

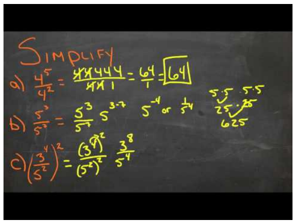
a) Group like terms together: $\frac{6x^2y^3}{2xy^2} = \frac{6}{2} \cdot \frac{x^2}{x} \cdot \frac{y^3}{y^2}$

Then reduce the numbers and apply the quotient rule on each fraction to get $3xy$.

b) Apply the quotient rule inside the parentheses first: $\left(\frac{2a^3b^3}{8a^7b}\right)^2 = \left(\frac{b^2}{4a^4}\right)^2$

Then apply the power rule for quotients: $\left(\frac{b^2}{4a^4}\right)^2 = \frac{b^4}{16a^8}$

Watch this video for help with the Examples above.



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CK-12 Foundation: Power of a Quotient

Vocabulary

- **Quotient of Powers Property:** For all real numbers x,

$\frac{x^n}{x^m} = x^{n-m}$.

- **Power of a Quotient Property:**

$\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n \cdot p}}{y^{m \cdot p}}$

Guided Practice

Simplify the following expressions.

a) $(x^2)^2 \cdot \frac{x^6}{x^4}$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Solution

In problems where we need to apply several rules together, we must keep the order of operations in mind.

a) We apply the power rule first on the first term:

$$(x^2)^2 \cdot \frac{x^6}{x^4} = x^4 \cdot \frac{x^6}{x^4}$$

Then apply the quotient rule to simplify the fraction:

$$x^4 \cdot \frac{x^6}{x^4} = x^4 \cdot x^2$$

And finally simplify with the product rule:

$$x^4 \cdot x^2 = x^6$$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Simplify inside the parentheses by reducing the numbers:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$$

Then apply the power rule to the first fraction:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}} = \frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}}$$

Group like terms together:

$$\frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}} = 64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}}$$

And apply the quotient rule to each fraction:

$$64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}} = \frac{64}{a^{10}b^{13}}$$

Explore More

Evaluate the following expressions.

1. $\left(\frac{3}{8}\right)^2$
2. $\left(\frac{2^2}{3^3}\right)^3$
3. $\left(\frac{2^3 \cdot 4^2}{2^4}\right)^2$

Simplify the following expressions.

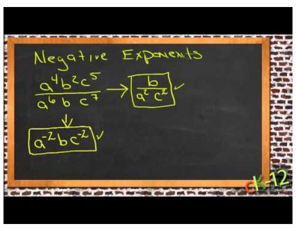
4. $\left(\frac{a^3b^4}{a^2b}\right)^3$
5. $\left(\frac{18a^4}{15a^{10}}\right)^4$
6. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
7. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
8. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$
9. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$ for $a = 2, b = 1$, and $c = 3$
10. $\left(\frac{3x^2y}{2z}\right)^3 \cdot \frac{z^2}{x}$ for $x = 1, y = 2$, and $z = -1$
11. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 2, y = -3$
12. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 0, y = 6$
13. If $a = 2$ and $b = 3$, simplify $\frac{(a^2b)(bc)^3}{a^3c^2}$ as much as possible.

6.5 Negative Exponents.

Here you'll learn how to simplify expressions that contain negative exponents.

What if you had a mathematical expression like $\frac{x^{-2}}{x^{-6}}$ that contained negative exponents? How could you simplify it so that none of the exponents were negative? After completing this Concept, you'll be able to simplify expressions with negative exponents like this one.

Watch This



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CK-12 Foundation: 0805S Negative Exponents

Guidance

The product and quotient rules for exponents lead to many interesting concepts. For example, so far we've mostly just considered positive, whole numbers as exponents, but you might be wondering what happens when the exponent isn't a positive whole number. What does it mean to raise something to the power of zero, or -1, or $\frac{1}{2}$? In this lesson, we'll find out.

Simplify Expressions With Negative Exponents

When we learned the quotient rule for exponents $\left(\frac{x^n}{x^m} = x^{(n-m)}\right)$, we saw that it applies even when the exponent in the denominator is bigger than the one in the numerator. Canceling out the factors in the numerator and denominator leaves the leftover factors in the denominator, and subtracting the exponents leaves a negative number. So negative exponents simply represent fractions with exponents in the denominator. This can be summarized in a rule:

Negative Power Rule for Exponents: $x^{-n} = \frac{1}{x^n}$, where $x \neq 0$

Negative exponents can be applied to products and quotients also. Here's an example of a negative exponent being applied to a product:

$(x^3y)^{-2} = x^{-6}y^{-2}$ using the power rule

$x^{-6}y^{-2} = \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2}$ using the negative power rule separately on each variable

And here's one applied to a quotient:

$\left(\frac{a}{b}\right)^{-3} = \frac{a^{-3}}{b^{-3}}$ using the power rule for quotients

$\frac{a^{-3}}{b^{-3}} = \frac{a^{-3}}{1} \cdot \frac{1}{b^{-3}} = \frac{1}{a^3} \cdot \frac{b^3}{1}$ using the negative power rule on each variable separately

$\frac{1}{a^3} \cdot \frac{b^3}{1} = \frac{b^3}{a^3}$ simplifying the division of fractions

$\frac{b^3}{a^3} = \left(\frac{b}{a}\right)^3$ using the power rule for quotients in reverse.

That last step wasn't really necessary, but putting the answer in that form shows us something useful: $\left(\frac{a}{b}\right)^{-3}$ is equal to $\left(\frac{b}{a}\right)^3$. This is an example of a rule we can apply more generally:

Negative Power Rule for Fractions: $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$, where $x \neq 0, y \neq 0$

This rule can be useful when you want to write out an expression without using fractions.

Example A

Write the following expressions without fractions.

a) $\frac{1}{x}$

b) $\frac{2}{x^2}$

c) $\frac{x^2}{y^3}$

d) $\frac{3}{xy}$

Solution

a) $\frac{1}{x} = x^{-1}$

b) $\frac{2}{x^2} = 2x^{-2}$

c) $\frac{x^2}{y^3} = x^2y^{-3}$

d) $\frac{3}{xy} = 3x^{-1}y^{-1}$

Example B

Simplify the following expressions and write them without fractions.

a) $\frac{4a^2b^3}{2a^5b}$

b) $\left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$

Solution

a) Reduce the numbers and apply the quotient rule to each variable separately:

$$\frac{4a^2b^3}{2a^5b} = 2 \cdot a^{2-5} \cdot b^{3-1} = 2a^{-3}b^2$$

b) Apply the power rule for quotients first:

$$\left(\frac{2x}{y^2}\right)^3 \cdot \frac{x^2y}{4} = \frac{8x^3}{y^6} \cdot \frac{x^2y}{4}$$

Then simplify the numbers, and use the product rule on the x's and the quotient rule on the y's:

$$\frac{8x^3}{y^6} \cdot \frac{x^2y}{4} = 2 \cdot x^{3+2} \cdot y^{1-6} = 2x^5y^{-5}$$

You can also use the negative power rule the other way around if you want to write an expression without negative exponents.

Example C

Write the following expressions without negative exponents.

a) $3x^{-3}$

b) $a^2b^{-3}c^{-1}$

c) $4x^{-1}y^3$

d) $\frac{2x^{-2}}{y^{-3}}$

Solution

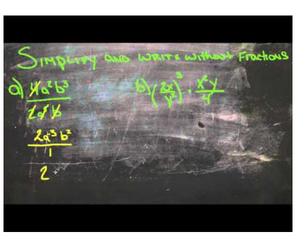
a) $3x^{-3} = \frac{3}{x^3}$

b) $a^2b^{-3}c^{-1} = \frac{a^2}{b^3c}$

c) $4x^{-1}y^3 = \frac{4y^3}{x}$

d) $\frac{2x^{-2}}{y^{-3}} = \frac{2y^3}{x^2}$

Watch this video for help with the Examples above.



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CK-12 Foundation: Negative Exponents

Vocabulary

- **Negative Power Rule for Exponents:** $x^{-n} = \frac{1}{x^n}$, where $x \neq 0$.

Guided Practice

Simplify the following expressions and write the answers without negative powers.

a) $\left(\frac{ab^{-2}}{b^3}\right)^2$

b) $\frac{x^{-3}y^2}{x^2y^{-2}}$

Solution

a) Apply the quotient rule inside the parentheses: $\left(\frac{ab^{-2}}{b^3}\right)^2 = (ab^{-5})^2$

Then apply the power rule: $(ab^{-5})^2 = a^2b^{-10} = \frac{a^2}{b^{10}}$

b) Apply the quotient rule to each variable separately: $\frac{x^{-3}y^2}{x^2y^{-2}} = x^{-3-2}y^{2-(-2)} = x^{-5}y^4 = \frac{y^4}{x^5}$

Explore More

Simplify the following expressions in such a way that there aren't any negative exponents in the answer.

1. $x^{-1}y^2$
2. x^{-4}
3. $\frac{x^{-3}}{x^{-7}}$
4. $\frac{x^{-3}y^{-5}}{z^{-7}}$
5. $\left(\frac{a}{b}\right)^{-2}$
6. $(3a^{-2}b^2c^3)^3$

Simplify the following expressions in such a way that there aren't any fractions in the answer.

7. $\frac{a^{-3}(a^5)}{a^{-6}}$
8. $\frac{5x^6y^2}{x^8y}$
9. $\frac{(4ab^6)^3}{(ab)^5}$
10. $\frac{(3x^3)(4x^4)}{(2y)^2}$
11. $\frac{a^{-2}b^{-3}}{c^{-1}}$

Independent Practice

12. b^0

13. c^{-1}

14. n^8p^{-3}

15. $2n^{-2}$

16. $(x^2y^3)^0$

17. $x^{-2} \bullet x^{-4}$

18. $a^{-1}b^{-2}c^3$

19. $\dfrac{75a^{-7}b^{-3}c^4}{25a^3b^{-6}c^5}$

20. $m^5 \bullet m^{-2}$

21. $k^0 \bullet k^7$

22. $s^{-2}t^{-2}u^4$

23. $\dfrac{(a^{-\frac{1}{3}})(b^{\frac{3}{4}})}{(a^{-\frac{2}{3}})(b^{-\frac{3}{4}})}$

24. $\dfrac{y^{-6}}{y^2}$

25. $\dfrac{fg^{-4}}{g^3}$

26. $\dfrac{-4y^{-7}}{5z^{-6}}$

27. $\dfrac{n^{-2}}{n^{-8}}$

FIGURE 1.7

28. $\left(\frac{x^{-4}y^{3}}{x^{-3}y^{4}}\right)^{0}$

29. $b^{-5} \bullet b^{5}$

30. $\frac{-2a^{4}}{a^{-2}b}$

31. $\frac{(2a^{3})(3a^{5})}{a^{-2}}$

32. $\frac{a^{-4}b^{-4}c^{0}}{a^{-6}b^{-2}}$

33. $\frac{12m^{-2}n^{4}}{3n}$

34. $\frac{x^{-c}}{y^{-d}}$

35. $\frac{-2w^{3} \bullet 2w^{4}}{w^{-5}}$

36. $\frac{6a^{-4x}d}{3a^{-2x}d^{-2}}$

37. $\frac{28y^{-c}}{7y^{c}}$

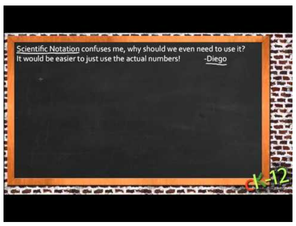
FIGURE 1.8

6.6 Scientific Notation

Here you'll learn how to write very large and very small numbers so that they are easier to work with and evaluate.

What if you knew that the population of the United States was 308,000,000? How could you simplify this number so that it is easier to work with? After completing this Concept, you'll be able to write very large and very small numbers like this one in scientific notation.

Watch This



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CK-12 Foundation: 0808S Scientific Notation

Guidance

Consider the number six hundred and forty three thousand, two hundred and ninety seven. We write it as 643,297 and each digit's position has a "value" assigned to it. You may have seen a table like this before:

hundred-thousands	ten-thousands	thousands	hundreds	tens	units
6	4	3	2	9	7

We've seen that when we write an exponent above a number, it means that we have to multiply a certain number of copies of that number together. We've also seen that a zero exponent always gives us 1, and negative exponents give us fractional answers.

Look carefully at the table above. Do you notice that all the column headings are powers of ten? Here they are listed:

$$\begin{aligned}100,000 &= 10^5\\10,000 &= 10^4\\1,000 &= 10^3\\100 &= 10^2\\10 &= 10^1\end{aligned}$$

Even the "units" column is really just a power of ten. ***Unit*** means 1, and 1 is 10^0.

If we divide 643,297 by 100,000 we get 6.43297; if we multiply 6.43297 by 100,000 we get 643, 297. But we have just seen that 100,000 is the same as 10^5, so if we multiply 6.43297 by 10^5 we should also get 643,297. In other words,

$$643,297 = 6.43297 \times 10^5$$

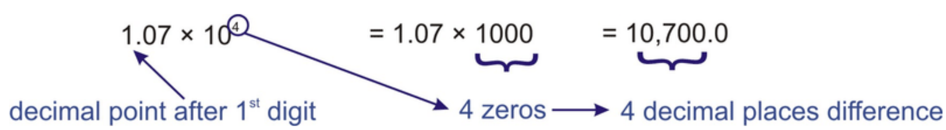
Writing Numbers in Scientific Notation

In scientific notation, numbers are always written in the form $a \times 10^b$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal). This notation is especially useful for numbers that are either very small or very large.

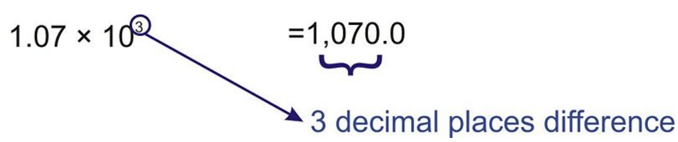
Here's a set of examples:

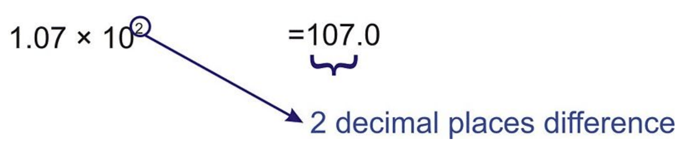
$$\begin{aligned}
1.07 \times 10^4 &= 10,700 \\
1.07 \times 10^3 &= 1,070 \\
1.07 \times 10^2 &= 107 \\
1.07 \times 10^1 &= 10.7 \\
1.07 \times 10^0 &= 1.07 \\
1.07 \times 10^{-1} &= 0.107 \\
1.07 \times 10^{-2} &= 0.0107 \\
1.07 \times 10^{-3} &= 0.00107 \\
1.07 \times 10^{-4} &= 0.000107
\end{aligned}$$

Look at the first example and notice where the decimal point is in both expressions.



So the exponent on the ten acts to move the decimal point over to the right. An exponent of 4 moves it 4 places and an exponent of 3 would move it 3 places.





This makes sense because each time you multiply by 10, you move the decimal point one place to the right. 1.07 times 10 is 10.7, times 10 again is 107.0, and so on.

Similarly, if you look at the later examples in the table, you can see that a negative exponent on the 10 means the decimal point moves that many places to the left. This is because multiplying by 10^{-1} is the same as multiplying by $\frac{1}{10}$, which is like dividing by 10. So instead of moving the decimal point one place to the right for every multiple of 10, we move it one place to the left for every multiple of $\frac{1}{10}$.

That's how to convert numbers from scientific notation to standard form. When we're converting numbers *to* scientific notation, however, we have to apply the whole process backwards. First we move the decimal point until it's immediately after the first nonzero digit; then we count how many places we moved it. If we moved it to the *left,* the exponent on the 10 is positive; if we moved it to the *right,* it's negative.

So, for example, to write 0.000032 in scientific notation, we'd first move the decimal five places to the right to get 3.2; then, since we moved it right, the exponent on the 10 should be *negative* five, so the number in scientific notation is 3.2×10^{-5}.

You can double-check whether you've got the right direction by comparing the number in scientific notation with the number in standard form, and thinking "Does this represent a *big* number or a *small* number?" A positive exponent on the 10 represents a number bigger than 10 and a negative exponent represents a number smaller than 10, and you can easily tell if the number in standard form is bigger or smaller than 10 just by looking at it.

For more practice, try the online tool at http://hotmath.com/util/hm_flash_movie.html?movie=/learning_activities/interactivities/sciNotation.swf . Click the arrow buttons to move the decimal point until the number in the middle is written in proper scientific notation, and see how the exponent changes as you move the decimal point.

Example A

Write the following numbers in scientific notation.

a) 63

b) 9,654

c) 653,937,000

d) 0.003

e) 0.000056

f) 0.00005007

Solution

a) $63 = 6.3 \times 10 = 6.3 \times 10^1$

b) $9,654 = 9.654 \times 1,000 = 9.654 \times 10^3$

c) $653,937,000 = 6.53937000 \times 100,000,000 = 6.53937 \times 10^8$

d) $0.003 = 3 \times \frac{1}{1000} = 3 \times 10^{-3}$

e) $0.000056 = 5.6 \times \frac{1}{100,000} = 5.6 \times 10^{-5}$

f) $0.00005007 = 5.007 \times \frac{1}{100,000} = 5.007 \times 10^{-5}$

Example B

Evaluate the following expressions and write your answer in scientific notation.

a) $(3.2 \times 10^6) \cdot (8.7 \times 10^{11})$

b) $(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19})$

c) $(1.7 \times 10^6) \cdot (2.7 \times 10^{-11})$

Solution

The key to evaluating expressions involving scientific notation is to group the powers of 10 together and deal with them separately.

a) $(3.2 \times 10^6)(8.7 \times 10^{11}) = \underbrace{3.2 \times 8.7}_{27.84} \times \underbrace{10^6 \times 10^{11}}_{10^{17}} = 27.84 \times 10^{17}$. But 27.84×10^{17} isn't in proper scientific notation, because it has more than one digit before the decimal point. We need to move the decimal point one more place to the left and add 1 to the exponent, which gives us 2.784×10^{18}.

b)

$$\begin{aligned}(5.2 \times 10^{-4})(3.8 \times 10^{-19}) &= \underbrace{5.2 \times 3.8}_{19.76} \times \underbrace{10^{-4} \times 10^{-19}}_{10^{-23}} \\ &= 19.76 \times 10^{-23} \\ &= 1.976 \times 10^{-22}\end{aligned}$$

c) $(1.7 \times 10^6)(2.7 \times 10^{-11}) = \underbrace{1.7 \times 2.7}_{4.59} \times \underbrace{10^6 \times 10^{-11}}_{10^{-5}} = 4.59 \times 10^{-5}$

When we use scientific notation in the real world, we often round off our calculations. Since we're often dealing with very big or very small numbers, it can be easier to round off so that we don't have to keep track of as many digits—and scientific notation helps us with that by saving us from writing out all the extra zeros. For example, if we round off 4,227, 457,903 to 4,200,000,000, we can then write it in scientific notation as simply 4.2×10^9.

When rounding, we often talk of **significant figures** or **significant digits**. Significant figures include

- all nonzero digits
- all zeros that come *before* a nonzero digit and *after* either a decimal point or another nonzero digit

For example, the number 4000 has one significant digit; the zeros don't count because there's no nonzero digit after them. But the number 4000.5 has five significant digits: the 4, the 5, and all the zeros in between. And the number 0.003 has three significant digits: the 3 and the two zeros that come between the 3 and the decimal point.

Example C

Evaluate the following expressions. Round to 3 significant figures and write your answer in scientific notation.

a) $(3.2 \times 10^6) \div (8.7 \times 10^{11})$

b) $(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19})$

Solution

It's easier if we convert to fractions and THEN separate out the powers of 10.

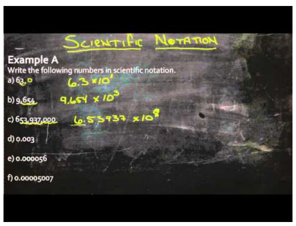
a)

$$\begin{aligned}(3.2 \times 10^6) \div (8.7 \times 10^{11}) &= \frac{3.2 \times 10^6}{8.7 \times 10^{11}} && - separate\ out\ the\ powers\ of\ 10: \\ &= \frac{3.2}{8.7} \times \frac{10^6}{10^{11}} && - evaluate\ each\ fraction\ (round\ to\ 3\ s.f.): \\ &= 0.368 \times 10^{(6-11)} \\ &= 0.368 \times 10^{-5} && - remember\ how\ to\ write\ scientific\ notation! \\ &= 3.68 \times 10^{-6}\end{aligned}$$

b)

$$\begin{aligned}
(5.2\times10^{-4})\div(3.8\times10^{-19}) &= \frac{5.2\times10^{-4}}{3.8\times10^{-19}} && \textit{– separate the powers of } 10: \\
&= \frac{5.2}{3.8}\times\frac{10^{-4}}{10^{-19}} && \textit{– evaluate each fraction (round to 3 s.f.)} \\
&= 1.37\times10^{((-4)-(-19))} \\
&= 1.37\times10^{15}
\end{aligned}$$

Watch this video for help with the Examples above.



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CK-12 Foundation: Scientific Notation

Vocabulary

- In **scientific notation**, numbers are always written in the form $a\times10^b$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal).

Guided Practice

Evaluate the following expression. Round to 3 significant figures and write your answer in scientific notation.

$(1.7\times10^6)\div(2.7\times10^{-11})$

Solution:

$$\begin{aligned}
(1.7\times10^{6})\div(2.7\times10^{-11}) &= \frac{1.7\times10^{6}}{2.7\times10^{-11}} && \textit{– next we separate the powers of } 10: \\
&= \frac{1.7}{2.7}\times\frac{10^{6}}{10^{-11}} && \textit{– evaluate each fraction (round to 3 s.f.)} \\
&= 0.630\times10^{(6-(-11))} \\
&= 0.630\times10^{17} \\
&= 6.30\times10^{16}
\end{aligned}$$

Note that we have to leave in the final zero to indicate that the result has been rounded.

Explore More

Write the numerical value of the following.

1. 3.102×10^2

2. 7.4×10^4
3. 1.75×10^{-3}
4. 2.9×10^{-5}
5. 9.99×10^{-9}

Write the following numbers in scientific notation.

6. 120,000
7. 1,765,244
8. 12
9. 0.00281
10. 0.000000027

How many significant digits are in each of the following?

11. 38553000
12. 2754000.23
13. 0.0000222
14. 0.0002000079

Round each of the following to two significant digits.

15. 3.0132
16. 82.9913

6.7 Graphs of Exponential Functions

Here you'll learn how to graph exponential functions and how to compare the graphs of exponential functions on the same coordinate axes.

What if you had an exponential function like $y = 2 \cdot (3^x)$.? How could you graph this function? After completing this Concept, you'll be able to graph and compare the graphs of exponential functions like this one.

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CK-12 Foundation: 0811S Exponential Functions

Guidance

A colony of bacteria has a population of three thousand at noon on Monday. During the next week, the colony's population doubles every day. What is the population of the bacteria colony just before midnight on Saturday?

At first glance, this seems like a problem you could solve using a geometric sequence. And you could, if the bacteria population doubled all at once every day; since it doubled every day for five days, the final population would be 3000 times 2^5.

But bacteria don't reproduce all at once; their population grows slowly over the course of an entire day. So how do we figure out the population after five *and a half* days?

Exponential Functions

Exponential functions are a lot like geometrical sequences. The main difference between them is that a geometric sequence is **discrete** while an exponential function is **continuous**.

Discrete means that the sequence has values only at distinct points (the 1st term, 2nd term, etc.)

Continuous means that the function has values for all possible values of x. The integers are included, but also all the numbers in between.

The problem with the bacteria is an example of a continuous function. Here's an example of a discrete function:

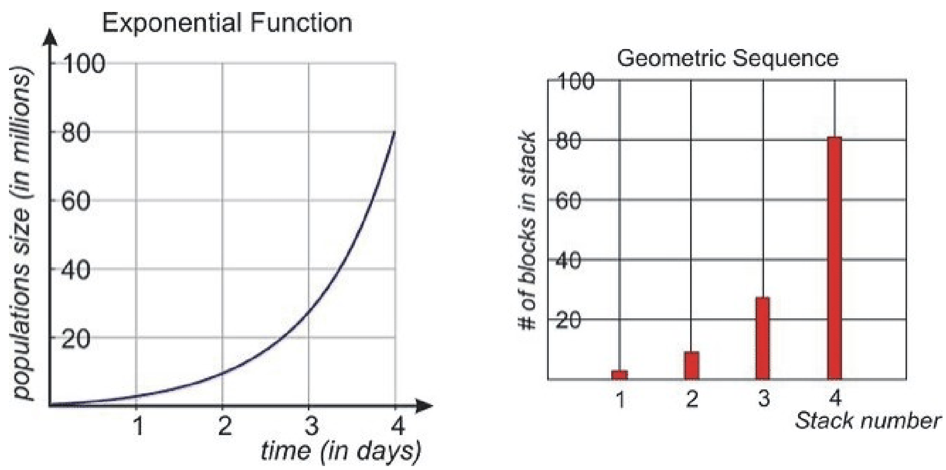
An ant walks past several stacks of Lego blocks. There is one block in the first stack, 3 blocks in the 2^{nd} *stack and 9 blocks in the* 3^{rd} *stack. In fact, in each successive stack there are triple the number of blocks than in the previous stack.*

In this example, each stack has a distinct number of blocks and the next stack is made by adding a certain number of whole pieces all at once. More importantly, however, there are no values of the sequence **between** the stacks. You can't ask how high the stack is between the 2^{nd} and 3^{rd} stack, as no stack exists at that position!

As a result of this difference, we use a geometric series to describe quantities that have values at discrete points, and we use exponential functions to describe quantities that have values that change continuously.

When we graph an exponential function, we draw the graph with a solid curve to show that the function has values at any time during the day. On the other hand, when we graph a geometric sequence, we draw discrete points to signify that the sequence only has value at those points but not in between.

Here are graphs for the two examples above:



The formula for an exponential function is similar to the formula for finding the terms in a geometric sequence. An exponential function takes the form

$$y = A \cdot b^x$$

where A is the starting amount and b is the amount by which the total is multiplied every time. For example, the bacteria problem above would have the equation $y = 3000 \cdot 2^x$.

Compare Graphs of Exponential Functions

Let's graph a few exponential functions and see what happens as we change the constants in the formula. The basic shape of the exponential function should stay the same—but it may become steeper or shallower depending on the constants we are using.

First, let's see what happens when we change the value of A.

Example A

Compare the graphs of $y = 2^x$ *and* $y = 3 \cdot 2^x$.

Solution

Let's make a table of values for both functions.

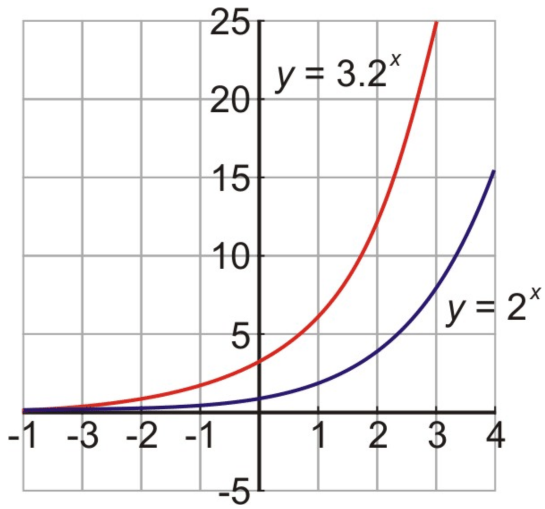
TABLE 1.1:

x	$y = 2^x$	$y = 3 \cdot 2^x$
-3	$\frac{1}{8}$	$y = 3 \cdot 2^{-3} = 3 \cdot \frac{1}{2^3} = \frac{3}{8}$
-2	$\frac{1}{4}$	$y = 3 \cdot 2^{-2} = 3 \cdot \frac{1}{2^2} = \frac{3}{4}$
-1	$\frac{1}{2}$	$y = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2^1} = \frac{3}{2}$
0	1	$y = 3 \cdot 2^0 = 3$

TABLE 1.1: (continued)

x	$y = 2^x$	$y = 3 \cdot 2^x$
1	2	$y = 3 \cdot 2^1 = 6$
2	4	$y = 3 \cdot 2^2 = 3 \cdot 4 = 12$
3	8	$y = 3 \cdot 2^3 = 3 \cdot 8 = 24$

Now let's use this table to graph the functions.



We can see that the function $y = 3 \cdot 2^x$ is bigger than the function $y = 2^x$. In both functions, the value of y doubles every time x increases by one. However, $y = 3 \cdot 2^x$ "starts" with a value of 3, while $y = 2^x$ "starts" with a value of 1, so it makes sense that $y = 3 \cdot 2^x$ would be bigger as its values of y keep getting doubled.

Similarly, if the starting value of A is smaller, the values of the entire function will be smaller.

Example B

Compare the graphs of $y = 2^x$ and $y = \frac{1}{3} \cdot 2^x$.

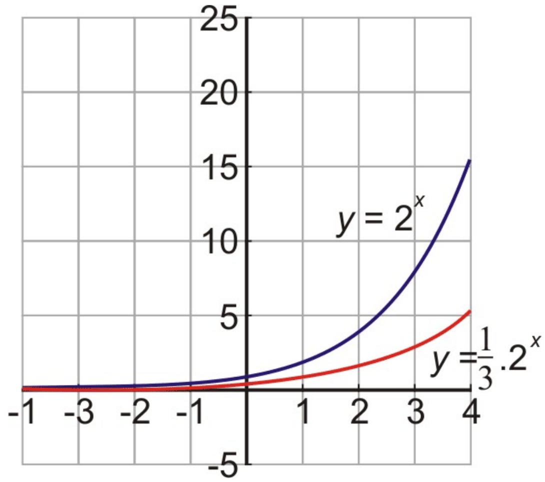
Solution

Let's make a table of values for both functions.

TABLE 1.2:

x	$y = 2^x$	$y = \frac{1}{3} \cdot 2^x$
-3	$\frac{1}{8}$	$y = \frac{1}{3} \cdot 2^{-3} = \frac{1}{3} \cdot \frac{1}{2^3} = \frac{1}{24}$
-2	$\frac{1}{4}$	$y = \frac{1}{3} \cdot 2^{-2} = \frac{1}{3} \cdot \frac{1}{2^2} = \frac{1}{12}$
-1	$\frac{1}{2}$	$y = \frac{1}{3} \cdot 2^{-1} = \frac{1}{3} \cdot \frac{1}{2^1} = \frac{1}{6}$
0	1	$y = \frac{1}{3} \cdot 2^0 = \frac{1}{3}$
1	2	$y = \frac{1}{3} \cdot 2^1 = \frac{2}{3}$
2	4	$y = \frac{1}{3} \cdot 2^2 = \frac{1}{3} \cdot 4 = \frac{4}{3}$
3	8	$y = \frac{1}{3} \cdot 2^3 = \frac{1}{3} \cdot 8 = \frac{8}{3}$

Now let's use this table to graph the functions.



As we expected, the exponential function $y = \frac{1}{3} \cdot 2^x$ is smaller than the exponential function $y = 2^x$.

So what happens if the starting value of A is negative? Let's find out.

Example C

Graph the exponential function $y = -5 \cdot 2^x$.

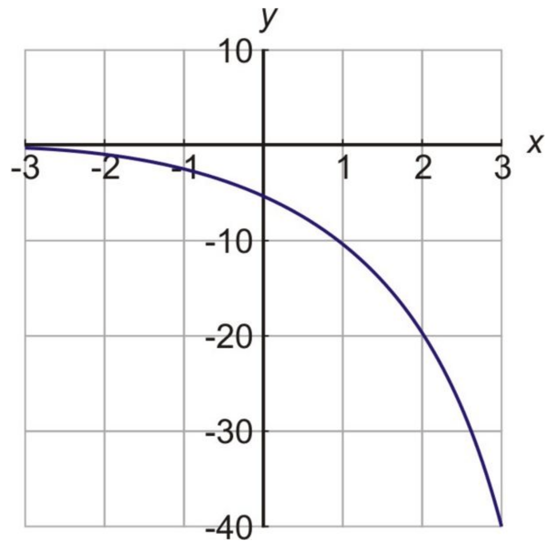
Solution

Let's make a table of values:

TABLE 1.3:

x	$y = -5 \cdot 2^x$
-2	$-\frac{5}{4}$
-1	$-\frac{5}{2}$
0	-5
1	-10
2	-20
3	-40

Now let's graph the function:



This result shouldn't be unexpected. Since the starting value is negative and keeps doubling over time, it makes sense that the value of y gets farther from zero, but in a negative direction. The graph is basically just like the graph of $y = 5 \cdot 2^x$, only mirror-reversed about the $x-$axis.

Now, let's compare exponential functions whose bases (b) are different.

Example D

Graph the following exponential functions on the same graph: $y = 2^x, y = 3^x, y = 5^x, y = 10^x$.

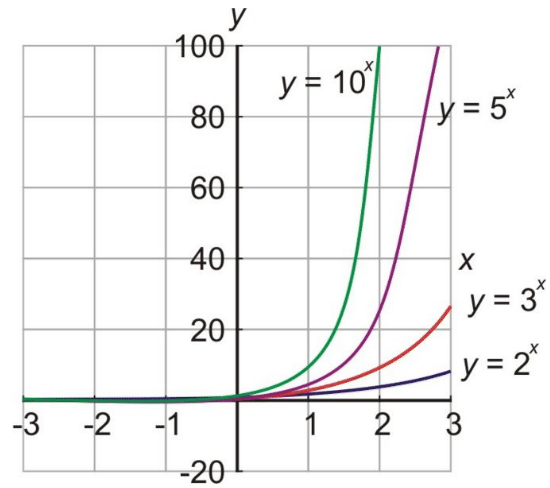
Solution

First we'll make a table of values for all four functions.

TABLE 1.4:

x	$y = 2^x$	$y = 3^x$	$y = 5^x$	$y = 10^x$
-2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$	$\frac{1}{100}$
-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{10}$
0	1	1	1	1
1	2	3	5	10
2	4	9	25	100
3	8	27	125	1000

Now let's graph the function:



Notice that for $x = 0$, all four functions equal 1. They all "start out" at the same point, but the ones with higher values for b grow faster when x is positive—and also shrink faster when x is negative.

Finally, let's explore what happens for values of b that are less than 1.

Example E

Graph the exponential function $y = 5 \cdot \left(\frac{1}{2}\right)^x$.

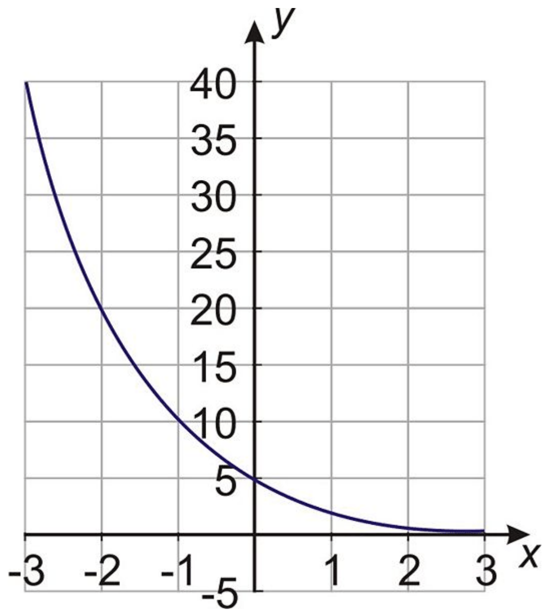
Solution

Let's start by making a table of values. (Remember that a fraction to a negative power is equivalent to its reciprocal to the same positive power.)

TABLE 1.5:

x	$y = 5 \cdot \left(\frac{1}{2}\right)^x$
-3	$y = 5 \cdot \left(\frac{1}{2}\right)^{-3} = 5 \cdot 2^3 = 40$
-2	$y = 5 \cdot \left(\frac{1}{2}\right)^{-2} = 5 \cdot 2^2 = 20$
-1	$y = 5 \cdot \left(\frac{1}{2}\right)^{-1} = 5 \cdot 2^1 = 10$
0	$y = 5 \cdot \left(\frac{1}{2}\right)^0 = 5 \cdot 1 = 5$
1	$y = 5 \cdot \left(\frac{1}{2}\right)^1 = \frac{5}{2}$
2	$y = 5 \cdot \left(\frac{1}{2}\right)^2 = \frac{5}{4}$

Now let's graph the function:



This graph looks very different than the graphs from the previous example! What's going on here?

When we raise a number greater than 1 to the power of x, it gets bigger as x gets bigger. But when we raise a number smaller than 1 to the power of x, it gets *smaller* as x gets bigger—as you can see from the table of values above. This makes sense because multiplying any number by a quantity less than 1 always makes it smaller.

So, when the base b of an exponential function is between 0 and 1, the graph is like an ordinary exponential graph, only decreasing instead of increasing. Graphs like this represent **exponential decay** instead of **exponential growth**. Exponential decay functions are used to describe quantities that decrease over a period of time.

When b can be written as a fraction, we can use the Property of Negative Exponents to write the function in a different form. For instance, $y = 5 \cdot \left(\frac{1}{2}\right)^x$ is equivalent to $5 \cdot 2^{-x}$. These two forms are both commonly used, so it's important to know that they are equivalent.

Watch this video for help with the Examples above.

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CK-12 Foundation: Exponential Functions

Vocabulary

- **General Form of an Exponential Function:** $y = A(b)^x$, where $A =$ *initial value* and $b =$ multiplication factor.

Guided Practice

a.) *Graph the exponential function* $y = 8 \cdot 3^{-x}$.

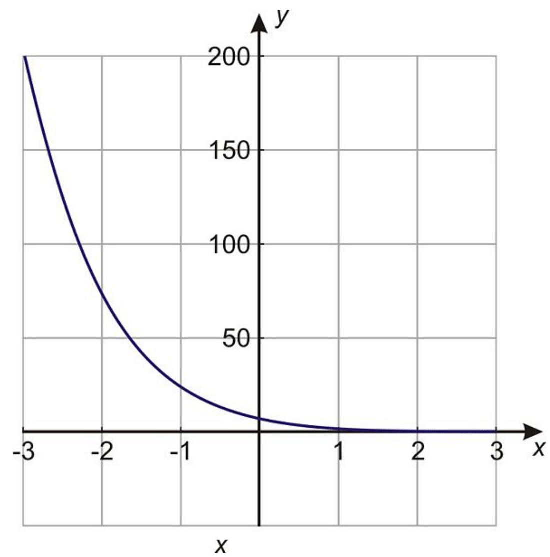
b.) *Graph the functions* $y = 4^x$ *and* $y = 4^{-x}$ *on the same coordinate axes.*

Solution:

a.) Here is our table of values and the graph of the function.

TABLE 1.6:

x	$y = 8 \cdot 3^{-x}$
-3	$y = 8 \cdot 3^{-(-3)} = 8 \cdot 3^3 = 216$
-2	$y = 8 \cdot 3^{-(-2)} = 8 \cdot 3^2 = 72$
-1	$y = 8 \cdot 3^{-(-1)} = 8 \cdot 3^1 = 24$
0	$y = 8 \cdot 3^0 = 8$
1	$y = 8 \cdot 3^{-1} = \frac{8}{3}$
2	$y = 8 \cdot 3^{-2} = \frac{8}{9}$

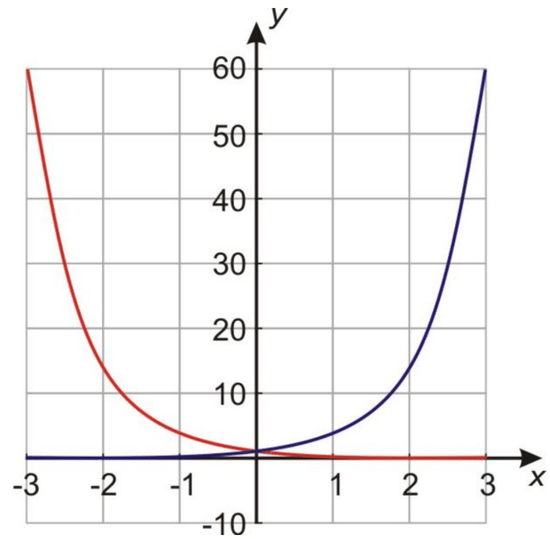


b.) Here is the table of values for the two functions. Looking at the values in the table, we can see that the two functions are "backwards" of each other, in the sense that the values for the two functions are reciprocals.

TABLE 1.7:

x	$y = 4^x$	$y = 4^{-x}$
-3	$y = 4^{-3} = \frac{1}{64}$	$y = 4^{-(-3)} = 64$
-2	$y = 4^{-2} = \frac{1}{16}$	$y = 4^{-(-2)} = 16$
-1	$y = 4^{-1} = \frac{1}{4}$	$y = 4^{-(-1)} = 4$
0	$y = 4^0 = 1$	$y = 4^0 = 1$
1	$y = 4^1 = 4$	$y = 4^{-1} = \frac{1}{4}$
2	$y = 4^2 = 16$	$y = 4^{-2} = \frac{1}{16}$
3	$y = 4^3 = 64$	$y = 4^{-3} = \frac{1}{64}$

Here is the graph of the two functions. Notice that the two functions are mirror images of each other if the mirror is placed vertically on the $y-$axis.



In the next lesson, you'll see how exponential growth and decay functions can be used to represent situations in the real world.

Explore More

Graph the following exponential functions by making a table of values.

1. $y = 3^x$
2. $y = 5 \cdot 3^x$
3. $y = 40 \cdot 4^x$
4. $y = 3 \cdot 10^x$

Graph the following exponential functions.

5. $y = \left(\frac{1}{5}\right)^x$
6. $y = 4 \cdot \left(\frac{2}{3}\right)^x$
7. $y = 3^{-x}$
8. $y = \frac{3}{4} \cdot 6^{-x}$
9. Which two of the eight graphs above are mirror images of each other?
10. What function would produce a graph that is the mirror image of the one in problem 4?
11. How else might you write the exponential function in problem 5?
12. How else might you write the function in problem 6?

Solve the following problems.

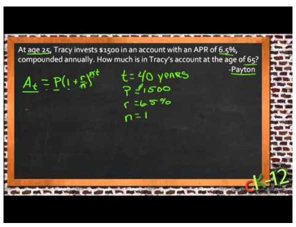
13. A chain letter is sent out to 10 people telling everyone to make 10 copies of the letter and send each one to a new person.
    a. Assume that everyone who receives the letter sends it to ten new people and that each cycle takes a week. How many people receive the letter on the sixth week?
    b. What if everyone only sends the letter to 9 new people? How many people will then get letters on the sixth week?
14. Nadia received \$200 for her 10^{th} birthday. If she saves it in a bank account with 7.5% interest compounded yearly, how much money will she have in the bank by her 21^{st} birthday?

6.8 Applications of Exponential Functions

Here you'll learn how to apply a problem-solving plan to problems involving exponential functions. You'll also learn how to solve real-world applications involving exponential growth and decay.

What if you won \$500 in a spelling bee competition and invested it into a mutual fund that pays 8% interest compounded annually? How much money would you have after 5 years? After completing this Concept, you'll be able to solve real-world problems like this one that involve exponential functions.

Watch This



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CK-12 Foundation: 0812S Applications of Exponential Functions

Guidance

For her eighth birthday, Shelley's grandmother gave her a full bag of candy. Shelley counted her candy and found out that there were 160 pieces in the bag. As you might suspect, Shelley loves candy, so she ate half the candy on the first day. Then her mother told her that if she eats it at that rate, the candy will only last one more day—so Shelley devised a clever plan. She will always eat half of the candy that is left in the bag each day. She thinks that this way she can eat candy every day and never run out.

How much candy does Shelley have at the end of the week? Will the candy really last forever?

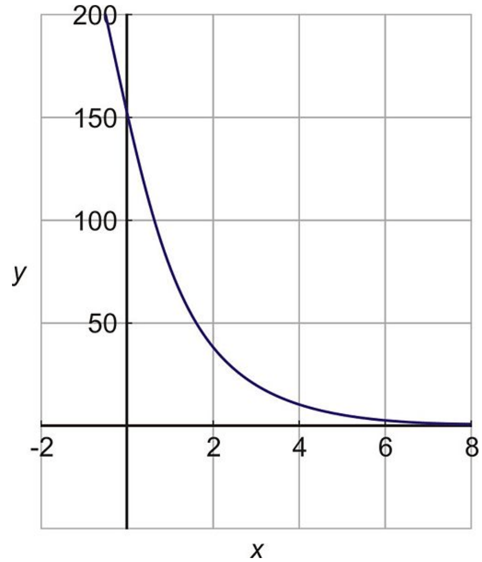
Let's make a table of values for this problem.

Day	0	1	2	3	4	5	6	7
# of candies	160	80	40	20	10	5	2.5	1.25

You can see that if Shelley eats half the candies each day, then by the end of the week she only has 1.25 candies left in her bag.

Let's write an equation for this exponential function. Using the formula $y = A \cdot b^x$, we can see that A is 160 (the number of candies she starts out with and b is $\frac{1}{2}$, so our equation is $y = 160 \cdot \left(\frac{1}{2}\right)^x$.)

Now let's graph this function. The resulting graph is shown below.



So, will Shelley's candy last forever? We saw that by the end of the week she has 1.25 candies left, so there doesn't seem to be much hope for that. But if you look at the graph, you'll see that the graph never really gets to zero. Theoretically there will always be *some* candy left, but Shelley will be eating very tiny fractions of a candy every day after the first week!

This is a fundamental feature of an exponential decay function. Its values get smaller and smaller but never quite reach zero. In mathematics, we say that the function has an **asymptote** at $y = 0$; in other words, it gets closer and closer to the line $y = 0$ but never quite meets it.

Problem-Solving Strategies

Remember our problem-solving plan from earlier?

1. Understand the problem.
2. Devise a plan –Translate.
3. Carry out the plan –Solve.
4. Look –Check and Interpret.

We can use this plan to solve application problems involving exponential functions. Compound interest, loudness of sound, population increase, population decrease or radioactive decay are all applications of exponential functions. In these problems, we'll use the methods of constructing a table and identifying a pattern to help us devise a plan for solving the problems.

Example A

Suppose \$4000 is invested at 6% interest compounded annually. How much money will there be in the bank at the end of 5 years? At the end of 20 years?

Solution

Step 1: Read the problem and summarize the information.

\$4000 is invested at 6% interest compounded annually; we want to know how much money we have in five years.

Assign variables:

Let $x =$ time in years

Let $y =$ amount of money in investment account

Step 2: Look for a pattern.

We start with \$4000 and each year we add 6% interest to the amount in the bank.

Start:	\$4000
1^{st} year:	Interest $= 4000 \times (0.06) = \$240$
	This is added to the previous amount: $\$4000 + \$4000 \times (0.06)$
	$= \$4000(1+0.06)$
	$= \$4000(1.06)$
	$= \$4240$
2^{nd} year	Previous amount $+$ interest on the previous amount
	$= \$4240(1+0.06)$
	$= \$4240(1.06)$
	$= \$4494.40$

The pattern is that each year we multiply the previous amount by the factor of 1.06.

Let's fill in a table of values:

Time (years)	0	1	2	3	4	5
Investments amount(\$)	4000	4240	4494.4	4764.06	5049.90	5352.9

We see that **at the end of five years we have \$5352.90 in the investment account**.

Step 3: Find a formula.

We were able to find the amount after 5 years just by following the pattern, but rather than follow that pattern for another 15 years, it's easier to use it to find a general formula. Since the original investment is multiplied by 1.06 each year, we can use exponential notation. Our formula is $y = 4000 \cdot (1.06)^x$, where x is the number of years since the investment.

To find the amount after 5 years we plug $x = 5$ into the equation:

$$y = 4000 \cdot (1.06)^5 = \$5352.90$$

To find the amount after 20 years we plug $x = 20$ into the equation:

$$y = 4000 \cdot (1.06)^{20} = \$12828.54$$

Step 4: Check.

Looking back over the solution, we see that we obtained the answers to the questions we were asked and the answers make sense.

To check our answers, we can plug some low values of x into the formula to see if they match the values in the table:

$x = 0: \quad y = 4000 \cdot (1.06)^0 = 4000$

$x = 1: \quad y = 4000 \cdot (1.06)^1 = 4240$

$x = 2: \quad y = 4000 \cdot (1.06)^2 = 4494.4$

The answers match the values we found earlier. The amount of increase gets larger each year, and that makes sense because the interest is 6% of an amount that is larger every year.

Example B

In 2002 the population of schoolchildren in a city was 90,000. This population decreases at a rate of 5% each year. What will be the population of school children in year 2010?

Solution

Step 1: Read the problem and summarize the information.

The population is 90,000; the rate of decrease is 5% each year; we want the population after 8 years.

Assign variables:

Let $x =$ time since 2002 (in years)

Let $y =$ population of school children

Step 2: Look for a pattern.

Let's start in 2002, when the population is 90,000.

The rate of decrease is 5% each year, so the amount in 2003 is 90,000 minus 5% of 90,000, or 95% of 90,000.

$$\begin{aligned} &\text{In 2003}: && \text{Population} = 90,000 \times 0.95 \\ &\text{In 2004}: && \text{Population} = 90,000 \times 0.95 \times 0.95 \end{aligned}$$

The pattern is that for each year we multiply by a factor of 0.95

Let's fill in a table of values:

Year	2002	2003	2004	2005	2006	2007
Population	90,000	85,500	81,225	77,164	73,306	69,640

Step 3: Find a formula.

Since we multiply by 0.95 every year, our exponential formula is $y = 90000 \cdot (0.95)^x$, where x is the number of years since 2002. To find the population in 2010 (8 years after 2002), we plug in $x = 8$:

$y = 90000 \cdot (0.95)^8 = 59,708$ schoolchildren.

Step 4: Check.

Looking back over the solution, we see that we answered the question we were asked and that it makes sense. The answer makes sense because the numbers decrease each year as we expected. We can check that the formula is correct by plugging in the values of x from the table to see if the values match those given by the formula.

$$\begin{aligned} &\text{Year } 2002, x = 0: && \text{Population} = y = 90000 \cdot (0.95)^0 = 90,000 \\ &\text{Year } 2003, x = 1: && \text{Population} = y = 90000 \cdot (0.95)^1 = 85,500 \\ &\text{Year } 2004, x = 2: && \text{Population} = y = 90000 \cdot (0.95)^2 = 81,225 \end{aligned}$$

Solve Real-World Problems Involving Exponential Growth

Now we'll look at some more real-world problems involving exponential functions. We'll start with situations involving exponential growth.

Example C

The population of a town is estimated to increase by 15% per year. The population today is 20 thousand. Make a graph of the population function and find out what the population will be ten years from now.

Solution

First, we need to write a function that describes the population of the town.

The general form of an exponential function is $y = A \cdot b^x$.

Define y as the population of the town.

Define x as the number of years from now.

A is the initial population, so $A = 20$ (thousand).

Finally we must find what b is. We are told that the population increases by 15% each year. To calculate percents we have to change them into decimals: 15% is equivalent to 0.15. So each year, the population increases by 15% of A, or $0.15A$.

To find the total population for the following year, we must add the *current* population to the *increase* in population. In other words, $A + 0.15A = 1.15A$. So the population must be multiplied by a factor of 1.15 each year. This means that the base of the exponential is $b = 1.15$.

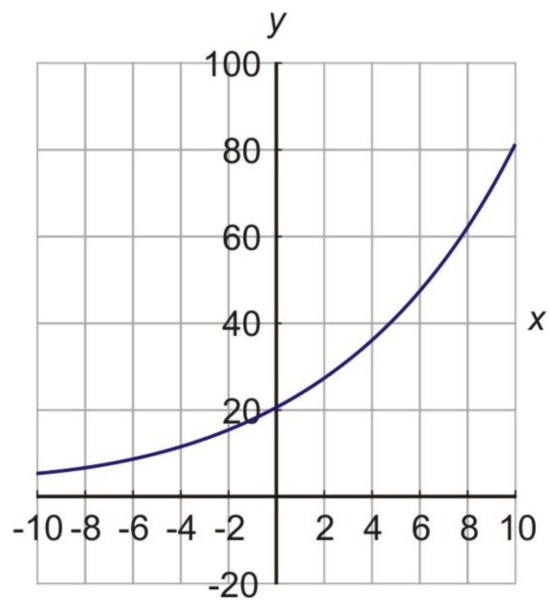
The formula that describes this problem is $y = 20 \cdot (1.15)^x$.

Now let's make a table of values.

TABLE 1.8:

x	$y = 20 \cdot (1.15)^x$
-10	4.9
-5	9.9
0	20
5	40.2
10	80.9

Now we can graph the function.



Notice that we used negative values of x in our table of values. Does it make sense to think of negative time? Yes; negative time can represent time in the past. For example, $x = -5$ in this problem represents the population from five years ago.

The question asked in the problem was: *what will be the population of the town ten years from now?* To find that number, we plug $x = 10$ into the equation we found: $y = 20 \cdot (1.15)^{10} = 80,911$.

The town will have 80,911 people ten years from now.

Example D

Peter earned \$1500 last summer. If he deposited the money in a bank account that earns 5% interest compounded yearly, how much money will he have after five years?

Solution

This problem deals with interest which is compounded yearly. This means that each year the interest is calculated on the amount of money you have in the bank. That interest is added to the original amount and next year the interest is calculated on this new amount, so you get paid interest on the interest.

Let's write a function that describes the amount of money in the bank.

The general form of an exponential function is $y = A \cdot b^x$.

Define y as the amount of money in the bank.

Define x as the number of years from now.

A is the initial amount, so $A = 1500$.

Now we have to find what b is.

We're told that the interest is 5% each year, which is 0.05 in decimal form. When we add $0.05A$ to A, we get $1.05A$, so that is the factor we multiply by each year. The base of the exponential is $b = 1.05$.

The formula that describes this problem is $y = 1500 \cdot 1.05^x$. To find the total amount of money in the bank at the end of five years, we simply plug in $x = 5$.

$$y = 1500 \cdot (1.05)^5 = \$1914.42$$

Solve Real-World Problems Involving Exponential Decay

Exponential decay problems appear in several application problems. Some examples of these are **half-life problems** and **depreciation problems**. Let's solve an example of each of these problems.

Example E

A radioactive substance has a half-life of one week. In other words, at the end of every week the level of radioactivity is half of its value at the beginning of the week. The initial level of radioactivity is 20 counts per second.

Draw the graph of the amount of radioactivity against time in weeks.

Find the formula that gives the radioactivity in terms of time.

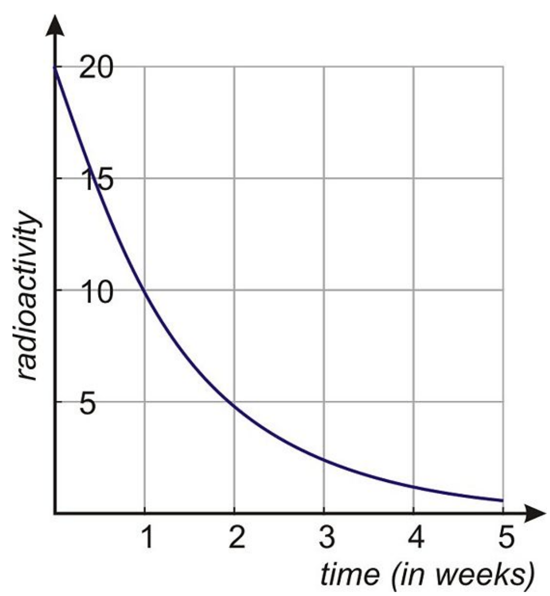
Find the radioactivity left after three weeks.

Solution

Let's start by making a table of values and then draw the graph.

TABLE 1.9:

Time	Radioactivity
0	20
1	10
2	5
3	2.5
4	1.25
5	0.625



Exponential decay fits the general formula $y = A \cdot b^x$. In this case:

y is the amount of radioactivity

x is the time in weeks

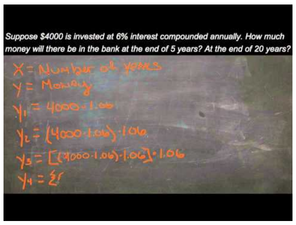
$A = 20$ is the starting amount

$b = \frac{1}{2}$ since the substance losses half its value each week

The formula for this problem is $y = 20 \cdot \left(\frac{1}{2}\right)^x$ or $y = 20 \cdot 2^{-x}$. To find out how much radioactivity is left after three weeks, we plug $x = 3$ into this formula.

$$y = 20 \cdot \left(\frac{1}{2}\right)^3 = 20 \cdot \left(\frac{1}{8}\right) = 2.5$$

Watch this video for help with the Examples above.



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CK-12 Foundation: Applications of Exponential Functions

Vocabulary

- **General Form of an Exponential Function:** $y = A(b)^x$, where $A =$ *initial value* and

$$b =$$

multiplication factor.

Guided Practice

The cost of a new car is \$32,000. It depreciates at a rate of 15% per year. This means that it loses 15% of each value each year.

Draw the graph of the car's value against time in year.

Find the formula that gives the value of the car in terms of time.

Find the value of the car when it is four years old.

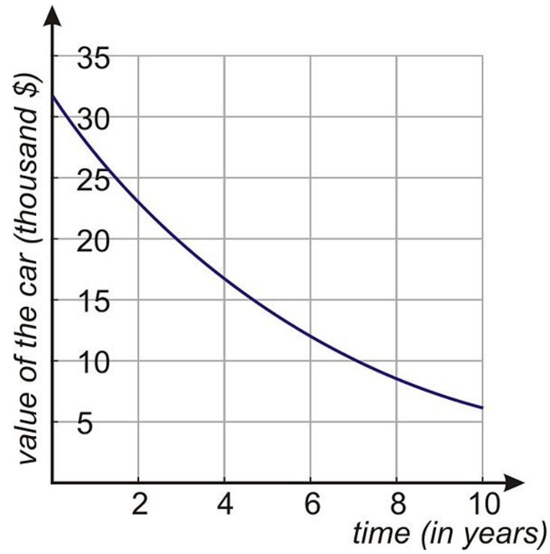
Solution

Let's start by making a table of values. To fill in the values we start with 32,000 at time $t = 0$. Then we multiply the value of the car by 85% for each passing year. (Since the car loses 15% of its value, that means it keeps 85% of its value). Remember that 85% means that we multiply by the decimal .85.

TABLE 1.10:

Time	Value (thousands)
0	32
1	27.2
2	23.1
3	19.7
4	16.7
5	14.2

Now draw the graph:



Let's start with the general formula $y = A \cdot b^x$

In this case:

y is the value of the car,

x is the time in years,

$A = 32$ is the starting amount in thousands,

$b = 0.85$ since we multiply the amount by this factor to get the value of the car next year

The formula for this problem is $y = 32 \cdot (0.85)^x$.

Finally, to find the value of the car when it is four years old, we plug $x = 4$ into that formula: $y = 32 \cdot (0.85)^4 = 16.7$ thousand dollars, or **$16,704** if we don't round.

Explore More

Solve the following problems involving exponential growth.

1. Nadia received $200 for her 10^{th} birthday. If she saves it in a bank with a 7.5% interest rate compounded yearly, how much money will she have in the bank by her 21^{st} birthday?
2. Suppose again that Nadia received $200 for her 10^{th} birthday. But what if she saves it in a bank, also with a 7.5% interest rate, but this bank compounds compounds quarterly - how much money will she have in the bank by her 21^{st} birthday?
3. The population of a city grows 15% each year. If the town started with 105 people, how many people will there be in 10 years?

4. **Half-life:** Suppose a radioactive substance decays at a rate of 3.5% per hour.
   a. What percent of the substance is left after 6 hours?
   b. What percent is left after 12 hours?
   c. The substance is safe to handle when at least 50% of it has decayed. Make a guess as to how many hours this will take.
   d. Test your guess. How close were you?

5. **Population decrease:** In 1990 a rural area has 1200 bird species.
   a. If species of birds are becoming extinct at the rate of 1.5% per decade (ten years), how many bird species will be left in the year 2020?
   b. At that same rate, how many were there in 1980?
6. **Growth:** Janine owns a chain of fast food restaurants that operated 200 stores in 1999. If the rate of increase is 8% annually, how many stores does the restaurant operate in 2007?
7. **Investment:** Paul invests $360 in an account that pays 7.25% compounded annually.
   a. What is the total amount in the account after 12 years?
   b. If Paul invests an equal amount in an account that pays 5% compounded quarterly (four times a year), what will be the amount in that account after 12 years?
   c. Which is the better investment?
8. The cost of a new ATV (all-terrain vehicle) is $7200. It depreciates at 18% per year.
   a. Draw the graph of the vehicle's value against time in years.
   b. Find the formula that gives the value of the ATV in terms of time.
   c. Find the value of the ATV when it is ten years old.
9. The percentage of light visible at d meters is given by the function $V(d) = 0.70^d$.
   a. What is the multiplication factor?
   b. What is the initial value?
   c. Find the percentage of light visible at 25 meters.
10. A person is infected by a certain bacterial infection. When he goes to the doctor the population of bacteria is 2 million. The doctor prescribes an antibiotic that reduces the bacteria population to $\frac{1}{4}$ of its size each day.
    a. Draw the graph of the size of the bacteria population against time in days.
    b. Find the formula that gives the size of the bacteria population in terms of time.
    c. Find the size of the bacteria population ten days after the drug was first taken.
    d. Find the size of the bacteria population after 2 weeks (14 days).

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook® resource, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9618 .

Polynomials

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CHAPTER 7 Polynomials

CHAPTER OUTLINE

7.1 Addition and Subtraction of Polynomials

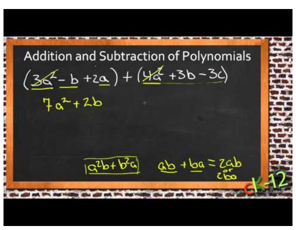
Here you'll learn how to add and subtract polynomials and simplify your answers. You'll also solve real-world problems using addition and subtraction of polynomials.

What if you had two polynomials like $4x^2 - 5$ and $13x + 2$? How could you add and subtract them? After completing this Concept, you'll be able to perform addition and subtraction on polynomials like these.

Try This

For more practice adding and subtracting polynomials, try playing the Battleship game at http://www.quia.com/ba/28820.html . (The problems get harder as you play; watch out for trick questions!)

Watch This



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CK-12 Foundation: 0902S Lesson Addition and Subtraction of Polynomials

Guidance

To add two or more polynomials, write their sum and then simplify by combining like terms.

Example A

Add and simplify the resulting polynomials.

a) Add $3x^2-4x+7$ and $2x^3-4x^2-6x+5$

b) Add $x^2-2xy+y^2$ and $2y^2-3x^2$ and $10xy+y^3$

Solution

a)

$$\begin{aligned} &(3x^2-4x+7)+(2x^3-4x^2-6x+5)\\ \text{Group like terms: } &= 2x^3+(3x^2-4x^2)+(-4x-6x)+(7+5)\\ \text{Simplify: } &= 2x^3-x^2-10x+12 \end{aligned}$$

b)

$$\begin{aligned} &(x^2-2xy+y^2)+(2y^2-3x^2)+(10xy+y^3)\\ \text{Group like terms: } &= (x^2-3x^2)+(y^2+2y^2)+(-2xy+10xy)+y^3\\ \text{Simplify: } &= -2x^2+3y^2+8xy+y^3 \end{aligned}$$

To subtract one polynomial from another, add the opposite of each term of the polynomial you are subtracting.

Example B

a) Subtract $x^3-3x^2+8x+12$ from $4x^2+5x-9$

b) Subtract $5b^2-2a^2$ from $4a^2-8ab-9b^2$

Solution

a)

$$\begin{aligned} (4x^2+5x-9)-(x^3-3x^2+8x+12) &= (4x^2+5x-9)+(-x^3+3x^2-8x-12)\\ \text{Group like terms: } &= -x^3+(4x^2+3x^2)+(5x-8x)+(-9-12)\\ \text{Simplify: } &= -x^3+7x^2-3x-21 \end{aligned}$$

b)

$$\begin{aligned} (4a^2-8ab-9b^2)-(5b^2-2a^2) &= (4a^2-8ab-9b^2)+(-5b^2+2a^2)\\ \text{Group like terms: } &= (4a^2+2a^2)+(-9b^2-5b^2)-8ab\\ \text{Simplify: } &= 6a^2-14b^2-8ab \end{aligned}$$

Note: An easy way to check your work after adding or subtracting polynomials is to substitute a convenient value in for the variable, and check that your answer and the problem both give the same value. For example, in part (b) above, if we let $a=2$ and $b=3$, then we can check as follows:

Given	Solution
$(4a^2-8ab-9b^2)-(5b^2-2a^2)$	$6a^2-14b^2-8ab$
$(4(2)^2-8(2)(3)-9(3)^2)-(5(3)^2-2(2)^2)$	$6(2)^2-14(3)^2-8(2)(3)$
$(4(4)-8(2)(3)-9(9))-(5(9)-2(4))$	$6(4)-14(9)-8(2)(3)$
$(-113)-37$	$24-126-48$
-150	-150

Since both expressions evaluate to the same number when we substitute in arbitrary values for the variables, we can be reasonably sure that our answer is correct.

Note: When you use this method, do not choose 0 or 1 for checking since these can lead to common problems.

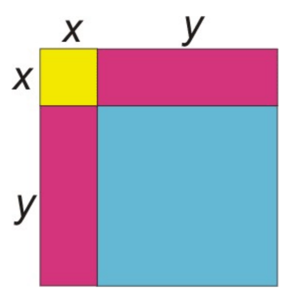
Problem Solving Using Addition or Subtraction of Polynomials

One way we can use polynomials is to find the area of a geometric figure.

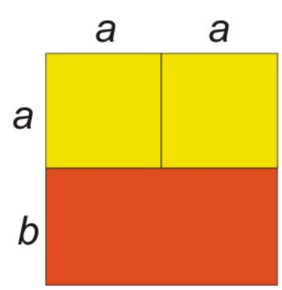
Example C

Write a polynomial that represents the area of each figure shown.

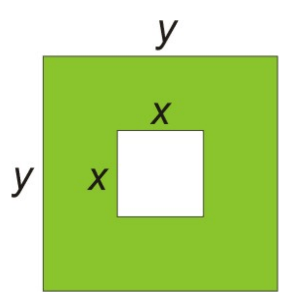
a)



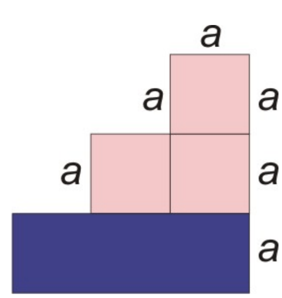
b)



c)



d)



Solution

a) This shape is formed by two squares and two rectangles.

$$\text{The blue square has area } y \times y = y^2.$$
$$\text{The yellow square has area } x \times x = x^2.$$
$$\text{The pink rectangles each have area } x \times y = xy.$$

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} Total\ area &= y^2 + x^2 + xy + xy \\ &= y^2 + x^2 + 2xy \end{aligned}$$

b) This shape is formed by two squares and one rectangle.

$$\text{The yellow squares each have area } a \times a = a^2.$$
$$\text{The orange rectangle has area } 2a \times b = 2ab.$$

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} Total\ area &= a^2 + a^2 + 2ab \\ &= 2a^2 + 2ab \end{aligned}$$

c) To find the area of the green region we find the area of the big square and subtract the area of the little square.

$$\text{The big square has area} : y \times y = y^2.$$
$$\text{The little square has area} : x \times x = x^2.$$
$$Area\ of\ the\ green\ region = y^2 - x^2$$

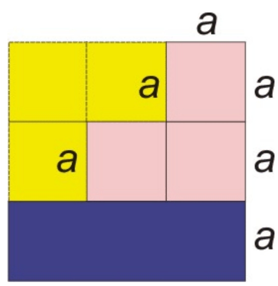
d) To find the area of the figure we can find the area of the big rectangle and add the areas of the pink squares.

$$\text{The pink squares each have area } a \times a = a^2.$$
$$\text{The blue rectangle has area } 3a \times a = 3a^2.$$

To find the total area of the figure we add all the separate areas:

$$Total\ area = a^2 + a^2 + a^2 + 3a^2 = 6a^2$$

Another way to find this area is to find the area of the big square and subtract the areas of the three yellow squares:

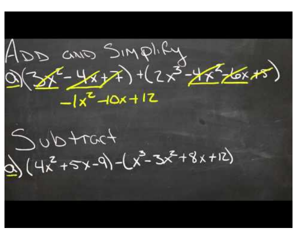


The big square has area $3a \times 3a = 9a^2$.
The yellow squares each have area $a \times a = a^2$.

To find the total area of the figure we subtract:

$$\begin{aligned} Area &= 9a^2 - (a^2 + a^2 + a^2) \\ &= 9a^2 - 3a^2 \\ &= 6a^2 \end{aligned}$$

Watch this video for help with the Examples above.



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CK-12 Foundation: Addition and Subtraction of Polynomials

Vocabulary

- A **polynomial** is an expression made with constants, variables, and *positive integer* exponents of the variables.

- In a polynomial, the number appearing in each term in front of the variables is called the **coefficient.**
- In a polynomial, the number appearing all by itself without a variable is called the **constant.**
- **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, but they can have different coefficients.

Guided Practice

Subtract $4t^2+7t^3-3t-5$ from $6t+3-5t^3+9t^2$.

Solution:

When subtracting polynomials, we have to remember to subtract each term. If the term is already negative, subtracting a negative term is the same thing as adding:

$$\begin{aligned}
6t+3-5t^3+9t^2-(4t^2+7t^3-3t-5) &= \\
6t+3-5t^3+9t^2-(4t^2)-(7t^3)-(-3t)-(-5) &= \\
6t+3-5t^3+9t^2-4t^2-7t^3+3t+5 &= \\
(6t+3t)+(3+5)+(-5t^3-7t^3)+(9t^2-4t^2) &= \\
9t+8-12t^3+5t^2 &= \\
-12t^3+5t^2+9t+8
\end{aligned}$$

The final answer is in standard form.

Explore More

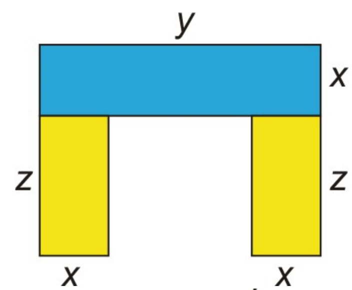
Add and simplify.

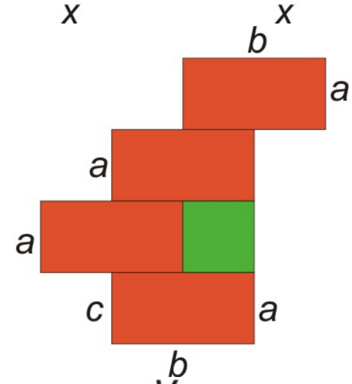
1. $(x+8)+(-3x-5)$
2. $(-2x^2+4x-12)+(7x+x^2)$
3. $(2a^2b-2a+9)+(5a^2b-4b+5)$
4. $(6.9a^2-2.3b^2+2ab)+(3.1a-2.5b^2+b)$
5. $\left(\frac{3}{5}x^2-\frac{1}{4}x+4\right)+\left(\frac{1}{10}x^2+\frac{1}{2}x-2\frac{1}{5}\right)$

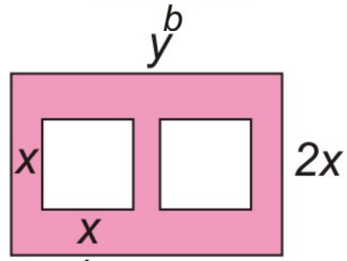
Subtract and simplify.

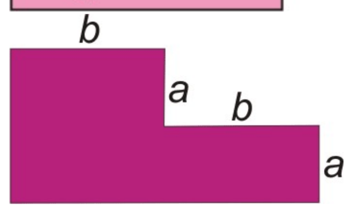
6. $(-t+5t^2)-(5t^2+2t-9)$
7. $(-y^2+4y-5)-(5y^2+2y+7)$
8. $(-5m^2-m)-(3m^2+4m-5)$
9. $(2a^2b-3ab^2+5a^2b^2)-(2a^2b^2+4a^2b-5b^2)$
10. $(3.5x^2y-6xy+4x)-(1.2x^2y-xy+2y-3)$

Find the area of the following figures.

11. 

12. 

13. 

14. 

Independent Practice

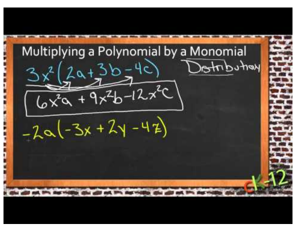
15. $(7x^4+8)+(8x^4-4)$
16. $(2x^3+8x^4)+(3x^4-4x^3)$
17. $(5x^3+3)-(4+6x^3)$
18. $(6+8a^2-8a^4)+(1-5a^2)$
19. $(4k^2+4-3K)-(1-8k^2)$
20. $(n+3n^3-6)-(1-5n^3)$
21. $(-4m^2+8+2m^4)+(m^3+2+6m^4)$
22. $(-8b^4-2b-2b^3)-(-3b^3+2b^4+4b^2)$

7.2 Multiplication of Monomials by Polynomials

Here you'll learn how to use the Distributive Property to multiply a polynomial by a monomial.

What if you had a monomial and polynomial like $3x^3$ and x^2+4? How could you multiply them? After completing this Concept, you'll be able multiply a polynomial by a monomial.

Watch This



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CK-12 Foundation: Multiplying a Polynomial by a Monomial

Guidance

Just as we can add and subtract polynomials, we can also multiply them. The Distributive Property and the techniques you've learned for dealing with exponents will be useful here.

Multiplying a Polynomial by a Monomial

When multiplying polynomials, we must remember the exponent rules that we learned in the last chapter. Especially important is the product rule: $x^n \cdot x^m = x^{n+m}$.

If the expressions we are multiplying have coefficients and more than one variable, we multiply the coefficients just as we would any number and we apply the product rule on each variable separately.

Example A

Multiply the following monomials.

a) $(2x^2)(5x^3)$

b) $(-3y^4)(2y^2)$

c) $(3xy^5)(-6x^4y^2)$

d) $(-12a^2b^3c^4)(-3a^2b^2)$

Solution

a) $(2x^2)(5x^3) = (2 \cdot 5) \cdot (x^2 \cdot x^3) = 10x^{2+3} = 10x^5$

b) $(-3y^4)(2y^2) = (-3 \cdot 2) \cdot (y^4 \cdot y^2) = -6y^{4+2} = -6y^6$

c) $(3xy^5)(-6x^4y^2) = -18x^{1+4}y^{5+2} = -18x^5y^7$

d) $(-12a^2b^3c^4)(-3a^2b^2) = 36a^{2+2}b^{3+2}c^4 = 36a^4b^5c^4$

To multiply a polynomial by a monomial, we have to use the **Distributive Property**. Remember, that property says that $a(b+c) = ab+ac$.

Example B

Multiply:

a) $3(x^2+3x-5)$

b) $4x(3x^2-7)$

c) $-7y(4y^2-2y+1)$

Solution

a) $3(x^2+3x-5) = 3(x^2)+3(3x)-3(5) = 3x^2+9x-15$

b) $4x(3x^2-7) = (4x)(3x^2)+(4x)(-7) = 12x^3-28x$

c)

$$\begin{aligned} -7y(4y^2-2y+1) &= (-7y)(4y^2)+(-7y)(-2y)+(-7y)(1) \\ &= -28y^3+14y^2-7y \end{aligned}$$

Notice that when we use the Distributive Property, the problem becomes a matter of just multiplying monomials by monomials and adding all the separate parts together.

Example C

Multiply:

a) $2x^3(-3x^4+2x^3-10x^2+7x+9)$

b) $-7a^2bc^3(5a^2-3b^2-9c^2)$

Solution

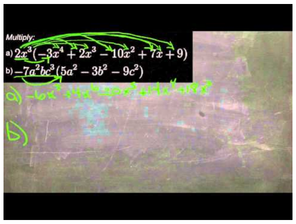
a)

$$\begin{aligned} 2x^3(-3x^4+2x^3-10x^2+7x+9) &= (2x^3)(-3x^4)+(2x^3)(2x^3)+(2x^3)(-10x^2)+(2x^3)(7x)+(2x^3)(9) \\ &= -6x^7+4x^6-20x^5+14x^4+18x^3 \end{aligned}$$

b)

$$\begin{aligned} -7a^2bc^3(5a^2-3b^2-9c^2) &= (-7a^2bc^3)(5a^2)+(-7a^2bc^3)(-3b^2)+(-7a^2bc^3)(-9c^2) \\ &= -35a^4bc^3+21a^2b^3c^3+63a^2bc^5 \end{aligned}$$

Watch this video for help with the Examples above.



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CK-12 Foundation: Multiplying a Polynomial by a Monomial

Vocabulary

- **Distributive Property:** For any expressions a, b, and c, $a(b+c) = ab+ac$.

Guided Practice

Multiply $-2a^2b^4(3ab^2+7a^3b-9a+3)$.

Solution:

Multiply the monomial by each term inside the parenthesis:

$$\begin{aligned}
&-2a^2b^4(3ab^2+7a^3b-9a+3)\\
&=(-2a^2b^4)(3ab^2)+(-2a^2b^4)(7a^3b)+(-2a^2b^4)(-9a)+(-2a^2b^4)(3)\\
&=-6a^3b^6-14a^5b^5+18a^5b^4-6a^2b^4
\end{aligned}$$

Explore More

Multiply the following monomials.

1. $(2x)(-7x)$
2. $(10x)(3xy)$
3. $(4mn)(0.5nm^2)$
4. $(-5a^2b)(-12a^3b^3)$
5. $(3xy^2z^2)(15x^2yz^3)$

Multiply and simplify.

6. $17(8x-10)$
7. $2x(4x-5)$
8. $9x^3(3x^2-2x+7)$
9. $3x(2y^2+y-5)$

10. $10q(3q^2r+5r)$
11. $-3a^2b(9a^2-4b^2)$

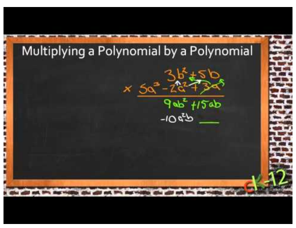
Independent Practice

12. $2n(4n+2)$
13. $3c(6c^2-5c)$
14. $-7x(3x-8)$
15. $6b^4(3b^2+6b+2b^3)$
16. $3a^2(a+4a^2-8)$
17. $5b^3(6b^2-2b+1)$
18. $-2x(5y-2)+3y(4x-5)$
19. $4a(-3b+2)-4b(2a+3)$
20. $-3g(2h+1)+2h(3g-1)-4(2+5g)$

7.3 Multiplication of Polynomials by Binomials.

What if you had two polynomials like $3x^3 + 2x^2$ and $x^2 - 1$? How could you multiply them? After completing this Concept, you'll be able to use the Distributive Property to multiply one polynomial by another.

Watch This



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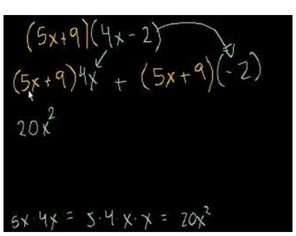
CK-12 Foundation: 0904S Multiplying a Polynomial by a Polynomial

This Khan Academy video shows how multiplying two binomials together is related to the distributive property.



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Khan Academy: Level 1 multiplying expressions

Guidance

Let's start by multiplying two binomials together. A binomial is a polynomial with two terms, so a product of two binomials will take the form $(a+b)(c+d)$.

We can still use the Distributive Property here if we do it cleverly. First, let's think of the first set of parentheses as one term. The Distributive Property says that we can multiply that term by c, multiply it by d, and then add those two products together: $(a+b)(c+d) = (a+b)\cdot c + (a+b)\cdot d$.

We can rewrite this expression as $c(a+b)+d(a+b)$. Now let's look at each half separately. We can apply the distributive property again to each set of parentheses in turn, and that gives us $c(a+b)+d(a+b) = ca+cb+da+db$.

What you should notice is that when multiplying any two polynomials, *every term in one polynomial is multiplied by every term in the other polynomial.*

Example A

Multiply and simplify: $(2x+1)(x+3)$

Solution

We must multiply each term in the first polynomial by each term in the second polynomial. Let's try to be systematic to make sure that we get all the products.

First, multiply the first term in the first set of parentheses by all the terms in the second set of parentheses.

$$(2x + 1)(x + 3) = (2x)(x) + (2x)(3) + \ldots$$

Now we're done with the first term. Next we multiply the second term in the first parentheses by all terms in the second parentheses and add them to the previous terms.

$$(2x + 1)(x + 3) = (2x)(x) + (2x)(3) + (1)(x) + (1)(3)$$

Now we're done with the multiplication and we can simplify:

$(2x)(x)+(2x)(3)+(1)(x)+(1)(3) = 2x^2+6x+x+3 = 2x^2+7x+3$

This way of multiplying polynomials is called **in-line** multiplication or **horizontal** multiplication. Another method for multiplying polynomials is to use **vertical** multiplication, similar to the vertical multiplication you learned with regular numbers.

Example B

Multiply and simplify:

a) $(4x-5)(x-20)$

b) $(3x-2)(3x+2)$

c) $(3x^2+2x-5)(2x-3)$

d) $(x^2-9)(4x^4+5x^2-2)$

Solution

a) With horizontal multiplication this would be

$$\begin{aligned}(4x-5)(x-20) &= (4x)(x)+(4x)(-20)+(-5)(x)+(-5)(-20)\\ &= 4x^2-80x-5x+100\\ &= 4x^2-85x+100\end{aligned}$$

To do vertical multiplication instead, we arrange the polynomials on top of each other with like terms in the same columns:

$$\begin{array}{r} 4x-5\\ \underline{x-20}\\ -80x+100\\ \underline{4x^2-5x\quad\quad}\\ 4x^2-85x+100\end{array}$$

Both techniques result in the same answer: $4x^2-85x+100$. We'll use vertical multiplication for the other problems.

b)

$$\begin{array}{r} 3x-2\\ \underline{3x+2}\\ 6x-4\\ \underline{9x^2-6x\quad}\\ 9x^2+0x-4\end{array}$$

The answer is $9x^2-4$.

c) It's better to place the smaller polynomial on the bottom:

$$\begin{array}{r} 3x^2+2x-5\\ \underline{2x-3}\\ -9x^2-6x+15\\ \underline{6x^3+4x^2-10x\quad\quad}\\ 6x^3-5x^2-16x+15\end{array}$$

The answer is $6x^3-5x^2-16x+15$.

d) Set up the multiplication vertically and leave gaps for missing powers of x:

$$\begin{array}{r} 4x^4+5x^2-2 \\ \underline{\qquad x^2-9} \\ -36x^4-45x^2+18 \\ \underline{4x^6+5x^4-2x^2 \qquad} \\ 4x^6-31x^4-47x^2+18 \end{array}$$

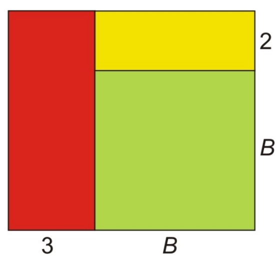
The answer is $4x^6-31x^4-47x^2+18$.

Solve Real-World Problems Using Multiplication of Polynomials

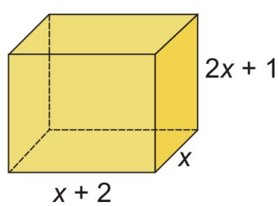
In this section, we'll see how multiplication of polynomials is applied to finding the areas and volumes of geometric shapes.

Example C

a) *Find the areas of the figure:*



b) *Find the volumes of the figure:*



Solutions:

a) We use the formula for the area of a rectangle: $\text{Area} = \text{length} \times \text{width}$.

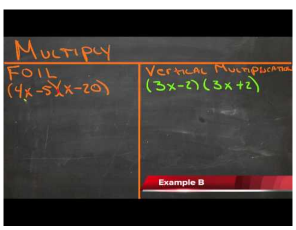
For the big rectangle:

$$\begin{aligned} \text{Length} &= b+3, \ \text{Width} = b+2 \\ \text{Area} &= (b+3)(b+2) \\ &= b^2+2b+3b+6 \\ &= b^2+5b+6 \end{aligned}$$

b) The volume of this shape = (area of the base)(height).

$$\begin{aligned}
\text{Area of the base} &= x(x+2)\\
&= x^2+2x\\
\text{Height} &= 2x+1\\
\text{Volume} &= (x^2+2x)(2x+1)\\
&= 2x^3+x^2+4x^2+2x\\
&= 2x^3+5x^2+2x
\end{aligned}$$

Watch this video for help with the Examples above.



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CK-12 Foundation: Multiplying a Polynomial by a Polynomial

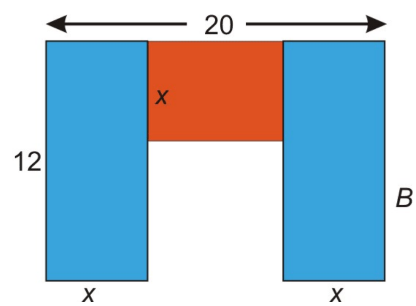
Vocabulary

- A ***binomial*** is a polynomial with two terms.
- **The Distributive Property for Binomials:** The Distributive Property says that the term in front of the parentheses multiplies with each term inside the parentheses separately. Then, we add the results of the products.

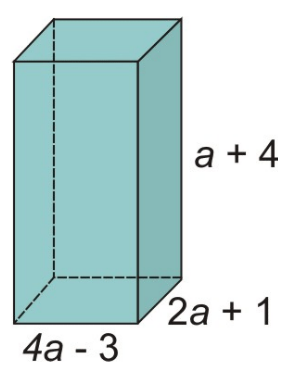
$$
\begin{aligned}
(a+b)(c+d) &= c\cdot(a+b)+d\cdot(a+b)\\
&= c\cdot a+c\cdot b+d\cdot a+d\cdot b\\
&= ca+cb+da+db
\end{aligned}
$$

Guided Practice

1. *Find the areas of the figure:*

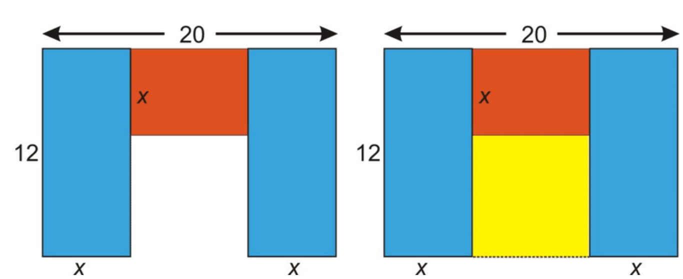


2. *Find the volumes of the figure:*



Solutions:

1. We could add up the areas of the blue and orange rectangles, but it's easier to just find the area of the whole *big* rectangle and subtract the area of the yellow rectangle.



$$\begin{aligned}
\text{Area of big rectangle} &= 20(12) = 240 \\
\text{Area of yellow rectangle} &= (12-x)(20-2x) \\
&= 240 - 24x - 20x + 2x^2 \\
&= 240 - 44x + 2x^2 \\
&= 2x^2 - 44x + 240
\end{aligned}$$

The desired area is the difference between the two:

$$\begin{aligned}
\text{Area} &= 240 - (2x^2 - 44x + 240) \\
&= 240 + (-2x^2 + 44x - 240) \\
&= 240 - 2x^2 + 44x - 240 \\
&= -2x^2 + 44x
\end{aligned}$$

2. The volume of this shape = (area of the base)(height).

$$\begin{aligned}
\text{Area of the base} &= (4a-3)(2a+1) \\
&= 8a^2 + 4a - 6a - 3 \\
&= 8a^2 - 2a - 3 \\
\text{Height} &= a + 4 \\
\text{Volume} &= (8a^2 - 2a - 3)(a+4)
\end{aligned}$$

Let’s multiply using the vertical method:

$$\begin{array}{rrrr}
 & 8a^2 & -2a & -3 \\
 & & a & +4 \\
\hline
 & 32a^2 & -8a & -12 \\
8a^3 & -2a^2 & -3a & \\
\hline
8a^3 & +30a^2 & -11a & -12
\end{array}$$

The volume is $8a^3 + 30a^2 - 11a - 12$.

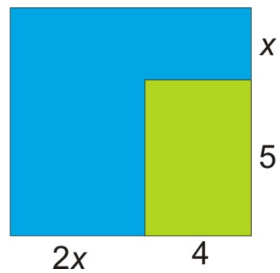
Explore More

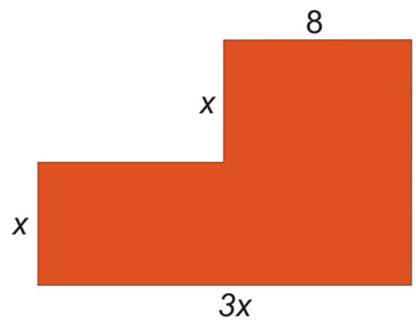
Multiply and simplify.

1. $(x-3)(x+2)$
2. $(a+b)(a-5)$
3. $(x+2)(x^2-3)$
4. $(a^2+2)(3a^2-4)$
5. $(7x-2)(9x-5)$

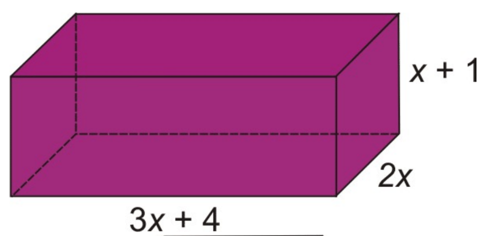
6. $(2x-1)(2x^2-x+3)$
7. $(3x+2)(9x^2-6x+4)$
8. $(a^2+2a-3)(a^2-3a+4)$
9. $3(x-5)(2x+7)$
10. $5x(x+4)(2x-3)$

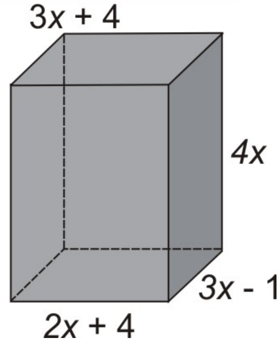
Find the areas of the following figures.

11. 

12. 

Find the volumes of the following figures.

13. 

14. 

Independent Practice

15. $(3n+2)(n+3)$

16. $(a-1)(2a-2)$
17. $(2x+3)(2x-3)$

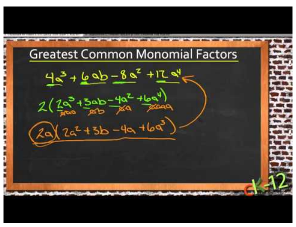
18. $(r+1)(r-3)$
19. $(2n+3)(2n+1)$
20. $((3p-3)(p-1)$
21. $(3m+3)^2$
22. $(k-2)^2$
23. $(x-4)(x+4)$
24. $(2x-3)(3x+3)$
25. $(4n+4)(5n-8)$
26. $(5w-2)(5w^2+2w-8)$
27. $((6b+2)(b^2+2b+8)$
28. $(g+4)(3g^3+4g^2-5g)$

7.4 Monomial Factors of Polynomials (Factor by Greatest Common Factor)

Here you'll learn how to factor out the greatest common monomial from a polynomial.

What if you had a polynomial like $3x^3 - 9x^2 + 6x$? How could you factor it completely? After completing this Concept, you'll be able to find a polynomial's greatest common monomial factor.

(Watch This



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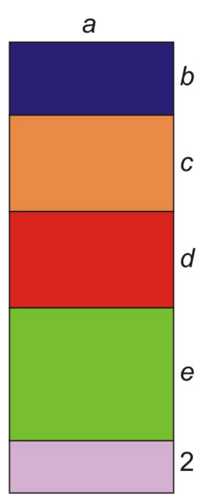
CK-12 Foundation: 0906S Greatest Common Monomial Factors

Guidance

In the last few sections, we learned how to multiply polynomials by using the Distributive Property. All the terms in one polynomial had to be multiplied by all the terms in the other polynomial. In this section, you'll start learning how to do this process in reverse. The reverse of distribution is called **factoring**.



The total area of the figure above can be found in two ways.

We could find the areas of all the small rectangles and add them: $ab + ac + ad + ae + 2a$.

Or, we could find the area of the big rectangle all at once. Its width is a and its length is $b + c + d + e + 2$, so its area is $a(b + c + d + e + 2)$.

Since the area of the rectangle is the same no matter what method we use, those two expressions must be equal.

$$ab+ac+ad+ae+2a = a(b+c+d+e+2)$$

To turn the right-hand side of this equation into the left-hand side, we would use the distributive property. To turn the left-hand side into the right-hand side, we would need to **factor** it. Since polynomials can be multiplied just like numbers, they can also be factored just like numbers—and we'll see later how this can help us solve problems.

Find the Greatest Common Monomial Factor

You will be learning several factoring methods in the next few sections. In most cases, factoring takes several steps to complete because we want to **factor completely**. That means that we factor until we can't factor any more.

Let's start with the simplest step: finding the greatest monomial factor. When we want to factor, we always look for common monomials first. Consider the following polynomial, written in expanded form:

$$ax+bx+cx+dx$$

A common factor is any factor that appears in all terms of the polynomial; it can be a number, a variable or a combination of numbers and variables. Notice that in our example, the factor x appears in all terms, so it is a common factor.

To factor out the x, we write it outside a set of parentheses. Inside the parentheses, we write what's left when we divide each term by x:

$$x(a+b+c+d)$$

Let's look at more examples.

Example A

Factor:

a) $2x+8$

b) $15x-25$

c) $3a+9b+6$

Solution

a) We see that the factor 2 divides evenly into both terms: $2x+8 = 2(x)+2(4)$

We factor out the 2 by writing it in front of a parenthesis: $2(\ )$

Inside the parenthesis we write what is left of each term when we divide by 2: $2(x+4)$

b) We see that the factor of 5 divides evenly into all terms: $15x-25 = 5(3x)-5(5)$

Factor out the 5 to get: $5(3x-5)$

c) We see that the factor of 3 divides evenly into all terms: $3a+9b+6 = 3(a)+3(3b)+3(2)$

Factor 3 to get: $3(a+3b+2)$

Example B

Find the greatest common factor:

a) $a^3 - 3a^2 + 4a$

b) $12a^4 - 5a^3 + 7a^2$

Solution

a) Notice that the factor a appears in all terms of $a^3 - 3a^2 + 4a$, but each term has a raised to a different power. The greatest common factor of all the terms is simply a.

So first we rewrite $a^3 - 3a^2 + 4a$ as $a(a^2) + a(-3a) + a(4)$.

Then we factor out the a to get $a(a^2 - 3a + 4)$.

b) The factor a appears in all the terms, and it's always raised to at least the second power. So the greatest common factor of all the terms is a^2.

We rewrite the expression $12a^4 - 5a^3 + 7a^2$ as $(12a^2 \cdot a^2) - (5a \cdot a^2) + (7 \cdot a^2)$

Factor out the a^2 to get $a^2(12a^2 - 5a + 7)$.

Example C

Factor completely:

a) $3ax + 9a$

b) $x^3y + xy$

c) $5x^3y - 15x^2y^2 + 25xy^3$

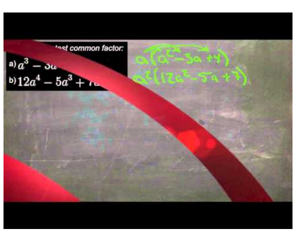
Solution

a) Both terms have a common factor of 3, but they also have a common factor of a. It's simplest to factor these both out at once, which gives us $3a(x + 3)$.

b) Both x and y are common factors. When we factor them both out at once, we get $xy(x^2 + 1)$.

c) The common factors are 5, x, and y. Factoring out $5xy$ gives us $5xy(x^2 - 3xy + 5xy^2)$.

Watch this video for help with the Examples above.



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CK-12 Foundation: Greatest Common Monomial Factors

Vocabulary

- A **common factor** can be a number, a variable, or a combination of numbers and variables that appear in **every term** of the polynomial.

Guided Practice

Find the greatest common factor.

$16x^2y^3z^2 + 4x^3yz + 8x^2y^4z^5$

Solution:

First, look at the coefficients to see if they share any common factors. They do: 4.

Next, look for the lowest power of each variable, because that is the most you can factor out. The lowest power of x is x^2. The lowest powers of y and z are to the first power.

This means we can factor out $4x^2yz$. Now, we have to determine what is left in each term after we factor out $4x^2yz$:

$16x^2y^3z^2 + 4x^3yz + 8x^2y^4z^5 = 4x^2yz(4y^2z + x + 2y^3z^4)$

Explore More

Factor out the greatest common factor in the following polynomials.

1. $2x^2 - 5x$
2. $3x^3 - 21x$
3. $5x^6 + 15x^4$
4. $4x^3 + 10x^2 - 2x$
5. $-10x^6 + 12x^5 - 4x^4$
6. $12xy + 24xy^2 + 36xy^3$
7. $5a^3 - 7a$
8. $3y + 6z$
9. $10a^3 - 4ab$
10. $45y^{12} + 30y^{10}$
11. $16xy^2z + 4x^3y$
12. $2a - 4a^2 + 6$
13. $5xy^2 - 10xy + 5y^2$

Independent Practice

14.	$2x^3 + 16x^4$	15.	$20a^3 + 25a^2 + 5a^4$	16.	$20p^2 - 16p^5$
17.	$6r^2s + 3rs^2 + 12$	18.	$32e^3 - 24ef$	19.	$16wv^4 - 12w^3v^2$
20.	$xy^2 + xy$	21.	$13x^4y^3 - 17x^2y^5$	22.	$35m^3n + 70m^2n^3$
23.	$15a^4b^5 + 3ab$	24.	$12c^2d + 30c^4 - 24cd$	25.	$6e^3f - 8ef$

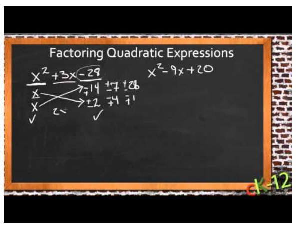
FIGURE 1.1

7.5 Factorization of Quadratic Expressions

Here you'll learn how to factor second-degree polynomials, also known as quadratic polynomials, in which all of the terms have positive coefficients.

What if you had a quadratic expression like $x^2+9x+14$ in which all the coefficients were positive? How could you factor that expression? After completing this Concept, you'll be able to factor quadratic expressions like this one with positive coefficient values.

Watch This



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CK-12 Foundation: 0908S Factoring Quadratic Expressions

Guidance

Quadratic polynomials are polynomials of the 2^{nd} degree. The standard form of a quadratic polynomial is written as

$$ax^2+bx+c$$

where a, b, and c stand for constant numbers. Factoring these polynomials depends on the values of these constants. In this section we'll learn how to factor quadratic polynomials for different values of a, b, and c. (When none of the coefficients are zero, these expressions are also called quadratic **trinomials**, since they are polynomials with three terms.)

You've already learned how to factor quadratic polynomials where $c=0$. For example, for the quadratic ax^2+bx, the common factor is x and this expression is factored as $x(ax+b)$. Now we'll see how to factor quadratics where c is nonzero.

Factor when a = 1, b is Positive, and c is Positive

First, let's consider the case where $a=1, b$ is positive and c is positive. The quadratic trinomials will take the form

$$x^2+bx+c$$

You know from multiplying binomials that when you multiply two factors $(x+m)(x+n)$, you get a quadratic polynomial. Let's look at this process in more detail. First we use distribution:

$$(x+m)(x+n) = x^2 + nx + mx + mn$$

Then we simplify by combining the like terms in the middle. We get:

$$(x+m)(x+n) = x^2 + (n+m)x + mn$$

So to factor a quadratic, we just need to do this process in reverse.

We see that	$x^2 + (n+m)x + mn$
is the same form as	$x^2 + bx + c$

This means that we need to find two numbers m and n where

$$n+m = b \qquad \text{and} \qquad mn = c$$

The factors of $x^2 + bx + c$ are always two binomials

$$(x+m)(x+n)$$

such that $n+m = b$ and $mn = c$.

Example A

Factor $x^2 + 5x + 6$.

Solution

We are looking for an answer that is a product of two binomials in parentheses:

$$(x \quad)(x \quad)$$

We want two numbers m and n that multiply to 6 and add up to 5. A good strategy is to list the possible ways we can multiply two numbers to get 6 and then see which of these numbers add up to 5:

$6 = 1 \cdot 6$	and	$1+6 = 7$	
$6 = 2 \cdot 3$	and	$2+3 = 5$	*This is the correct choice.*

So the answer is $(x+2)(x+3)$.

We can check to see if this is correct by multiplying $(x+2)(x+3)$:

$$\begin{array}{r} x+2 \\ \underline{x+3} \\ 3x+6 \\ \underline{x^2+2x} \\ x^2+5x+6 \end{array}$$

The answer checks out.

Example B

Factor $x^2+7x+12$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 12 can be written as the product of the following numbers:

$$\begin{array}{llll} 12 = 1 \cdot 12 & \text{and} & 1+12 = 13 & \\ 12 = 2 \cdot 6 & \text{and} & 2+6 = 8 & \\ 12 = 3 \cdot 4 & \text{and} & 3+4 = 7 & \textit{This is the correct choice.} \end{array}$$

The answer is $(x+3)(x+4)$.

Example C

Factor $x^2+8x+12$.

Solution

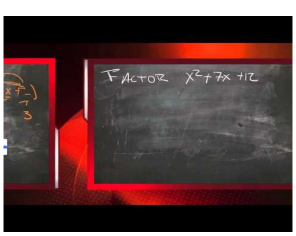
We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 12 can be written as the product of the following numbers:

$$\begin{array}{llll} 12 = 1 \cdot 12 & \text{and} & 1+12 = 13 & \\ 12 = 2 \cdot 6 & \text{and} & 2+6 = 8 & \textit{This is the correct choice.} \\ 12 = 3 \cdot 4 & \text{and} & 3+4 = 7 & \end{array}$$

The answer is $(x+2)(x+6)$.

Watch this video for help with the Examples above.



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CK-12 Foundation: Factoring Quadratic Expressions

Vocabulary

- A quadratic of the form x^2+bx+c factors as a product of two binomials in parentheses: $(x+m)(x+n)$
- If b and c are positive, then both m and n are positive.

Guided Practice

Factor $x^2+12x+36$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 36 can be written as the product of the following numbers:

$$\begin{aligned}
36 &= 1 \cdot 36 && \text{and} && 1+36=37 \\
36 &= 2 \cdot 18 && \text{and} && 2+18=20 \\
36 &= 3 \cdot 12 && \text{and} && 3+12=15 \\
36 &= 4 \cdot 9 && \text{and} && 4+9=13 \\
36 &= 6 \cdot 6 && \text{and} && 6+6=12 \qquad \textit{This is the correct choice.}
\end{aligned}$$

The answer is $(x+6)(x+6)$.

Explore More

Factor the following quadratic polynomials.

1. $x^2+10x+9$
2. $x^2+15x+50$
3. $x^2+10x+21$
4. $x^2+16x+48$
5. $x^2+14x+45$
6. $x^2+27x+50$
7. $x^2+22x+40$
8. $x^2+15x+56$
9. x^2+2x+1
10. $x^2+10x+24$
11. $x^2+17x+72$
12. $x^2+25x+150$

7.6 Factoring Trinomials

Independent Practice.

Factor the following trianomials in the form ax^2+bx+c,

1. $x^2 + 11x + 18$
2. $4x^2 + 8x + 3$
3. $6x^2 + 5x - 6$
4. $x^2 + 12x - 28$
5. $6x^2 - 5x - 4$
6. $x^2 - 5x + 6$
7. $x^2 + 2x - 15$
8. $8x^2 + 18x + 9$

FIGURE 1.2

9. $3x^2 - 5x - 2$
10. $x^2 - 8x - 20$
11. $x^2 + 11x + 18$
12. $2x^2 - 5x + 3$
13. $x^2 + 2x - 15$
14. $10x^2 - 17x + 3$
15. $3x^2 - 2x - 21$
16. $2x^2 - 11x + 12$
17. $x^2 + 10x + 16$
18. $6x^2 - 7x - 5$
19. $x^2 + 3x - 10$
20. $4x^2 + 7x + 3$

FIGURE 1.3

7.7 Dividing Polynomials

Independent Practice.

Dividing Polynomials

1. $\dfrac{(x^2 + 7x - 18)}{(x + 9)}$

2. $\dfrac{(r^2 + 3r - 54)}{(r - 6)}$

3. $\dfrac{(7n^2 - 53n - 24)}{(n - 8)}$

4. $\dfrac{(2x^2 + 15x + 27)}{(x + 3)}$

5. $\dfrac{(n^2 - 4n - 45)}{(n - 9)}$

6. $\dfrac{(4b^2 + 31b - 45)}{(4b - 5)}$

7. $\dfrac{(30x^2 + 50x + 20)}{(6x + 4)}$

8. $\dfrac{(v^2 + v - 2)}{(v + 2)}$

9. $\dfrac{(3k^2 - 7k - 40)}{(3k + 8)}$

10. $\dfrac{(14x^2 + 41x - 28)}{(7x - 4)}$

FIGURE 1.4

7.8 Applications Using Factoring

7.9 Chapter 7 Review

Quadratics

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CHAPTER 8 Quadratics

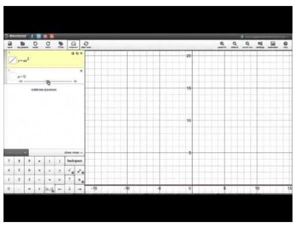
CHAPTER OUTLINE

8.1 Quadratic Functions and Their Graphs

Here you'll learn how to make a table of values to graph the curved lines of quadratic functions called parabolas. You'll also learn how to describe what parabolas will look like.

What if you had a quadratic function like $5+2x-3x^2$? What would its graph look like? Would the graph of $5+2x-x^2$ be wider or narrower than it? After completing this Concept, you'll be able to graph and compare graphs of quadratic functions like these.

Watch This



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CK-12 Foundation: 1001S Graphs of Quadratic Functions

Try This

Meanwhile, if you want to explore further what happens when you change the coefficients of a quadratic equation, the page at http://www.analyzemath.com/quadraticg/quadraticg.htm has an applet you can use. Click on the "Click here to start" button in section A, and then use the sliders to change the values of a, b, and c.

Guidance

The graphs of quadratic functions are curved lines called **parabolas**. You don't have to look hard to find parabolic shapes around you. Here are a few examples:

- The path that a ball or a rocket takes through the air.
- Water flowing out of a drinking fountain.
- The shape of a satellite dish.
- The shape of the mirror in car headlights or a flashlight.
- The cables in a suspension bridge.

Example A

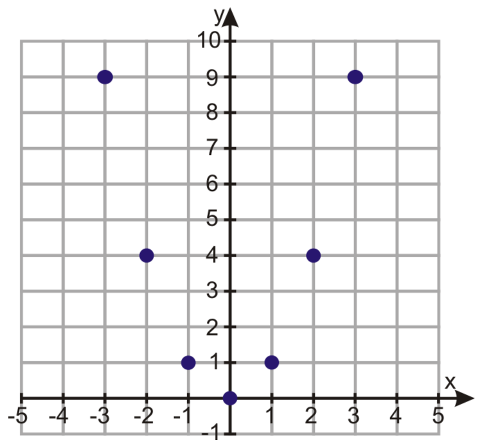
Let's see what a parabola looks like by graphing the simplest quadratic function, $y=x^2$.

We'll graph this function by making a table of values. Since the graph will be curved, we need to plot a fair number of points to make it accurate.

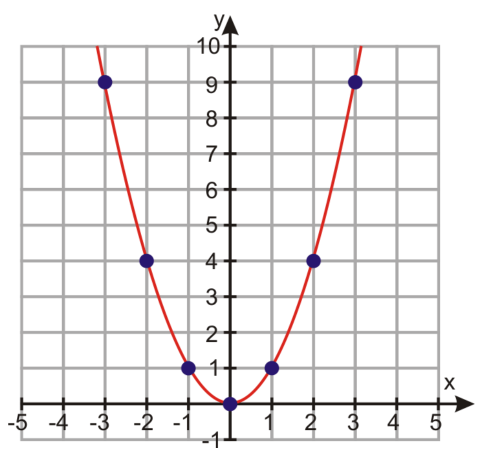
TABLE 1.1:

x	$y = x^2$
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$

Here are the points plotted on a coordinate graph:



To draw the parabola, draw a smooth curve through all the points. (Do not connect the points with straight lines).



Let's graph a few more examples.

Example B

Graph the following parabolas.

a) $y = 2x^2 + 4x + 1$

b) $y = -x^2 + 3$

c) $y = x^2 - 8x + 3$

Solution

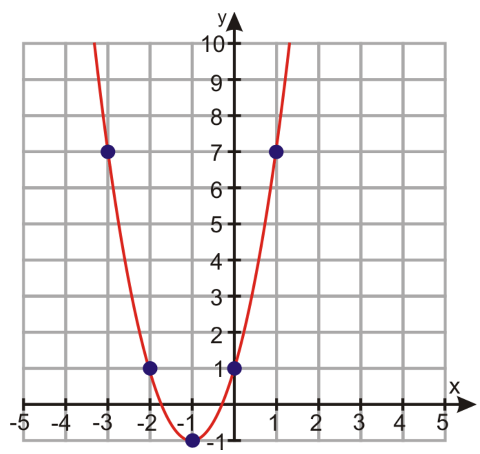
a) $y = 2x^2 + 4x + 1$

Make a table of values:

TABLE 1.2:

x	$y = 2x^2 + 4x + 1$
-3	$2(-3)^2 + 4(-3) + 1 = 7$
–2	$2(-2)^2 + 4(-2) + 1 = 1$
–1	$2(-1)^2 + 4(-1) + 1 = -1$
0	$2(0)^2 + 4(0) + 1 = 1$
1	$2(1)^2 + 4(1) + 1 = 7$
2	$2(2)^2 + 4(2) + 1 = 17$
3	$2(3)^2 + 4(3) + 1 = 31$

Notice that the last two points have very large $y-$values. Since we don't want to make our $y-$scale too big, we'll just skip graphing those two points. But we'll plot the remaining points and join them with a smooth curve.



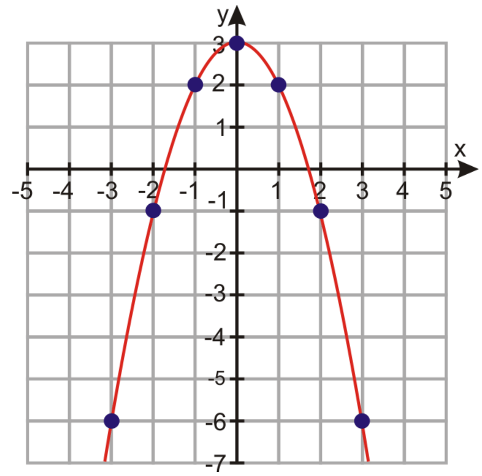
b) $y = -x^2 + 3$

Make a table of values:

TABLE 1.3:

x	$y = -x^2 + 3$
-3	$-(-3)^2 + 3 = -6$
–2	$-(-2)^2 + 3 = -1$
–1	$-(-1)^2 + 3 = 2$
0	$-(0)^2 + 3 = 3$
1	$-(1)^2 + 3 = 2$
2	$-(2)^2 + 3 = -1$
3	$-(3)^2 + 3 = -6$

Plot the points and join them with a smooth curve.



Notice that this time we get an "upside down" parabola. That's because our equation has a negative sign in front of the x^2 term. The sign of the coefficient of the x^2 term determines whether the parabola turns up or down: the parabola turns up if it's positive and down if it's negative.

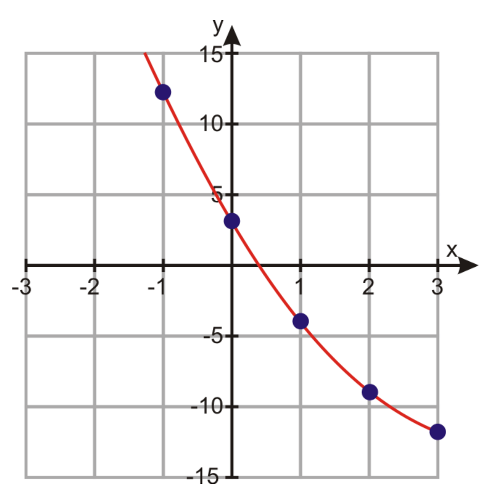
c) $y = x^2 - 8x + 3$

Make a table of values:

TABLE 1.4:

x	$y = x^2 - 8x + 3$
-3	$(-3)^2 - 8(-3) + 3 = 36$
-2	$(-2)^2 - 8(-2) + 3 = 23$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$

Let's not graph the first two points in the table since the values are so big. Plot the remaining points and join them with a smooth curve.



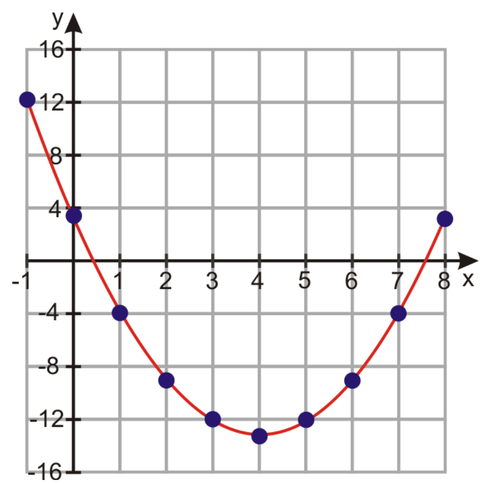
Wait—this doesn't look like a parabola. What's going on here?

Maybe if we graph more points, the curve will look more familiar. For negative values of x it looks like the values of y are just getting bigger and bigger, so let's pick more positive values of x beyond $x = 3$.

TABLE 1.5:

x	$y = x^2 - 8x + 3$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$
4	$(4)^2 - 8(4) + 3 = -13$
5	$(5)^2 - 8(5) + 3 = -12$
6	$(6)^2 - 8(6) + 3 = -9$
7	$(7)^2 - 8(7) + 3 = -4$
8	$(8)^2 - 8(8) + 3 = 3$

Plot the points again and join them with a smooth curve.



Now we can see the familiar parabolic shape. And now we can see the drawback to graphing quadratics by making a table of values—if we don't pick the right values, we won't get to see the important parts of the graph.

In the next couple of lessons, we'll find out how to graph quadratic equations more efficiently—but first we need to learn more about the properties of parabolas.

Compare Graphs of Quadratic Functions

The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

Here a, b and c are the **coefficients.** Remember, a coefficient is just a number (a constant term) that can go before a variable or appear alone.

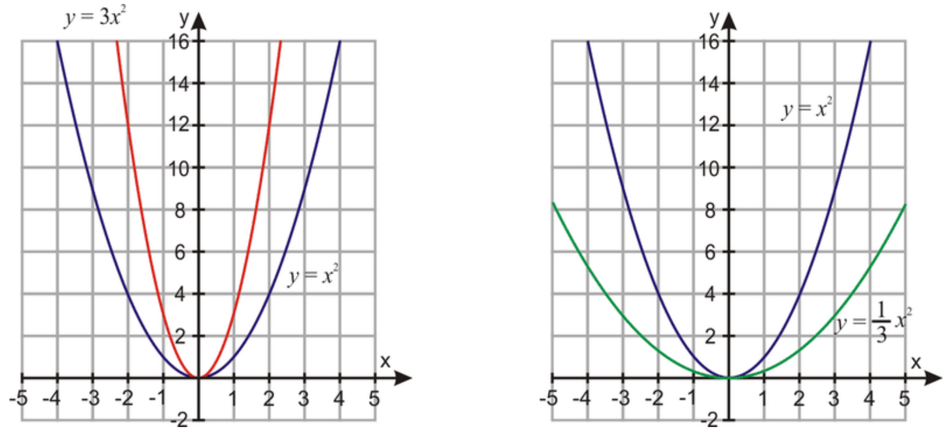
Although the graph of a quadratic equation in standard form is always a parabola, the shape of the parabola depends on the values of the coefficients a, b and c. Let's explore some of the ways the coefficients can affect the graph.

Dilation

Changing the value of a makes the graph "fatter" or "skinnier". Let's look at how graphs compare for different positive values of a.

Example C

The plot on the left shows the graphs of $y = x^2$ and $y = 3x^2$. The plot on the right shows the graphs of $y = x^2$ and $y = \frac{1}{3}x^2$.



Notice that the larger the value of a is, the skinnier the graph is –for example, in the first plot, the graph of $y = 3x^2$ is skinnier than the graph of $y = x^2$. Also, the smaller a is, the fatter the graph is –for example, in the second plot, the graph of $y = \frac{1}{3}x^2$ is fatter than the graph of $y = x^2$. This might seem counterintuitive, but if you think about it, it should make sense. Let's look at a table of values of these graphs and see if we can explain why this happens.

TABLE 1.6:

x	$y = x^2$	$y = 3x^2$	$y = \frac{1}{3}x^2$
-3	$(-3)^2 = 9$	$3(-3)^2 = 27$	$\frac{(-3)^2}{3} = 3$
-2	$(-2)^2 = 4$	$3(-2)^2 = 12$	$\frac{(-2)^2}{3} = \frac{4}{3}$
-1	$(-1)^2 = 1$	$3(-1)^2 = 3$	$\frac{(-1)^2}{3} = \frac{1}{3}$
0	$(0)^2 = 0$	$3(0)^2 = 0$	$\frac{(0)^2}{3} = 0$
1	$(1)^2 = 1$	$3(1)^2 = 3$	$\frac{(1)^2}{3} = \frac{1}{3}$
2	$(2)^2 = 4$	$3(2)^2 = 12$	$\frac{(2)^2}{3} = \frac{4}{3}$
3	$(3)^2 = 9$	$3(3)^2 = 27$	$\frac{(3)^2}{3} = 3$

From the table, you can see that the values of $y = 3x^2$ are bigger than the values of $y = x^2$. This is because each value of y gets multiplied by 3. As a result the parabola will be skinnier because it grows three times faster than $y = x^2$. On the other hand, you can see that the values of $y = \frac{1}{3}x^2$ are smaller than the values of $y = x^2$, because each value of y gets divided by 3. As a result the parabola will be fatter because it grows at one third the rate of $y = x^2$.

Orientation

As the value of a gets smaller and smaller, then, the parabola gets wider and flatter. What happens when a gets all the way down to zero? What happens when it's negative?

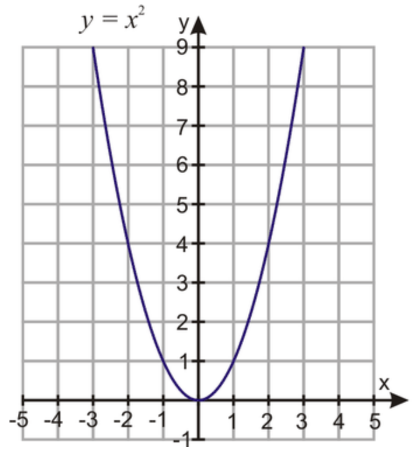
Well, when $a = 0$, the x^2 term drops out of the equation entirely, so the equation becomes linear and the graph is just a straight line. For example, we just saw what happens to $y = ax^2$ when we change the value of a; if we tried to graph $y = 0x^2$, we would just be graphing $y = 0$, which would be a horizontal line.

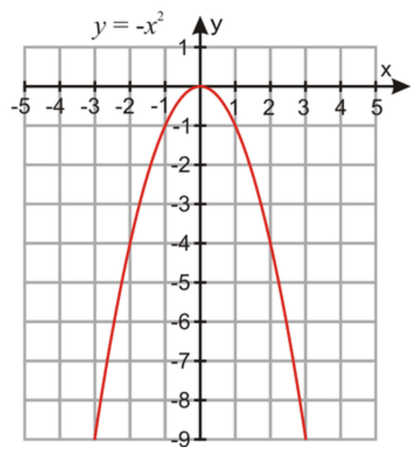
So as a gets smaller and smaller, the graph of $y = ax^2$ gets flattened all the way out into a horizontal line. Then, when a becomes negative, the graph of $y = ax^2$ starts to curve again, only it curves downward instead of upward. This fits with what you've already learned: the graph opens upward if a is positive and downward if a is negative.

Example D

What do the graphs of $y = x^2$ and $y = -x^2$ look like?

Solution:





You can see that the parabola has the same shape in both graphs, but the graph of $y = x^2$ is right-side-up and the graph of $y = -x^2$ is upside-down.

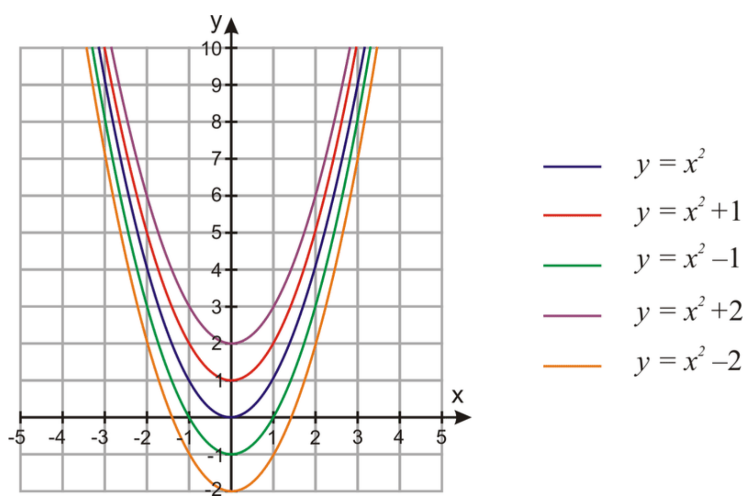
Vertical Shifts

Changing the constant c just shifts the parabola up or down.

Example E

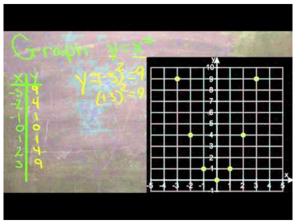
What do the graphs of $y = x^2, y = x^2+1, y = x^2-1, y = x^2+2$, and $y = x^2-2$ look like?

Solution:



You can see that when c is positive, the graph shifts up, and when c is negative the graph shifts down; in either case, it shifts by $|c|$ units. In one of the later Concepts, we'll learn about **horizontal shift** (i.e. moving to the right or to the left). Before we can do that, though, we need to learn how to rewrite quadratic equations in different forms - our objective for the next Concept.

Watch this video for help with the Examples above.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/133105

CK-12 Foundation: 1001 Graphs of Quadratic Functions

Vocabulary

- The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

Here a, b and c are the **coefficients.**

- Changing the value of a makes the graph "fatter" or "skinnier". This is called **dilation.**

- The vertical movement along a parabola's line of symmetry is called a ***vertical shift.***

Guided Practice

Graph the quadratic function, $y = -x^2 + 2$.

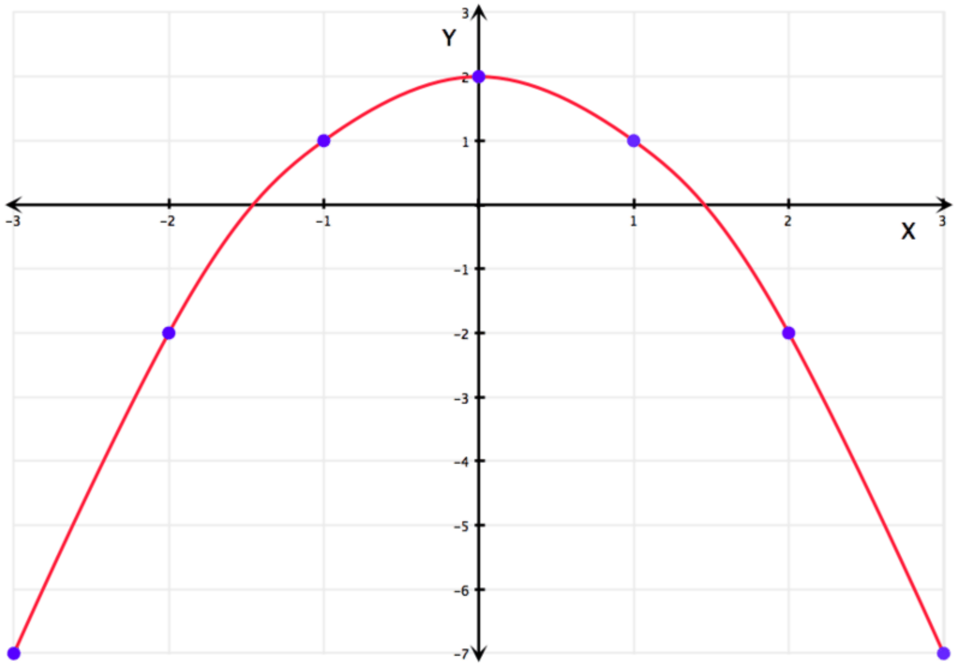
Solution:

We'll graph this function by making a table of values. Since the graph will be curved, we need to plot a fair number of points to make it accurate.

TABLE 1.7:

x	$y = x^2$
-3	$-(-3)^2 + 2 = -7$
-2	$-(-2)^2 + 2 = -2$
-1	$-(-1)^2 + 2 = 1$
0	$-(0)^2 + 2 = 2$
1	$-(1)^2 + 2 = 1$
2	$-(2)^2 + 2 = -2$
3	$-(3)^2 + 2 = -7$

Plot the points and connect them with a smooth curve:



Explore More

For 1-5, does the graph of the parabola turn up or down?

1. $y = -2x^2 - 2x - 3$
2. $y = 3x^2$
3. $y = 16 - 4x^2$
4. $y = -100 + 0.25x^2$
5. $y = 3x^2 - 2x - 4x^2 + 3$

For 6-10, which parabola is wider?

6. $y = x^2$ or $y = 4x^2$
7. $y = 2x^2 + 4$ or $y = \frac{1}{2}x^2 + 4$
8. $y = -2x^2 - 2$ or $y = -x^2 - 2$
9. $y = x^2 + 3x^2$ or $y = x^2 + 3$
10. $y = -x^2$ or $y = \frac{1}{10}x^2$

8.2 Use Graphs to Solve Quadratic Equations

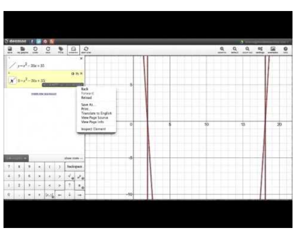
Here you'll learn how to identify the number of solutions to a quadratic equation and how to find those solutions. You'll also learn how to use a graphing calculator to find the roots and the vertex of polynomials. Finally, you'll solve real-world problems by graphing quadratic functions.

What if you had a quadratic function like $y = 2x^2 + 5x + 3$? How could you graph it to find its roots? After completing this Concept, you'll be able to determine the number of solutions for a quadratic equation like this one and you'll find the solutions by graphing.

Try This

Now that you've learned how to solve quadratic equations by graphing them, you can sharpen your skills even more by learning how to find an equation from the graph alone. Go to the page linked in the previous section, http://www.analyzemath.com/quadraticg/quadraticg.htm , and scroll down to section E. Read the example there to learn how to find the equation of a quadratic function by reading off a few key values from the graph; then click the "Click here to start" button to try a problem yourself. The "New graph" button will give you a new problem when you finish the first one.

Watch This



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URL: https://www.ck12.org/flx/render/embeddedobject/133102

CK-12 Foundation: 1002S Lesson Solving Quadratic Equations by Graphing

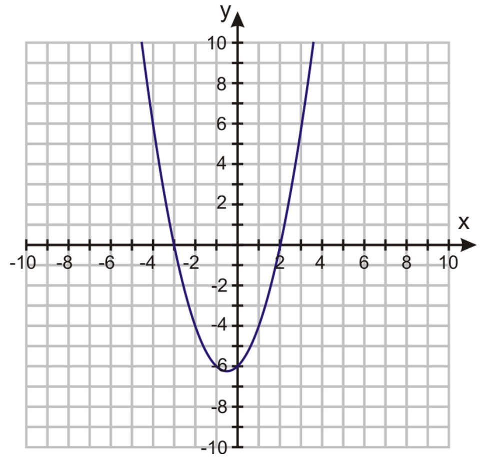
Guidance

Solving a quadratic equation means finding the $x-$values that will make the quadratic function equal zero; in other words, it means finding the points where the graph of the function crosses the $x-$axis. The solutions to a quadratic equation are also called the **roots** or **zeros** of the function, and in this section we'll learn how to find them by graphing the function.

Identify the Number of Solutions of a Quadratic Equation

Three different situations can occur when graphing a quadratic function:

Case 1: The parabola crosses the $x-$axis at two points. An example of this is $y = x^2 + x - 6$:

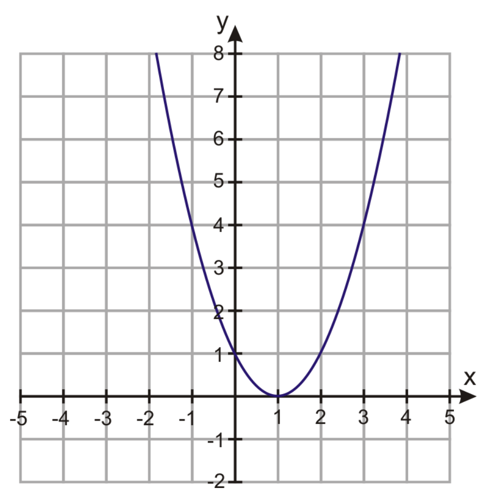


Looking at the graph, we see that the parabola crosses the $x-$axis at $x = -3$ and $x = 2$.

We can also find the solutions to the equation $x^2 + x - 6 = 0$ by setting $y = 0$. We solve the equation by factoring: $(x+3)(x-2) = 0$, so $x = -3$ or $x = 2$.

When the graph of a quadratic function crosses the $x-$axis at two points, we get **two distinct solutions** to the quadratic equation.

Case 2: The parabola touches the $x-$axis at one point. An example of this is $y = x^2 - 2x + 1$:

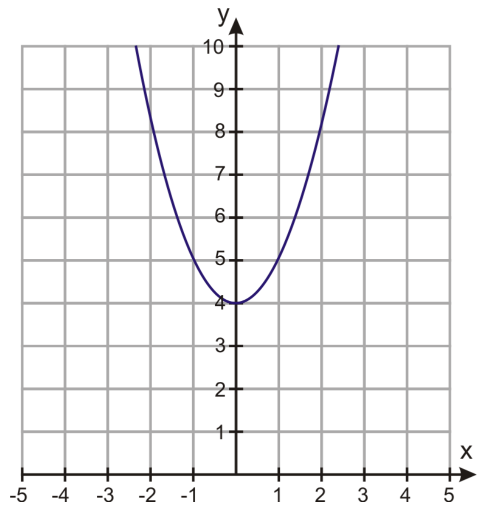


We can see that the graph touches the $x-$axis at $x = 1$.

We can also solve this equation by factoring. If we set $y = 0$ and factor, we obtain $(x-1)^2 = 0$, so $x = 1$. Since the quadratic function is a perfect square, we get only one solution for the equation—it's just the same solution repeated twice over.

When the graph of a quadratic function touches the $x-$axis at one point, the quadratic equation has one solution and the solution is called a **double root.**

Case 3: The parabola does not cross or touch the $x-$axis. An example of this is $y = x^2 + 4$:



If we set $y = 0$ we get $x^2 + 4 = 0$. This quadratic polynomial does not factor.

When the graph of a quadratic function does not cross or touch the $x-$axis, the quadratic equation has **no real solutions.**

Solve Quadratic Equations by Graphing

So far we've found the solutions to quadratic equations using factoring. However, in real life very few functions factor easily. As you just saw, graphing a function gives a lot of information about the solutions. We can find exact or approximate solutions to a quadratic equation by graphing the function associated with it.

Example A

Find the solutions to the following quadratic equations by graphing.

a) $-x^2 + 3 = y$

b) $-x^2 + x - 3 = y$

c) $y = -x^2 + 4x - 4$

Solution

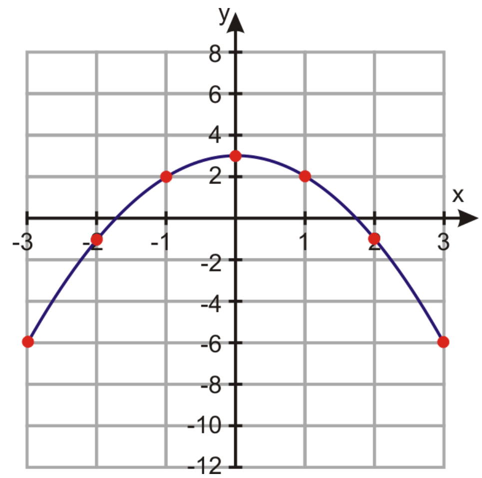
Since we can't factor any of these equations, we won't be able to graph them using intercept form (if we could, we wouldn't need to use the graphs to find the intercepts!) We'll just have to make a table of arbitrary values to graph each one.

a)

TABLE 1.8:

x	$y = -x^2 + 3$
-3	$y = -(-3)^2 + 3 = -6$
-2	$y = -(-2)^2 + 3 = -1$
-1	$y = -(-1)^2 + 3 = 2$
0	$y = -(0)^2 + 3 = 3$
1	$y = -(1)^2 + 3 = 2$
2	$y = -(2)^2 + 3 = -1$
3	$y = -(3)^2 + 3 = -6$

We plot the points and get the following graph:



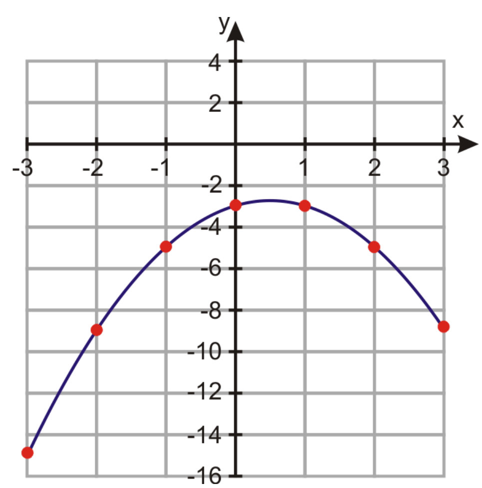
From the graph we can read that the $x-$intercepts are approximately $x = 1.7$**and** $x = -1.7$. These are the solutions to the equation.

b)

TABLE 1.9:

x	$y = -x^2 + x - 3$
-3	$y = -(-3)^2 + (-3) - 3 = -15$
–2	$y = -(-2)^2 + (-2) - 3 = -9$
–1	$y = -(-1)^2 + (-1) - 3 = -5$
0	$y = -(0)^2 + (0) - 3 = -3$
1	$y = -(1)^2 + (1) - 3 = -3$
2	$y = -(2)^2 + (2) - 3 = -5$
3	$y = -(3)^2 + (3) - 3 = -9$

We plot the points and get the following graph:



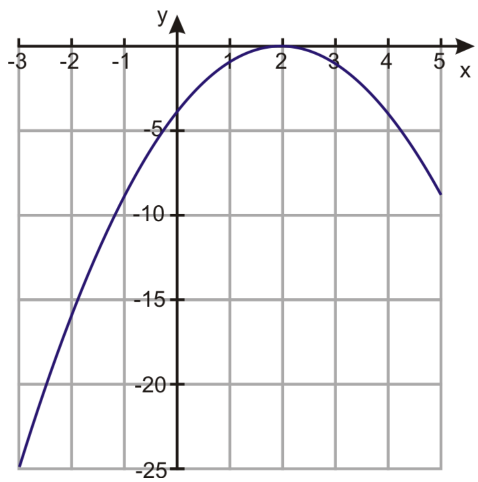
The graph curves up toward the $x-$axis and then back down without ever reaching it. This means that the graph never intercepts the $x-$axis, and so the corresponding equation has **no real solutions.**

c)

TABLE 1.10:

x	$y = -x^2 + 4x - 4$
-3	$y = -(-3)^2 + 4(-3) - 4 = -25$
–2	$y = -(-2)^2 + 4(-2) - 4 = -16$
–1	$y = -(-1)^2 + 4(-1) - 4 = -9$
0	$y = -(0)^2 + 4(0) - 4 = -4$
1	$y = -(1)^2 + 4(1) - 4 = -1$
2	$y = -(2)^2 + 4(2) - 4 = 0$
3	$y = -(3)^2 + 4(3) - 4 = -1$
4	$y = -(4)^2 + 4(4) - 4 = -4$
5	$y = -(5)^2 + 4(5) - 4 = -9$

Here is the graph of this function:



The graph just touches the $x-$axis at $x = 2$, so the function has a **double root** there. $x = 2$ is the only solution to the equation.

'Analyze Quadratic Functions Using a Graphing Calculator

A graphing calculator is very useful for graphing quadratic functions. Once the function is graphed, we can use the calculator to find important information such as the roots or the vertex of the function.

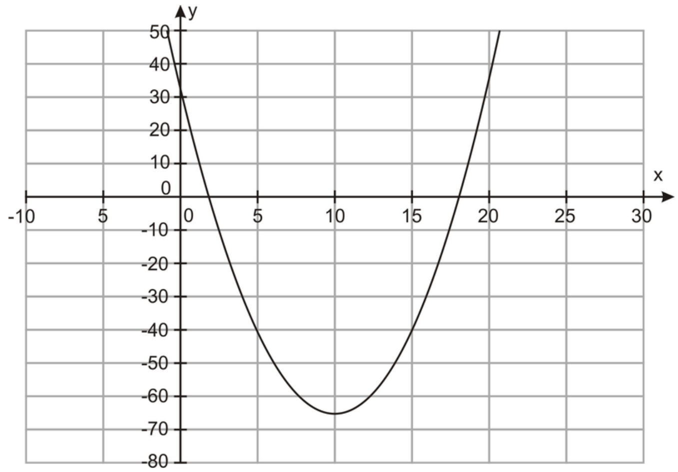
Example B

Use a graphing calculator to analyze the graph of $y = x^2 - 20x + 35$.

Solution

1. **Graph** the function.

Press the **[Y=]** button and enter "$x^2 - 20x + 35$" next to $[Y_1 =]$. Press the **[GRAPH]** button. This is the plot you should see:



If this is not what you see, press the **[WINDOW]** button to change the window size. For the graph shown here, the $x-$values should range from -10 to 30 and the $y-$values from -80 to 50.

2. Find the **roots.**

There are at least three ways to find the roots:

Use **[TRACE]** to scroll over the $x-$intercepts. The approximate value of the roots will be shown on the screen. You can improve your estimate by zooming in.

OR

Use **[TABLE]** and scroll through the values until you find values of y equal to zero. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** (i.e. 'calc' button) and use option 'zero'.

Move the cursor to the left of one of the roots and press **[ENTER]**.

Move the cursor to the right of the same root and press **[ENTER]**.

Move the cursor close to the root and press **[ENTER]**.

The screen will show the value of the root. Repeat the procedure for the other root.

Whichever technique you use, you should get about $x = 1.9$**and** $x = 18$ for the two roots.

3. Find the **vertex.**

There are three ways to find the vertex:

Use **[TRACE]** to scroll over the highest or lowest point on the graph. The approximate value of the roots will be shown on the screen.

OR

Use **[TABLE]** and scroll through the values until you find values the lowest or highest value of y. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** and use the option 'maximum' if the vertex is a maximum or 'minimum' if the vertex is a minimum.

Move the cursor to the left of the vertex and press **[ENTER]**.

Move the cursor to the right of the vertex and press **[ENTER]**.

Move the cursor close to the vertex and press **[ENTER]**.

The screen will show the $x-$ and $y-$values of the vertex.

Whichever method you use, you should find that the vertex is at **(10, -65)**.

Solve Real-World Problems by Graphing Quadratic Functions

Here's a real-world problem we can solve using the graphing methods we've learned.

Example C

Andrew is an avid archer. He launches an arrow that takes a parabolic path. The equation of the height of the ball with respect to time is $y = -4.9t^2 + 48t$, *where* y *is the height of the arrow in meters and* t *is the time in seconds since Andrew shot the arrow. Find how long it takes the arrow to come back to the ground.*

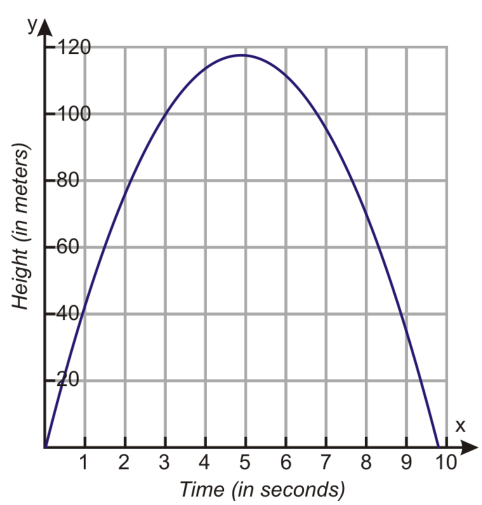
Solution

Let's graph the equation by making a table of values.

TABLE 1.11:

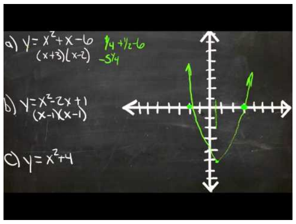
t	$y = -4.9t^2 + 48t$
0	$y = -4.9(0)^2 + 48(0) = 0$
1	$y = -4.9(1)^2 + 48(1) = 43.1$
2	$y = -4.9(2)^2 + 48(2) = 76.4$
3	$y = -4.9(3)^2 + 48(3) = 99.9$
4	$y = -4.9(4)^2 + 48(4) = 113.6$
5	$y = -4.9(5)^2 + 48(5) = 117.6$
6	$y = -4.9(6)^2 + 48(6) = 111.6$
7	$y = -4.9(7)^2 + 48(7) = 95.9$
8	$y = -4.9(8)^2 + 48(8) = 70.4$
9	$y = -4.9(9)^2 + 48(9) = 35.1$
10	$y = -4.9(10)^2 + 48(10) = -10$

Here's the graph of the function:



The roots of the function are approximately $x = 0$ sec and $x = 9.8$ sec. The first root tells us that the height of the arrow was 0 meters when Andrew first shot it. The second root says that it takes approximately **9.8 seconds** for the arrow to return to the ground.

Watch this video for help with the Examples above.



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CK-12 Foundation: 1003 Solving Quadratic Equations by Graphing

Vocabulary

- The **solutions of a quadratic equation** are often called the ***roots*** or ***zeros.***

Guided Practice

Find the solutions to $2x^2+5x-7=0$ *by graphing.*

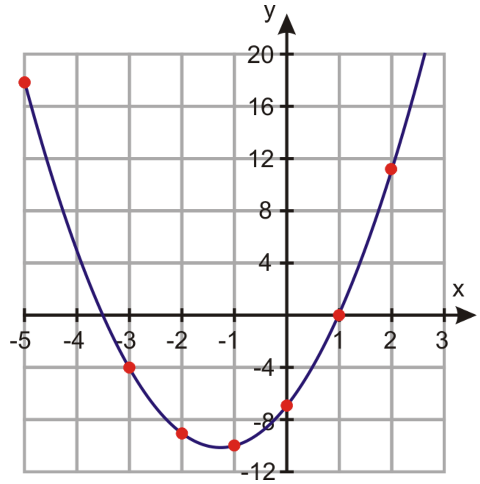
Solution

Since we can't factor this equation, we won't be able to graph it using intercept form (if we could, we wouldn't need to use the graphs to find the intercepts!) We'll just have to make a table of arbitrary values to graph the equation.

TABLE 1.12:

x	$y=2x^2+5x-7$
-5	$y=2(-5)^2+5(-5)-7=18$
-4	$y=2(-4)^2+5(-4)-7=5$
-3	$y=2(-3)^2+5(-3)-7=-4$
-2	$y=2(-2)^2+5(-2)-7=-9$
-1	$y=2(-1)^2+5(-1)-7=-10$
0	$y=2(0)^2+5(0)-7=-7$
1	$y=2(1)^2+5(1)-7=0$
2	$y=2(2)^2+5(2)-7=11$
3	$y=2(3)^2+5(3)-7=26$

We plot the points and get the following graph:



From the graph we can read that the $x-$intercepts are $x = 1$**and** $x = -3.5$. These are the solutions to the equation.

Explore More

For 1-6, find the solutions of the following equations by graphing.

1. $x^2 + 3x + 6 = 0$
2. $-2x^2 + x + 4 = 0$
3. $x^2 - 9 = 0$
4. $x^2 + 6x + 9 = 0$
5. $10x - 3x^2 = 0$
6. $\frac{1}{2}x^2 - 2x + 3 = 0$

For 7-12, find the roots of the following quadratic functions by graphing.

7. $y = -3x^2 + 4x - 1$
8. $y = 9 - 4x^2$
9. $y = x^2 + 7x + 2$
10. $y = -x^2 - 10x - 25$
11. $y = 2x^2 - 3x$
12. $y = x^2 - 2x + 5$

For 13-18, use your graphing calculator to find the roots and the vertex of each polynomial.

13. $y = x^2 + 12x + 5$
14. $y = x^2 + 3x + 6$
15. $y = -x^2 - 3x + 9$
16. $y = -x^2 + 4x - 12$
17. $y = 2x^2 - 4x + 8$
18. $y = -5x^2 - 3x + 2$
19. Graph the equations $y = 2x^2 - 4x + 8$ and $y = x^2 - 2x + 4$ on the same screen. Find their roots and vertices.
    a. What is the same about the graphs? What is different?
    b. How are the two equations related to each other? (Hint: factor them.)
    c. What might be another equation with the same roots? Graph it and see.
20. Graph the equations $y = x^2 - 2x + 2$ and $y = x^2 - 2x + 4$ on the same screen. Find their roots and vertices.

a. What is the same about the graphs? What is different?
b. How are the two equations related to each other?

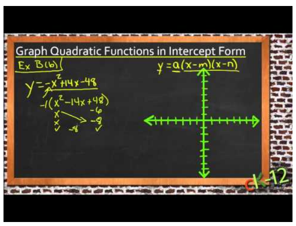
21. Phillip throws a ball and it takes a parabolic path. The equation of the height of the ball with respect to time is $y = -16t^2 + 60t$, where y is the height in feet and t is the time in seconds. Find how long it takes the ball to come back to the ground.
22. Use your graphing calculator to solve Ex. C. You should get the same answers as we did graphing by hand, but a lot quicker!

8.3 Graphs of Quadratic Functions in Intercept Form

Here you'll learn how to write and graph quadratic functions in intercept form. You'll also learn how to find the x-intercepts and vertex of quadratic functions.

What if you had a quadratic function like $y = x^2 + 3x + 2$? How could you find its x-intercepts and vertex to help you graph it? After completing this Concept, you'll be able to use the intercept form of quadratic functions to solve problems like this one.

Watch This



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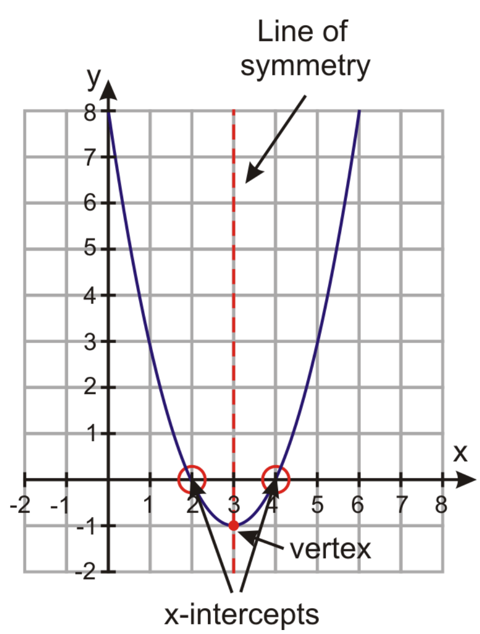
CK-12 Foundation: 1002S Graph Quadratic Functions in Intercept Form

Guidance

Now it's time to learn how to graph a parabola without having to use a table with a large number of points.

Let's look at the graph of $y = x^2 - 6x + 8$.



There are several things we can notice:

- The parabola crosses the $x-$axis at two points: $x = 2$ and $x = 4$. These points are called the $x-$**intercepts** of the parabola.
- The lowest point of the parabola occurs at (3, -1).
  - This point is called the **vertex** of the parabola.
  - The vertex is the lowest point in any parabola that turns upward, or the highest point in any parabola that turns downward.
  - The vertex is ***exactly halfway between the two $x-$intercepts.*** This will always be the case, and you can find the vertex using that property.
- The parabola is **symmetric.** If you draw a vertical line through the vertex, you see that the two halves of the parabola are mirror images of each other. This vertical line is called the **line of symmetry.**

We said that the general form of a quadratic function is $y = ax^2 + bx + c$. When we can factor a quadratic expression, we can rewrite the function in **intercept form:**

$$y = a(x-m)(x-n)$$

This form is very useful because it makes it easy for us to find the $x-$intercepts and the vertex of the parabola. The $x-$intercepts are the values of x where the graph crosses the $x-$axis; in other words, they are the values of x when $y = 0$. To find the $x-$intercepts from the quadratic function, we set $y = 0$ and solve:

$$0 = a(x-m)(x-n)$$

Since the equation is already factored, we use the zero-product property to set each factor equal to zero and solve the individual linear equations:

$$\begin{array}{ccc} x-m=0 & & x-n=0 \\ & \text{or} & \\ x=m & & x=n \end{array}$$

So the $x-$intercepts are at points $(m, 0)$ and $(n, 0)$.

Once we find the $x-$intercepts, it's simple to find the vertex. The $x-$value of the vertex is halfway between the two $x-$intercepts, so we can find it by taking the average of the two values: $\frac{m+n}{2}$. Then we can find the $y-$value by plugging the value of x back into the equation of the function.

Example A

Find the $x-$intercepts and the vertex of the following quadratic functions:

a) $y = x^2 - 8x + 15$

b) $y = 3x^2 + 6x - 24$

Solution

a) $y = x^2 - 8x + 15$

Write the quadratic function in intercept form by factoring the right hand side of the equation. Remember, to factor we need two numbers whose product is 15 and whose sum is –8. These numbers are –5 and –3.

The function in intercept form is $y = (x-5)(x-3)$

We find the $x-$intercepts by setting $y = 0$.

We have:

$$0 = (x-5)(x-3)$$

$$x-5 = 0 \quad \text{or} \quad x-3 = 0$$

$$x = 5 \qquad\qquad x = 3$$

So **the $x-$intercepts are (5, 0) and (3, 0).**

The vertex is halfway between the two $x-$intercepts. We find the $x-$value by taking the average of the two $x-$intercepts: $x = \frac{5+3}{2} = 4$

We find the $y-$value by plugging the $x-$value we just found into the original equation:

$$y = x^2 - 8x + 15 \Rightarrow y = 4^2 - 8(4) + 15 = 16 - 32 + 15 = -1$$

So **the vertex is (4, -1).**

b) $y = 3x^2 + 6x - 24$

Re-write the function in intercept form.

Factor the common term of 3 first: $y = 3(x^2 + 2x - 8)$

Then factor completely: $y = 3(x+4)(x-2)$

Set $y = 0$ and solve:

$$0 = 3(x+4)(x-2) \Rightarrow \quad \begin{array}{c} x+4 = 0 \\ x = -4 \end{array} \quad \text{or} \quad \begin{array}{c} x-2 = 0 \\ x = 2 \end{array}$$

The $x-$intercepts are (-4, 0) and (2, 0).

For the vertex,

$x = \frac{-4+2}{2} = -1$ and $y = 3(-1)^2 + 6(-1) - 24 = 3 - 6 - 24 = -27$

The vertex is: (-1, -27)

Knowing the vertex and $x-$intercepts is a useful first step toward being able to graph quadratic functions more easily. Knowing the vertex tells us where the middle of the parabola is. When making a table of values, we can make sure to pick the vertex as a point in the table. Then we choose just a few smaller and larger values of x. In this way, we get an accurate graph of the quadratic function without having to have too many points in our table.

Example B

Find the $x-$ intercepts and vertex. Use these points to create a table of values and graph each function.

a) $y = x^2 - 4$

b) $y = -x^2 + 14x - 48$

Solution

a) $y = x^2 - 4$

Let's find the $x-$intercepts and the vertex:

Factor the right-hand side of the function to put the equation in intercept form:

$$y = (x-2)(x+2)$$

Set $y = 0$ and solve:

$$0 = (x-2)(x+2)$$

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 2 \quad\quad x = -2$$

The $x-$intercepts are (2, 0) and (-2, 0).

Find the vertex:

$$x = \frac{2-2}{2} = 0 \quad y = (0)^2 - 4 = -4$$

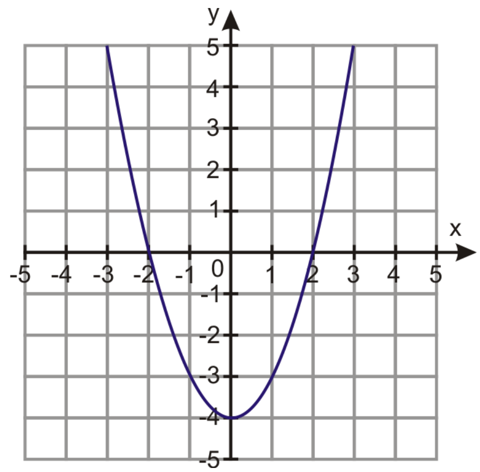
The vertex is (0, -4).

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 0$. Include the $x-$intercepts in the table.

TABLE 1.13:

x	$y = x^2 - 4$	
-3	$y = (-3)^2 - 4 = 5$	
–2	$y = (-2)^2 - 4 = 0$	$x-$intercept
–1	$y = (-1)^2 - 4 = -3$	
0	$y = (0)^2 - 4 = -4$	vertex
1	$y = (1)^2 - 4 = -3$	
2	$y = (2)^2 - 4 = 0$	$x-$intercept
3	$y = (3)^2 - 4 = 5$	

Then plot the graph:



b) $y = -x^2 + 14x - 48$

Let's find the $x-$intercepts and the vertex:

Factor the right-hand-side of the function to put the equation in intercept form:

$$y = -(x^2 - 14x + 48) = -(x-6)(x-8)$$

Set $y = 0$ and solve:

$$0 = -(x-6)(x-8)$$

$$x - 6 = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = 6 \qquad\qquad x = 8$$

The $x-$intercepts are (6, 0) and (8, 0).

Find the vertex:

$$x = \frac{6+8}{2} = 7 \quad y = -(7)^2 + 14(7) - 48 = 1$$

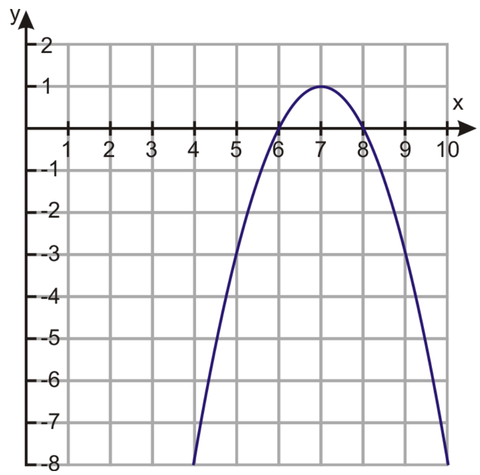
The vertex is (7, 1).

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 7$. Include the $x-$intercepts in the table.

TABLE 1.14:

x	$y = -x^2 + 14x - 48$
4	$y = -(4)^2 + 14(4) - 48 = -8$
5	$y = -(5)^2 + 14(5) - 48 = -3$
6	$y = -(6)^2 + 14(6) - 48 = 0$
7	$y = -(7)^2 + 14(7) - 48 = 1$
8	$y = -(8)^2 + 14(8) - 48 = 0$
9	$y = -(9)^2 + 14(9) - 48 = -3$
10	$y = -(10)^2 + 14(10) - 48 = -8$

Then plot the graph:



Applications of Quadratic Functions to Real-World Problems

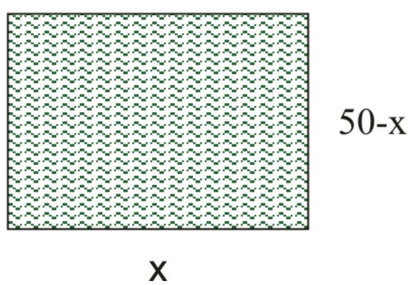
As we mentioned in a previous concept, parabolic curves are common in real-world applications. Here we will look at a few graphs that represent some examples of real-life application of quadratic functions.

Example C

Andrew has 100 feet of fence to enclose a rectangular tomato patch. What should the dimensions of the rectangle be in order for the rectangle to have the greatest possible area?

Solution

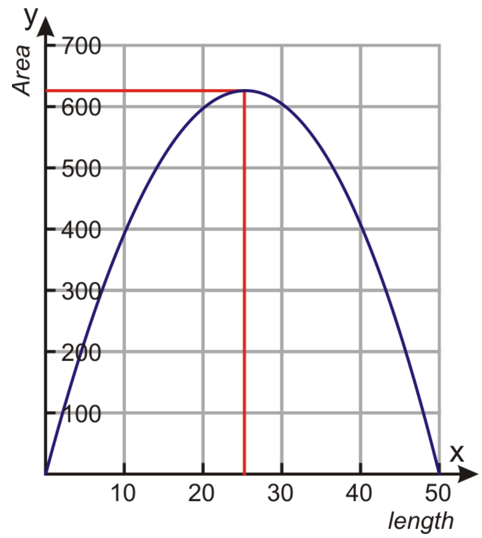
Drawing a picture will help us find an equation to describe this situation:



If the length of the rectangle is x, then the width is $50-x$. (The length and the width add up to 50, not 100, because two lengths and two widths together add up to 100.)

If we let y be the area of the triangle, then we know that the area is length $\times$ width, so $y = x(50-x) = 50x - x^2$.

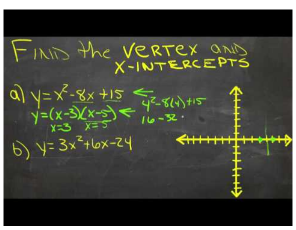
Here's the graph of that function, so we can see how the area of the rectangle depends on the length of the rectangle:



We can see from the graph that the highest value of the area occurs when the length of the rectangle is 25. The area of the rectangle for this side length equals 625. (Notice that the width is also 25, which makes the shape a square with side length 25.)

This is an example of an *optimization problem.* These problems show up often in the real world, and if you ever study calculus, you'll learn how to solve them without graphs.

Watch this video for help with the Examples above.



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CK-12 Foundation: 1002 Graph Quadratic Functions in Intercept Form

Vocabulary

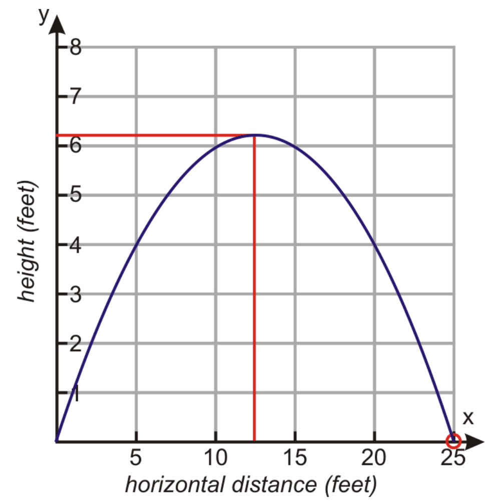
- A parabola can be divided in half by a vertical line. Because of this, parabolas have ***symmetry***. The vertical line dividing the parabola into two equal portions is called the line of ***symmetry***.
- The point where the parabola crosses the $x-$axis is called the $x-$**intercepts** of the parabola.
- All parabolas have a ***vertex***, the ordered pair that represents the bottom (or the top) of the curve. The line of symmetry always goes through the vertex.
  - The vertex is the lowest point in any parabola that turns upward, or the highest point in any parabola that turns downward.
  - The vertex is ***exactly halfway between the two*** $x-$***intercepts.*** This will always be the case, and you can find the vertex using that property.
- If the $x-$intercepts are at points $(m,0)$ and $(n,0)$. The $x-$value of the **vertex** is halfway between the two $x-$intercepts, so we can find it by taking the average of the two values: $\frac{m+n}{2}$. Then we can find the $y-$value by plugging the value of x back into the equation of the function.

Guided Practice

Anne is playing golf. On the 4^{th} tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation $y = x - 0.04x^2$, where y is the ball's height in the air and x is the horizontal distance it has traveled from the tee. The distances are measured in feet. How far from the tee does the ball hit the ground? At what distance from the tee does the ball attain its maximum height? What is the maximum height?

Solution

Let's graph the equation of the path of the ball:



$x(1 - 0.04x) = 0$ has solutions $x = 0$ and $x = 25$.

From the graph, we see that the ball hits the ground **25 feet from the tee.** (The other $x-$intercept, $x = 0$, tells us that the ball was also on the ground when it was on the tee!)

We can also see that the ball reaches its maximum height of **about 6.25 feet** when it is **12.5 feet from the tee.**

Explore More

For 1-4, rewrite the following functions in intercept form. Find the $x-$intercepts and the vertex.

1. $y = x^2 - 2x - 8$
2. $y = -x^2 + 10x - 21$
3. $y = 2x^2 + 6x + 4$
4. $y = 3(x + 5)(x - 2)$

For 5-8, the vertex of which parabola is higher?

5. $y = x^2 + 4$ or $y = x^2 + 1$
6. $y = -2x^2$ or $y = -2x^2 - 2$
7. $y = 3x^2 - 3$ or $y = 3x^2 - 6$
8. $y = 5 - 2x^2$ or $y = 8 - 2x^2$

For 9-14, graph the following functions by making a table of values. Use the vertex and $x-$intercepts to help you pick values for the table.

9. $y = 4x^2 - 4$

10. $y = -x^2 + x + 12$
11. $y = 2x^2 + 10x + 8$
12. $y = \frac{1}{2}x^2 - 2x$
13. $y = x - 2x^2$
14. $y = 4x^2 - 8x + 4$

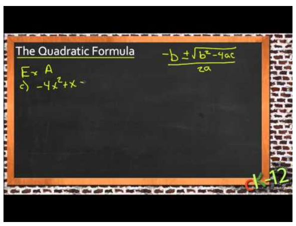
15. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $y = 4 + 2x - 0.16x^2$. Here y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet.
    a. How far from Nadia does the ball hit the ground?
    b. At what distance x from Nadia, does the ball attain its maximum height?
    c. What is the maximum height?
16. Jasreel wants to enclose a vegetable patch with 120 feet of fencing. He wants to put the vegetable against an existing wall, so he only needs fence for three of the sides. The equation for the area is given by $A = 120x - x^2$. From the graph, find what dimensions of the rectangle would give him the greatest area.

8.4 Quadratic Formula

Here you'll learn how to use the quadratic formula to find the vertex and solution of quadratic equations.

What if you had a quadratic equation like x^2+5x+2 that you could not easily factor? How could you use its coefficient values to solve it? After completing this Concept, you'll be able to use the quadratic formula to solve equations like this one.

Watch This



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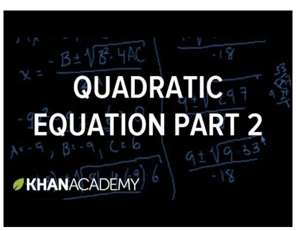
CK-12 Foundation: 1008S The Quadratic Formula

For more examples of solving quadratic equations using the quadratic formula, see the Khan Academy video at



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Guidance

The **Quadratic Formula** is probably the most used method for solving quadratic equations. For a quadratic equation in standard form, $ax^2+bx+c=0$, the quadratic formula looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation: $ax^2+bx+c=0$

Subtract the constant term from both sides: $ax^2+bx=-c$

Divide by the coefficient of the x^2 term:

$x^2+\frac{b}{a}x=-\frac{c}{a}$

Rewrite:

$x^2+2\left(\frac{b}{2a}\right)x=-\frac{c}{a}$

Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides:

$x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$

Factor the perfect square trinomial:

$\left(x+\frac{b}{2a}\right)^2=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}$

Simplify:

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Take the square root of both sides:

$x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{4a^2}}$ and $x+\frac{b}{2a}=-\sqrt{\frac{b^2-4ac}{4a^2}}$

Simplify:

$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a}$ and $x+\frac{b}{2a}=-\frac{\sqrt{b^2-4ac}}{2a}$

$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$ and $x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$

$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$

This can be written more compactly as $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious, so the quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

To use the quadratic formula, just plug in the values of a, b, and c.

Example A

Solve the following quadratic equations using the quadratic formula.

a) $2x^2+3x+1=0$

b) $x^2-6x+5=0$

c) $-4x^2+x+1=0$

Solution

Start with the quadratic formula and plug in the values of a, b and c.

a)

Quadratic formula:	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=2,\ b=3,\ c=1$	$x=\dfrac{-3\pm\sqrt{(3)^2-4(2)(1)}}{2(2)}$
Simplify:	$x=\dfrac{-3\pm\sqrt{9-8}}{4}=\dfrac{-3\pm\sqrt{1}}{4}$
Separate the two options:	$x=\dfrac{-3+1}{4}$ and $x=\dfrac{-3-1}{4}$
Solve:	$x=\dfrac{-2}{4}=-\dfrac{1}{2}$ and $x=\dfrac{-4}{4}=-1$

Answer: $x=-\frac{1}{2}$ and $x=-1$

b)

Quadratic formula:	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=1,\ b=-6,\ c=5$	$x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(1)(5)}}{2(1)}$
Simplify:	$x=\dfrac{6\pm\sqrt{36-20}}{2}=\dfrac{6\pm\sqrt{16}}{2}$
Separate the two options:	$x=\dfrac{6+4}{2}$ and $x=\dfrac{6-4}{2}$
Solve:	$x=\dfrac{10}{2}=5$ and $x=\dfrac{2}{2}=1$

Answer: $x=5$ and $x=1$

c)

Quadratic formula:	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=-4,\ b=1,\ c=1$	$x=\dfrac{-1\pm\sqrt{(1)^2-4(-4)(1)}}{2(-4)}$
Simplify:	$x=\dfrac{-1\pm\sqrt{1+16}}{-8}=\dfrac{-1\pm\sqrt{17}}{-8}$
Separate the two options:	$x=\dfrac{-1+\sqrt{17}}{-8}$ and $x=\dfrac{-1-\sqrt{17}}{-8}$
Solve:	$x=-.39$ and $x=.64$

Answer: $x=-.39$ and $x=.64$

Often when we plug the values of the coefficients into the quadratic formula, we end up with a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced math classes, you'll learn how to work with "complex" (or "imaginary") solutions to quadratic equations.

Example B

Use the quadratic formula to solve the equation $x^2+2x+7=0$.

Solution

$$\text{Quadratic formula:} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = 1,\ b = 2,\ c = 7 \qquad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(7)}}{2(1)}$$

$$\text{Simplify:} \qquad x = \frac{-2 \pm \sqrt{4 - 28}}{2} = \frac{-2 \pm \sqrt{-24}}{2}$$

Answer: There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, that means we have to start by rewriting the equation.

Finding the Vertex of a Parabola with the Quadratic Formula

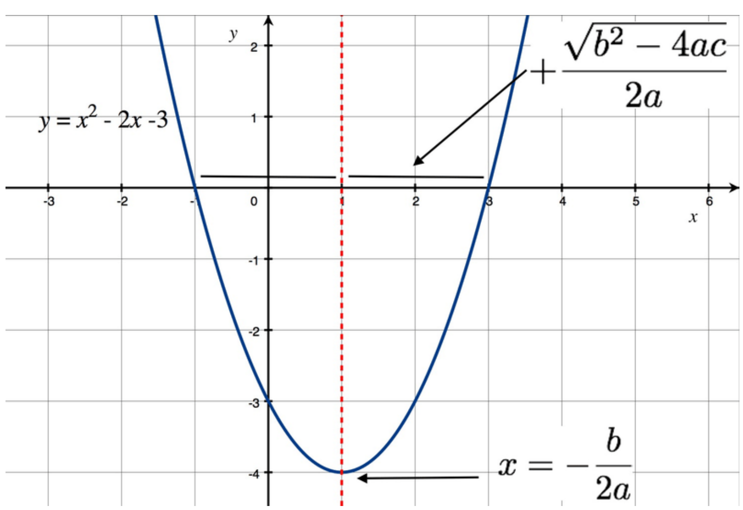
Sometimes a formula gives you even more information than you were looking for. For example, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

Remember that the quadratic formula tells us the **roots** or **solutions** of the equation $ax^2 + bx + c = 0$. Those roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,, and we can rewrite that as $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.

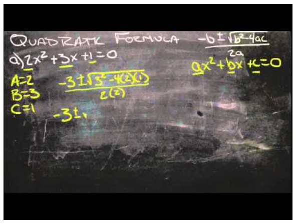
Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2 - 4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x = \frac{-b}{2a}$.

Example C

In the equation $x^2 - 2x - 3 = 0$, the roots -1 and 3 are both 2 units from the vertical line $x = 1$, as you can see in the graph below:



Watch this video for help with the Examples above.



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CK-12 Foundation: 1008 The Quadratic Formula

Vocabulary

- For a **quadratic equation** in standard form, $ax^2+bx+c=0$, the **quadratic formula** looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- The quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and we can rewrite that as $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$.
- The roots are symmetric about the **vertex**. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x=\frac{-b}{2a}$.

Guided Practice

Solve the following equations using the quadratic formula.

a) $x^2-6x=10$

b) $-8x^2=5x+6$

Solution

a)

Re-write the equation in standard form:	$x^2-6x-10=0$
Quadratic formula:	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=1,\ b=-6,\ c=-10$	$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-10)}}{2(1)}$
Simplify:	$x=\frac{6\pm\sqrt{36+40}}{2}=\frac{6\pm\sqrt{76}}{2}$
Separate the two options:	$x=\frac{6+\sqrt{76}}{2}$ and $x=\frac{6-\sqrt{76}}{2}$
Solve:	$x=7.36$ and $x=-1.36$

Answer: $x=7.36$ and $x=-1.36$

b)

Re-write the equation in standard form:	$8x^2+5x+6=0$
Quadratic formula:	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Plug in the values $a=8,\ b=5,\ c=6$	$x=\dfrac{-5\pm\sqrt{(5)^2-4(8)(6)}}{2(8)}$
Simplify:	$x=\dfrac{-5\pm\sqrt{25-192}}{16}=\dfrac{-5\pm\sqrt{-167}}{16}$

Answer: no real solutions

Explore More

Solve the following quadratic equations using the quadratic formula.

1. $x^2+4x-21=0$
2. $x^2-6x=12$
3. $3x^2-\frac{1}{2}x=\frac{3}{8}$
4. $2x^2+x-3=0$
5. $-x^2-7x+12=0$
6. $-3x^2+5x=2$
7. $4x^2=x$
8. $x^2+2x+6=0$
9. $5x^2-2x+100=0$
10. $100x^2+10x+70=0$